HAESE MATHEMATICS

Mathematics

Applications and interpretation HL 2



Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo

for use with

IB Diploma Programme

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Specialists in mathematics education

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MATHEMATICS: APPLICATIONS AND INTERPRETATION HL

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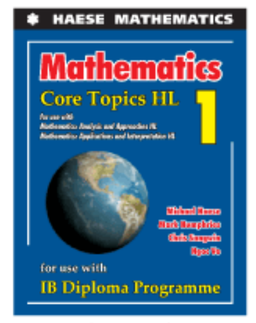
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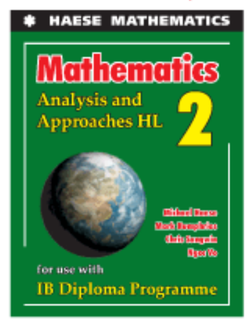
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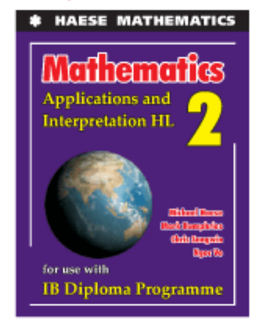
FOREWORD

This book has been written for the International Baccalaureate Diploma Programme course *Mathematics: Applications and Interpretation HL*, for first teaching in August 2019, and first assessment in May 2021.

HL Mathematics







This book is designed to complete the course in conjunction with the **Mathematics: Core Topics HL** textbook. It is expected that students will start using this book approximately 6-7 months into the two-year course, upon the completion of the Mathematics: Core Topics HL textbook.

The *Mathematics: Applications and Interpretation* courses have a focus on technology, and the book has been written with this focus in mind. Where appropriate, graphics calculator screenshots and instructions have been provided to help students use technology to solve problems. An algebraic approach to solving the problem may be included for completeness, and to help students enhance their understanding. The material is presented in a clear, easy-to-follow style, free from unnecessary distractions, while effort has been made to contextualise questions so that students can relate concepts to everyday use.

Each chapter begins with an Opening Problem, offering an insight into the application of the mathematics that will be studied in the chapter. Important information and key notes are highlighted, while worked examples provide step-by-step instructions with concise and relevant explanations. Discussions, Activities, and Investigations are used throughout the chapters to develop understanding, problem solving, and reasoning. A large number of more difficult questions have been included to challenge the HL student.

Time constraints may not allow for each chapter to be worked through in full. Teachers must select exercises carefully, according to the abilities and prior knowledge of their students, to make the most efficient use of time and give as thorough coverage of work as possible.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

Email: info@haesemathematics.com Web: www.haesemathematics.com

PMH, MAH, CS, NV

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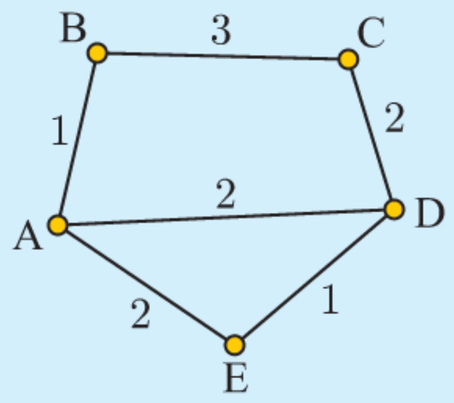
SELF TUTOR

Simply 'click' on the Self Tutor (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Example 6 Self Tutor

Solve the Chinese Postman Problem for this weighted graph.



The vertices A and D have odd degree, so the graph is semi-Eulerian.

We need to walk twice between these vertices. The most efficient way to achieve this is to traverse the edge AD twice.

A route that traverses every edge at least once will have weight equal to the sum of the weights of all the edges, plus the weight of edge AD. The total weight is $11 + 2 = 13$.

An example solution is $A \to B \to C \to D \to A \to E \to D \to A$.

See **Chapter 15, Graph theory**, p. 404

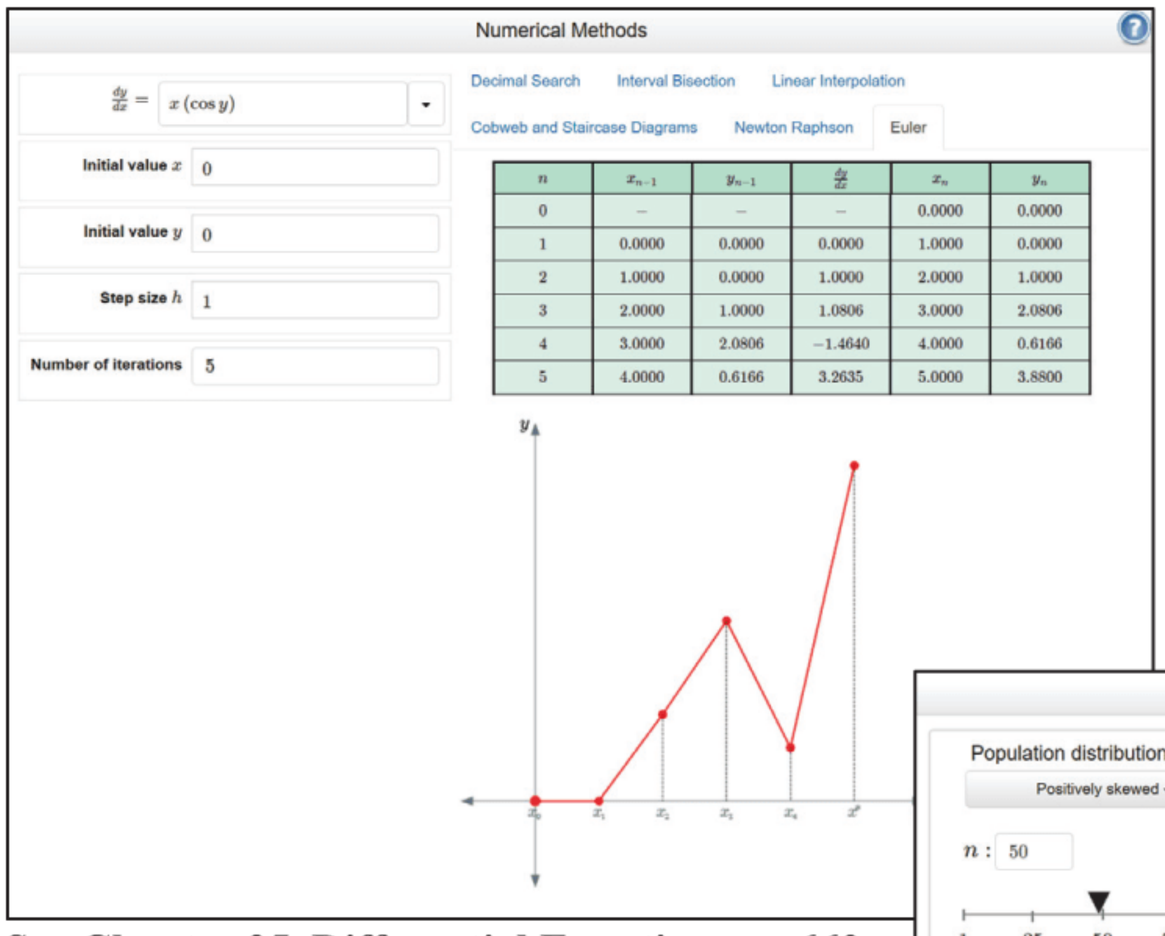
INTERACTIVE LINKS

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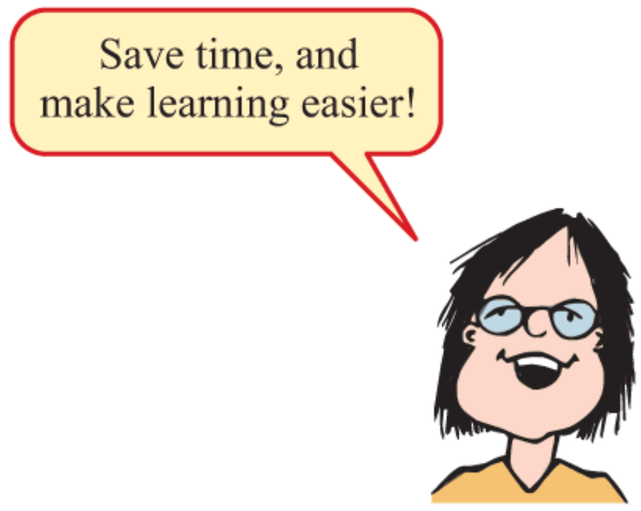
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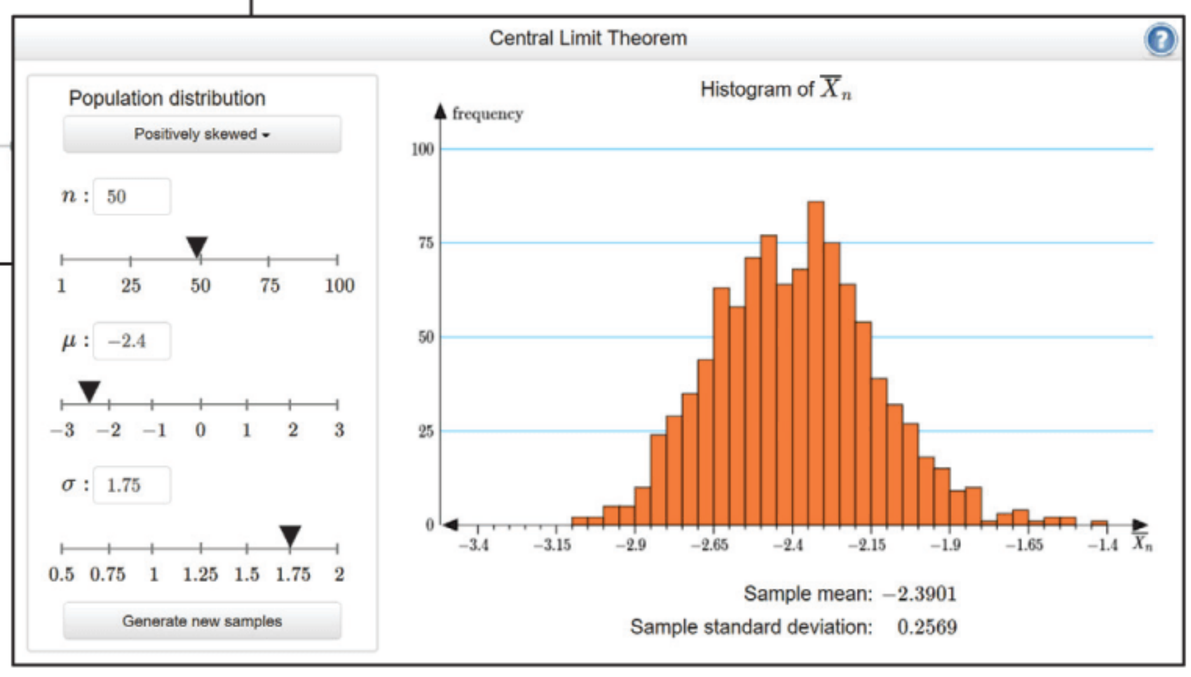


- interactive demonstrations to illustrate and animate concepts
- games and other tools for practising your skills
- graphing and statistics packages which are fast, powerful alternatives to using a graphics calculator
- printable pages to save class time.



See **Chapter 25**, **Differential Equations**, p. 663





See **Chapter 29**, **Estimation and Confidence Intervals**, p. 762

GRAPHICS CALCULATOR INSTRUCTIONS

Graphics calculator instruction booklets are available for the **Casio fx-CG50**, **TI-84 Plus CE**, **TI-*n*spire**, and the **HP Prime**. Click on the relevant icon below.









When additional calculator help may be needed, specific instructions are available from icons within the text.

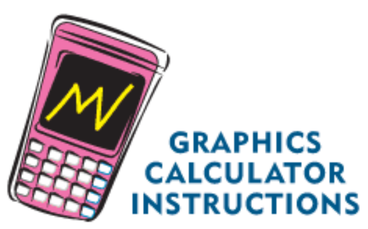


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SYMBOLS AND NOTATION USED IN THIS COURSE

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}'$	the set of irrational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers $\{a + bi \mid a, b \in \mathbb{R}\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\lvert z \rvert$	the modulus of z
$\arg z$	the argument of z
$\mathcal{Re}(z)$	the real part of z
$\mathcal{Im}(z)$	the imaginary part of z
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$ or $\{\ \}$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$a^{\frac{1}{n}}, \sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a
$a^{\frac{1}{2}}, \sqrt{a}$	a to the power $\frac{1}{2}$, square root of a
$\lvert x \rvert$	the modulus or absolute value of x, that is $\begin{cases} x & \text{for } x \geqslant 0, \quad x \in \mathbb{R} \\ -x & \text{for } x < 0, \quad x \in \mathbb{R} \end{cases}$
$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$[a, b]$	the closed interval $a \leqslant x \leqslant b$
$]a, b[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence

S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞ or S	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$n!$	$n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1$
$\binom{n}{r}$ or ${}^nC_r$	the rth binomial coefficient, $r = 0, 1, 2, ...$ in the expansion of $(a+b)^n$
$f : x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\frac{dy}{dx}$	the derivative of y with respect to x
$\frac{d}{dx}(....)$	the derivative of .... with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\frac{d^2y}{dx^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\frac{d^ny}{dx^n}$	the nth derivative of y with respect to x
$f^{(n)}(x)$	the nth derivative of $f(x)$ with respect to x
$\dot{x}$	the first derivative of $x(t)$ with respect to time (t)
$\ddot{x}$	the second derivative of $x(t)$ with respect to time (t)
$\int y\, dx$	the indefinite integral of y with respect to x
$\int_a^b y\, dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log_a x$	the logarithm in base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
$\sin, \cos, \tan$	the circular functions
$\sin^{-1}, \cos^{-1}, \tan^{-1}$	the inverse circular functions
$\operatorname{cis}\theta$	$\cos\theta + i\sin\theta$

$\mathrm{A}(x, y)$	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
PB(A, B)	the perpendicular bisector of [AB]
$\widehat{A}$	the angle at A
CAB	the angle between [CA] and [AB]
$\triangle$ABC	the triangle whose vertices are A, B, and C
$\parallel$	is parallel to
$\perp$	is perpendicular to
$\mathbf{v}$	the vector $\mathbf{v}$
$\overrightarrow{\mathrm{AB}}$	the displacement vector represented in magnitude and direction by the directed line segment from A to B
$\mathbf{a}$	the position vector $\overrightarrow{\mathrm{OA}}$
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert\,\mathbf{a}\,\rvert$	the magnitude of vector $\mathbf{a}$
$\lvert\,\overrightarrow{\mathrm{AB}}\,\rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\mathbf{a} \bullet \mathbf{b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \times \mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$
a_{ij}	the element in the ith row and jth column of matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	the inverse of the non-singular matrix $\mathbf{A}$
$\det \mathbf{A}$, $\lvert\,\mathbf{A}\,\rvert$	the determinant of the square matrix $\mathbf{A}$
$\mathbf{I}$	the identity matrix
$\mathbf{O}$	the zero matrix
$\mathbf{A}_G$	the adjacency matrix of a graph G
k_n	a complete graph with n vertices
$p(\lambda) = \lvert\,\lambda\mathbf{I} - \mathbf{A}\,\rvert$	the characteristic polynomial of matrix $\mathbf{A}$
$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$	the diagonalisation of matrix $\mathbf{A}$
$\mathbf{T}$	a transition matrix
$\mathbf{s}_0$	the initial state matrix
$\mathbf{s}_n$	the state matrix at time n
$\mathbf{s}$	the steady state matrix
$\mathrm{P}(A)$	probability of event A
$\mathrm{P}(A')$	probability of the event "not A"
$\mathrm{P}(A \mid B)$	probability of the event A given B
$x_1, x_2, \ldots$	observations of a variable
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, x_3, \ldots$ occur
$p_1, p_2, \ldots$	probabilities with which the observations $x_1, x_2, x_3, \ldots$ occur

$\mathrm{P}(X = x)$, $P(x)$	probability mass function of the discrete random variable X
$\mathrm{E}(X)$	the expected value of the random variable X
$\mathrm{Var}\,(X)$	the variance of the random variable X
μ	population mean
σ	population standard deviation
σ^2	population variance
$\overline{x}$	sample mean
s^2	sample variance
s	sample standard deviation
$\overline{X}_n$	the sample mean random variable for a sample of size n
S^2_{n-1}	the sample variance random variable for a sample of size n
$\mathrm{B}(n, p)$	binomial distribution with n trials and probability of success p
$\mathrm{Po}(\lambda)$	Poisson distribution with rate λ
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$\sim$	is distributed as
z	z-score, $z = \dfrac{x - \mu}{\sigma}$
r	Pearson's product-moment correlation coefficient
ρ	the population product-moment correlation coefficient
H_0	the null hypothesis
H_1	the alternative hypothesis
$\mathcal{C}$	the critical region of a hypothesis test
c	the critical value of a hypothesis test
$\mathcal{A}$	acceptance region of a hypothesis test
α	significance level of a hypothesis test
β	probability of a Type II error
df, ν	degrees of freedom
t_{n-1}	Student's t-distribution with $n - 1$ degrees of freedom
$a_1X_1 + \ldots + a_nX_n$	a linear combination of random variables
z_α	$\mathrm{P}(Z \geqslant z_\alpha) = \alpha$ where $Z \sim \mathrm{N}(0, 1^2)$
$t_{\alpha,\,\nu}$	$\mathrm{P}(T \geqslant t_{\alpha,\,\nu}) = \alpha$ where $T \sim t_\nu$
χ^2_{calc}	calculated chi-squared value
χ^2_{crit}	critical value of the chi-squared distribution
f_{obs}	observed frequency
f_{exp}	expected frequency

THEORY OF KNOWLEDGE

Theory of Knowledge is a Core requirement in the International Baccalaureate Diploma Programme.

Students are encouraged to think critically and challenge the assumptions of knowledge. Students should be able to analyse different ways of knowing and areas of knowledge, while considering different cultural and emotional perceptions, fostering an international understanding.

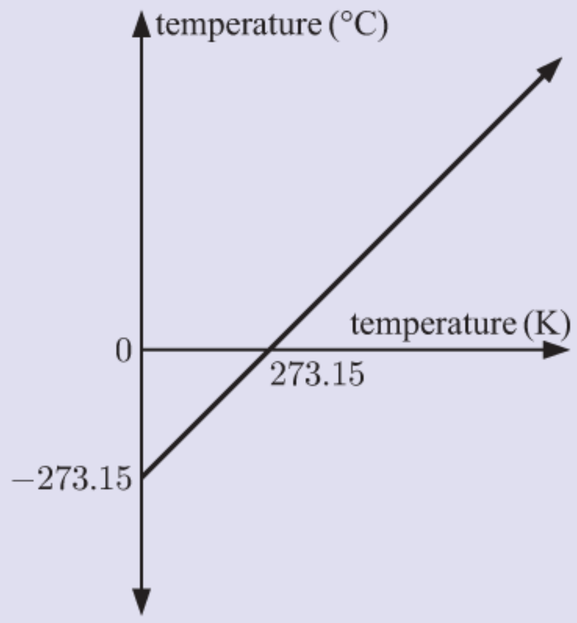
The activities and discussion topics in the below table aim to help students discover and express their views on knowledge issues.

Chapter 2: Logarithms p. 55	**IS MATHEMATICS AN INVENTION OR A DISCOVERY?**
Chapter 5: Modelling p. 97	**MOVEMENT OF THE PLANETS**
Chapter 6: Direct and inverse variation p. 121	**MEASURES OF TEMPERATURE**
Chapter 7: Bivariate statistics p. 158	**MATHEMATICAL EXTRAPOLATION**
Chapter 10: Vector applications p. 260	**INDEPENDENT DEVELOPMENT OF MATHEMATICS**
Chapter 15: Graph theory p. 384	**HENRY BECK'S LONDON UNDERGROUND MAP**
Chapter 17: Introduction to differential calculus p. 457	**ZENO'S PARADOXES**
Chapter 20: Applications of differentiation p. 537	**OPTIMISATION**
Chapter 25: Differential equations p. 665	**NUMERICAL METHODS**
Chapter 29: Estimation and confidence intervals p. 767	**THE CENTRAL LIMIT THEOREM**
Chapter 30: Hypothesis testing p. 792	**SIGNIFICANCE LEVELS IN RESEARCH**
Chapter 31: χ^2 Hypothesis tests p. 838	**STATISTICAL FALLACIES**

THEORY OF KNOWLEDGE

In 1848, **William Thomson** (1824 - 1907), also known as **Lord Kelvin**, proposed the need for a temperature scale starting at *absolute zero*. His idea stemmed from research showing a proportional relationship between the kinetic energy of a system and its temperature. By extrapolating his results to a point where the kinetic energy of a system was zero, Thomson was able to predict *absolute zero* as about $-273°C$.

The SI unit for temperature is the kelvin (K), named in Thomson's honour. *Absolute zero* is regarded as 0 kelvin, and is defined as $-273.15°C$. An increase of 1 kelvin corresponds to an increase of $1°C$, so $0°C$ is equivalent to 273.15 K, and $100°C$ is equivalent to 373.15 K.



1. In what ways is it useful to use variables in direct proportion?
2. Which measure of temperature is most convenient?
3. What is the most *natural* measure of temperature?

See **Chapter 6**, **Direct and inverse variation**, p. 121

Chapter 1

Exponentials

Contents:

A Rational exponents
B Algebraic expansion and factorisation
C Exponential functions
D Graphing exponential functions from a table of values
E Graphs of exponential functions
F Exponential equations
G Growth and decay
H The natural exponential
I The logistic model



OPENING PROBLEM

At an antiques fair, Bernard purchases a clock for £500 and a vase for £400. The clock increases in value by 5% each year, and the vase increases in value by 7% each year.



Things to think about:

a What is the value of each item 1 year after purchase?

b Can you write a formula for the value of each item t years after purchase?

c Which item is more valuable 15 years after purchase?

d How can we determine when the items are equal in value?

We have seen previously how exponents are used to indicate when a number is raised to a power.

For a positive integer exponent, the exponent tells us how many of the base are multiplied together.

Any non-zero base to the power 0 is defined as 1, to give consistency to the exponent laws.

For a negative integer exponent, we take the reciprocal of the corresponding positive integer power.

$$\begin{cases} 2^3 = 2 \times 2 \times 2 & = 8 \\ 2^2 = 2 \times 2 & = 4 \\ 2^1 = 2 & = 2 \end{cases}$$

$$2^0 = 1 \qquad = 1$$

$$\begin{cases} 2^{-1} = \frac{1}{2} & = \frac{1}{2} \\ 2^{-2} = \frac{1}{2 \times 2} & = \frac{1}{4} \\ 2^{-3} = \frac{1}{2 \times 2 \times 2} & = \frac{1}{8} \end{cases}$$

In this Chapter we give meaning to exponents which are **rational**, allowing us to start filling in the gaps between the integer exponents. This will allow us to consider **exponential functions** for which the variable appears in an exponent.

A RATIONAL EXPONENTS

The **laws of exponents** used previously can also be applied to **rational exponents**, or exponents $\in \mathbb{Q}$.

For any $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {exponent laws}

and $\sqrt{a} \times \sqrt{a} = a$ also.

Likewise, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$

and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$ also.

By direct comparison, we conclude that $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ and $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$.

In general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads "the nth root of a" for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$

$= a^{\frac{m}{n}}$

$\therefore \quad \boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \text{for} \quad a > 0, \ n \in \mathbb{Z}^+, \ m \in \mathbb{Z}}$

Example 1 **Self Tutor**

Write as a single power of 2:

a $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2} = 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}} = \dfrac{1}{2^{\frac{1}{2}}} = 2^{-\frac{1}{2}}$

c $\sqrt[5]{4} = (2^2)^{\frac{1}{5}} = 2^{2\times\frac{1}{5}} = 2^{\frac{2}{5}}$

EXERCISE 1A

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

3 Write in the form a^k, where a is a prime number and k is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

4 Write in the form x^k, where k is rational:

a $\sqrt{x}$ **b** $x\sqrt{x}$ **c** $\dfrac{1}{\sqrt{x}}$ **d** $x^2\sqrt{x}$ **e** $\dfrac{1}{x\sqrt{x}}$

5 Use your calculator to find, correct to 3 significant figures:

a $3^{\frac{3}{4}}$ **b** $4^{-\frac{3}{5}}$ **c** $\sqrt[4]{8}$ **d** $\sqrt[5]{27}$ **e** $\dfrac{1}{\sqrt[3]{7}}$



GRAPHICS CALCULATOR INSTRUCTIONS

6 Write *without* rational exponents:

a $5^{\frac{1}{3}}$ **b** $3^{-\frac{1}{2}}$ **c** $3^{\frac{5}{2}}$ **d** $m^{\frac{3}{2}}$ **e** $x^{\frac{7}{2}}$

Example 2 — Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a
$$\begin{aligned} & 8^{\frac{4}{3}} \\ &= (2^3)^{\frac{4}{3}} \\ &= 2^{3\times\frac{4}{3}} \qquad \{(a^m)^n = a^{mn}\} \\ &= 2^4 \\ &= 16 \end{aligned}$$

b
$$\begin{aligned} & 27^{-\frac{2}{3}} \\ &= (3^3)^{-\frac{2}{3}} \\ &= 3^{3\times(-\frac{2}{3})} \\ &= 3^{-2} \\ &= \tfrac{1}{9} \end{aligned}$$

7 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

B ALGEBRAIC EXPANSION AND FACTORISATION

We can use the standard rules of algebra, together with the laws of exponents, to simplify expressions containing rational or variable exponents:

$$\begin{aligned} a(b+c) &= ab + ac \\ (a+b)(c+d) &= ac + ad + bc + bd \\ (a+b)(a-b) &= a^2 - b^2 \\ (a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2 \end{aligned}$$

Example 3 — Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$

$$\begin{aligned} & x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}) \\ &= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} + x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times 3x^{-\frac{1}{2}} && \{\text{each term is multiplied by } x^{-\frac{1}{2}}\} \\ &= x^1 + 2x^0 - 3x^{-1} && \{\text{adding indices}\} \\ &= x + 2 - \frac{3}{x} \end{aligned}$$

EXERCISE 1B

1 Simplify:

a $x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$ **b** $x^{\frac{3}{2}} \times x^{-\frac{1}{2}}$ **c** $x^2 \times x^{-\frac{3}{2}}$

2 Expand and simplify:

a $x^2(x^3 + 2x^2 + 1)$ **b** $2^x(2^x + 1)$ **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$

d $7^x(7^x + 2)$ **e** $3^x(2 - 3^{-x})$ **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + 3x^{-\frac{1}{2}})$

g $2^{-x}(2^x + 5)$ **h** $5^{-x}(5^{2x} + 5^x)$ **i** $x^{-\frac{1}{2}}(x^2 + x + x^{\frac{1}{2}})$

j $3^x(3^x + 5 + 3^{-x})$ **k** $x^{-\frac{1}{2}}(2x^2 - x + 5x^{\frac{1}{2}})$ **l** $2^{2x}(2^x - 3 - 2^{-2x})$

Example 4 Self Tutor

Expand and simplify:

a $(2^x + 3)(2^x + 1)$ **b** $(7^x + 7^{-x})^2$

a
$$\begin{aligned}&(2^x + 3)(2^x + 1)\\ &= 2^x \times 2^x + 2^x + 3 \times 2^x + 3\\ &= 2^{2x} + 4 \times 2^x + 3\end{aligned}$$

b
$$\begin{aligned}&(7^x + 7^{-x})^2\\ &= (7^x)^2 + 2 \times 7^x \times 7^{-x} + (7^{-x})^2\\ &= 7^{2x} + 2 \times 7^0 + 7^{-2x}\\ &= 7^{2x} + 2 + 7^{-2x}\end{aligned}$$

3 Expand and simplify:

a $(2^x - 1)(2^x + 3)$ **b** $(3^x + 2)(3^x + 5)$ **c** $(5^x - 2)(5^x - 4)$

d $(2^x + 3)^2$ **e** $(3^x - 1)^2$ **f** $(4^x + 7)^2$

g $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$ **h** $(2^x + 3)(2^x - 3)$ **i** $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$

j $\left(x + \frac{2}{x}\right)^2$ **k** $(7^x - 7^{-x})^2$ **l** $(5 - 2^{-x})^2$

Example 5 Self Tutor

Factorise:

a $2^{n+3} + 2^n$ **b** $2^{n+3} + 8$ **c** $2^{3n} + 2^{2n}$

a
$$\begin{aligned}&2^{n+3} + 2^n\\ &= 2^n 2^3 + 2^n\\ &= 2^n(2^3 + 1)\\ &= 2^n \times 9\end{aligned}$$

b
$$\begin{aligned}&2^{n+3} + 8\\ &= 2^n 2^3 + 8\\ &= 8(2^n) + 8\\ &= 8(2^n + 1)\end{aligned}$$

c
$$\begin{aligned}&2^{3n} + 2^{2n}\\ &= 2^{2n} 2^n + 2^{2n}\\ &= 2^{2n}(2^n + 1)\end{aligned}$$

4 Factorise:

a $5^{2x} + 5^x$ **b** $3^{n+2} + 3^n$ **c** $7^n + 7^{3n}$

d $5^{n+1} - 5$ **e** $6^{n+2} - 6$ **f** $4^{n+2} - 16$

g $2^{2n} - 2^{n+3}$ **h** $2^{n+1} + 2^{n-1}$ **i** $4^{n+1} + 2^{2n-1}$

Example 6 — Self Tutor

Factorise:

a $4^x - 9$ **b** $9^x + 4(3^x) + 4$

a
$$\begin{aligned} & 4^x - 9 \\ &= (2^x)^2 - 3^2 && \{\text{compare } a^2 - b^2 = (a+b)(a-b)\} \\ &= (2^x + 3)(2^x - 3) \end{aligned}$$

b
$$\begin{aligned} & 9^x + 4(3^x) + 4 \\ &= (3^x)^2 + 4(3^x) + 4 && \{\text{compare } a^2 + 4a + 4\} \\ &= (3^x + 2)^2 && \{\text{as } a^2 + 4a + 4 = (a+2)^2\} \end{aligned}$$

5 Factorise:

a $9^x - 4$ **b** $4^x - 25$ **c** $16 - 9^x$

d $25 - 4^x$ **e** $9^x - 4^x$ **f** $4^x + 6(2^x) + 9$

g $9^x + 10(3^x) + 25$ **h** $4^x - 14(2^x) + 49$ **i** $25^x - 4(5^x) + 4$

6 Factorise:

a $(2^x)^2 - 2^x - 2$ **b** $(3^x)^2 + 3^x - 6$ **c** $4^x - 7(2^x) + 12$

d $4^x + 9(2^x) + 18$ **e** $4^x - 2^x - 20$ **f** $9^x + 9(3^x) + 14$

g $9^x + 4(3^x) - 5$ **h** $25^x + 5^x - 2$ **i** $49^x - 7^{x+1} + 12$

Example 7 — Self Tutor

Simplify:

a $\dfrac{6^n}{3^n}$ **b** $\dfrac{4^n}{6^n}$

a
$$\begin{aligned} & \frac{6^n}{3^n} \quad \textbf{or} \quad \frac{6^n}{3^n} \\ &= \frac{2^n \cancel{3^n}}{\cancel{3^n}_1} && = \left(\frac{6}{3}\right)^n \\ &= 2^n && = 2^n \end{aligned}$$

b
$$\begin{aligned} & \frac{4^n}{6^n} \quad \textbf{or} \quad \frac{4^n}{6^n} \\ &= \frac{\cancel{2^n} 2^n}{\cancel{2^n} 3^n} && = \left(\frac{4}{6}\right)^n \\ &= \frac{2^n}{3^n} && = \left(\frac{2}{3}\right)^n \end{aligned}$$

7 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$

d $\dfrac{4^n}{20^n}$ **e** $\dfrac{35^x}{7^x}$ **f** $\dfrac{6^a}{8^a}$

g $\dfrac{24^k}{9^k}$ **h** $\dfrac{5^{n+1}}{5^n}$ **i** $\dfrac{5^{n+1}}{5}$

Example 8 — Self Tutor

Simplify:

a $\dfrac{3^n + 6^n}{3^n}$ **b** $\dfrac{2^{m+2} - 2^m}{2^m}$ **c** $\dfrac{2^{m+3} + 2^m}{9}$

a
$$\frac{3^n + 6^n}{3^n} = \frac{3^n + 2^n 3^n}{3^n} = \frac{\cancel{3^n}(1 + 2^n)}{\cancel{3^n}_1} = 1 + 2^n$$

b
$$\frac{2^{m+2} - 2^m}{2^m} = \frac{2^m 2^2 - 2^m}{2^m} = \frac{\cancel{2^m}(4 - 1)}{\cancel{2^m}_1} = 3$$

c
$$\frac{2^{m+3} + 2^m}{9} = \frac{2^m 2^3 + 2^m}{9} = \frac{2^m(8 + 1)}{\cancel{9}_1} = 2^m$$

8 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$ **b** $\dfrac{2^n + 12^n}{2^n}$ **c** $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$ **e** $\dfrac{6^n + 12^n}{1 + 2^n}$ **f** $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$ **h** $\dfrac{4^n - 2^n}{2^n}$ **i** $\dfrac{2^n - 2^{n-1}}{2^n}$

9 Simplify:

a $2^n(n + 1) + 2^n(n - 1)$ **b** $3^n\left(\dfrac{n-1}{6}\right) - 3^n\left(\dfrac{n+1}{6}\right)$

C EXPONENTIAL FUNCTIONS

An **exponential function** is a function in which the variable appears in the exponent.

For example, $f(x) = 5^x$ and $g(x) = 3 \times 2^{-x} + 1$ are exponential functions.

Example 9 — Self Tutor

For the function $f(x) = 3^x + 5$, find:

a $f(6)$ **b** $f(0)$ **c** $f(-2)$

a $f(6) = 3^6 + 5 = 729 + 5 = 734$

b $f(0) = 3^0 + 5 = 1 + 5 = 6$

c $f(-2) = 3^{-2} + 5 = \dfrac{1}{3^2} + 5 = 5\frac{1}{9}$

EXERCISE 1C

1 Decide whether each function is an exponential function:

a $y = x^3 - 2$ **b** $f(x) = 7 - 2^{-x}$ **c** $g(x) = \sqrt{x} + 5$ **d** $f(x) = -3 \times 5^{\frac{x}{2}}$

2 If $f(x) = 2^x - 3$, find:

a $f(2)$ **b** $f(1)$ **c** $f(0)$ **d** $f(-1)$ **e** $f(-2)$

3 If $f(x) = 5 \times 3^x$, find:

a $f(1)$ **b** $f(3)$ **c** $f(0)$ **d** $f(-4)$ **e** $f(-1)$

4 If $f(x) = 2 \times 2^x$, find:

a $f(4)$ **b** $f(0)$ **c** $f(1)$ **d** $f(-1)$ **e** $f(-5)$

5 If $g(x) = 5^{-x}$, find:

a $g(1)$ **b** $g(3)$ **c** $g(0)$ **d** $g(-2)$ **e** $g(-3)$

6 If $h(x) = 3 \times (1.1)^x$, use your calculator to evaluate:

a $h(0)$ **b** $h(1)$ **c** $h(5)$ **d** $h(-2)$ **e** $h(3.8)$

7 Determine whether the given point satisfies the exponential function:

a $y = 2^x + 1$ $(3, 9)$ **b** $f(x) = 5^{2x}$ $(1, 5)$

c $f(x) = 3^{-x} - 2$ $(0, -1)$ **d** $y = 6 \times 2^x$ $(-1, 3)$

e $f(x) = -4 \times 3^x + 1$ $(2, -13)$ **f** $y = 4^{-3x} + 2$ $(-1, 66)$

8 For the function $f(x) = 3^x - 1$, show that the axes intercepts are both zero.

9 For the function $f(x) = 2^{-x} - 8$, show that:

a the y-intercept is -7 **b** the x-intercept is -3.

D GRAPHING EXPONENTIAL FUNCTIONS FROM A TABLE OF VALUES

We can construct a table of values to help graph exponential functions.

For example, the table below gives values of $y = 2^x$.

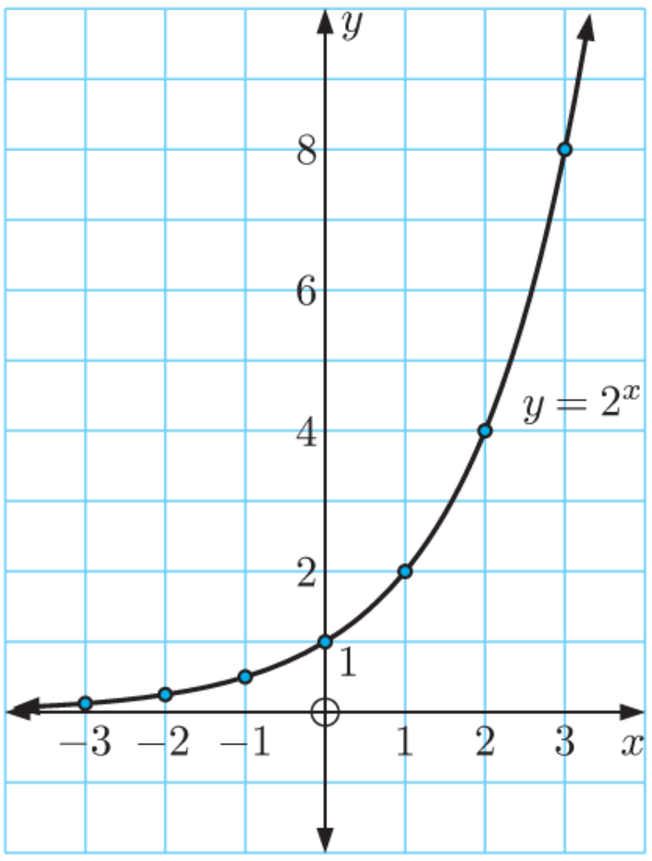
x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Drawing a smooth curve through the points, we see that as x becomes large and negative, the graph approaches the x-axis.

For example: when $x = -10$, $y = 2^{-10} \approx 0.001$

when $x = -50$, $y = 2^{-50} \approx 8.88 \times 10^{-16}$.

We conclude that $y = 0$ is a **horizontal asymptote** of the function.



We write "as $x \to -\infty$, $y \to 0$" to mean "as x approaches minus infinity, y approaches zero".

We also notice that as x becomes large and positive, y also becomes large and positive. We write "as $x \to \infty$, $y \to \infty$".

Drawing a smooth curve through the points also gives meaning to 2^x for irrational values of x.

EXERCISE 1D

1 Consider the exponential function $y = 4^x$.

x	-3	-2	-1	0	1	2	3
y			$\frac{1}{4}$	1	4		

a Copy and complete the table of values.

b Complete the following statements:

i If x is increased by 1, the value of y is .....

ii If x is decreased by 1, the value of y is .....

c Use your table of values to draw the graph of $y = 4^x$.

d Copy and complete:

i As $x \to \infty$, $y \to$ ..... **ii** As $x \to -\infty$, $y \to$ .....

e Find the horizontal asymptote of $y = 4^x$.

2 Consider the exponential function $y = \left(\frac{1}{3}\right)^x$.

x	-3	-2	-1	0	1	2	3
y							

a Copy and complete the table of values.

b Use your table of values to draw the graph of $y = \left(\frac{1}{3}\right)^x$.

c Is the graph of $y = \left(\frac{1}{3}\right)^x$ increasing or decreasing?

d Copy and complete:

i As $x \to \infty$, $y \to$ ..... **ii** As $x \to -\infty$, $y \to$ .....

e Find the horizontal asymptote of $y = \left(\frac{1}{3}\right)^x$.

3 Construct a table of values for $x = -3, -2, -1, 0, 1, 2, 3,$ then use your table to graph each function:

GRAPHING PACKAGE

a $y = 2^x + 3$ **b** $y = 3^x - 4$ **c** $y = 4^{-x}$ **d** $y = 5 \times 2^x$

Use the **graphing package** or your **graphics calculator** to check your answers.

4 **a** Construct a table of values for $x = -3, -2, -1, 0, 1, 2, 3$ for the function:

i $y = 2^{-x}$ **ii** $y = \left(\frac{1}{2}\right)^x$

Comment on your answers.

b Explain why $2^{-x} = \left(\frac{1}{2}\right)^x$. **c** Sketch the graph of $f(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$.

E GRAPHS OF EXPONENTIAL FUNCTIONS

INVESTIGATION 1 GRAPHS OF EXPONENTIAL FUNCTIONS

In this Investigation we examine the graphs of various families of exponential functions.

GRAPHING PACKAGE

You can use the **graphing package** or your calculator.

What to do:

1 **a** State the transformation which maps $y = a^x$ to $y = a^x + c$.

b Predict the effect, if any, this transformation will have on the:

i shape of the graph **ii** position of the graph

iii horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$ on the same set of axes.

2 **a** State the transformation which maps $y = a^x$ to $y = k \times a^x$, $k > 0$.

b Predict the effect, if any, this transformation will have on the:

i shape of the graph **ii** position of the graph

iii horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$ on the same set of axes.

3 **a** State the transformation which maps $y = a^x$ to $y = -a^x$.

b Predict what the graph of $y = -2^x$ will look like, and check your answer using technology.

c For the family $y = k \times a^x$, what effect does the sign of k have?

4 **a** State the transformation which maps $y = a^x$ to $y = a^{qx}$, $q > 0$.

b Predict the effect, if any, this transformation will have on the:

i shape of the graph **ii** position of the graph

iii horizontal asymptote.

c Notice that $2^{2x} = (2^2)^x = 4^x$ and $2^{3x} = (2^3)^x = 8^x$.

Check your predictions by graphing $y = 2^x$, $y = 4^x$, and $y = 8^x$ on the same set of axes.

5 **a** State the transformation which maps $y = a^x$ to $y = a^{-x}$.

b Notice that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$.

Predict what the graph of $y = \left(\frac{1}{2}\right)^x$ will look like, and check your answer using technology.

c Explain why $a^{-x} = \left(\frac{1}{a}\right)^x$ for all $a > 0$, $a \neq 1$. Hence describe how the graphs of $y = a^{-x}$ and $y = \left(\frac{1}{a}\right)^x$ are related.

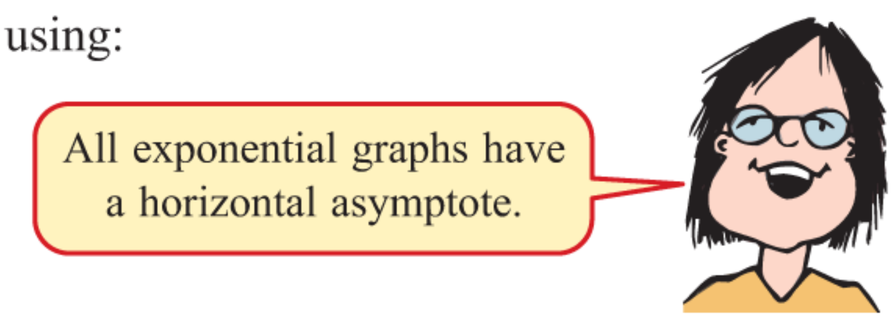
From the **Investigation** you should have discovered that:

For the general exponential functions $\boldsymbol{y = ka^x + c}$ and $\boldsymbol{y = ka^{-x} + c}$:

- c controls the vertical position of the graph.
- The horizontal asymptote is $y = c$.
- a and k both affect how steeply the graph increases or decreases.
- The sign of k determines whether the graph lies above or below the asymptote.

We can sketch the graph of an exponential function using:

- the horizontal asymptote
- the y-intercept
- two other points.



Example 10 **Self Tutor**

Sketch the graph of $y = 2^{-x} - 3$. Hence state the domain and range of $f(x) = 2^{-x} - 3$.

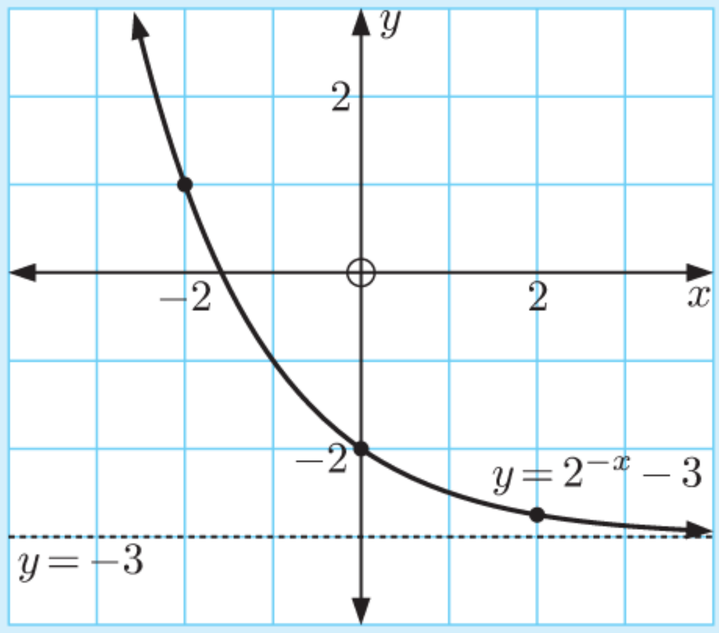
For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3 = 1 - 3 = -2$
$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3 = \frac{1}{4} - 3 = -2\frac{3}{4}$

When $x = -2$, $y = 2^2 - 3 = 1$

The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y > -3\}$.



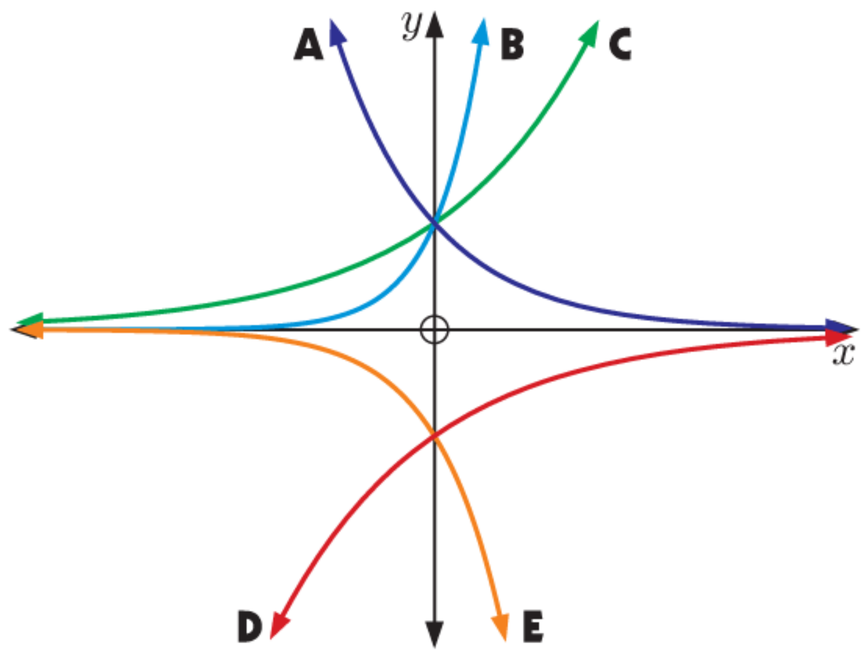
EXERCISE 1E

1 Match each function with its graph:

a $y = 2^x$ **b** $y = 10^x$

c $y = -5^x$ **d** $y = \left(\frac{1}{3}\right)^x$

e $y = -\left(\frac{1}{2}\right)^x$



2 Sketch each pair of functions on the same set of axes:

a $y = 2^x$ and $y = 2^x + 3$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 5^x$ **d** $y = 2^x$ and $y = 2 \times 2^x$

3 Sketch each pair of functions on the same set of axes:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = \frac{1}{2} \times 3^x$

GRAPHING PACKAGE

4 State the equation of the horizontal asymptote of:

a $y = 5^x - 1$ **b** $y = 2^{-x} + 4$ **c** $y = 3 \times 4^x + 1$ **d** $y = -\left(\frac{1}{2}\right)^x - 5$

5 Find the y-intercept of:

a $y = 3^x + 4$ **b** $y = 6^{-x} - 2$ **c** $y = 3 \times 2^x + 7$ **d** $y = -\frac{1}{2} \times 5^x + 6$

6 Consider the exponential function $f(x) = 3^x - 2$.

a Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

7 Consider the function $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$.

a Find: **i** $g(0)$ **ii** $g(2)$ **iii** $g(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

8 For each of the following exponential functions:

i Calculate the y-intercept. **ii** State the equation of the horizontal asymptote.

iii Find the values of y when $x = 2$ and $x = -2$.

iv Sketch the function using the information from **i** to **iii**.

v State the domain and range of the function.

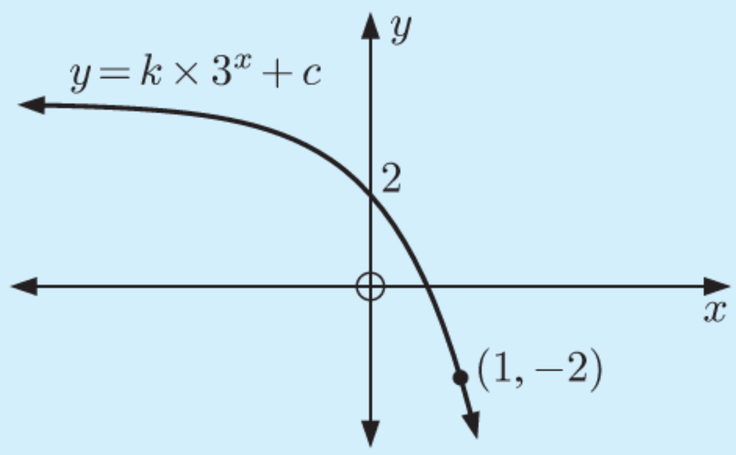
a $y = 2^x + 1$ **b** $y = 3^{-x} + 4$ **c** $y = \left(\frac{2}{5}\right)^x$ **d** $y = \left(\frac{1}{2}\right)^x - 3$

e $y = 2 - 2^x$ **f** $y = 4^{-x} + 3$ **g** $y = 3 - 2^{-x}$ **h** $y = -\frac{1}{2} \times 3^{-x} + 1$

Example 11 **Self Tutor**

This graph shows the curve $y = k \times 3^x + c$, where k and c are constants.

Find the values of k and c.



Substituting $(0, 2)$ into the equation gives $2 = k \times 3^0 + c$ $\therefore$ $k + c = 2$ .... (1)

Substituting $(1, -2)$ into the equation gives $-2 = k \times 3^1 + c$ $\therefore$ $3k + c = -2$ .... (2)

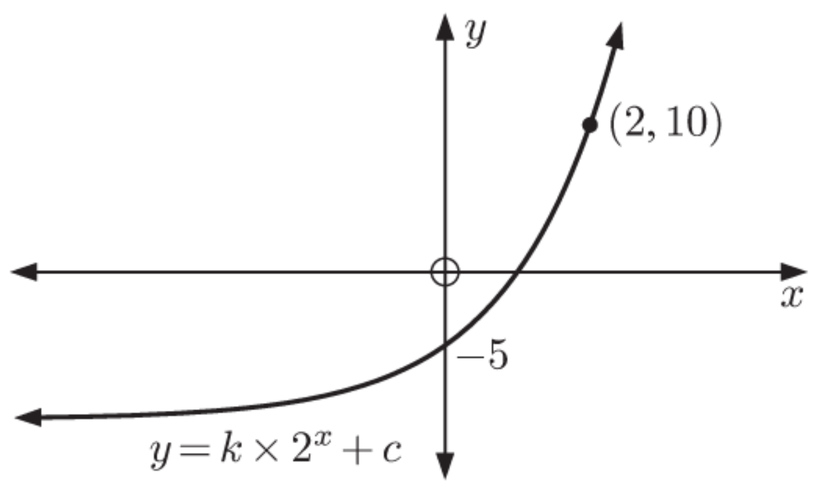
Now (2) − (1) gives $2k = -4$

$\therefore$ $k = -2$ and so $c = 4$.

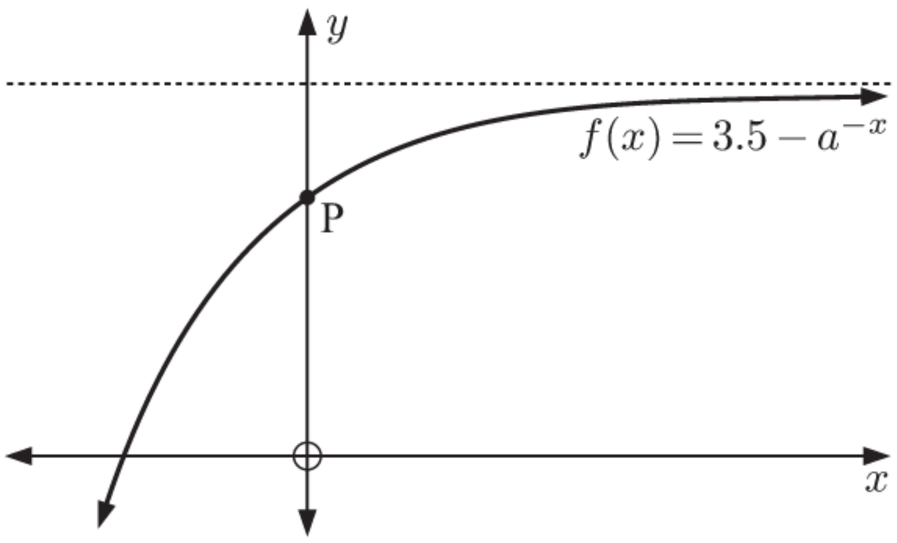
9 The graph alongside shows the curve $y = k \times 2^x + c$, where k and c are constants.

a Find the values of k and c.

b Find y when $x = 6$.



10



This graph shows the function $f(x) = 3.5 - a^{-x}$, where a is a positive constant.

The point $(-1, 2)$ lies on the graph.

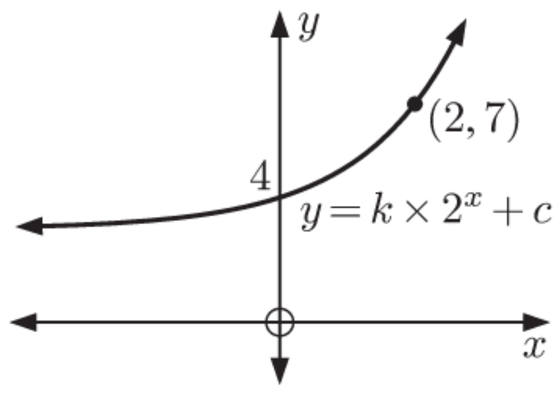
a Write down the coordinates of the y-intercept, P.

b Find the value of a.

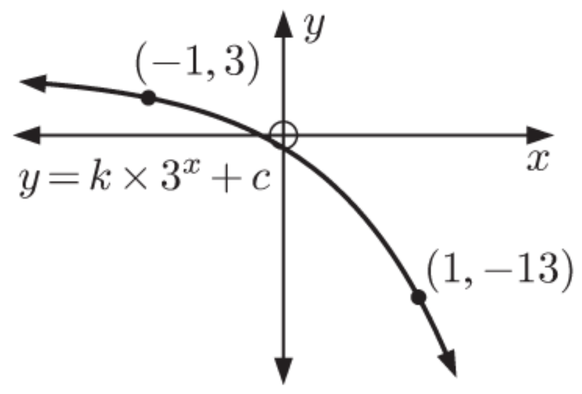
c Find the equation of the horizontal asymptote.

11 Find the exponential function in each graph:

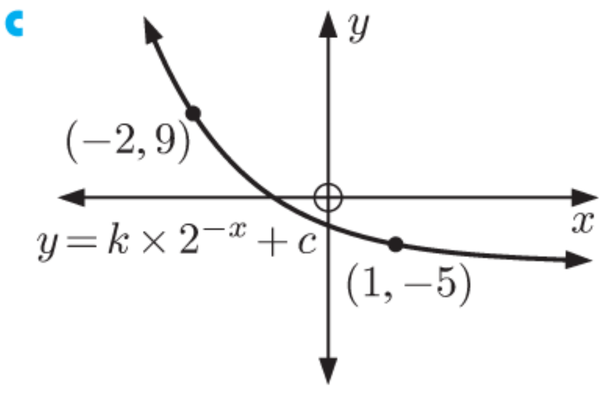
a



b



c



12 The graph of $y = k \times 4^x + c$ is shown alongside.

a Explain why:

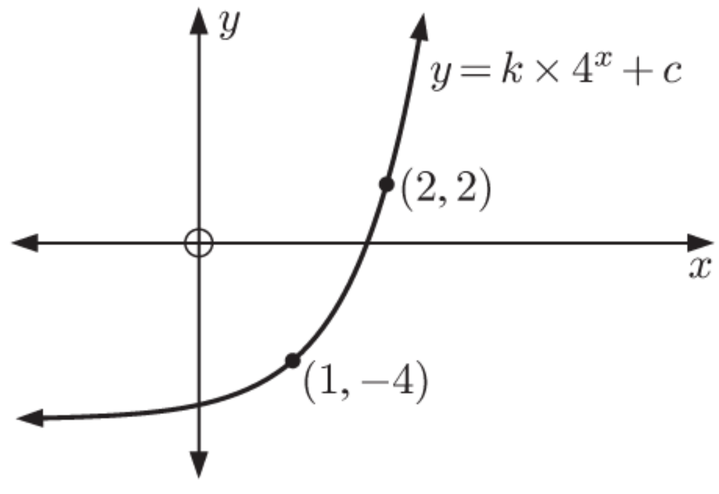
i k must be positive

ii c must be negative.

b Find the equation of the exponential function.

c Find the y-intercept.

d State the horizontal asymptote.



13 An exponential function of the form $f(x) = k \times a^x + c$ has y-intercept 10, horizontal asymptote $y = 2$, and passes through the point $(5, 258)$. Find the exponential function.

14 An exponential function of the form $f(x) = k \times a^{-x} + c$ has y-intercept 1, horizontal asymptote $y = 4$, and passes through the point $(1, 2)$. Find the exponential function.

15 An exponential function of the form $f(x) = ka^x + c$ passes through the points $(1, 11)$, $(2, 17)$, and $(3, 29)$. Find the y-intercept of the function.

16 An exponential function of the form $f(x) = ka^{-x} + c$ passes through the points $(-2, 21)$, $(1, 0)$, and $(2, -\frac{3}{2})$. Find:

a the y-intercept **b** the horizontal asymptote.

DISCUSSION

- For the exponential function $y = a^x$, why do we choose to specify:
  - **a** $a \neq 1$ **b** $a > 0$?
- What does the graph of $y = (-2)^x$ look like? What is its domain and range?

F EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the exponent or index.

For example, $2^x = 50$ and $7^{1-x} = 40$ are both exponential equations.

To solve an exponential equation, we graph each side on the same set of axes. The x-coordinate of the intersection point is the solution to the equation.

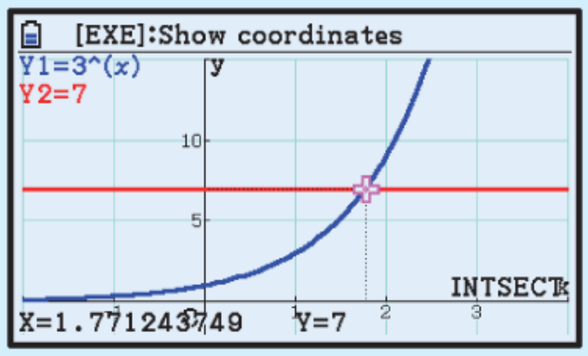
Example 12

Self Tutor

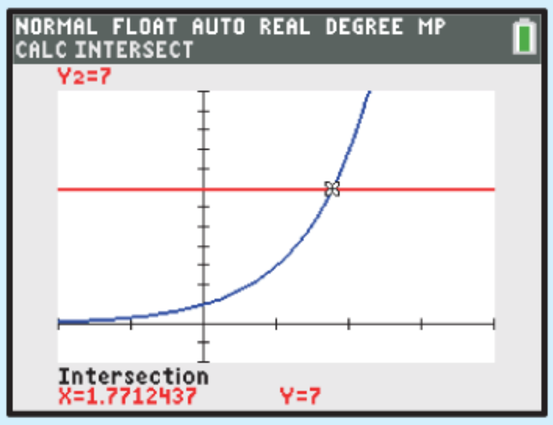
Use technology to solve $3^x = 7$.

We graph $Y_1 = 3^X$ and $Y_2 = 7$ on the same set of axes, and find their point of intersection.

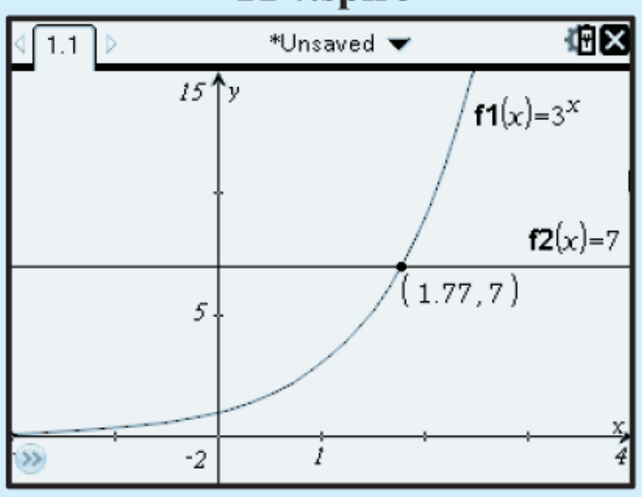
Casio fx-CG50



TI-84 Plus CE



TI-*n*spire



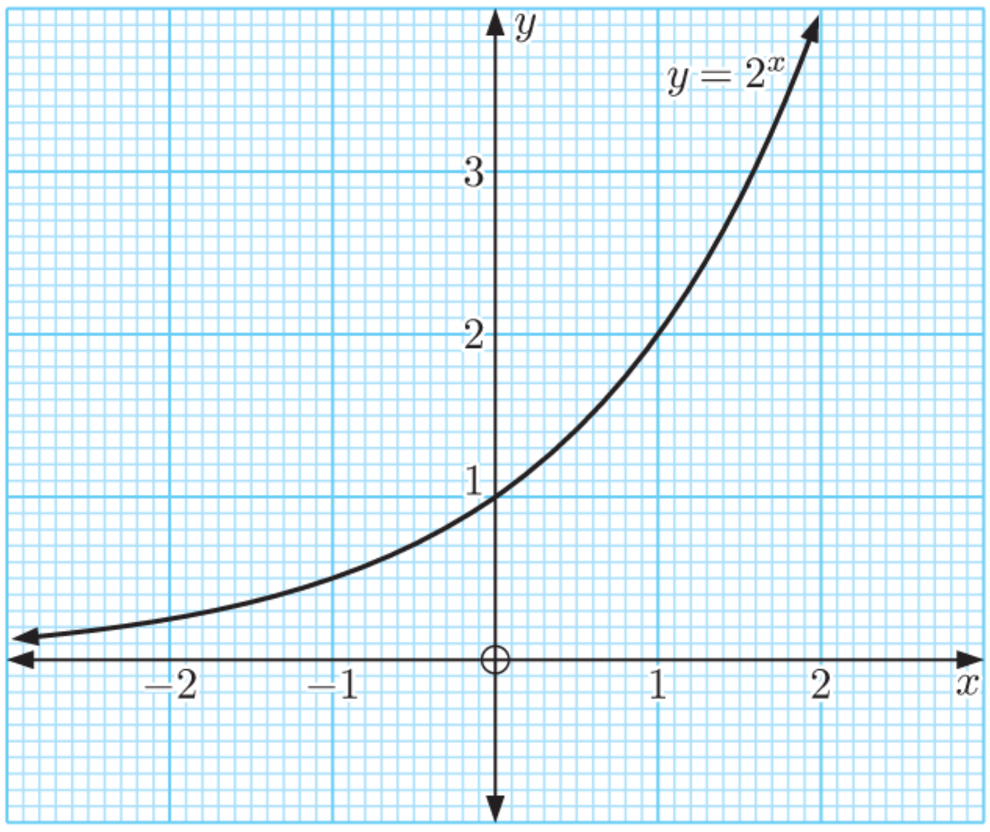
The solution is $x \approx 1.77$.

EXERCISE 1F

1 **a** Use the given graph of $y = 2^x$ to estimate the solution to:

i $2^x = 3$ **ii** $2^x = 0.6$

b Use technology to check your answer.



2 Solve using technology:

a $2^x = 20$ **b** $4^x = 100$ **c** $3^x = 30$

d $(1.2)^x = 3$ **e** $(1.04)^x = 4.238$ **f** $(0.9)^x = 0.5$

3 Solve using technology:

a $3 \times 2^x = 93$ **b** $40 \times (0.8)^x = 10$ **c** $8 \times 3^x = 120$

d $21 \times (1.05)^x = 34$ **e** $500 \times (0.95)^x = 350$ **f** $250 \times (1.125)^x = 470$

g $3^x + 5 = 50$ **h** $60 + 10 \times (1.5)^x = 80$ **i** $20 + 80 \times (0.75)^x = 30$

4 An exponential function of the form $y = 4 \times a^x - 7$, $a > 0$, passes through $(-2, 5)$. Find:

a the value of a **b** the x-intercept of the function.

5 For what values of k does the equation $10 - 8 \times (0.5)^x = k$ have:

a 1 solution **b** no solutions?

6 An exponential function of the form $y = k \times 2^x + c$ passes through $(-1, -\frac{7}{8})$ and $(5, 7)$. Find the value of x when $y = 10$.

ACTIVITY 1 EQUATING INDICES

In the previous Exercise we used technology to solve exponential equations. However, in some cases we can solve exponential equations algebraically.

If both sides of an exponential equation are written as powers with the same base numbers, we can **equate indices**.

$$\text{If } a^x = a^k \text{ then } x = k.$$

For example, if $2^x = 8$ then $2^x = 2^3$. Thus $x = 3$, and this is the only solution.

What to do:

1 Solve algebraically:

a $2^x = 32$	**b** $5^x = 25$	**c** $3^x = 81$	**d** $7^x = 1$
e $3^x = \frac{1}{3}$	**f** $2^x = \sqrt{2}$	**g** $5^x = \frac{1}{125}$	**h** $4^x = 64$
i $2^{x-2} = \frac{1}{32}$	**j** $3^{x+1} = \frac{1}{27}$	**k** $7^{x+1} = \frac{1}{\sqrt{7}}$	**l** $5^{1-2x} = \frac{1}{5}$

2 Discuss the advantages and disadvantages of using this algebraic method to solve exponential equations.

G GROWTH AND DECAY

In this Section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay** modelling, and occur frequently in the world around us.

Populations of animals, people, and bacteria usually grow exponentially until they become limited by resources. We also see exponential growth in the value of items which *appreciate* over time.

We observe exponential *decay* in radioactive substances, in the temperature of objects as they cool, and in the value of items which *depreciate* over time.

For the exponential function $y = k \times a^x + c$ where $a, k > 0$, $a \neq 1$, we see:

- growth if $a > 1$
- decay if $0 < a < 1$.

GROWTH

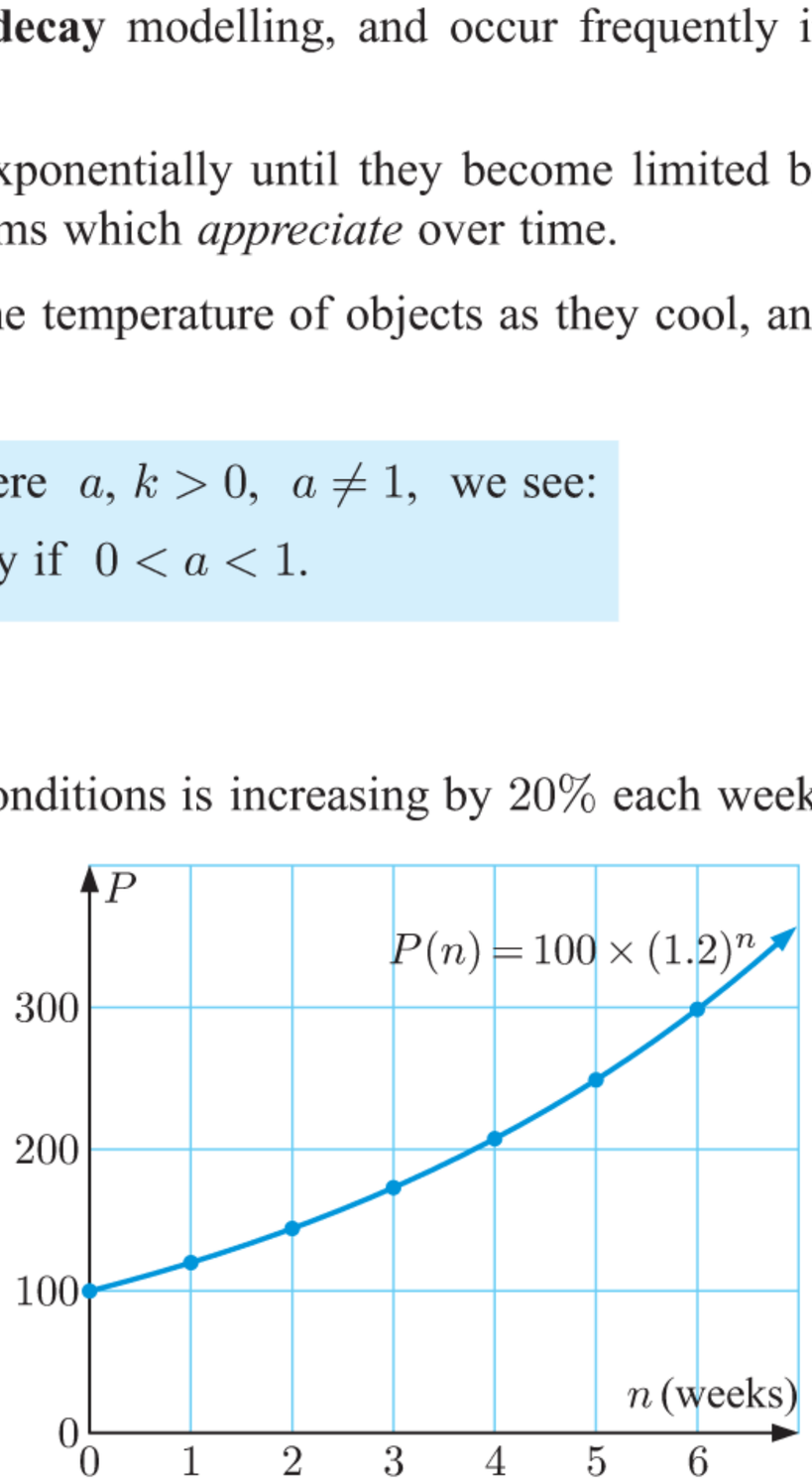
Consider a population of 100 mice which under favourable conditions is increasing by 20% each week.

To increase a quantity by 20%, we multiply it by 1.2.

If P_n is the population after n weeks, then:

$P_0 = 100$ {the *original* population}
$P_1 = P_0 \times 1.2 = 100 \times 1.2$
$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$
$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$, $n \in \mathbb{Z}$, which is a geometric sequence.



However, while the population of mice must always be an integer, we expect that the population will grow continuously throughout the year, rather than in big, discrete jumps. We therefore expect it will be well approximated by the corresponding exponential function $P(n) = 100 \times (1.2)^n$, $n \in \mathbb{R}$.

Example 13 **Self Tutor**

A scientist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A(n) = 1000 \times (1.15)^n$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after:

i 5 weeks **ii** 10 weeks.

c Draw the graph of the affected area over time.

d Use your graph or technology to find how long it will take for the affected area to reach 8000 hectares.

a $A(0) = 1000 \times 1.15^0 = 1000$

$\therefore$ the original affected area was 1000 hectares.

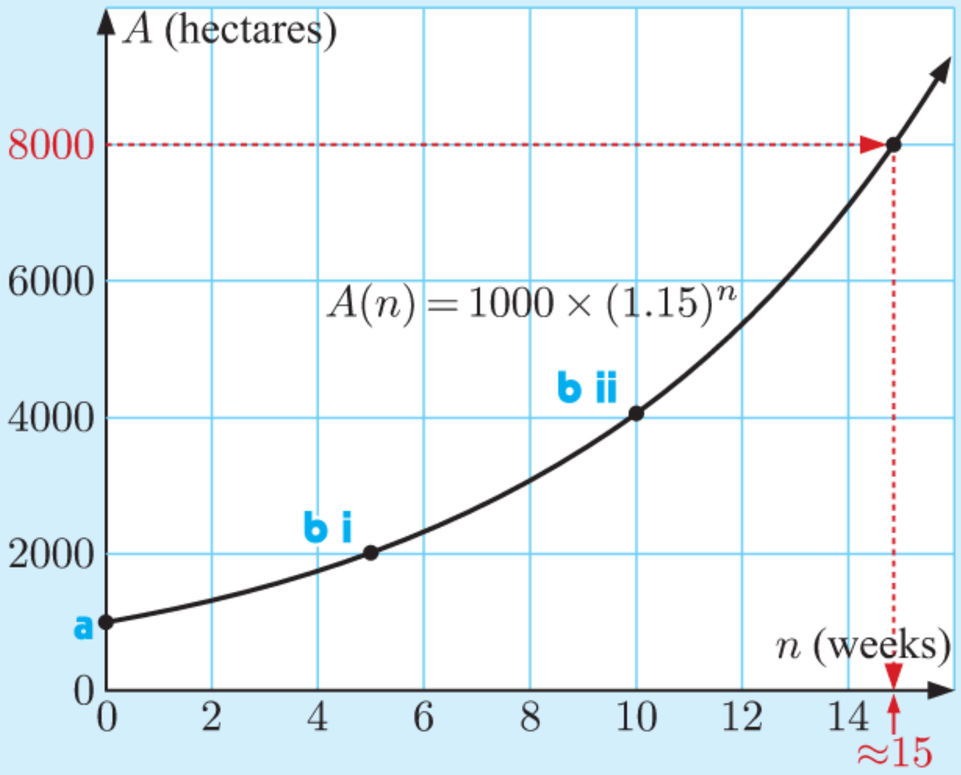
b **i** $A(5) = 1000 \times 1.15^5 \approx 2010$

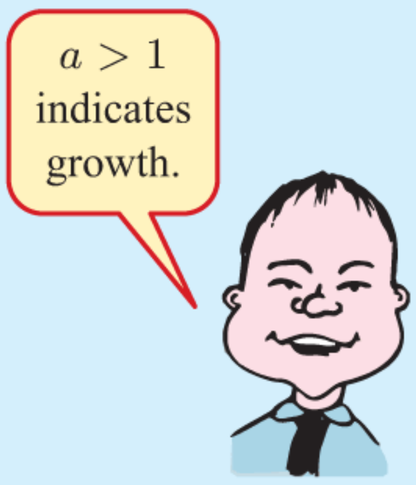
The affected area is about 2010 hectares.

ii $A(10) = 1000 \times 1.15^{10} \approx 4050$

The affected area is about 4050 hectares.

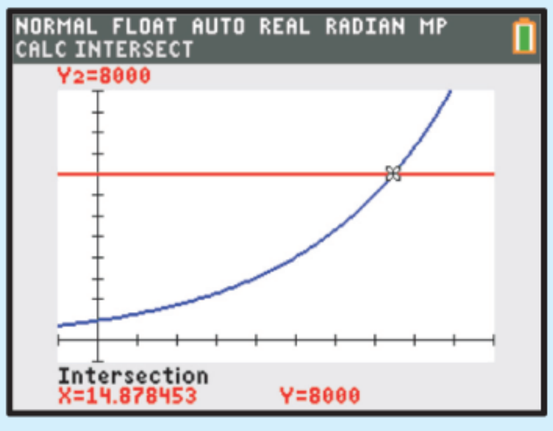
c





d From the graph in **c**, it would take about 15 weeks for the affected area to reach 8000 hectares.

or Using technology, the solution is ≈ 14.9 weeks.



EXERCISE 1G.1

1 A weed in a field covers an area of $A(t) = 3 \times (1.08)^t$ square metres after t days.

a Find the initial area covered by the weed.

b By what percentage does the area increase each day?

c Find the area covered after:

i 2 days **ii** 10 days **iii** 30 days.

d Sketch the graph of $A(t)$ using the results of **a** and **c** only.

GRAPHING PACKAGE

Use technology to graph $Y_1 = 3 \times (1.08)^X$ and hence check your answers.

2 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times (1.07)^t$ grams.

a Find the initial weight.

b Interpret the value 1.07 in the model.

c Find the weight after:

i 4 hours **ii** 10 hours **iii** 24 hours.

d Sketch the graph of the bacteria weight over time using the results of **a** and **c** only.

Use technology to graph $Y_1 = 100 \times (1.07)^X$ and hence check your answers.

3 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times (1.23)^n$.

a What is the value of P_0?

b What is the expected population after:

i 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of the population over time using **a** and **b** only.

d How long will it take for the population to reach 500?

Use technology to graph $Y_1 = 50 \times (1.23)^X$ and hence check your answers.

4 A flu virus spreads in a school. The number of people N infected after t days is given by $N = 4 \times 1.332^t, \ t \geqslant 0$.

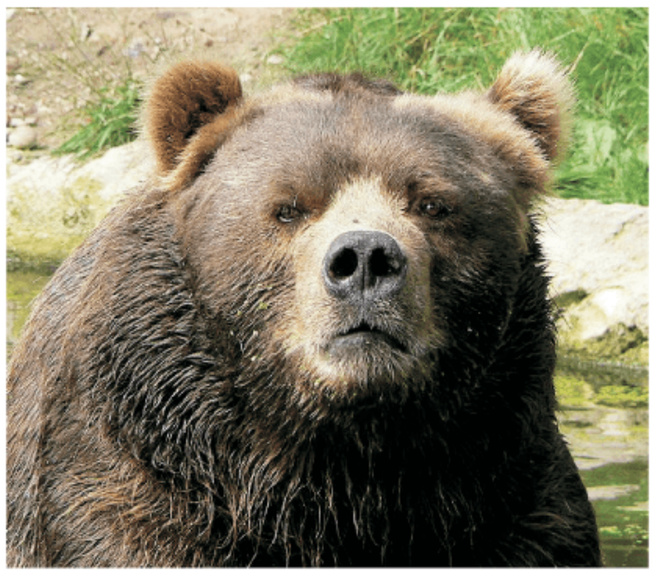
a Find the number of people initially infected.

b Calculate the number of people infected after 16 days.

c There are 1200 people in the school. Estimate the time it will take for everybody in the school to catch the flu.

d For what values of t is it reasonable to use this model?

5 In 1998, 200 bears were introduced to a large island off Alaska where previously there were no bears. The population increased exponentially according to $B(t) = B_0 \times a^t$, where $a > 0$ is a constant and t is the time in years since the introduction.



a Find B_0.

b In 2000 there were 242 bears. Find a, and interpret your answer.

c Find the expected bear population in 2018.

d Find the expected percentage increase in population from 2008 to 2018.

e How long will it take for the population to reach 2000?

6 The speed V of a chemical reaction is given by $V(t) = V_0 \times 2^{0.05t}$ where t is the temperature in °C.

a Find the reaction speed at: **i** 0°C **ii** 20°C.

b Find the percentage increase in reaction speed at 20°C compared with 0°C.

c Find $\left(\dfrac{V(50) - V(20)}{V(20)}\right) \times 100\%$ and explain what this calculation means.

7 Kayla deposited £5000 into an account. The amount in the account increases by 10% each year.

a Write a formula for the amount $A(t)$ in the account after t years.

b Find the amount in the account after: **i** 2 years **ii** 5 years.

c Sketch the graph of $A(t)$.

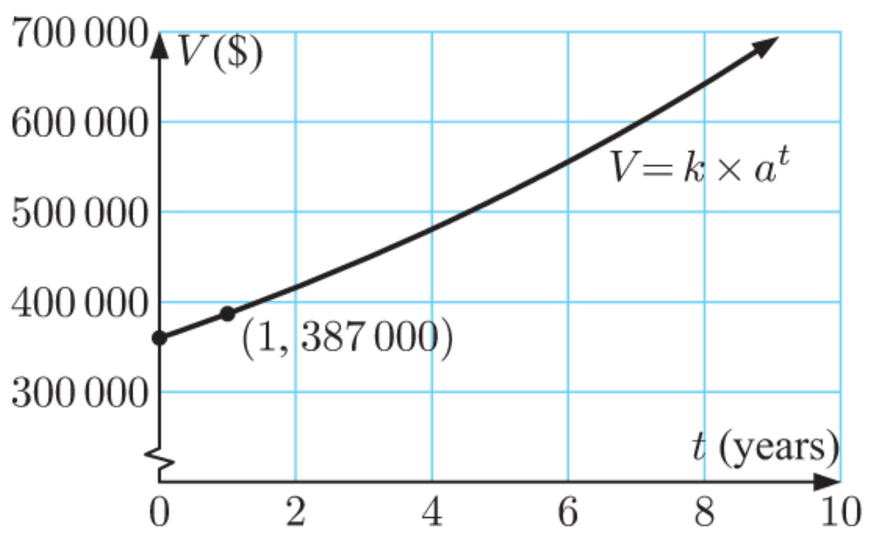
d How long will it take for the amount in the account to reach £8000?

8 A house is expected to increase in value by 7.5% per year. Its expected value in t years' time is given by the exponential model $V = k \times a^t$ dollars, where $t \geqslant 0$. The model is graphed alongside.

a State the value of a.

b Find the value of k, and interpret your answer.

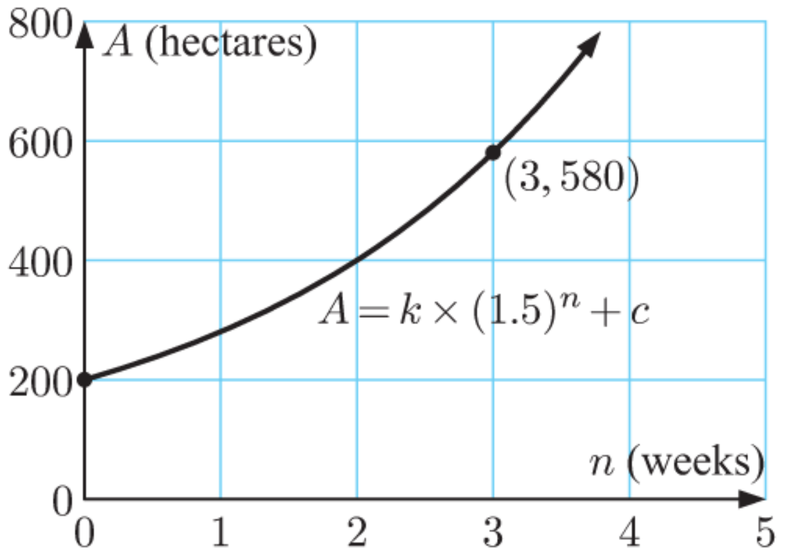
c How long will it take for the house's value to reach \$550 000?



9 A biologist is modelling an infestation of fire ants. He determines that the area affected by the ants is given by $A(n) = k \times (1.5)^n + c$ hectares, where n is the number of weeks after the initial observation, and k and c are constants.
The model is graphed alongside.

a Use technology to find k and c.

b Estimate the time taken for the infested area to reach 1000 hectares.



10 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = c - k \times (0.8)^t$ m s^{-1} where c and k are constants, and t is the time in seconds.

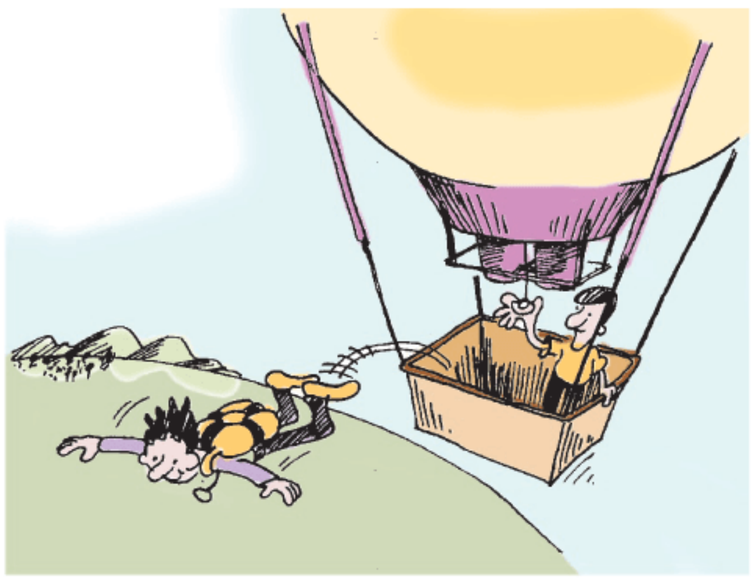
a Explain why $c = k$.

b After 2 seconds, the parachutist has speed 21.6 m s^{-1}. Find the exponential model connecting V and t.

c Find the speed of the parachutist after 4 seconds.

d Sketch the graph of V against t.

e Describe how the speed of the parachutist varies over time.



PUZZLE

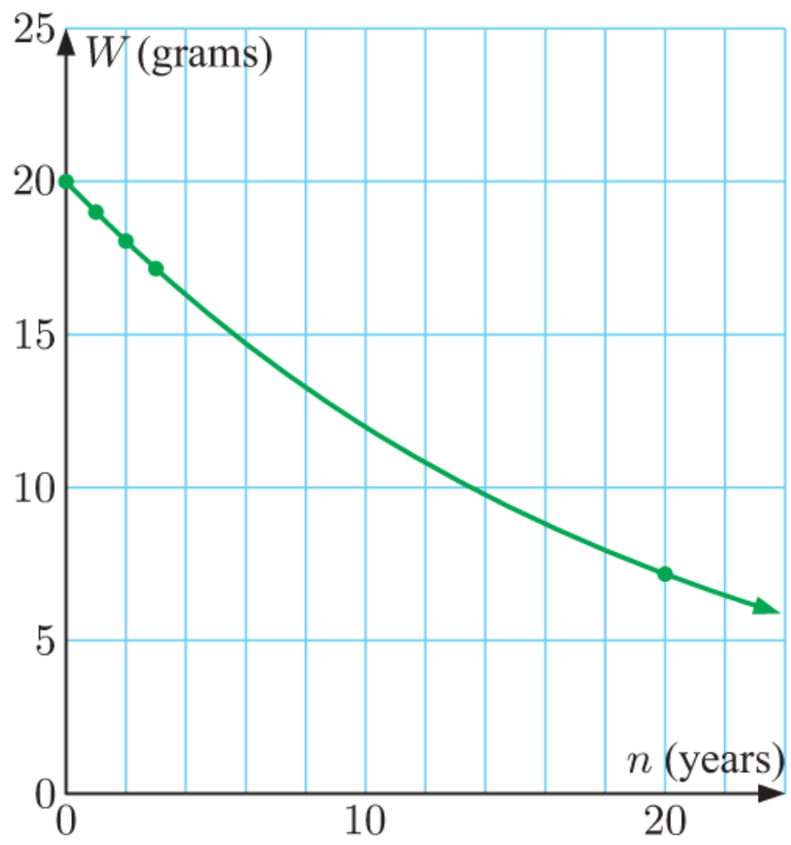
A population of insects doubles in size each week, and exhausts the food supply after 6 months. At what time does $\frac{3}{4}$ of the food remain?

DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier for this is 95% or 0.95 .

If W_n is the weight after n years, then:

$$W_0 = 20 \text{ grams}$$
$$W_1 = W_0 \times 0.95 = 20 \times 0.95 \text{ grams}$$
$$W_2 = W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams}$$
$$W_3 = W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams}$$
$$\vdots$$
$$W_{20} = 20 \times (0.95)^{20} \approx 7.2 \text{ grams}$$
$$\vdots$$
$$W_{100} = 20 \times (0.95)^{100} \approx 0.1 \text{ grams.}$$



From this pattern we see that $W_n = 20 \times (0.95)^n$, $n \in \mathbb{Z}$, which is again a geometric sequence.

However, we know that radioactive decay is a continuous process, so the weight remaining will actually be given by the smooth exponential curve $W(n) = 20 \times (0.95)^n$, $n \in \mathbb{R}$.

Example 14 **Self Tutor**

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

a Find $I(t)$ when $t = 0$, 1, 2, and 3.

b What current flowed in the generator at the instant it was switched off?

c Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.

d Use your graph or technology to find how long it takes for the current to reach 4 amps.

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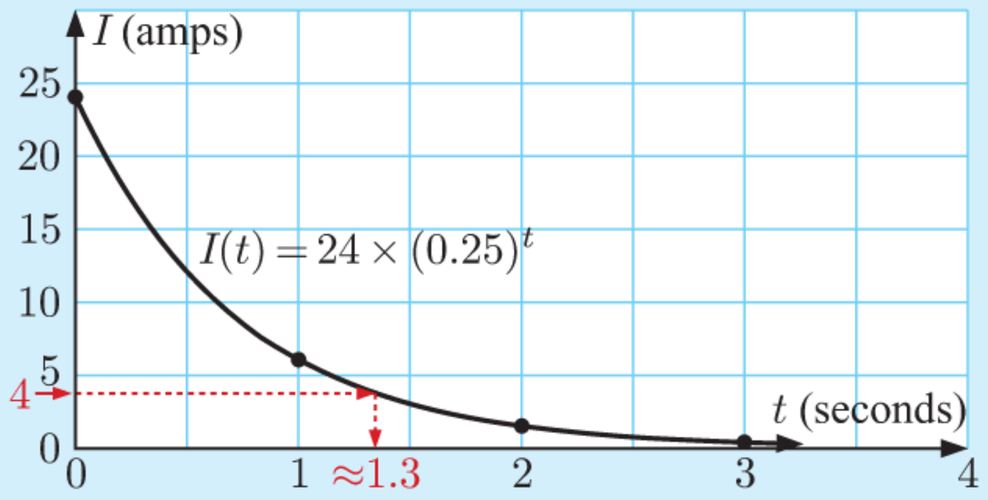
a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b $I(0) = 24$

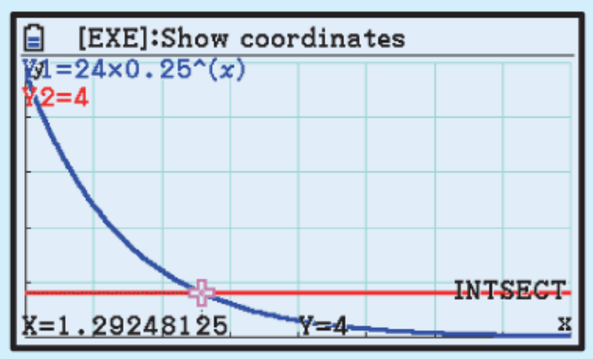
When the generator was switched off, 24 amps of current flowed in the circuit.

c



d From the graph on the previous page, the time to reach 4 amps is about 1.3 seconds.

or Using technology, the solution is ≈ 1.29 seconds.



EXERCISE 1G.2

1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

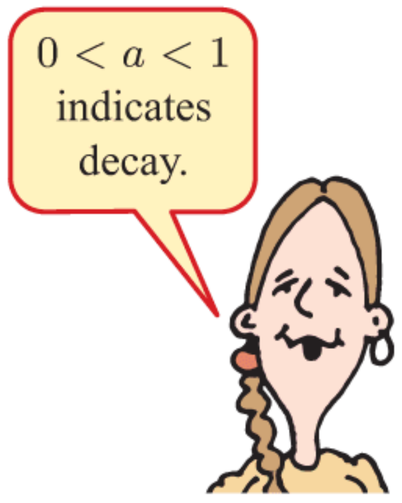
a How much radioactive substance was initially set aside?

b Determine the weight of the substance after:

i 400 years **ii** 800 years **iii** 1200 years.

c Sketch the graph of $W(t)$ for $t \geqslant 0$ using **a** and **b** only.

d Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.



2 The temperature of a liquid t minutes after it has been placed in a refrigerator, is given by $T(t) = 100 \times (1.02)^{-t}$ °C.

a Find the initial temperature.

b Find the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

d How long will it take for the temperature to fall to 15°C?

3 An initial count of orangutans in a forest found that the forest contained 400 orangutans. Since then, the destruction of their habitat has caused the population to fall by 8% each year.

a Write a formula for the population P of orangutans t years after the initial count.

b Find the population of orangutans after:

i 1 year **ii** 5 years.

c Sketch the graph of the population over time.

d How long will it take for the population to fall to 200?



4 The intensity of light L diminishes below the surface of the sea according to the formula $L = 10 \times a^d$ units, where $a > 0$ is a constant and d is the depth in metres measured from the surface of the sea.

a Would you expect that $0 < a < 1$ or $a > 1$? Explain your answer.

b The intensity of light 1 m below the surface is 9.5 units. Find the value of a.

c Find the intensity of light 25 m below the surface.

d A light intensity of 4 units is considered adequate for divers to be able to see clearly. Calculate the depth corresponding to this intensity of light.

e Calculate the range of depths for which the light intensity is between 1 and 3 units.

5 A group of turtles was observed in a lake in 2005. The population of turtles has decreased by 5% each year since. In 2006 the population was 323.

Suppose the population is given by $P = k \times a^t$ where t is the number of years since 2005.

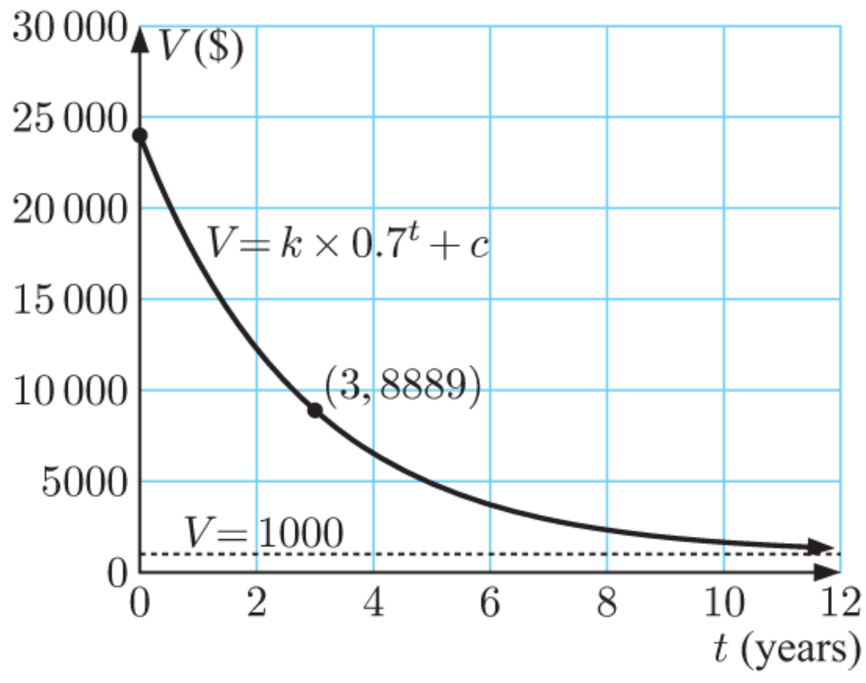
a State the value of a. **b** Find the value of k, and interpret this value.

c Find the population of turtles in 2015.

d Do you think it reasonable to apply this model for negative values of t? Explain your answer.

6 The value of a car after t years is given by the exponential model $V = k \times 0.7^t + c$ dollars, where k and c are constants, and $t \geqslant 0$.

The model is graphed alongside.



a State the value of c. Interpret this value in the context of the situation.

b Find the value of k.

c Find the initial value of the car.

d How long will it take for the value of the car to reduce to \$5000? Give your answer to the nearest year.

e Does the car depreciate in value by the same percentage each year? Explain your answer.

7 When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = c + k \times 0.875^t$ °C, where c and k are constants.

The temperature of the peas is 18°C after 1 minute, and 14.5°C after 2 minutes.

a Use technology to find c and k.

b What do you think is the temperature of the interior of the freezer? Explain your answer.

c What was the temperature of the packet of peas:

i when first placed in the freezer **ii** after 5 minutes **iii** after 10 minutes?

d Sketch the graph of $T(t)$.

e How long does it take for the temperature of the packet of peas to fall to 0°C?

Example 15 **Self Tutor**

The weight W of radioactive substance remaining after t years is given by $W(t) = 1000 \times (0.979)^t$ grams.

a Find the initial weight of the radioactive substance.

b Find the half-life of the substance.

a $W(0) = 1000 \times (0.979)^0 = 1000$

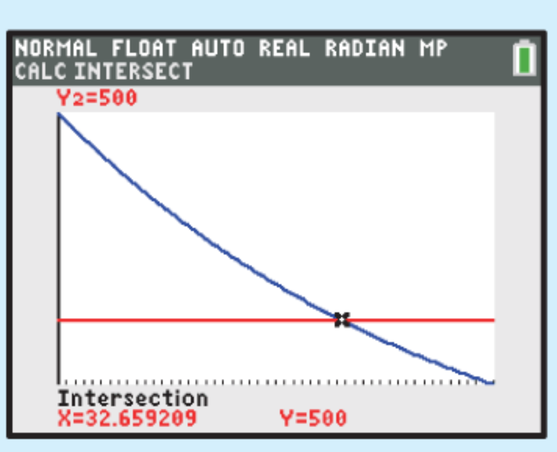
The initial weight was 1000 g.

b Half of the initial weight is 500 g.

$\therefore \quad 1000 \times (0.979)^t = 500$

Using technology, it takes ≈ 32.7 years for the substance's weight to fall to 500 g.

So, the half-life of the substance ≈ 32.7 years.



The **half-life** of a quantity is the time it takes to halve in value.



8 When medication is taken by a patient, it is slowly used by their body. After t hours, the amount of drug remaining is given by $D(t) = 120 \times (0.9)^t$ mg.

a Find the initial drug dose. **b** Find the half-life of the drug.

c Find the time when there is only 25 mg of the drug left in the body.

9 The weight W of radioactive substance remaining after t years is given by $W(t) = 800 \times (0.96)^t$ grams.

a Find the initial weight of the substance. **b** Find the half-life of the substance.

c Without using technology, determine the time it takes for the weight of the substance to reach 50 g.

10 The radioactive isotope fermium-253 has a half-life of 3 days. The weight of fermium-253 detected t days after an explosion is $W(t) = 10 \times a^t$ mg.

a Interpret the value 10 in this model.

b Calculate the value of a, correct to 4 decimal places, and interpret this value.

c Find the weight of fermium-253 after 2 days.

d How long will it take for the weight of fermium-253 to fall to:

i 3 mg **ii** 1.25 mg?

11 The weight of a substance after t hours is given by the exponential model $W = k \times a^t$ grams, $k > 0$, $0 < a < 1$.
The substance weighs 300 g after 2 hours, and 100 g after 6 hours.
Find the half-life of the substance.

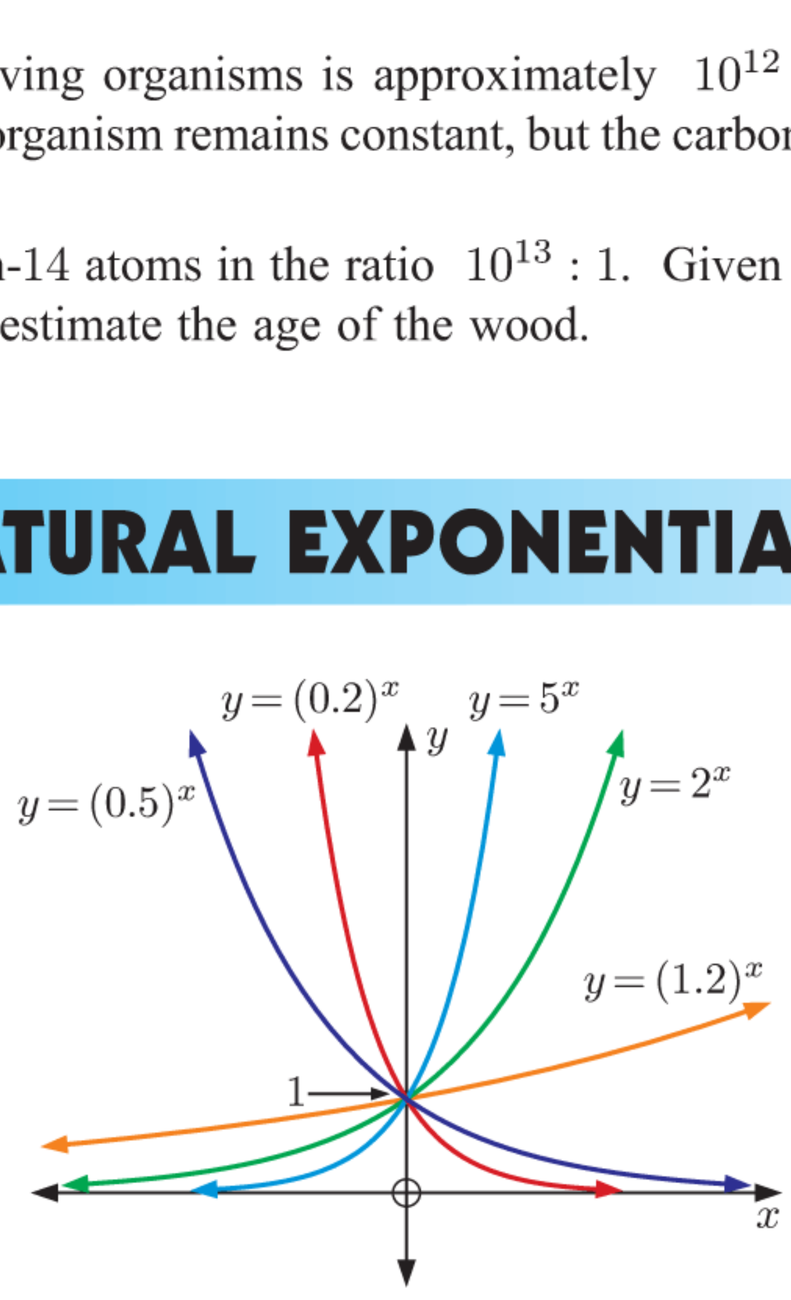
12 The ratio of carbon-12 atoms to carbon-14 atoms in living organisms is approximately $10^{12} : 1$. When an organism dies, the amount of carbon-12 in the organism remains constant, but the carbon-14 begins to decay.

A piece of wood is found to have carbon-12 and carbon-14 atoms in the ratio $10^{13} : 1$. Given that the half-life of carbon-14 is approximately 5700 years, estimate the age of the wood.

H THE NATURAL EXPONENTIAL

We have seen that the simplest exponential functions have the form $f(x) = a^x$ where $a > 0$, $a \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base a, the graph is always positive.

Hence $a^x > 0$ for all $a > 0$.

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Investigation**.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A discrete formula for calculating the amount to which an investment grows under compound interest is $u_n = u_0(1+i)^n$ where u_n is the final amount, u_0 is the initial amount, i is the interest rate per compounding period, and n is the number of periods, or times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to become extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

a annually $(n = 1,\ i = 6\% = 0.06)$

b quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$

c monthly

d daily

e by the second

f by the millisecond.

Comment on your answers.

2 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

If we let $a = \frac{N}{r}$, show that the growth formula becomes $u_n = u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt}$.

3 For continuous compound growth, the number of interest payments per year N gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

4 You should have found that for very large values of a, $\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$

Use the e^x key of your calculator to find the value of e^1. What do you notice?

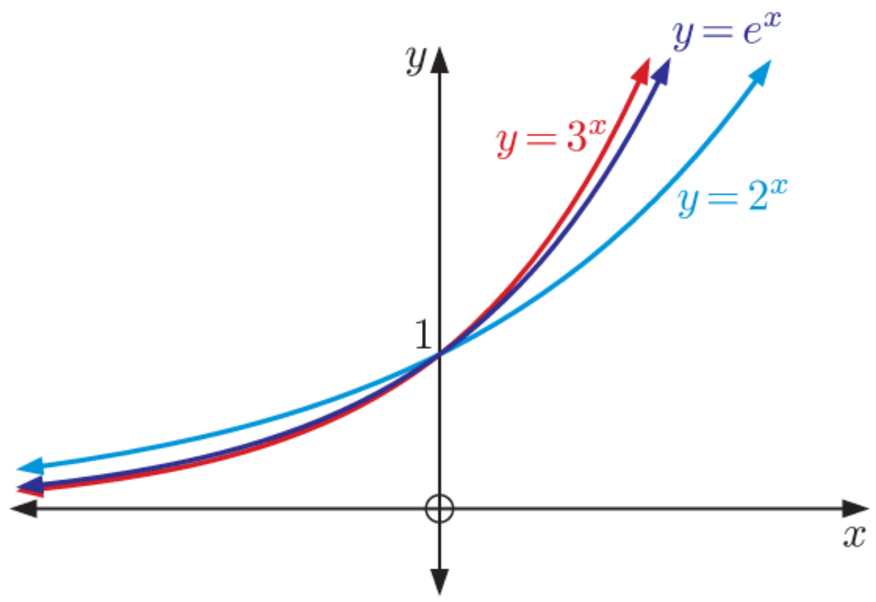
5 For continuous growth, $u_n = u_0e^{rt}$ where u_0 is the initial amount, r is the annual percentage rate, and t is the number of years.

Use this formula to find the final amount if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is paid continuously.

From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate per annum and t is the number of years.

Since $e \approx 2.718\,28$, the graph of $y = e^x$ lies between the graphs of $y = 2^x$ and $y = 3^x$.

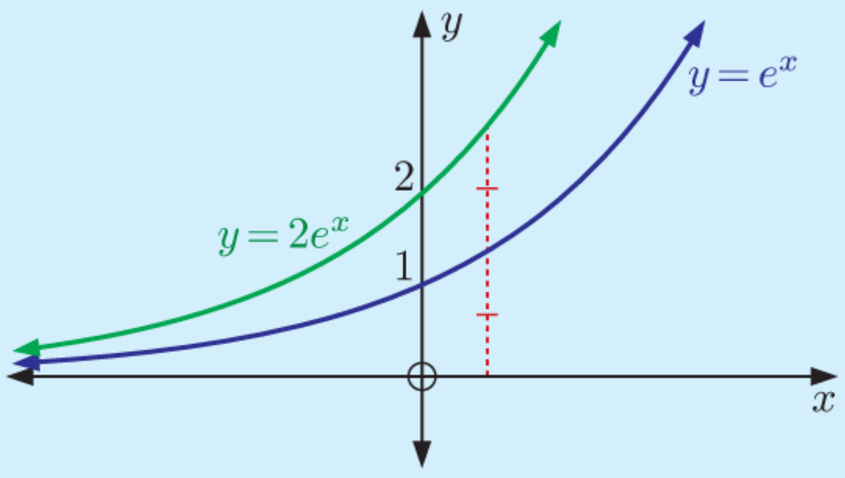


Example 16 — Self Tutor

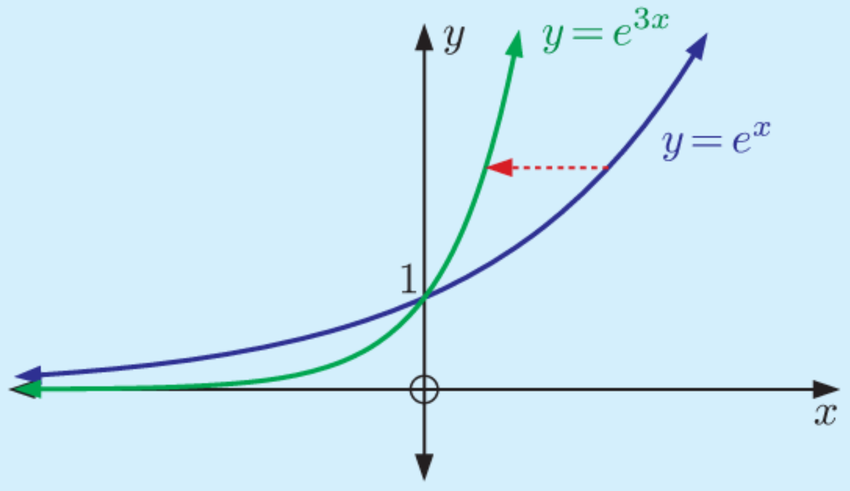
Sketch, on the same set of axes, the graphs of:

a $y = e^x$ and $y = 2e^x$ **b** $y = e^x$ and $y = e^{3x}$

a $y = 2e^x$ is a vertical dilation of $y = e^x$ with scale factor 2.



b $y = e^{3x}$ is a horizontal dilation of $y = e^x$ with scale factor $\frac{1}{3}$.



EXERCISE 1H

1 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = -e^x$. What is the geometric connection between these two graphs?

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

GRAPHING PACKAGE

3 Sketch, on the same set of axes as the graph of $y = e^x$:

a $y = e^x + 2$ **b** $y = e^x - 3$ **c** $y = 3e^x$

d $y = \frac{1}{2}e^x$ **e** $y = e^{2x}$ **f** $y = e^{\frac{x}{2}}$

For each graph, state the y-intercept and equation of the horizontal asymptote.

4 For the general exponential function $y = ke^{rx} + c$, what is the y-intercept?

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\frac{1}{\sqrt{e}}$ **c** $\frac{1}{e^2}$ **d** $e\sqrt{e}$

7 Evaluate, to five significant figures:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$ **d** $e^{-4.829}$

e $50e^{-0.1764}$ **f** $80e^{-0.6342}$ **g** $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

8 Expand and simplify:

a $(e^x + 1)^2$ **b** $(1 + e^x)(1 - e^x)$ **c** $e^x(e^{-x} - 3)$

9 Factorise:

a $e^{2x} + e^x$ **b** $e^{2x} - 16$ **c** $e^{2x} - 8e^x + 12$

10 **a** On the same set of axes, sketch and clearly label the graphs of $f(x) = e^x$, $g(x) = -e^x$, and $h(x) = 10 - e^x$.

b State the domain and range of each function.

c Describe the behaviour of each function as $x \to \pm\infty$.

11 The weight of bacteria in a culture is given by $W(t) = 2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

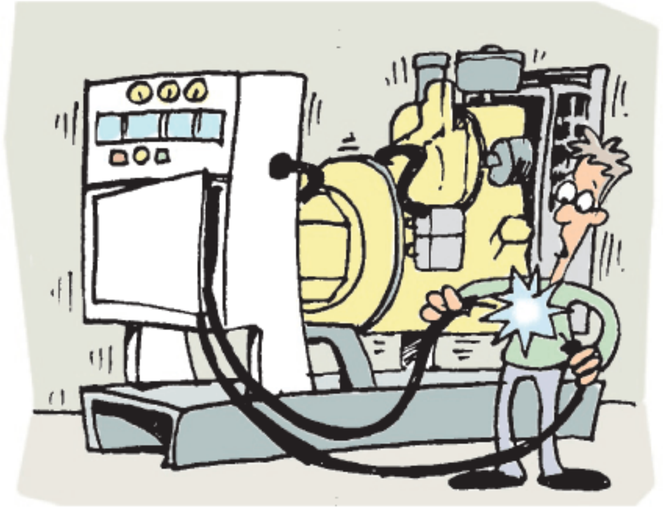
a Find the weight of the culture:

i initially **ii** after 30 minutes

iii after $1\frac{1}{2}$ hours **iv** after 6 hours.

b Hence sketch the graph of $W(t) = 2e^{\frac{t}{2}}$.

12 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 75e^{-0.15t}$ amps.



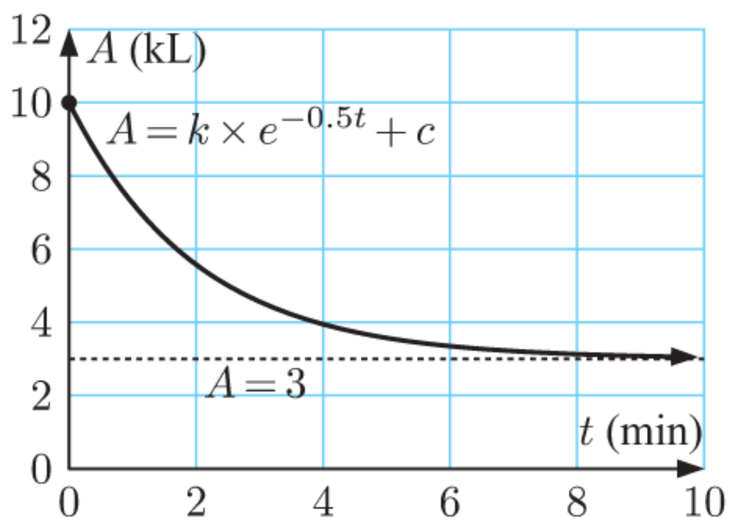
a What current is still flowing in the circuit after:

i 1 second **ii** 10 seconds?

b Use your graphics calculator to help sketch the graph of $I(t) = 75e^{-0.15t}$.

c How long will it take for the current to fall to 1 amp?

13 This graph shows the amount of water in a tank t minutes after it is punctured by a drill.



a Find the exponential model connecting A and t.

b Do you think the hole is in the bottom or the side of the tank? Explain your answer.

c Find the amount of water in the tank after 2 minutes.

d How long will it take for the tank to lose 6 kL of water?

14 A meteor hurtling through the atmosphere has speed of descent given by

$$V(t) = 650(4 + 2 \times e^{-0.1t}) \text{ m s}^{-1}$$

where t is the time in seconds after the meteor is sighted.

a Is the meteor's speed increasing or decreasing?

b Find the speed of the meteor:

i when it was first sighted **ii** after 2 minutes.

c How long will it take for the meteor's speed to reach 3000 m s^{-1}?

15 Consider the function $f(x) = e^x$.

a On the same set of axes, sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.

b State the domain and range of f^{-1}.

16 It can be shown that $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times3}x^3 + \frac{1}{2\times3\times4}x^4 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ which is an infinite polynomial expansion.

Check this statement by finding an approximation for e^1 using its first 20 terms.

ACTIVITY 2

Click on the icon to run a card game for exponential functions.



HISTORICAL NOTE

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Investigation 2**.

The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748, Euler evaluated e correct to 18 decimal places.



Leonhard Euler

Euler also discovered some patterns in **continued fraction** expansions of e. He wrote that

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+....}}}}} \quad \text{and} \quad e - 1 = 1 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+....}}}}}}}$$

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

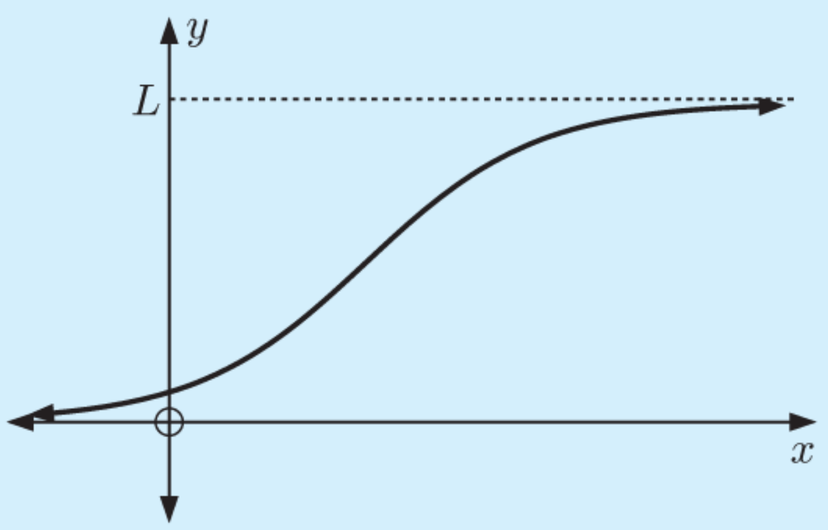
I THE LOGISTIC MODEL

Some quantities grow exponentially at first, and then level off to approach a maximum value. For example, the population of animals on an island may grow exponentially initially, but then approach a maximum value due to limited resources of land or food.

This model of growth is called the **logistic model**.

The **logistic model** has the form $y = \dfrac{L}{1 + Ce^{-kx}}$

where $L > 0$ is the limiting value, and C and k are positive constants.



Notice that as $x \to \infty$, $e^{-kx} \to 0$, and so $y \to \dfrac{L}{1+0} = L$.

Example 17 Self Tutor

The weight of bacteria in a petri dish after t hours is $W(t) = \dfrac{30}{1 + 29e^{-0.7t}}$ grams.

a Find the initial weight of bacteria.

b Find the weight of the bacteria after: i 2 hours ii 6 hours.

c Sketch the graph of W against t.

d Discuss the weight of the bacteria as $t \to \infty$.

e How long did it take for the weight to reach 25 g?

a $W(0) = \dfrac{30}{1 + 29} = 1$

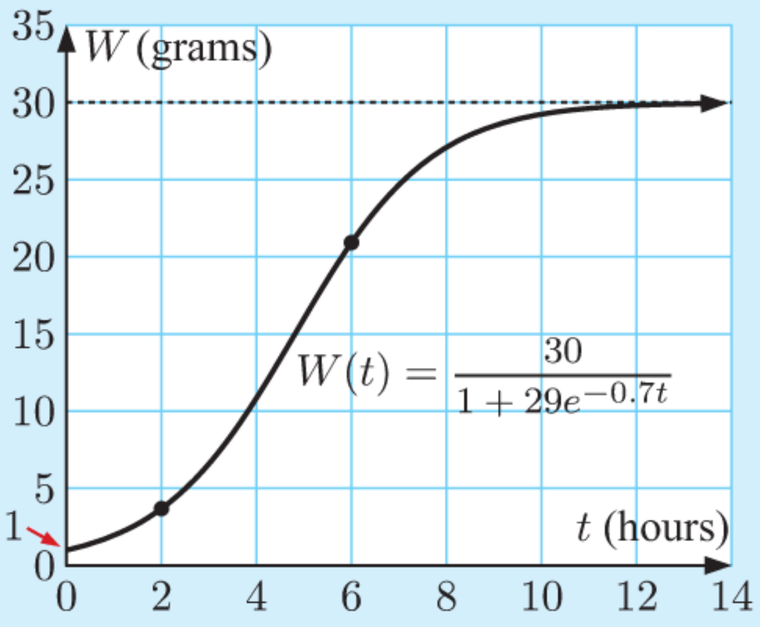
The initial weight is 1 g.

b i $W(2) = \dfrac{30}{1 + 29e^{-0.7(2)}} \approx 3.68$

After 2 hours, the bacteria weighs ≈ 3.68 g.

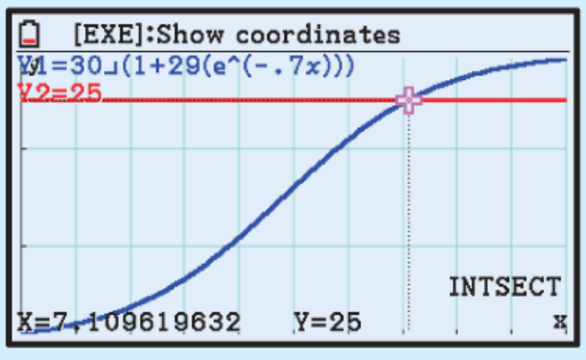
ii $W(6) = \dfrac{30}{1 + 29e^{-0.7(6)}} \approx 20.9$

After 6 hours, the bacteria weighs ≈ 20.9 g.

c 

d As $t \to \infty$, the weight of the bacteria approaches 30 g.

e Using technology, it took ≈ 7.11 hours for the weight to reach 25 g.



EXERCISE 1I

1 The population of an island t years after it is first inhabited, is given by $P(t) = \dfrac{5000}{1 + 24e^{-0.1t}}$.

a How many people initially inhabited the island?

b Find the population after:

i 20 years **ii** 60 years.

c State the limiting population of the island.

d Use technology to help sketch the graph of P against t.

e How long did it take for the population to reach 3000?

2 A contagious virus infects the members of a sports club. The number of members infected after t days is given by $N = \dfrac{80}{1 + Ce^{-t}}$.

a Given that only one member was initially infected, find C.

b Discuss the behaviour of N as $t \to \infty$.

c Find the number of people infected after:

i 5 days **ii** 2 weeks.

d Sketch the graph of N against t.

3 Paige's height at age t years is given by $H(t) = \dfrac{L}{1 + 2e^{-\frac{t}{4}}}$ cm.

a Given that Paige was 54 cm tall at birth, find L.

b Find Paige's height when she was:

i 3 years old **ii** 10 years old.

c How tall is Paige now that she is fully grown?

d The height of Paige's twin sister Piper also follows a logistic model. Piper had the same birth height as Paige, and as adults they have the same height. However, at age 3 Piper was 81 cm tall. Find Piper's height at age 10.

4 In a small country town, rumours spread very fast. At 8 am on Monday, a rumour begins with 2 people. By 10 am, 10 people have heard the rumour, and by noon, 45 people have heard the rumour.

a Find the logistic model for the number of people N who have heard the rumour t hours after 8 am.

b How many people have heard the rumour by 3 pm?

c How many people do you think live in the town?

d At what time would 200 people have heard the rumour?

REVIEW SET 1A

1 Write as a single power of 2:

a $\sqrt[4]{2}$ **b** $8\sqrt{2}$ **c** $\dfrac{1}{\sqrt[3]{4}}$

2 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$ **c** $81^{-\frac{1}{4}}$

3 Expand and simplify:

a $e^x(e^{-x} + e^x)$ **b** $(2^x + 5)^2$ **c** $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

4 Determine whether the given point satisfies the exponential function:

a $y = 4^x - 1 \quad (2, 15)$ **b** $f(x) = 5 \times 3^{-x} \quad (-1, \frac{5}{3})$

5 Consider the exponential function $y = \left(\frac{1}{2}\right)^x + 2$.

a Copy and complete the table of values alongside.

x	-3	-2	-1	0	1	2	3
y							

b Hence graph the function.

c Copy and complete:

i As $x \to \infty$, $y \to .....$ **ii** As $x \to -\infty$, $y \to .....$

d Find the horizontal asymptote.

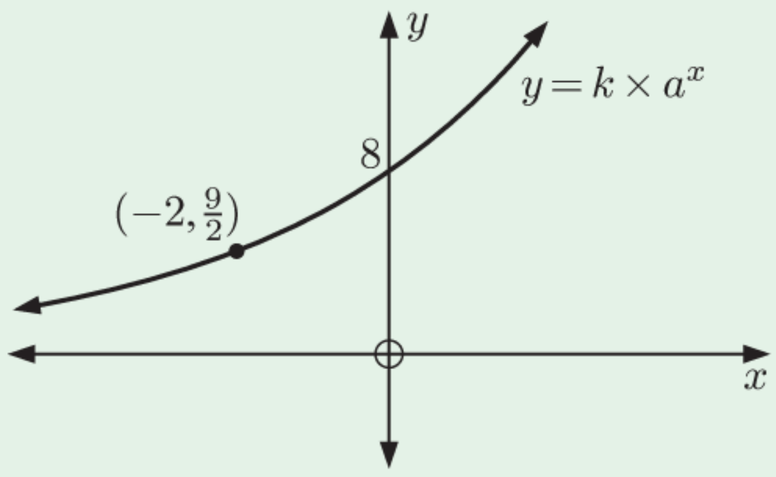
6 On the same set of axes, draw the graphs of $y = 2^x$ and $y = 2^x - 4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

7 Consider $y = 3 - 2^{-x}$.

a Find y when $x = 0, \pm 1, \pm 2$. **b** Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3 - 2^{-x}$. **d** State the equation of any asymptote.

8 This graph shows the exponential function $y = k \times a^x$, where a and k are constants, $a > 0$.



a Find k and a.

b State the equation of the horizontal asymptote.

c Find the value of y when $x = 2$.

9 Use technology to solve:

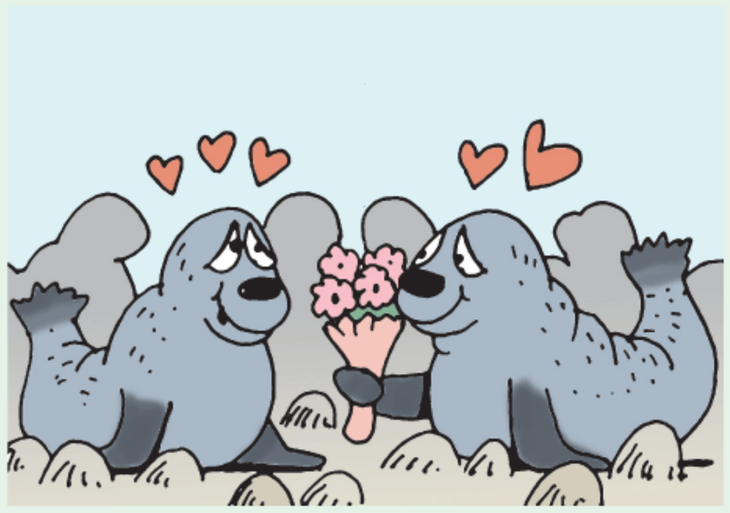
a $5^x = 1000$ **b** $15 \times (1.6)^x = 80$ **c** $400 \times (0.98)^x = 70$

10 A population of seals is given by $P(t) = 80 \times (1.15)^t$ where t is the time in years, $t \geqslant 0$.

a Find the initial population.

b Find the time required for the population to double in size.

c Find the percentage increase in population during the first 4 years.



11 Find, to 3 significant figures, the value of:

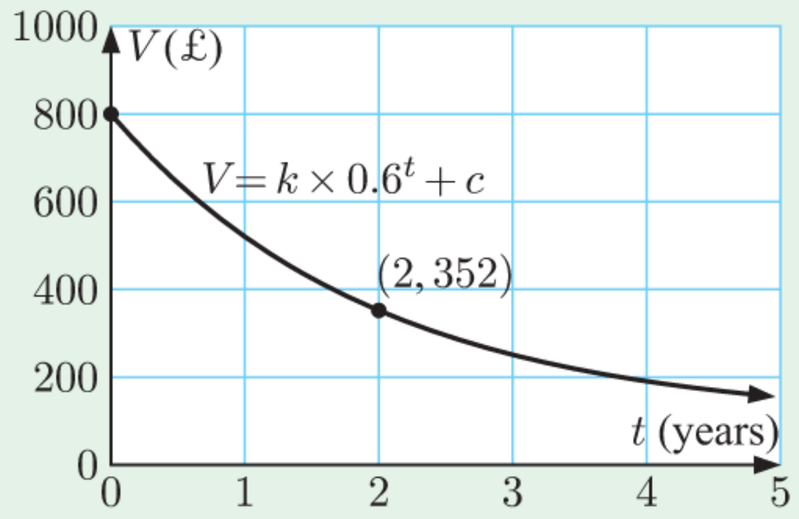
a e^4 **b** e^{-2} **c** $10e^{3.5}$ **d** $40e^{-2.53}$

12 Sketch, on the same set of axes, the graphs of:

a $y = e^x$ and $y = e^x - 2$ **b** $y = e^x$ and $y = 4e^x$

c $y = e^x$ and $y = -\frac{1}{2}e^x$ **d** $y = e^x$ and $y = e^{\frac{x}{3}}$

13 The value of a computer t years after it is purchased is given by $V = k \times 0.6^t + c$ pounds, where k and c are constants.



a Use technology to find k and c.

b Warren thinks that the value of the computer decreases by 40% each year. Is he correct? Explain your answer.

c Find the value of the computer after 3 years.

d State the equation of the horizontal asymptote and explain what it means in this situation.

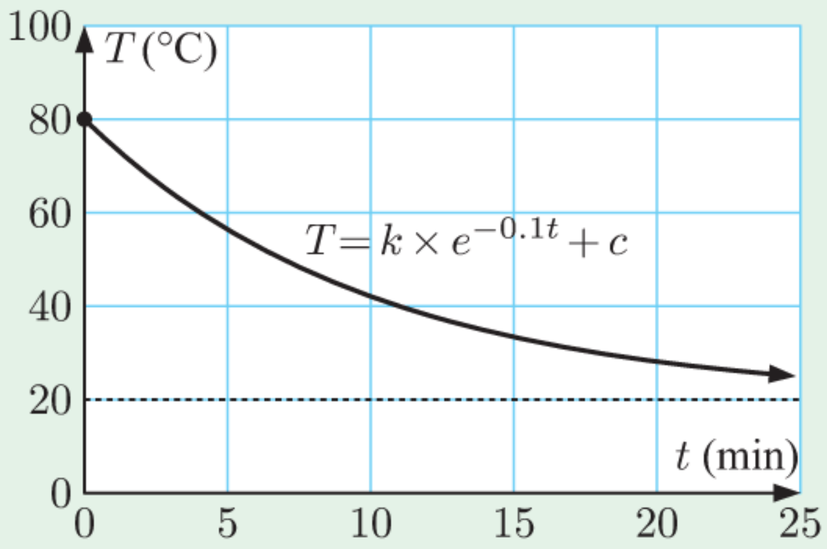
14 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the half-life of the substance.

c Find the time taken for the weight to reduce to 375 grams.

15 This graph shows the temperature of a mug of water t minutes after it has been poured from a kettle.





a Find the exponential model connecting T and t.

b Find the temperature of the water after 10 minutes.

c How long will it take for the temperature of the water to fall to 30°C?

16 The height of a seedling t weeks after it is planted is given by the logistic model

$$H(t) = \frac{L}{1 + 5e^{-\frac{t}{2}}} \text{ cm.}$$

a Given that the seedling was 10 cm tall when it was planted, find the value of L.

b Find the height of the seedling after 5 weeks.

c Sketch the graph of H against t.

d Find the time taken for the seedling's height to reach 50 cm.

REVIEW SET 1B

1 Write in the form x^k, where k is rational:

a $\sqrt[3]{x}$ **b** $x^3\sqrt{x}$ **c** $\dfrac{1}{x^2\sqrt{x}}$

2 Evaluate, correct to 3 significant figures:

a $3^{\frac{5}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

3 Expand and simplify:

a $(3 - e^x)^2$ **b** $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$ **c** $2^{-x}(2^{2x} + 2^x)$

4 Factorise:

a $3^{x+2} - 3^x$ **b** $4^x - 2^x - 12$ **c** $e^{2x} + 2e^x - 15$

5 Consider $y = 2e^{-x} + 1$.

a Find y when $x = 0, \pm1, \pm2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 2e^{-x} + 1$.

d State the equation of any asymptote.

6 Answer the **Opening Problem** on page **14**.

7 Let $f(x) = 3^x$.

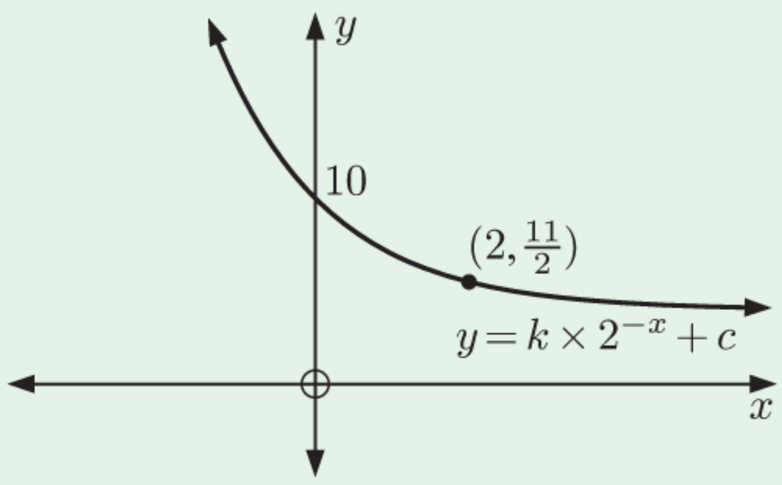
a Write down the value of:

i $f(4)$ **ii** $f(-1)$

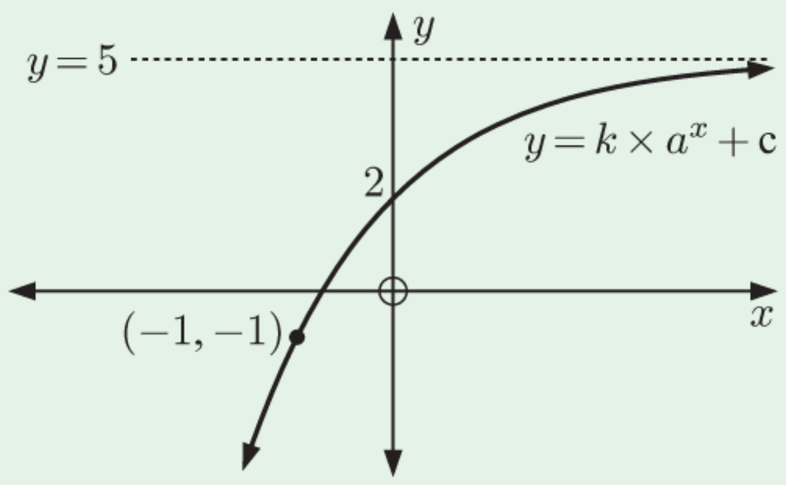
b Find the value of k such that $f(x + 2) = k\,f(x)$, $k \in \mathbb{Z}$.

8 Find the exponential model in the graph:

a



b



9 An exponential model of the form $y = k \times 5^{-x} + c$ has x-intercept 1 and y-intercept 8.

a Find the exponential model. **b** Find the value of y when $x = -1$.

10 Solve using technology:

a $4^x = 30$ **b** $2 \times (1.6)^x = 7$ **c** $50 \times (0.65)^x = 15$

11 For what values of k does the equation $4 \times (0.3)^x - 2 = k$ have:

a 1 solution **b** no solutions?

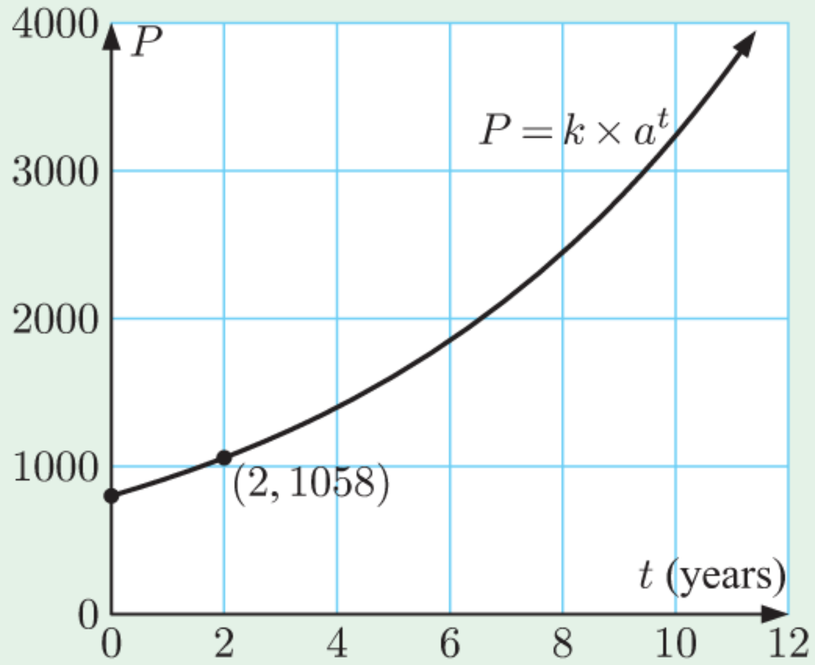
12 When a species of bird is introduced onto an island, its population increases by 15% each year. The population t years after introduction is given by $P = k \times a^t$, where k and a are constants, and $t \geqslant 0$.

a State the value of a.

b Find the value of k and interpret your answer.

c Find the population after 5 years.

d Do you think this model is likely to be accurate for very large values of t? Explain your answer.

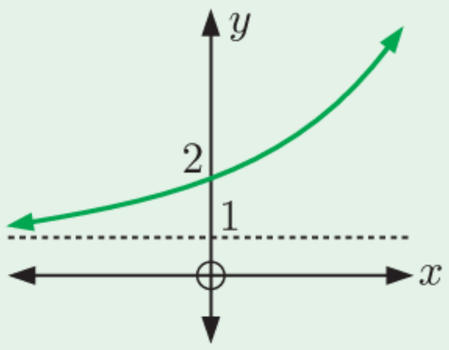


13 The weight of a substance after t days is given by $W = k \times a^t$ grams.
The substance has a half-life of 5 days, and it weighs 400 g after 2 days.
Find the weight of the substance after 9 days.

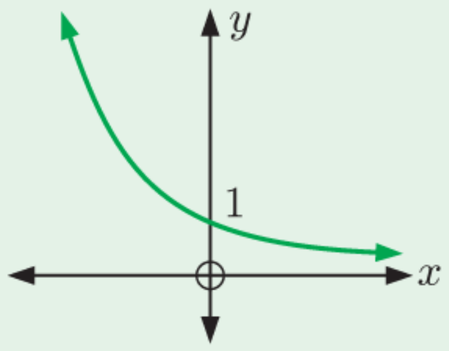
14 Match each equation to its corresponding graph:

a $y = -e^x$ **b** $y = 3 \times 2^x$ **c** $y = e^x + 1$ **d** $y = 3^{-x}$ **e** $y = -e^{-x}$

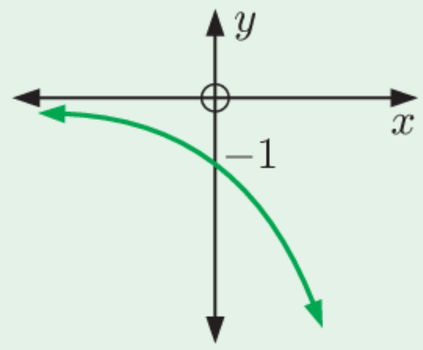
A



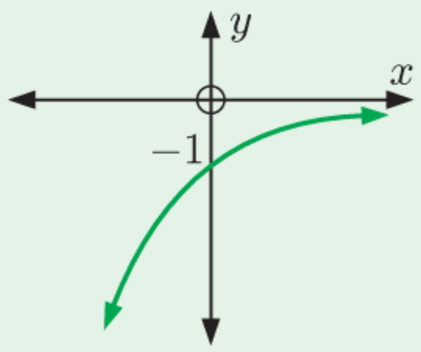
B



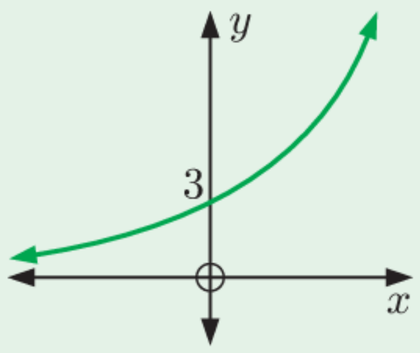
C



D



E



15 **a** Use technology to help sketch the graph of $f(x) = 3 - e^x$.

b State the domain and range of the function.

c Describe the behaviour of the function as $x \to \pm\infty$.

16 A small island has an abundance of insects and fruits which are food for an endangered species of mammal. A small group of the species is introduced to the island including 10 adult females.

The number of adult females on the island after t months is given by $N = \dfrac{L}{1 + Ce^{-\frac{t}{10}}}$.

a Given that there were 20 adult females on the island after 10 months, find L and C.

b Find the number of adult females on the island after 15 months.

c Find the time taken for the number of adult females to reach 40.

Chapter 2

Logarithms

Contents:

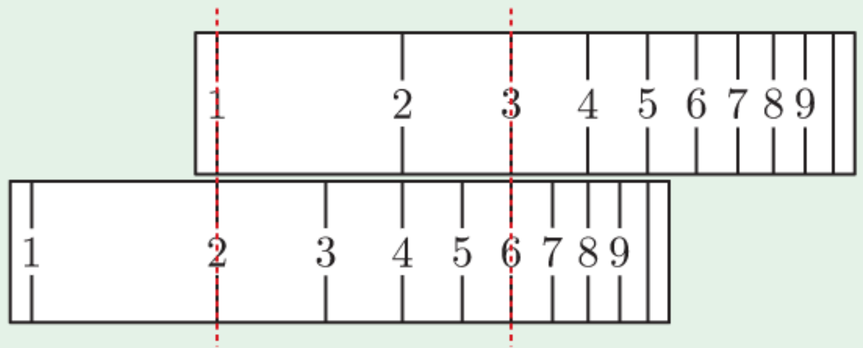
A Logarithms in base 10
B Laws of logarithms
C Natural logarithms
D Logarithmic functions
E Logarithmic scales



OPENING PROBLEM

The **slide rule** was invented by **William Oughtred** in 1622 as a tool to perform multiplication and division more quickly.

A simplified slide rule is shown alongside.



To multiply 2 and 3, we align the "1" on the top scale with the "2" on the bottom scale. We then notice that the "3" on the top scale aligns with the "6" on the bottom scale. This tells us that $2 \times 3 = 6$.

Things to think about:

a Are the integers on the scales equally spaced?

b Let $L(X)$ be the distance between 1 and X on these scales.

i Explain why $L(1) = 0$.

ii For this multiplication method to work, explain why $L(A) + L(B) = L(AB)$.

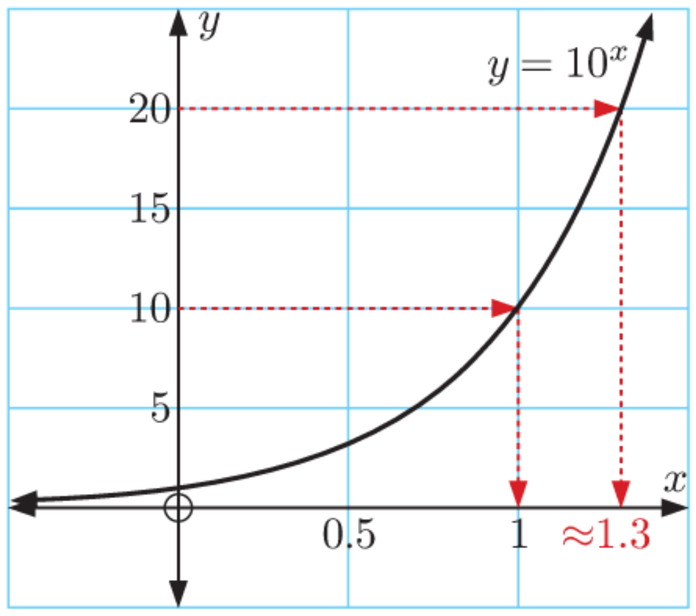
c Suppose we write X as a power of 10, so $X = 10^x$, and let $L(X) = x$. Show that this function satisfies both properties in **b**.

d *Describe* how the integer markings are spaced on the scale.

e Describe how you would perform division using the slide rule.

A LOGARITHMS IN BASE 10

Consider the graph of $y = 10^x$ shown.



Notice that the range of the function is $\{y \mid y > 0\}$. This means that every positive number y can be written in the form 10^x.

For example:

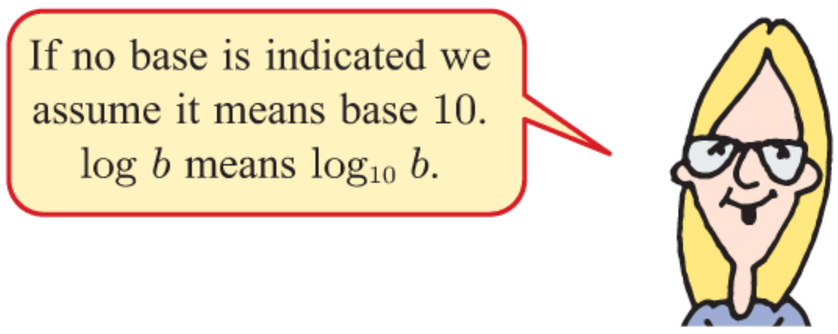
- When $y = 10$, $x = 1$, so $10 = 10^1$.
- When $y = 20$, $x \approx 1.3$, so $20 \approx 10^{1.3}$.

When we write a positive number y in the form 10^x, we say that x is the **logarithm in base 10**, of y.

The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain that number.

For example:

- The logarithm in base 10 of 1000 is 3, since $1000 = 10^3$.
  We write $\log_{10} 1000 = 3$ or simply $\log 1000 = 3$.
- $\log(0.01) = -2$ since $0.01 = 10^{-2}$.



By observing that $\log 1000 = \log(10^3) = 3$ and $\log(0.01) = \log(10^{-2}) = -2$, we conclude that $\log 10^x = x$ for any $x \in \mathbb{R}$.

Example 1 — Self Tutor

Find:

a $\log 100$ **b** $\log(0.1)$ **c** $\log \sqrt[4]{10}$

a $\log 100 = \log(10^2)$
$= 2$

b $\log(0.1) = \log(10^{-1})$
$= -1$

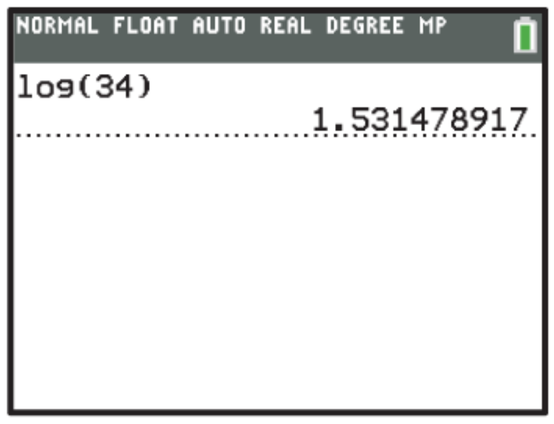
c $\log \sqrt[4]{10} = \log(10^{\frac{1}{4}})$
$= \frac{1}{4}$

The logarithms in **Example 1** can be found by hand because it is easy to write 100, 0.1, and $\sqrt[4]{10}$ as powers of 10. The logarithms of most values, however, can only be found using a calculator.

For example, $\log 34 \approx 1.53$
so $34 \approx 10^{1.53}$



GRAPHICS CALCULATOR INSTRUCTIONS



Logarithms allow us to write any number as a power of 10. In particular:

$$x = 10^{\log x} \quad \text{for any} \quad x > 0.$$

Example 2 — Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

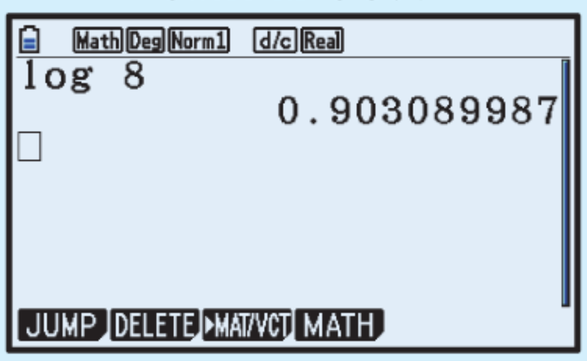
a 8 **b** 800 **c** 0.08

a $8 = 10^{\log 8}$
$\approx 10^{0.9031}$

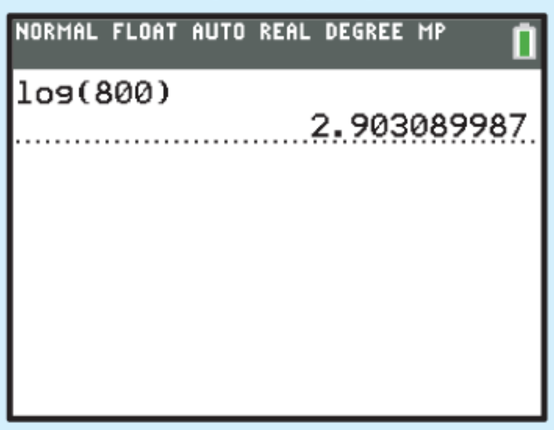
b $800 = 10^{\log 800}$
$\approx 10^{2.9031}$

c $0.08 = 10^{\log 0.08}$
$\approx 10^{-1.0969}$

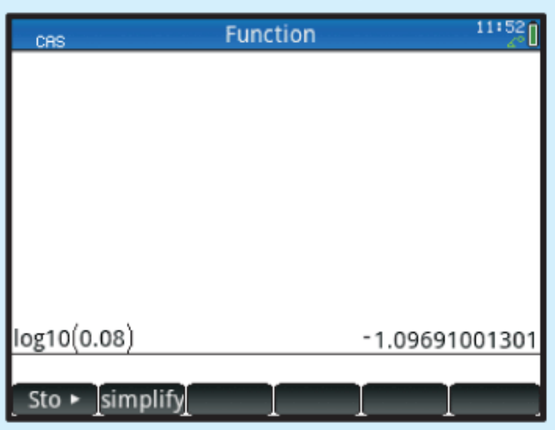
Casio fx-CG50



TI-84 Plus CE



HP Prime



EXERCISE 2A

1 Without using a calculator, find:

a $\log 10\,000$ **b** $\log 0.001$ **c** $\log 10$ **d** $\log 1$

e $\log \sqrt{10}$ **f** $\log \sqrt[3]{10}$ **g** $\log\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\log\left(10\sqrt{10}\right)$

i $\log \sqrt[3]{100}$ **j** $\log\left(\frac{100}{\sqrt{10}}\right)$ **k** $\log\left(10 \times \sqrt[3]{10}\right)$ **l** $\log\left(1000\sqrt{10}\right)$

Check your answers using your calculator.

2 Simplify:

a $\log(10^n)$ **b** $\log(10^a \times 100)$ **c** $\log\left(\frac{10}{10^m}\right)$ **d** $\log\left(\frac{10^a}{10^b}\right)$

3 **a** Explain why $\log 237$ must lie between 2 and 3.

b Use your calculator to evaluate $\log 237$ correct to 2 decimal places.

4 **a** Between which two consecutive whole numbers does $\log(0.6)$ lie?

b Check your answer by evaluating $\log(0.6)$ correct to 2 decimal places.

5 Use your calculator to evaluate, correct to 2 decimal places:

a $\log 76$ **b** $\log 114$ **c** $\log 3$ **d** $\log 831$

e $\log(0.4)$ **f** $\log 3247$ **g** $\log(0.008)$ **h** $\log(-7)$

6 For what values of x is $\log x$:

a positive **b** zero **c** negative **d** undefined?

7 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

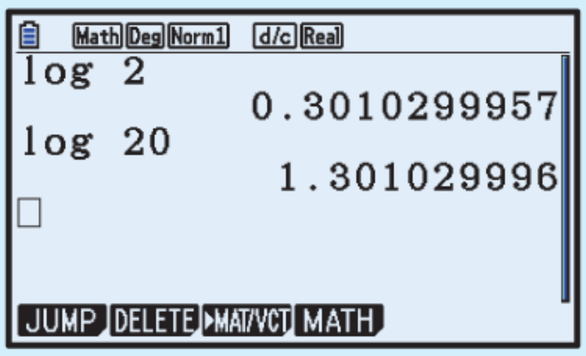
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 3 — Self Tutor

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$

b Explain why $\log 20 = \log 2 + 1$.

a



i $\log 2 \approx 0.3010$

ii $\log 20 \approx 1.3010$

b $\log 20 = \log(2 \times 10)$

$= \log(10^{\log 2} \times 10^1)$ $\{x = 10^{\log x}\}$

$= \log(10^{\log 2 + 1})$ {adding indices}

$= \log 2 + 1$

8 **a** Use your calculator to find: **i** $\log 3$ **ii** $\log 300$

b Explain why $\log 300 = \log 3 + 2$.

9 **a** Use your calculator to find: **i** $\log 5$ **ii** $\log(0.05)$

b Explain why $\log(0.05) = \log 5 - 2$.

Example 4 — Self Tutor

Find x such that:

a $\log x = 3$ **b** $\log x \approx -0.271$

a $\log x = 3$

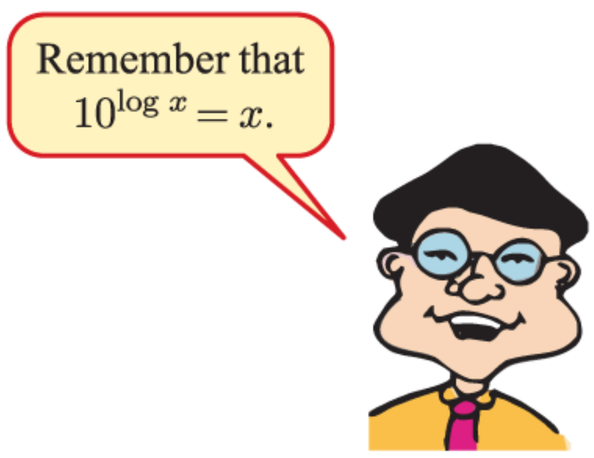
$\therefore \ 10^{\log x} = 10^3$

$\therefore \ x = 1000$

b $\log x \approx -0.271$

$\therefore \ 10^{\log x} \approx 10^{-0.271}$

$\therefore \ x \approx 0.536$



10 Find x such that:

a $\log x = 2$ **b** $\log x = 1$ **c** $\log x = 0$

d $\log x = -1$ **e** $\log x = \frac{1}{2}$ **f** $\log x = -\frac{1}{2}$

g $\log x = 4$ **h** $\log x = -5$ **i** $\log x \approx 0.8351$

j $\log x \approx 2.1457$ **k** $\log x \approx -1.378$ **l** $\log x \approx -3.1997$

11 Suppose the point (a, b) lies on the graph of $y = 10^x$.

a Write an equation connecting a and b.

b Hence show that $\log b = a$.

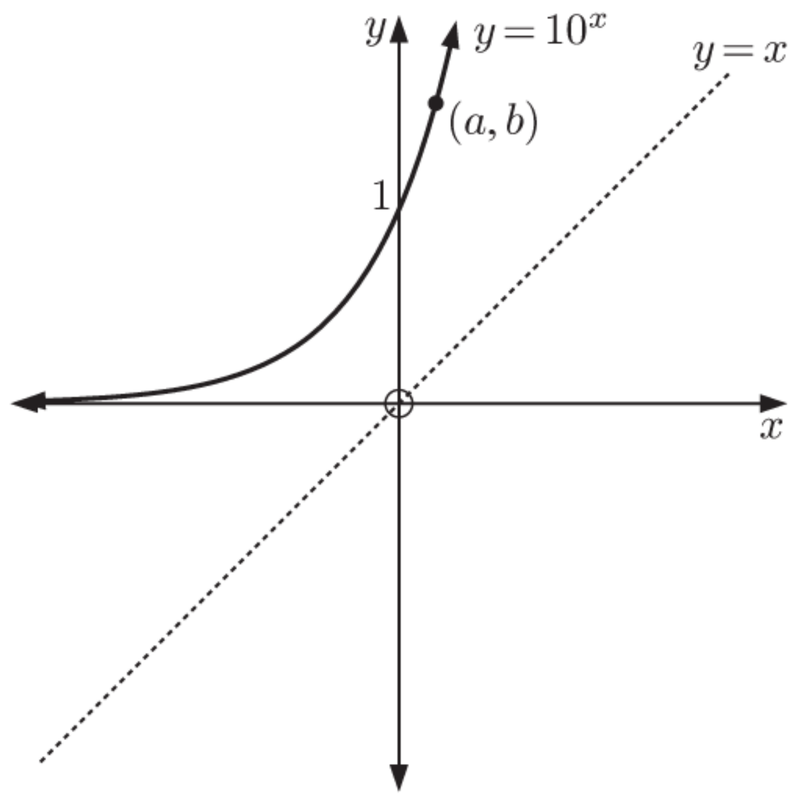
c Explain why the point (b, a) lies on the graph of $y = \log x$.

d What is the relationship between the functions $y = 10^x$ and $y = \log x$?

e Copy the graph of $y = 10^x$, and sketch the graph of $y = \log x$ on the same set of axes.

f Find the x-intercept of $y = \log x$.

g State the domain and range of $y = \log x$.



B LAWS OF LOGARITHMS

INVESTIGATION 1 DISCOVERING THE LAWS OF LOGARITHMS

What to do:

1 **a** Use your calculator to find:

i $\log 2 + \log 3$ **ii** $\log 3 + \log 7$ **iii** $\log 4 + \log 20$

iv $\log 6$ **v** $\log 21$ **vi** $\log 80$

b From your answers, suggest a possible simplification for $\log m + \log n$.

2 **a** Use your calculator to find:

i $\log 6 - \log 2$ **ii** $\log 12 - \log 3$ **iii** $\log 3 - \log 5$

iv $\log 3$ **v** $\log 4$ **vi** $\log(0.6)$

b From your answers, suggest a possible simplification for $\log m - \log n$.

3 **a** Use your calculator to find:

i $3 \log 2$ **ii** $2 \log 5$ **iii** $-4 \log 3$

iv $\log(2^3)$ **v** $\log(5^2)$ **vi** $\log(3^{-4})$

b From your answers, suggest a possible simplification for $m \log b$.

From the **Investigation**, you should have discovered the three important **laws of logarithms**:

- $\log m + \log n = \log(mn)$ for $m, n > 0$
- $\log m - \log n = \log\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \log b = \log(b^m)$ for $b > 0$

Proof:

- $\log(mn)$
  $= \log\left(10^{\log m} \times 10^{\log n}\right)$
  $= \log\left(10^{\log m + \log n}\right)$
  $= \log m + \log n$
- $\log\left(\frac{m}{n}\right)$
  $= \log\left(\frac{10^{\log m}}{10^{\log n}}\right)$
  $= \log\left(10^{\log m - \log n}\right)$
  $= \log m - \log n$
- $\log(b^m)$
  $= \log\left(\left(10^{\log b}\right)^m\right)$
  $= \log\left(10^{m \log b}\right)$
  $= m \log b$

Example 5 — Self Tutor

Use the laws of logarithms to write as a single logarithm or as an integer:

a $\log 5 + \log 3$ b $\log 80 - \log 8$ c $\log 7 - 1$

a
$\log 5 + \log 3$
$= \log(5 \times 3)$
$= \log 15$

b
$\log 80 - \log 8$
$= \log\left(\frac{80}{8}\right)$
$= \log 10$
$= 1$

c
$\log 7 - 1$
$= \log 7 - \log 10$
$= \log\left(\frac{7}{10}\right)$

Example 6 — Self Tutor

Simplify by writing as a single logarithm or as a rational number:

a $2 \log 7 - 3 \log 2$ b $2 \log 3 + 3$ c $\frac{\log 8}{\log 4}$

a
$2 \log 7 - 3 \log 2$
$= \log(7^2) - \log(2^3)$
$= \log 49 - \log 8$
$= \log\left(\frac{49}{8}\right)$

b
$2 \log 3 + 3$
$= \log(3^2) + \log(10^3)$
$= \log 9 + \log 1000$
$= \log 9000$

c
$\frac{\log 8}{\log 4} = \frac{\log(2^3)}{\log(2^2)}$
$= \frac{3 \log 2}{2 \log 2}$
$= \frac{3}{2}$

EXERCISE 2B

1 Write as a single logarithm or as an integer:

a $\log 8 + \log 2$ b $\log 4 + \log 5$ c $\log 40 - \log 5$

d $\log p - \log m$ e $\log 8 - \log 2$ f $\log 5 + \log(0.4)$

g $\log 250 + \log 4$ h $\log 100 - \log 4$ i $\log 2 + \log 3 + \log 4$

j $\log 5 + \log 4 - \log 2$ k $\log 6 - \log 2 - \log 3$ l $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$

2 Write as a single logarithm:

a $\log 7 + 2$ **b** $\log 4 - 1$ **c** $1 + \log 3$

d $\log 5 - 2$ **e** $2 + \log 2$ **f** $\log 50 - 4$

g $t + \log w$ **h** $\log 40 - 2$ **i** $3 - \log 50$

3 Write as a single logarithm or integer:

a $5\log 2 + \log 3$ **b** $2\log 3 + 3\log 2$ **c** $3\log 4 - \log 8$

d $2\log 5 - 3\log 2$ **e** $\frac{1}{2}\log 4 + \log 3$ **f** $\frac{1}{3}\log\left(\frac{1}{8}\right)$

g $3 - \log 4 - 2\log 5$ **h** $1 - 3\log 2 + \log 20$ **i** $2 - \frac{1}{2}\log 4 - \log 5$

4 Simplify without using a calculator:

a $\dfrac{\log 4}{\log 2}$ **b** $\dfrac{\log 27}{\log 9}$ **c** $\dfrac{\log 8}{\log 2}$ **d** $\dfrac{\log 3}{\log 9}$ **e** $\dfrac{\log 25}{\log(0.2)}$ **f** $\dfrac{\log 8}{\log(0.25)}$

Example 7 **Self Tutor**

Show that:

a $\log\left(\frac{1}{9}\right) = -2\log 3$ **b** $\log 500 = 3 - \log 2$

a
$$\log\left(\tfrac{1}{9}\right) = \log(3^{-2})$$
$$= -2\log 3$$

b
$$\log 500 = \log\left(\tfrac{1000}{2}\right)$$
$$= \log 1000 - \log 2$$
$$= \log(10^3) - \log 2$$
$$= 3 - \log 2$$

5 Show that:

a $\log 9 = 2\log 3$ **b** $\log\sqrt{2} = \frac{1}{2}\log 2$ **c** $\log\left(\frac{1}{8}\right) = -3\log 2$

d $\log\left(\frac{1}{5}\right) = -\log 5$ **e** $\log 5 = 1 - \log 2$ **f** $\log 5000 = 4 - \log 2$

6 The number $a \times 10^k$ where $1 \leqslant a < 10$, $k \in \mathbb{Z}$ is written in standard form. Show that $\log(a \times 10^k) = \log a + k$.

7 Suppose $p = \log 2$, $q = \log 3$, and $r = \log 5$. Write in terms of p, q, and r:

a $\log 6$ **b** $\log 45$ **c** $\log 108$

d $\log\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log\left(\frac{5}{32}\right)$ **f** $\log\left(\frac{2}{9}\right)$

8 Suppose $\log P = x$, $\log Q = y$, and $\log R = z$. Write in terms of x, y, and z:

a $\log(PR)$ **b** $\log(RQ^2)$ **c** $\log\left(\dfrac{PR}{Q}\right)$

d $\log\left(P^2\sqrt{Q}\right)$ **e** $\log\left(\dfrac{Q^3}{\sqrt{R}}\right)$ **f** $\log\left(\dfrac{R^2\sqrt{Q}}{P^3}\right)$

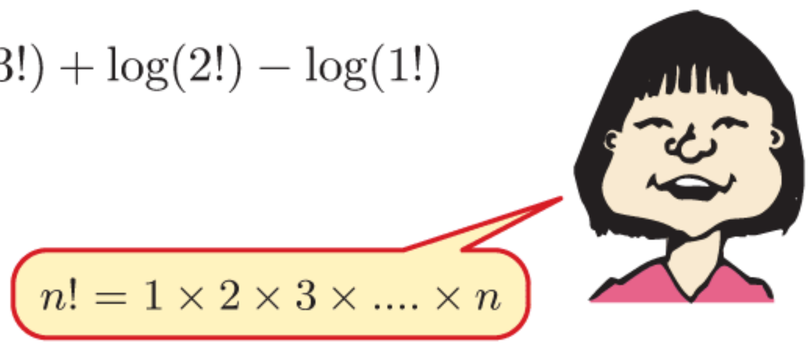
9 If $\log M = 1.29$ and $\log(N^2) = 1.72$, find:

a $\log N$ **b** $\log(MN)$ **c** $\log\left(\dfrac{N^2}{\sqrt{M}}\right)$

10 Write as a single logarithm:

$\log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!)$

11 Write $\log(5!)$ in the form $a + \log b$, where $a, b \in \mathbb{Z}$ and b is as small as possible.



C NATURAL LOGARITHMS

In the previous Section we considered the logarithm in base 10.

The logarithm in base e is called the **natural logarithm**.

The **natural logarithm** of a positive number is the power that e must be raised to in order to obtain that number.

The natural logarithm is represented as $\log_e x$, or more commonly as $\ln x$.

For all $x > 0$: $\ln e^x = x$ and $e^{\ln x} = x$.

Example 8 — Self Tutor

Find:

a $\ln(e^3)$ **b** $\ln\left(\frac{1}{e^4}\right)$ **c** $\ln\sqrt{e}$ **d** $e^{2\ln 5}$

a $\ln(e^3) = 3$

b $\ln\left(\frac{1}{e^4}\right) = \ln(e^{-4}) = -4$

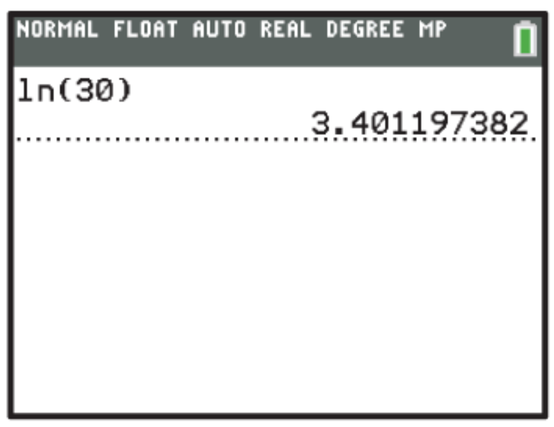
c $\ln\sqrt{e} = \ln(e^{\frac{1}{2}}) = \frac{1}{2}$

d $e^{2\ln 5} = (e^{\ln 5})^2 = 5^2 = 25$

As with base 10 logarithms, we can use our calculator to find natural logarithms.

For example, $\ln 30 \approx 3.40$, which means that $30 \approx e^{3.40}$.





LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

- $\ln m + \ln n = \ln(mn)$ for $m, n > 0$
- $\ln m - \ln n = \ln\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \ln b = \ln(b^m)$ for $b > 0$

EXERCISE 2C

1 Without using a calculator, find:

a $\ln(e^2)$ **b** $\ln(e^4)$ **c** $\ln\left((\sqrt{e})^3\right)$ **d** $\ln 1$

e $\ln\left(\frac{1}{e}\right)$ **f** $\ln \sqrt[3]{e}$ **g** $\ln\left(\frac{1}{e^2}\right)$ **h** $\ln\left(\frac{1}{\sqrt{e}}\right)$

Check your answers using a calculator.

2 Simplify:

a $e^{\ln 3}$ **b** $e^{2\ln 3}$ **c** $e^{-\ln 5}$ **d** $e^{-2\ln 2}$

e $\ln e^a$ **f** $\ln(e \times e^a)$ **g** $\ln(e^a \times e^b)$ **h** $\ln\left((e^a)^b\right)$

3 **a** Use your calculator to evaluate e^3 and e^4 correct to 4 decimal places.

b Between which two consecutive whole numbers does $\ln 40$ lie?

c Find $\ln 40$ correct to 3 decimal places.

4 Use your calculator to find, correct to 3 decimal places:

a $\ln 12$ **b** $\ln 68$ **c** $\ln(1.4)$ **d** $\ln(0.7)$ **e** $\ln 500$

5 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

Example 9 — Self Tutor

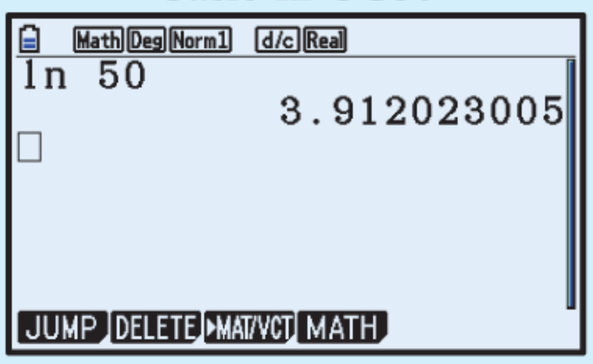
Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

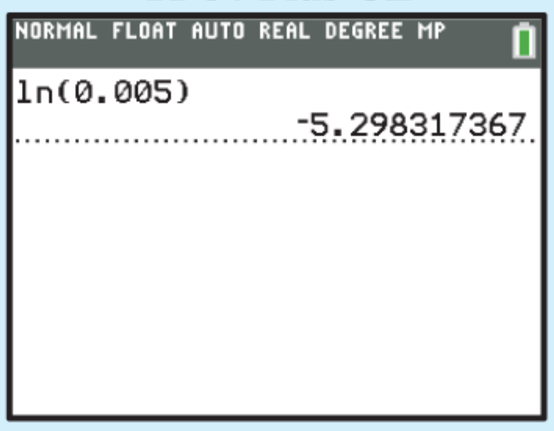
a $50 = e^{\ln 50}$ $\{x = e^{\ln x}\}$
$\approx e^{3.9120}$

b $0.005 = e^{\ln 0.005}$
$\approx e^{-5.2983}$

Casio fx-CG50



TI-84 Plus CE



6 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006

f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 10 — Self Tutor

Find x if:

a $\ln x = 2.17$ **b** $\ln x = -0.384$

a $\ln x = 2.17$

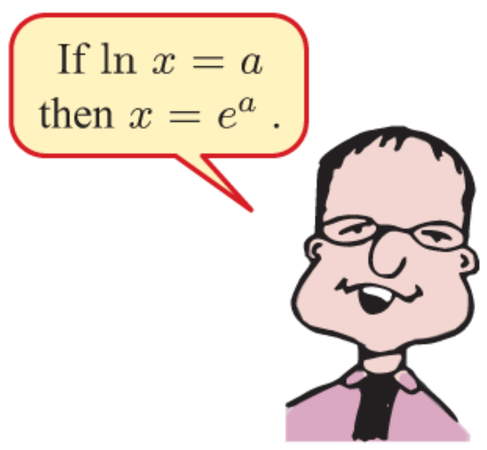
$\therefore\ x = e^{2.17}$

$\therefore\ x \approx 8.76$

b $\ln x = -0.384$

$\therefore\ x = e^{-0.384}$

$\therefore\ x \approx 0.681$



7 Find x if:

a $\ln x = 3$ **b** $\ln x = 1$ **c** $\ln x = 0$ **d** $\ln x = -1$

e $\ln x = -5$ **f** $\ln x \approx 0.835$ **g** $\ln x \approx 2.145$ **h** $\ln x \approx -3.2971$

8 **a** Write in simplest form:

i $\ln(e^x)$ **ii** $e^{\ln x}$

b What does this tell us about the functions $y = e^x$ and $y = \ln x$?

c Sketch the graphs of $y = e^x$ and $y = \ln x$ on the same set of axes.

d State the domain and range of $y = \ln x$.

Example 11 — Self Tutor

Use the laws of logarithms to write as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a $\ln 5 + \ln 3$

$= \ln(5 \times 3)$

$= \ln 15$

b $\ln 24 - \ln 8$

$= \ln\left(\frac{24}{8}\right)$

$= \ln 3$

c $\ln 5 - 1$

$= \ln 5 - \ln(e^1)$

$= \ln\left(\frac{5}{e}\right)$

9 Write as a single logarithm or integer:

a $\ln 15 + \ln 3$ **b** $\ln 15 - \ln 3$ **c** $\ln 20 - \ln 5$

d $\ln 4 + \ln 6$ **e** $\ln 5 + \ln(0.2)$ **f** $\ln 2 + \ln 3 + \ln 5$

g $1 + \ln 4$ **h** $\ln 6 - 1$ **i** $\ln 5 + \ln 8 - \ln 2$

j $2 + \ln 4$ **k** $\ln 20 - 2$ **l** $\ln 12 - \ln 4 - \ln 3$

Example 12 — Self Tutor

Use the laws of logarithms to simplify:

a $2\ln 7 - 3\ln 2$ **b** $2\ln 3 + 3$

a $2\ln 7 - 3\ln 2 = \ln(7^2) - \ln(2^3)$

$= \ln 49 - \ln 8$

$= \ln\left(\frac{49}{8}\right)$

b $2\ln 3 + 3 = \ln(3^2) + \ln(e^3)$

$= \ln 9 + \ln(e^3)$

$= \ln(9e^3)$

10 Write in the form $\ln a$, $a \in \mathbb{R}$:

a $5\ln 3 + \ln 4$ **b** $3\ln 2 + 2\ln 5$ **c** $3\ln 2 - \ln 8$

d $3\ln 4 - 2\ln 2$ **e** $\frac{1}{3}\ln 8 + \ln 3$ **f** $\frac{1}{3}\ln\left(\frac{1}{27}\right)$

g $-\ln 2$ **h** $-\ln\left(\frac{1}{2}\right)$ **i** $-2\ln\left(\frac{1}{4}\right)$

j $4\ln 2 + 2$ **k** $\frac{1}{2}\ln 9 - 1$ **l** $-3\ln 2 + \frac{1}{2}$

11 Show that:

a $\ln 27 = 3\ln 3$ **b** $\ln\sqrt{3} = \frac{1}{2}\ln 3$ **c** $\ln\left(\frac{1}{16}\right) = -4\ln 2$

d $\ln\left(\frac{1}{6}\right) = -\ln 6$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

g $\ln(6e) = \ln 6 + 1$ **h** $\ln\sqrt[3]{5} = \frac{1}{3}\ln 5$ **i** $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5}\ln 2$

j $\ln\left(\frac{e^2}{8}\right) = 2 - 3\ln 2$ **k** $\ln\left(\frac{\sqrt{3}}{e^4}\right) = \frac{1}{2}\ln 3 - 4$ **l** $\ln\left(\frac{1}{16 \times \sqrt[3]{e}}\right) = -4\ln 2 - \frac{1}{3}$

THEORY OF KNOWLEDGE

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and techniques such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms*. John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.



Johannes Kepler

Logarithms were an extremely important development, and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used logarithms to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".

1 Can anyone claim to have *invented* logarithms?

2 Can we consider the process of mathematical discovery as an *evolution* of ideas?

3 Has modern computing effectively doubled the life of a mathematician?

Many areas of mathematics have been developed over centuries as several mathematicians have worked in a particular area, or taken the knowledge from one area and applied it to another field. Sometimes the process is held up because a method for solving a particular class of problem has not yet been found. In other cases, pure mathematicians have published research papers on seemingly useless mathematical ideas, which have then become vital in applications much later.

In *Everybody Counts*: *A report to the nation on the future of Mathematical Education* by the National Academy of Sciences (National Academy Press, 1989), there is an excellent section on the Nature of Mathematics. It includes:

> "Even the most esoteric and abstract parts of mathematics - number theory and logic, for example - are now used routinely in applications (for example, in computer science and cryptography). Fifty years ago, the leading British mathematician G.H. Hardy could boast that number theory was the most pure and least useful part of mathematics. Today, Hardy's mathematics is studied as an essential prerequisite to many applications, including control of automated systems, data transmission from remote satellites, protection of financial records, and efficient algorithms for computation."

4 Should we only study the mathematics required to enter our chosen profession?

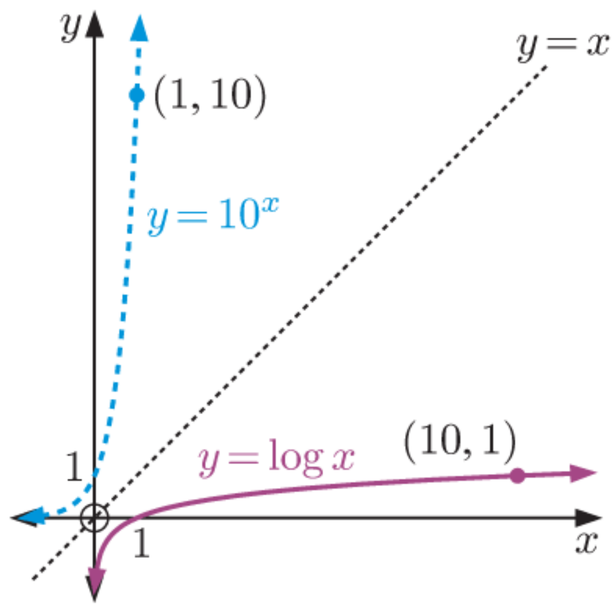
5 Why should we explore mathematics for its own sake, rather than to address the needs of science?

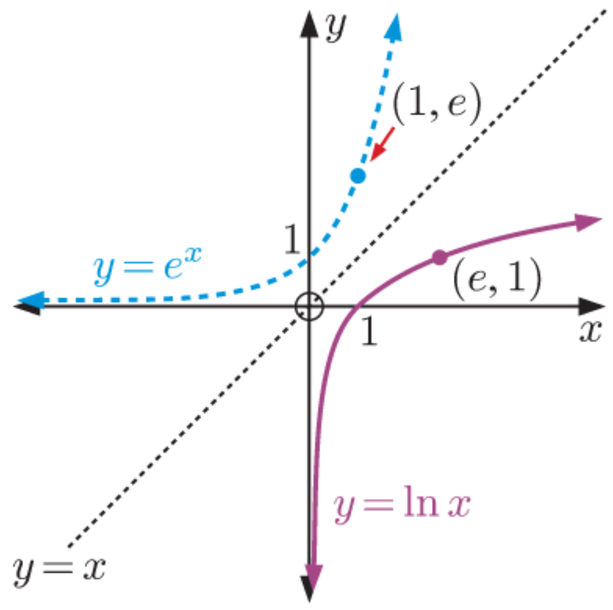
D LOGARITHMIC FUNCTIONS

We have seen that $\log 10^x = 10^{\log x} = x$ for all $x > 0$

and that $\ln e^x = e^{\ln x} = x$ for all $x > 0$.

The logarithmic functions $\log x$ and $\ln x$ are therefore the **inverses** of the exponential functions 10^x and e^x respectively. We can graph the logarithmic functions by reflecting the appropriate exponential function in the line $y = x$.





We can compare the properties of basic exponential and logarithmic functions as follows:

Function	$y = 10^x$ $y = e^x$	$y = \log x$ $y = \ln x$
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$
Asymptote	horizontal $y = 0$	vertical $x = 0$

Example 13 **Self Tutor**

Consider the function $f(x) = 2\ln x - 1$.

a State the transformations which map $y = \ln x$ to $y = f(x)$.

b Find the domain and range of f.

c Find: i $f(1)$ ii $f(e)$

d Find the x-intercept, using technology if necessary.

e Sketch the graph of $y = f(x)$ showing all important features.

f Find the inverse function f^{-1}.

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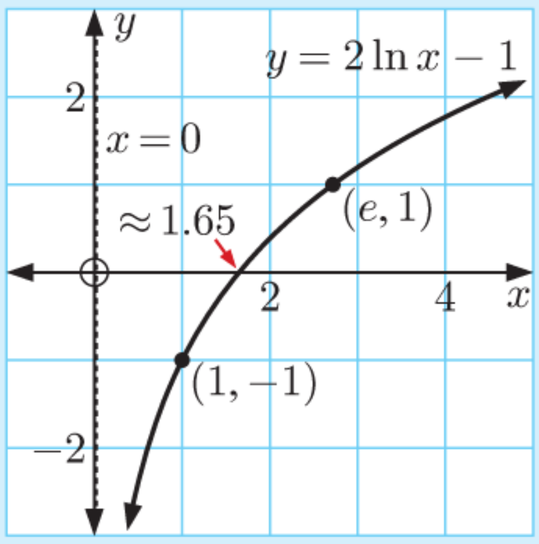
a To transform $y = \ln x$ to $y = 2\ln x - 1$, we perform a vertical stretch with scale factor 2, followed by a translation through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

b The domain is $x > 0$ and the range is $y \in \mathbb{R}$.

c i $f(1) = 2\ln 1 - 1$
$= -1$

ii $f(e) = 2\ln e - 1$
$= 1$

d When $f(x) = 0$, $2\ln x - 1 = 0$
$\therefore \ln x = \frac{1}{2}$
$\therefore x = e^{\frac{1}{2}} \approx 1.65$

e 

f f is defined by $y = 2\ln x - 1$
$\therefore f^{-1}$ is defined by $x = 2\ln y - 1$
$\therefore x + 1 = 2\ln y$
$\therefore \frac{x+1}{2} = \ln y$
$\therefore e^{\frac{x+1}{2}} = y$
$\therefore f^{-1}(x) = e^{\frac{x+1}{2}}$

EXERCISE 2D

1 For each of the functions f:

i State the transformation(s) which map $y = \ln x$ to $y = f(x)$.

ii State the domain and range.

iii Find $f(1)$ and $f(e)$.

iv Find the x-intercept, using technology if necessary.

v Sketch the graph of $y = f(x)$, showing all important features.

vi Find the inverse function f^{-1}.

a $f(x) = \ln x - 4$

b $f(x) = \ln(x - 1) + 2$

c $f(x) = 3\ln x - 1$

d $f(x) = 3 - \ln x$

2 For each of the following functions f:

i Find the domain and range. **ii** Find any asymptotes and axes intercepts.

iii Sketch the graph of $y = f(x)$, showing all important features.

a $f(x) = \log x - 2$ **b** $f(x) = \log(x + 1)$

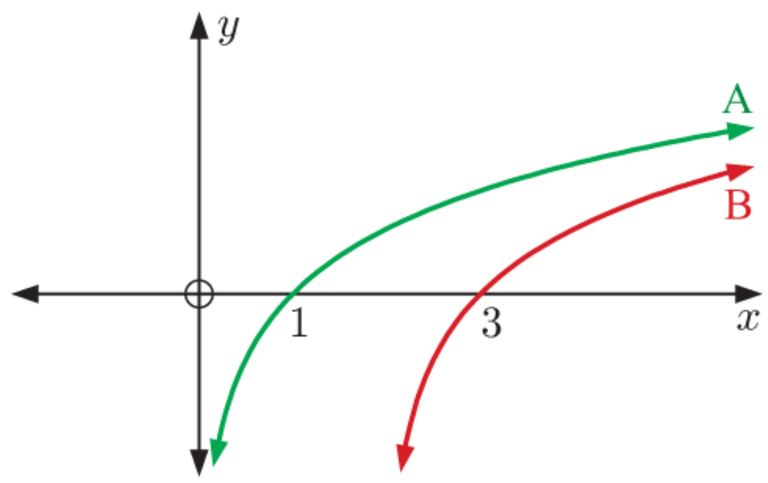
c $f(x) = 2\log x + 1$ **d** $f(x) = 1 - \log x$

3 Consider the curves A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x - 2)$.

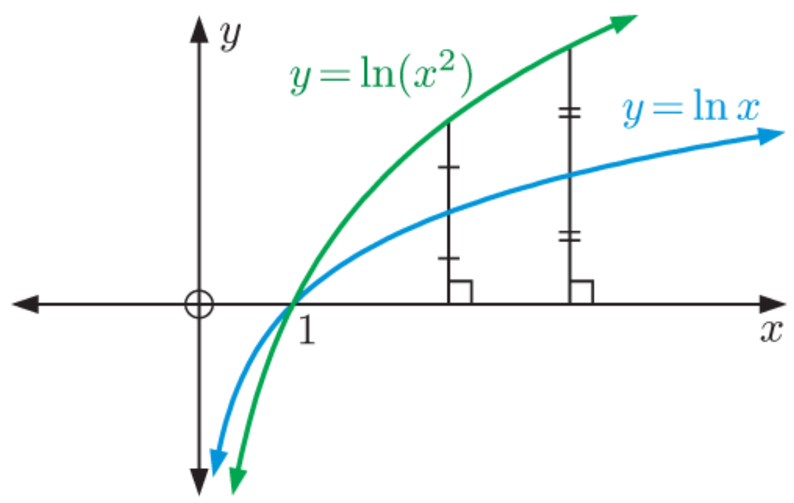
a Identify which curve is which, giving evidence for your answer.

b Copy the graphs onto a new set of axes, and then draw the graph of $y = \ln(x + 2)$.

c Find the equation of the vertical asymptote for each graph.



4 Kelly said that in order to graph $y = \ln(x^2)$, $x > 0$, you could first graph $y = \ln x$ and then double the distance of each point on the graph from the x-axis. Is Kelly correct? Explain your answer.



5 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x^3)$ **b** $y = \ln x$ and $y = \ln\left(\frac{1}{x}\right)$

c $y = \ln x$ and $y = \ln(x + e)$ **d** $y = \ln x$ and $y = \ln(x - 2) - 3$

e $y = 2\ln x$ and $y = \ln(x^2) + 2$

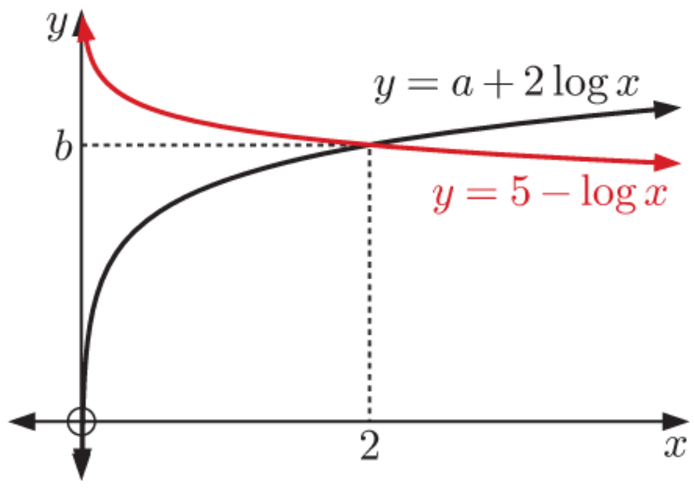
6 **a** Write $\ln(ex^2)$ in the form $a + b\ln x$, $x > 0$.

b Hence sketch the graph of $y = \ln(ex^2)$, $x > 0$.

7 The graphs of $y = a + 2\log x$ and $y = 5 - \log x$ are shown alongside.

Find, in the form $\log k$ where $k \in \mathbb{R}$:

a b **b** a



8 Given $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$, find:

a $(f^{-1} \circ g)(x)$ **b** $(g \circ f)^{-1}(x)$

E LOGARITHMIC SCALES

With some real-world quantities, there is enormous variation in the size of measurements we make.

For example, the table below shows the energy associated with some physical processes.

Energy (J)	*Process*
3×10^{-15}	average kinetic energy of a human red blood cell
7×10^{-9}	energy needed to raise a grain of sand by 1 mm
2×10^{0}	kinetic energy when a 200 g apple falls 1 m
1.4×10^{3}	solar energy received by 1 m^2 of the Earth's surface directly under the Sun in 1 second
4×10^{7}	energy from the combustion of 1 m^3 of natural gas
6×10^{20}	total world energy consumption in 2016

If we tried to place these values on a standard number line where 1 mm represented 1 J of energy, then the smallest two values would be indistinguishable from zero, and the largest value would be placed about 4 million times the distance from the Earth to the Sun.

Clearly we cannot use a standard number line to compare these numbers, so instead we *scale* them using logarithms.

To form a simple logarithmic scale, we take the base 10 logarithm of each value. The major tick marks on the scale are equally spaced, and correspond to a power of 10. We often say there is an **order of magnitude** difference between each mark.

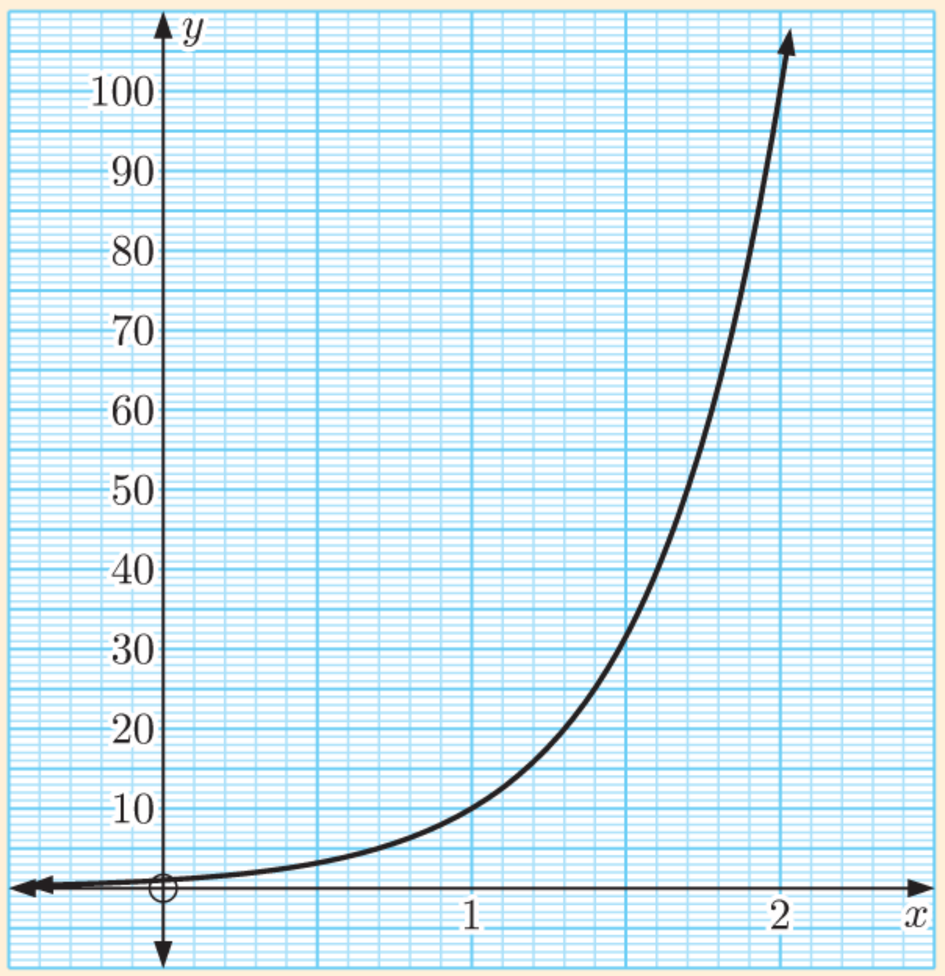


INVESTIGATION 2 LOGARITHMIC SCALES

We know that if $y = 10^x$ then $x = \log y$.

We can therefore use the graph of $y = 10^x$ to generate a logarithmic scale with minor tick marks, and locate values on the scale.

PRINTABLE GRAPH



What to do:

1 Print the graph. Mark, as accurately as possible, the values on the x-axis corresponding to y being 1, 2, 3, 4, ..., 9, and 11, 12, 13, 14, ..., 19.

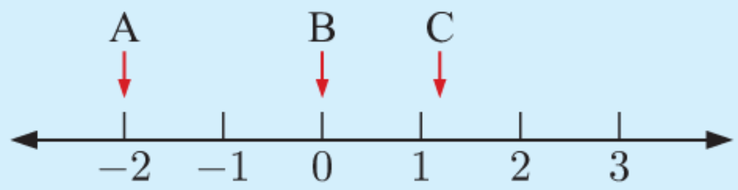
2 Comment on the spacing of your minor tick marks. How does this compare with the slide rules shown in the **Opening Problem**?

3 Measure the distance on your logarithmic scale between the marks corresponding to measurements (y-values) of 1 and 10, 2 and 20, and 5 and 50. Use the logarithm laws to explain why these distances should all be the same.

4 On the logarithmic scale, which measurement is placed further to the right, 5×10^2 or $10^{2.5}$? Explain your answer.

Example 14 **Self Tutor**

On the logarithmic scale alongside, values have been scaled using the function $y = \log x$.



a State the values corresponding to points A and B.

b Identify the position on the scale which represents the value:

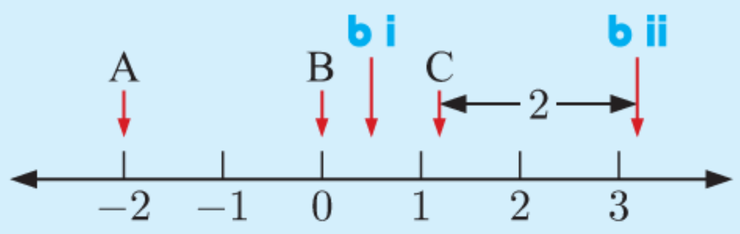
i $\sqrt{10}$ ii 100 times larger than the value corresponding to C.

a A has value $10^{-2} = \frac{1}{100}$

B has value $10^0 = 1$

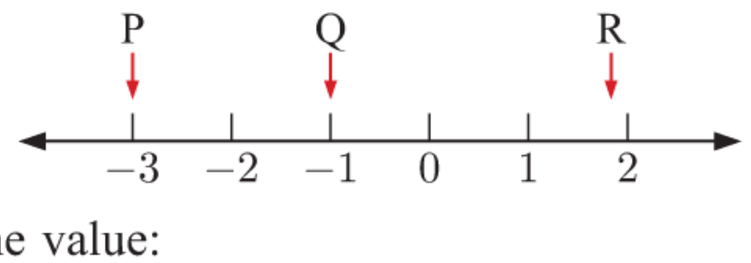
b i $\log \sqrt{10} = \log 10^{\frac{1}{2}}$
$= \frac{1}{2}$

ii $\log(100C) = \log 100 + \log C$
$= 2 + \log C$



EXERCISE 2E

1 On the logarithmic scale alongside, values have been scaled using the function $y = \log x$.



a State the values corresponding to points P and Q.

b Identify the position on the scale which represents the value:

i $\frac{1}{\sqrt{10}}$ ii $\frac{1}{10}$ as large as R.

c Is it possible to represent 0 on the number line? Explain your answer.

2 a Find the logarithm of each energy value in the table at the start of this Section.

b Illustrate the values on a single logarithmic scale.

3 The populations of some Asian countries are shown alongside.

Country	*Population*
Cambodia	16 482 646
China	1 420 062 022
India	1 368 737 513
Indonesia	269 536 482
Nauru	11 260
Singapore	5 836 496
Thailand	69 306 160
Vanuatu	288 017

a Scale these values using the function $y = \log x$, and display the values on a logarithmic scale.

b For each country, divide the population by the population of Singapore, then scale your results using the function $y = \log x$.

c Use a logarithmic scale to display each country's population *relative to Singapore*.

d Compare your scales from **a** and **c**. Explain your result.

4 Earthquakes can range from microscopic tremors to huge natural disasters. The magnitude of earthquakes is measured on the **Richter scale** which relates to the energy released by the earthquake. The magnitude is calculated as $M = \log\left(\frac{I}{I_0}\right)$, where I is the earthquake intensity and I_0 is a reference intensity level.

a What does it mean for a tremor to have magnitude:

i 0 **ii** 1?

b Explain why an earthquake of magnitude 6 is *not* twice as intense as a magnitude 3 earthquake. How much more intense is it?

c Find the magnitude of an earthquake which has:

i 10% of the intensity of a magnitude 5 earthquake

ii half the intensity of a magnitude 6 earthquake.

d In January 2018 a 7.5 magnitude earthquake hit Honduras, and in August 2018 an 8.2 magnitude earthquake hit Fiji. How many times more intense was the Fiji earthquake than the Honduras earthquake?

5 The *acidity* of a solution is determined by the concentration of hydronium ions (H_3O^+), measured in moles per litre ($\text{mol}\,\text{L}^{-1}$). The higher the concentration of H_3O^+, the more acidic it is. The opposite of acidic is *alkaline*.

a In extremely acidic solutions, the concentration of H_3O^+ is typically more than 10^{-3} $\text{mol}\,\text{L}^{-1}$. In very alkaline solutions, it is usually less than 10^{-12} $\text{mol}\,\text{L}^{-1}$.
Explain why a logarithmic scale would be useful in describing the acidity of a solution.

b In chemistry, the **pH** scale is used to measure acidity. The pH of a solution is given by $\text{pH} = -\log C$, where C is the concentration of H_3O^+. Find:

i the pH of a solution with H_3O^+ concentration $0.000\,234$ $\text{mol}\,\text{L}^{-1}$

ii the H_3O^+ concentration in a solution with pH 7.

c Milk has pH 6.6, and vinegar has pH 2.4.

i Which is more acidic?

ii Find the pH of a substance which has 200 times the H_3O^+ concentration in milk.

iii How many times greater is the H_3O^+ concentration in vinegar than in the substance in **ii**?

d Is it possible for a solution to have a negative pH? Explain what this means in terms of the concentration of H_3O^+.

Example 15 **Self Tutor**

The magnitude of an earthquake which releases E joules of energy is given by $M = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right)$.

a Find the magnitude of an earthquake which releases 5.1×10^{10} joules of energy.

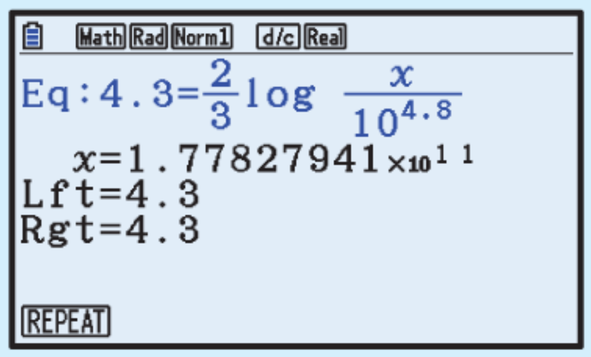
b How much energy is released by a magnitude 4.3 earthquake?

a When $E = 5.1 \times 10^{10}$, $M = \frac{2}{3}\log\left(\frac{5.1 \times 10^{10}}{10^{4.8}}\right)$

≈ 3.94

b When $M = 4.3$, $4.3 = \frac{2}{3}\log\left(\frac{E}{10^{4.8}}\right)$

$\therefore$ using technology, $E \approx 1.78 \times 10^{11}$ joules.



6 Use the formula in the Example above to find:

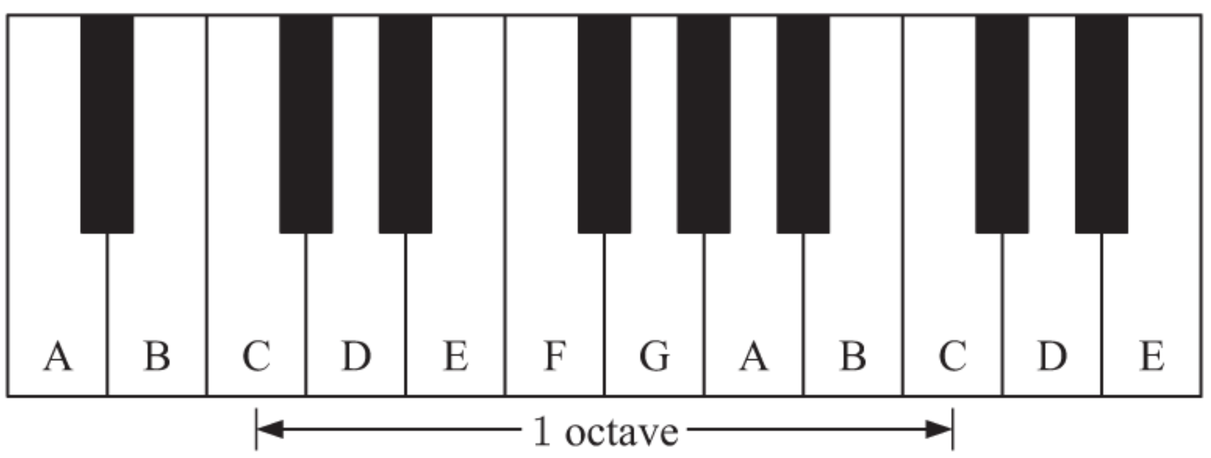
a the magnitude of an earthquake which releases 6.2×10^{13} joules of energy

b the energy released by a magnitude 5.1 earthquake.

7 Musical notes are named according to the frequency of their sound waves. They are labelled with letters of the alphabet.

A note which has *twice* the frequency of another is said to be one **octave** higher than it. So, one C is an octave below the next C.

A note n octaves above "Middle C" has frequency f Hz. The variables n and f are related by the equation $n \approx 3.322\log\left(\frac{f}{261.6}\right)$.



a Find the frequency of "Middle C".

b How many octaves above "Middle C" is a note with frequency 784 Hz?

c Find the frequency of the note:

i 3 octaves above "Middle C" **ii** 1 octave below "Middle C".

d There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.

8 In the **Krubein phi scale** for particle size, a particle with diameter D is assigned the value $\phi = -\log_2\frac{D}{D_0}$, where $D_0 = 1$ mm is a reference diameter.

a Show that if D is measured in mm, then $\phi = -\frac{\log D}{\log 2}$.

b Find the diameter of a gravel particle with $\phi = -3.4$.

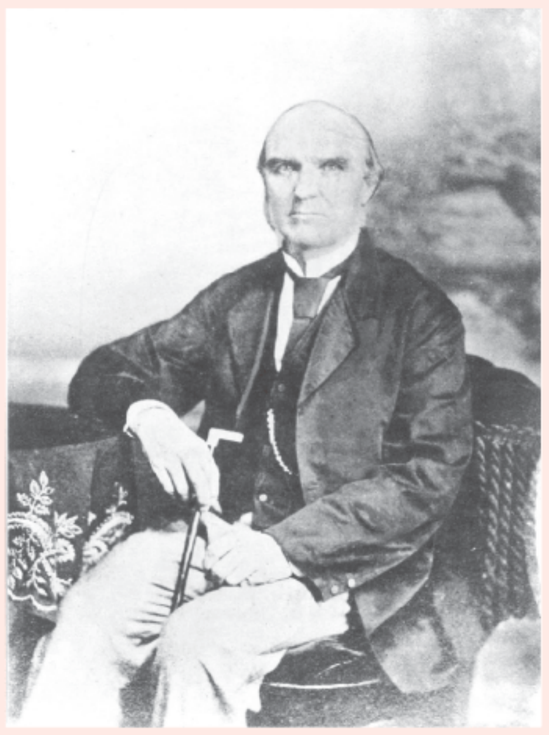
ACTIVITY THE BRIGHTNESS OF THE STARS

The brightness of an astronomical object such as a star, planet, or moon, is measured in two ways:

- the **apparent magnitude** is the object's brightness as seen from Earth
- the **absolute magnitude** is the apparent magnitude the object would have, if it were 10 parsecs from Earth.

The scale for the apparent magnitude of stars originated in Ancient Greece. The brightest stars were said to have magnitude 1, and the faintest were said to have magnitude 6. Each grade of magnitude was considered twice the brightness of the following grade.

This system was formalised in 1856 by the English astronomer **Norman Pogson**, who defined a magnitude 1 star to be 100 times brighter than a magnitude 6 star.



Norman Pogson

What to do:

1 Explain why, under Pogson's definition, a magnitude increase of 1 corresponds to a decrease in brightness by the factor $\sqrt[5]{100} \approx 2.512$.



2 The apparent magnitude M of an object's brightness is defined by $M = -2.5 \log\left(\frac{F}{F_0}\right)$ where F is the observed flux density of the object and F_0 is a reference flux density.

a Does a higher apparent magnitude indicate a brighter or fainter object?

b Let F_1 and F_6 be the observed flux densities of stars with apparent magnitudes 1 and 6 respectively. Use the apparent magnitude formula to show that $\frac{F_1}{F_6} = 100$, in agreement with Pogson's definition.

c What does it mean for a star to have a negative apparent magnitude?

d The Sun has apparent magnitude -26.74, and the full Moon has apparent magnitude -12.74.

i When viewed from Earth, how many times brighter is the Sun than the full Moon?

ii Find the apparent magnitude of a star which is 1 000 000 times fainter than the full Moon.

3 The brightness of a star varies inversely with the square of its distance from the observer.

a Stars A and B have the same apparent magnitude when viewed from Earth, but star B is twice as far away.

i Explain why star B has the higher absolute magnitude.

ii If star A has absolute magnitude -6, find the absolute magnitude of star B.

b On average, the Sun is 4.848×10^{-6} parsecs from Earth. Find the absolute magnitude of the Sun.

REVIEW SET 2A

1 Find:

a $\log\sqrt{10}$ **b** $\log\left(\frac{1}{\sqrt[3]{10}}\right)$ **c** $\log(10^a \times 10^{b+1})$

2 Use your calculator to evaluate, correct to 3 decimal places:

a $\log 27$ **b** $\log(0.58)$ **c** $\log 400$ **d** $\ln 40$

3 Write in the form 10^x, giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

4 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

5 Write as a single logarithm:

a $\log 16 + 2\log 3$ **b** $\log 16 - 2\log 3$ **c** $2 + \log 5$

6 Suppose $A = \log 2$ and $B = \log 3$. Write in terms of A and B:

a $\log 36$ **b** $\log 54$ **c** $\log\left(8\sqrt{3}\right)$

d $\log\left(\sqrt{6}\right)$ **e** $\log(20.25)$ **f** $\log\left(\frac{8}{9}\right)$

7 Find x if:

a $\log x = -3$ **b** $\ln x = 5$

8 Consider the function $f(x) = 3\ln x + 3$.

a State the transformations which map $y = \ln x$ to $y = f(x)$.

b Find $f(1)$. **c** Find any axes intercepts.

d Sketch the graph of $y = f(x)$. **e** Find the inverse function $f^{-1}(x)$.

9 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \frac{1}{2}\ln x$ **b** $y = \ln x$ and $y = \ln(ex)$

c $y = \log x$ and $y = 2\log(x - 2)$

10 Suppose $\ln k > 1$ but $\log k < 1$. Between which two values must k lie?

11 The point P has x-coordinate 2. It lies on the graph of the function $f(x) = 10^x$.

a State the y-coordinate of P.

b State the coordinates of the corresponding point on the inverse function $f^{-1}(x) = \log x$.

12 The wavelengths of various types of electromagnetic radiation are shown alongside.

Scale these values using the function $y = \log x$, and display the results on a logarithmic scale.

Electromagnetic radiation type	*Wavelength* (m)
X-rays	0.000 000 001
Ultraviolet	0.000 000 05
Infrared	0.000 04
High frequency microwaves	30
Super low frequency radio waves	7 000 000

REVIEW SET 2B

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\dfrac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\dfrac{e}{e^x}\right)$

3 Simplify:

a $\dfrac{\log 25}{\log 125}$ **b** $\dfrac{\log 64}{\log 32}$ **c** $\dfrac{\log 81}{\log \sqrt{3}}$

4 Simplify:

a $e^{4\ln x}$ **b** $\ln(e^5)$ **c** $\ln(\sqrt{e})$

d $10^{\log x + \log 3}$ **e** $\ln\left(\dfrac{1}{e^x}\right)$ **f** $\dfrac{\log(x^2)}{\log(0.01)}$

5 Write in the form e^x, where x is correct to 4 decimal places:

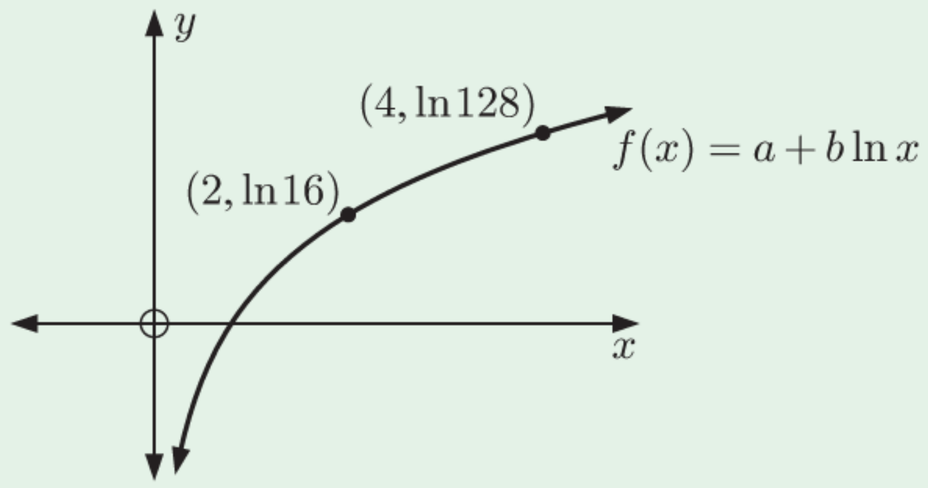
a 20 **b** 3000 **c** 0.075

6 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

7 For what value(s) of x is $\log x = \ln x$?

8 Find a and b.



9 **a** Given $f : x \mapsto e^{5x-1}$, write $f^{-1}(x)$ in the form $f^{-1}(x) = a + b\ln x$.

b Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

10 In acoustics, the intensity of sound is measured in **decibels** (dB).

The **sound intensity level** (SIL) is given by $L = 10\log\left(\dfrac{I}{I_0}\right)$ dB, where I is the sound intensity, and I_0 is the reference sound intensity.

a How many times more intense is a 50 dB sound than a 30 dB sound?

b The *perceived loudness* of a sound doubles for every 10 dB increase in the SIL.
Suppose a jackhammer has SIL 90 dB, and a passing car has 2% of the sound intensity of the jackhammer.

i Find, in decibels, the SIL of the car.

ii How much louder will the jackhammer be perceived than the car?

11 If a substance decays by $k\%$ per year, its half-life is given by $H = \dfrac{\ln 0.5}{\ln(1 - \frac{k}{100})}$ years.

a Find the half-life of a substance which decays by:

i 5% per year **ii** 2.2% per year.

b Substance A decays by 1.5% per year, whereas substance B decays by 0.7% per year. How much longer is substance B's half-life than substance A's half-life?

c A substance has a half-life of 30 years. By what percentage does it decay each year?

Chapter 3

Approximations and error

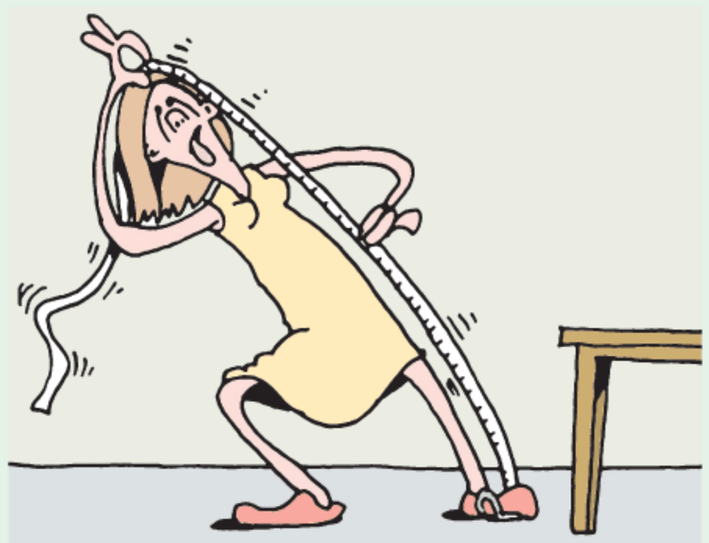
Contents:

A Errors in measurement
B Absolute and percentage error



OPENING PROBLEM

After using a tape measure to measure her height, Abigail stated that she is 173 cm tall.



Things to think about:

a Do you think that Abigail is *exactly* 173 cm tall? Is it *possible* for Abigail to be exactly 173 cm tall?

b In what range of values could Abigail's actual height lie?

c Abigail uses the same tape measure to measure the length of her finger as 6 cm.

i In what range of values do you think the actual length of Abigail's finger lies?

ii Which of the two measurements would you consider to be "more accurate"?

When describing quantities, we often give an **approximate** answer rather than an exact answer.

For example:

- Suppose a survey found that a total of 6428 flights were taken by 825 people last year. It is not sensible to give the average number of flights per person as 7.791 515 152. An approximate answer of 7.8 is more appropriate.
- In the **Opening Problem**, Abigail's height is a continuous variable, so it is impossible to find her *exact* height. Instead, her height is rounded to the nearest centimetre.

In this Chapter, we will study the **errors** that arise when we approximate values.

DISCUSSION

Why is it important to understand errors?

What things can go wrong if people measure inaccurately or round off incorrectly?

You may wish to consider cases such as:

- If a pilot flies off-course by 0.1° for 1000 km, how far away from his target will he be?
- What happens to a patient if a doctor injects 2 mg of a drug instead of 2 μg?

A ERRORS IN MEASUREMENT

When we take measurements, we are usually reading some sort of scale.

The scale of a ruler may have centimetres marked on it, but when we measure the length of an object, it is likely to fall between two divisions. We **approximate** the length of the object by recording the value at the nearest centimetre mark. In doing so our answer may be inaccurate by up to a half a centimetre.

A measurement is accurate to $\pm\frac{1}{2}$ of the smallest division on the scale.

Example 1 **Self Tutor**

Ling uses a ruler to measure the length l of her pencil case. She records the length as 18.7 cm.

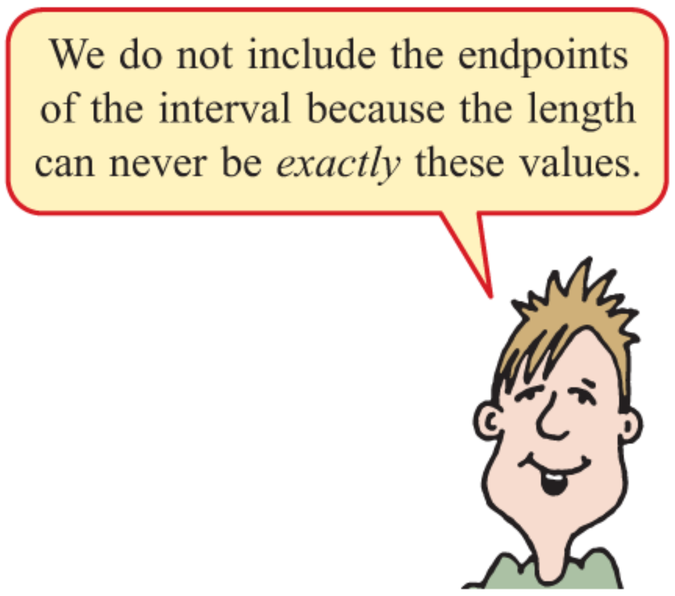
Find the range of values in which the length may lie.

18.7 cm is 187 mm, so the measuring device must be accurate to the nearest half mm.

$\therefore$ the range of values is $187 \pm \frac{1}{2}$ mm

The actual length is in the range $186\frac{1}{2}$ mm to $187\frac{1}{2}$ mm.

$\therefore$ $18.65 \text{ cm} < l < 18.75 \text{ cm}$.



EXERCISE 3A

1 State the accuracy of the following measuring devices:

a a tape measure marked in cm

b a measuring cylinder with 1 mL graduations

c a beaker with 100 mL graduations

d a set of scales with marks every 500 g

e a thermometer with marks every 0.1°C.

2 Find the range of possible values corresponding to the following measurements:

a 27 mm **b** 38.3 cm **c** 4.8 m

d 1.5 kg **e** 25 g **f** 3.75 kg

3 Tom's digital thermometer said his temperature was 36.4°C. In what range of values did Tom's actual temperature T lie?

4 Joanne's exercise watch displays the distance she has run to 3 significant figures. State the *least* distance Joanne could have run, if the watch displays:

a 1.06 km **b** 9.72 km **c** 10.1 km

Comment on the accuracy of the watch.

5 Four students measured the width of their classroom using the same tape measure. The measurements were 6.1 m, 6.4 m, 6.0 m, 6.1 m.

a Which measurement is likely to be incorrect? Explain your answer.

b What answer would you give for the width of the classroom? Explain your answer.

c What graduations do you think were on the tape measure?

6 Hasan has many lengths of rope. He has measured each length to be 2.4 m.

a In what range of values does the actual length of a rope l lie?

b If Hasan carefully places n of his ropes end to end, in what range of values will the total length of rope L lie?

7 In the 800 m race at the sports carnival, the times recorded for Jiao and Liang were 2 min 8 s and 2 min 13 s respectively.

Find the range of possible values for the time t by which Jiao beat Liang.

Example 2 **Self Tutor**

A rectangular board was measured as 78 cm by 24 cm. Find the boundary values for its perimeter.

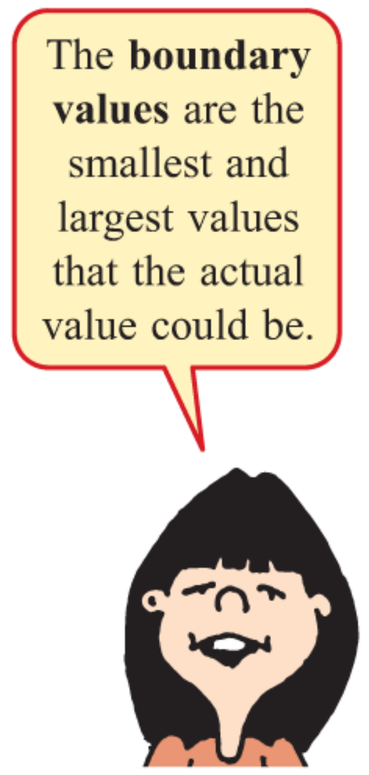
The length of the board could be from $77\frac{1}{2}$ cm to $78\frac{1}{2}$ cm.

The width of the board could be from $23\frac{1}{2}$ cm to $24\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the perimeter is $2 \times 77\frac{1}{2} + 2 \times 23\frac{1}{2} = 202$ cm

and the upper boundary of the perimeter is $2 \times 78\frac{1}{2} + 2 \times 24\frac{1}{2} = 206$ cm

The perimeter is between 202 cm and 206 cm, which is 204 ± 2 cm.



8 A rectangular bath mat was measured as 86 cm by 38 cm. Find the boundary values of its perimeter.

9 A rectangular garden bed is measured as 252 cm by 143 cm. Find the range of possible values for the total length of edging l required to border the garden bed.

Example 3 **Self Tutor**

A paving brick is measured as 18 cm $\times$ 10 cm. What are the boundary values for its actual area?

The length of the paving brick could be from $17\frac{1}{2}$ cm to $18\frac{1}{2}$ cm.

The width of the paving brick could be from $9\frac{1}{2}$ cm to $10\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $17\frac{1}{2} \times 9\frac{1}{2} = 166.25$ cm^2

and the upper boundary of the area is $18\frac{1}{2} \times 10\frac{1}{2} = 194.25$ cm^2.

The area is between 166.25 cm^2 and 194.25 cm^2.

This could also be represented as $\dfrac{166.25 + 194.25}{2} \pm \dfrac{194.25 - 166.25}{2}$ cm^2

which is 180.25 ± 14 cm^2.

10 A rectangle is measured to be 6 cm by 8 cm. Find:

a the largest area it could have **b** the smallest area it could have.

11 Find the boundary values for the actual area of a glass window measured as 42 cm by 26 cm.

12 The base of a triangle is measured as 9 cm and its height is measured as 8 cm. In what range of values does its actual area A lie?

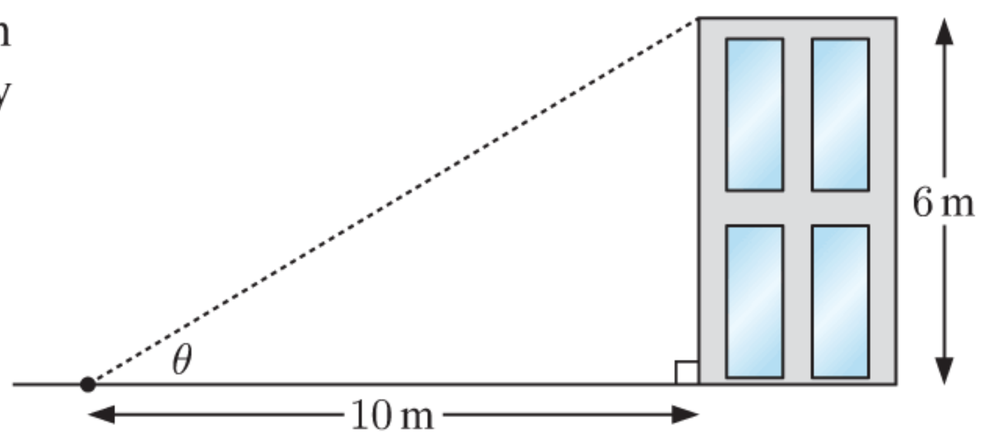
13 Find the boundary values for the actual volume of a box measuring 4 cm by 8 cm by 6 cm.

14 Find the range of values in which the actual volume V of a house brick measuring 21.3 cm by 9.8 cm by 7.3 cm must lie.

15 A cylinder is measured to have radius 5 cm and height 15 cm. Find the boundary values for the cylinder's volume.

16 A cone is measured to have radius 8.4 cm and height 4.6 cm. Find the boundary values for the cone's volume.

17 The distances recorded alongside have been rounded to the nearest metre. Find the boundary values for the angle of elevation θ.



18 Tahlia measures the lengths of two sides of a triangle as 5.1 cm and 7.2 cm, rounded to the nearest millimetre. She measures the included angle as 12°, rounded to the nearest degree. Find the boundary values for the triangle's:

a area **b** perimeter.

19 Eko measures the diameter of a ball to be 18.2 cm. Do you expect the rounding in Eko's measurement to have more effect on a calculation of the ball's surface area, or a calculation of its volume? Explain your answer.

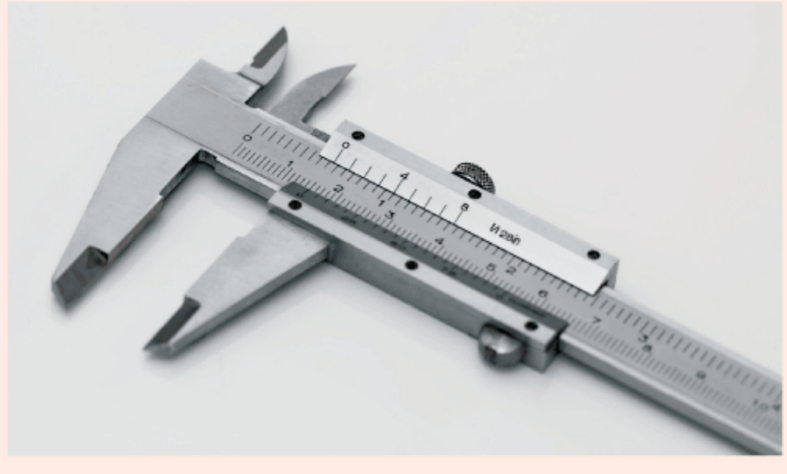
20 Rachel measures the base side lengths of a square-based pyramid to be 4.6 cm, and its height to be 5.2 cm. Find the boundary values for the pyramid's:

a volume **b** surface area.

RESEARCH **VERNIER SCALES**

A **vernier scale** is used to measure the length of objects with a high degree of accuracy.

Research how vernier scales work.



B ABSOLUTE AND PERCENTAGE ERROR

Whenever we measure a quantity there is almost always a difference between our measurement and the actual value. We call this difference the **error**.

The *size* or *magnitude* of the error, whether the measured or estimated value is too high or too low, is called the **absolute error**.

If the actual or exact value is V_E and the approximate value is V_A then the

$$\textbf{absolute error} = |V_A - V_E|$$

Error is often expressed as a percentage of the exact value:

$$\textbf{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$

Example 4 — Self Tutor

Suppose you estimate a fence's length to be 70 m, whereas its true length is 78.3 m. Find, correct to one decimal place, the:

a absolute error **b** percentage error.

a absolute error $= |V_A - V_E|$
$= |70 - 78.3|$ m
$= |-8.3|$ m
$= 8.3$ m

b percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{8.3}{78.3} \times 100\%$
$\approx 10.6\%$

EXERCISE 3B

1 Find the absolute error and percentage error in rounding:

a the yearly profit of €1 367 540 made by a company to €1.37 million
b a population of 31 467 people to 31 000 people
c a retail sales figure of \$458 110 to \$460 000
d the number of new cars sold in a year from 2811 to 3000.

2 Find the absolute error and percentage error if you estimate:

a the mass of a brick to be 5 kg when its actual mass is 6.238 kg
b the perimeter of a property to be 100 m when its actual perimeter is 97.6 m
c the capacity of a container to be 20 L when its actual capacity is 23.8 L
d the time to write a computer program to be 50 hours when it actually takes 72 hours.

Example 5 — Self Tutor

Alan wants to lay carpet on his 4.2 m by 5.1 m lounge room floor. He estimates the area of the lounge room by rounding each measurement to the nearest metre.

a Find Alan's estimate of the lounge room area.
b The carpet costs \$39 per square metre. Find the cost of the carpet using Alan's estimate of the area.
c Find the actual area of Alan's lounge room.
d Find the percentage error in Alan's estimation.
e Will Alan have enough carpet to cover his lounge room? How should he have rounded the measurements?

a Area ≈ 4 m $\times 5$ m
≈ 20 m^2

b Cost $= 20 \times \$39$
$= \$780$

c Actual area $= 4.2 \times 5.1$
$= 21.42$ m^2

d Percentage error $= \dfrac{|V_A - V_E|}{V_E} \times 100\%$
$= \dfrac{|20 - 21.42|}{21.42} \times 100\%$
$\approx 6.63\%$

e Alan only has 20 m^2 of carpet, so he will not have enough to cover his lounge room. He should have rounded the measurements *up* to make sure he had enough carpet.

3 Jon's apartment is a 10.3 m by 9.7 m rectangle.

a Find the actual area of the apartment.

b Estimate the floor area by rounding each length to the nearest metre.

c Find the absolute error and percentage error in your estimate.

4 The cost of freight for a parcel is dependent on its volume. Justine lists the dimensions of a parcel as 24 cm by 15 cm by 9 cm on the consignment note. The actual dimensions are 23.9 cm $\times$ 14.8 cm $\times$ 9.2 cm.

a Calculate the actual volume of the parcel.

b Estimate the volume using the dimensions given on the consignment note.

c Find the absolute error and percentage error in the calculation using the consignment note.

5 Consider the **Opening Problem** on page **68**.

a Find the maximum absolute error and percentage error in Abigail's estimate of:

i her height **ii** the length of her finger.

b Which estimate do you consider to be more accurate? Explain your answer.

6 A hotel wants to cover an 8.2 m by 9.4 m rectangular courtyard with synthetic grass. The manager estimates the area by rounding each measurement to the nearest metre.

a Find the manager's estimate of the area.

b The synthetic grass costs \$85 per square metre. Find its cost using the manager's estimate.

c Find the actual area of the rectangle.

d Calculate the percentage error in the manager's estimate.

e Will the hotel have enough grass to cover the courtyard?

f Find the cost of the grass if the manager had rounded each measurement *up* to the next metre.

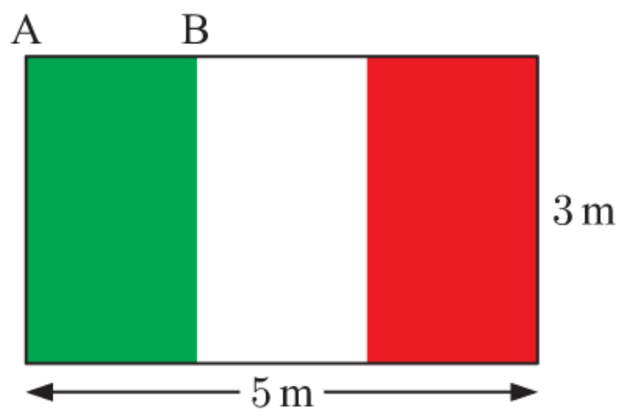
7 The Italian flag has three different regions of equal size.

a Find the area of the green section exactly.

b Find the length AB correct to 1 decimal place.

c Hence estimate the area of the green section.

d Find the percentage error in your estimate.



8 Hemi estimates that she can drive at an average speed of 70 km h^{-1} between her house and the beach, 87 km away. One particular journey took her 1 hour and 20 minutes.

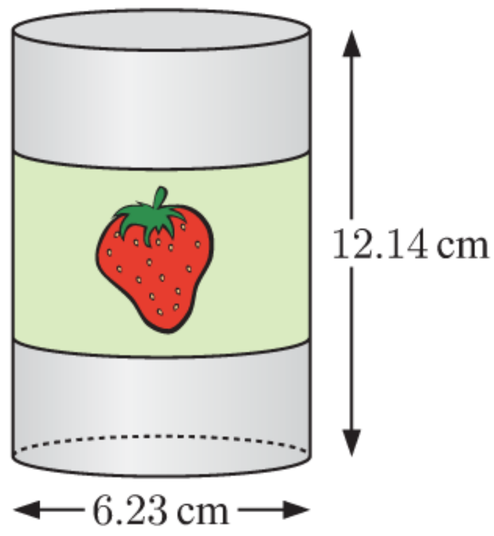
a Calculate Hemi's average speed for this journey.

b Find the absolute error and percentage error in her estimate.

9 A tinned fruit can has the dimensions shown.
Suppose the dimensions are rounded to the nearest millimetre. Find the resulting percentage error in an estimate of:

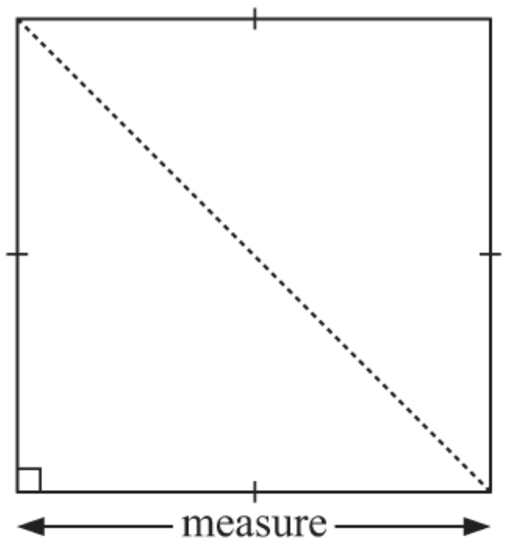
a the surface area of the can

b the capacity of the can.



10 The sentence below is translated from an ancient Indian text, the *Śulba sūtra*:

> *The measure is to be increased by its third and this (third) again by its own fourth less the thirtyfourth part (of that fourth); this is (the value of) the diagonal of a square (whose side is the measure).*



a Show that the diagonal of a square is $\sqrt{2}$ times the measure of its side.

b Hence show that the text estimates $\sqrt{2} \approx \frac{577}{408}$.

c Find the percentage error for this estimate, giving your answer in scientific notation.

Example 6 **Self Tutor**

The side length of a square is measured as 22 cm, rounded to the nearest centimetre.

a Use this measurement to estimate the area of the square.

b Find the boundary values for the area of the square.

c Hence find the maximum percentage error in the estimate.

a Area ≈ 22 cm $\times$ 22 cm ≈ 484 cm^2

b The side length of the square could be from $21\frac{1}{2}$ cm to $22\frac{1}{2}$ cm.

$\therefore$ the lower boundary of the area is $21\frac{1}{2} \times 21\frac{1}{2} = 462.25$ cm^2

and the upper boundary of the area is $22\frac{1}{2} \times 22\frac{1}{2} = 506.25$ cm^2.

c If the exact area V_E was 462.25 cm^2, the

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$
$$= \frac{|484 - 462.25|}{462.25} \times 100\%$$
$$\approx 4.71\%$$

If the exact area V_E was 506.25 cm^2, the

$$\text{percentage error} = \frac{|V_A - V_E|}{V_E} \times 100\%$$
$$= \frac{|484 - 506.25|}{506.25} \times 100\%$$
$$\approx 4.40\%$$

$\therefore$ the maximum percentage error in the estimate $\approx 4.71\%$.

11 The side lengths of a rectangle are measured as 2.3 m and 1.4 m, rounded to one decimal place.

a Use these measurements to estimate the area of the rectangle.

b Find the boundary values for the area of the rectangle.

c Hence find the maximum percentage error in the estimate.

12 Jasper measured the dimensions of his cylindrical can of tuna. He found the radius was 4 cm and the height was 5 cm, rounded to the nearest centimetre.

a Use these measurements to estimate the volume of the can.

b Find the boundary values for the volume of the can.

c Hence find the maximum percentage error in the estimate.

13 Carolina completed a 250 km car trip (rounded to the nearest km). The GPS in her car displays the average speed for the trip as 56.8 km h^{-1} (rounded to 1 decimal place).

a Estimate the time it took Carolina to complete the trip.

b Find the maximum possible percentage error in the estimate.

REVIEW SET 3A

1 **a** How accurate is a tape measure marked in cm?

b Find the range of possible values for a measurement of 36 cm.

c If the sides of a square are measured to be 36 cm, in what range of values must the actual area A lie?

2 A photograph was measured as 15 cm by 10 cm. Find the boundary values for its perimeter.

3 A triangular garden is measured to have sides of length 8 m, 12 m, and 14 m, rounded to the nearest metre. Find the range of possible values for the area A of the garden.

4 Find the absolute error and percentage error if you:

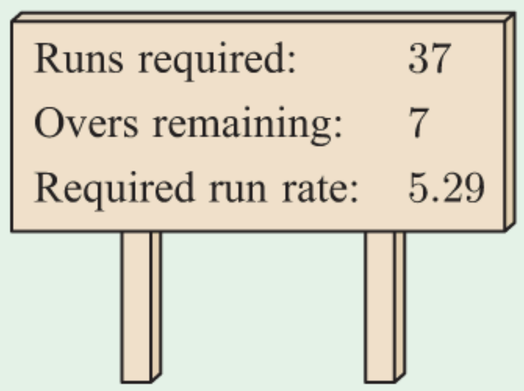
a estimate your credit card balance to be \$2000 when it is \$2590

b round 26.109 cm to 26 cm

c estimate the number of people at a music festival to be 4000 when there are 4386 in attendance.

5 In limited overs cricket, teams must score as many runs as possible within a particular number of overs. For the team batting second, the *required run rate* is found by dividing the number of *runs required* by the number of *overs remaining*.

On this scoreboard, the required run rate has been rounded to 2 decimal places. Find the percentage error in the calculation.



6 A circular gazebo has diameter 2.8 m. In calculating the area of the gazebo, the diameter is rounded to the nearest metre.

a Find the actual area of the gazebo.

b Find the calculated area of the gazebo.

c Find the absolute error and percentage error in the calculation.



REVIEW SET 3B

1 Dafne competed in the 100 m sprint at her school sports carnival. Her time for the race, rounded to 1 decimal place, was recorded as 14.9 seconds.

Find the range of possible values for:

a her time t seconds

b her average speed s m s^{-1}.

2 A box has dimensions 5 cm by 7 cm by 10 cm, rounded to the nearest centimetre. Find the boundary values for the surface area A of the box.

3 Boats A and B depart from a port at the same time. Boat A travels at 20 km h^{-1} and Boat B travels at 15 km h^{-1}. The bearing of each boat is recorded as 035° and 278° respectively, rounded to the nearest degree. Find the boundary values for the distance D between the boats after 2 hours.

4 Edward measures the width and height of a television screen to the nearest 10 cm. The width is approximately 150 cm and the height is approximately 90 cm.

a Use Edward's measurements to estimate the length of the diagonal of the screen.

b Given that the diagonal has length 177.8 cm, find the absolute error and percentage error of the estimate.

5 An architect designs a support beam to be $\sqrt{5}$ metres long. The builder working from the architect's plans converts this length to a decimal number.

a Write down the length of the support beam correct to the nearest:

i metre **ii** centimetre **iii** millimetre.

b The architect insists that there be no more than 1% error. Which of the approximations in **a**, if any, will satisfy this?

6 In order to quickly estimate areas and volumes, people use various approximations for π. Find the percentage error if π is approximated by:

a 3 **b** 3.1 **c** 3.14 **d** $\frac{22}{7}$ **e** $\frac{355}{113}$

7 The radius of a circle is measured as 3.5 cm, rounded to 1 decimal place.

a Use this measurement to estimate the area of the circle.

b Find the boundary values for the area of the circle.

c Find the maximum percentage error in the estimate.

Chapter 4

Loans and annuities

Contents:

A Loans
B Annuities



OPENING PROBLEM

Diane is about to retire at the age of 68. Over her career, she has accumulated £600 000 in savings which she places in an *annuity* fund paying 4% p.a. interest compounded monthly. Her plan is to make monthly withdrawals from the account to live on for the next 25 years.



Things to think about:

a What is meant by an annuity fund? How does it differ from the standard compound interest investments we have studied previously?

b Diane reasons that for her money to last for 25 years or 300 months, she should be able to withdraw at least $\frac{£600\,000}{300} = £2000$ each month. Is Diane correct? Explain your answer.

c If Diane's investment is to be completely used up by regular withdrawals over 25 years, how much can she withdraw each month?

Loans and **annuities** play an important role in modern life. Understanding these processes allows you to make informed financial decisions.

HISTORICAL NOTE

In 14th century Italy, merchants would set up stalls in local markets and lend money to customers. Interest was applied, and the borrower was expected to repay the money in regular intervals. The word "bank" is derived from the Italian word *banca*, which describes the benches on which the merchants sat.

If a banker suffered a series of bad loans and was unable to repay his own creditors, the creditors would break up his bench, showing that the banker was no longer in business. This was known as *banca rotta*, meaning "broken bench". Over time, the Italian word for "broken" was replaced with the Latin word *rupta*, giving *banca rupta*, the basis of the modern term "bankrupt".

A LOANS

A common way to borrow money to finance larger purchases such as houses, cars, renovations, education expenses, and share portfolios, is to take out a **personal loan**. The financial institution lends an amount of money to the borrower, who repays the amount plus interest by a series of regular repayments over a given time period.

The process of repaying a loan with a series of regular repayments is called **amortisation**.

Interest is calculated on the **reducing balance** of the loan, so the interest gradually reduces as the loan is repaid.

Because of the interest, the total repayment will be greater than the amount originally borrowed. The difference between these values is the interest paid.

interest paid = total repayments − amount borrowed

CALCULATING REPAYMENTS

We can use a **table of repayments** to work out the repayments required for a loan. The table below shows the amount you need to repay each month for every \$1000 borrowed.

Table of monthly repayments per \$1000 borrowed							
Loan term (years)	*Annual interest rate*						
	4%	4.5%	5%	5.5%	6%	6.5%	7%
1	85.1499	85.3785	85.6075	85.8368	86.0664	86.2964	86.5267
1.5	57.3314	57.5557	57.7805	58.0059	58.2317	58.4581	58.6850
2	43.4249	43.6478	43.8714	44.0957	44.3206	44.5463	44.7726
3	29.5240	29.7469	29.9709	30.1959	30.4219	30.6490	30.8771
4	22.5791	22.8035	23.0293	23.2565	23.4850	23.7150	23.9462
5	18.4165	18.6430	18.8712	19.1012	19.3328	19.5661	19.8012
10	10.1245	10.3638	10.6066	10.8526	11.1021	11.3548	11.6108
15	7.39688	7.64993	7.90794	8.17083	8.43857	8.71107	8.98828
20	6.05980	6.32649	6.59956	6.87887	7.16431	7.45573	7.75299

We can also calculate the monthly repayments using the TVM solver on a calculator.



In this case, you are receiving the amount loaned, so PV is positive. PMT is the amount to be repaid each time period, so it is negative.

Example 1 **Self Tutor**

Erica takes out a personal loan of \$16 500 to buy a car. She negotiates a term of 4 years at 5.5% p.a. interest.

a Calculate the monthly repayments using the table. Check your answer using technology.

b Hence calculate: i the total repayment ii the total interest charged.

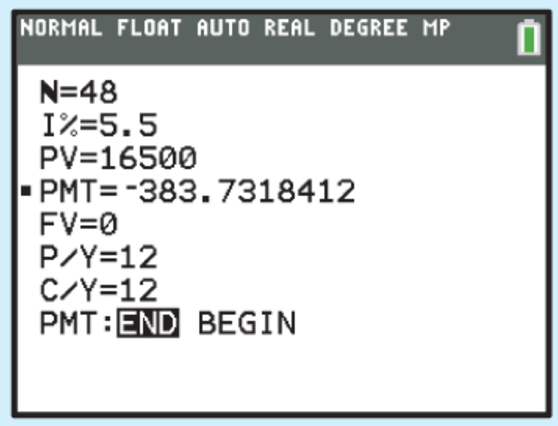
a From the table, the monthly repayments on each \$1000 for 4 years at 5.5% p.a. $=$ \$23.2565

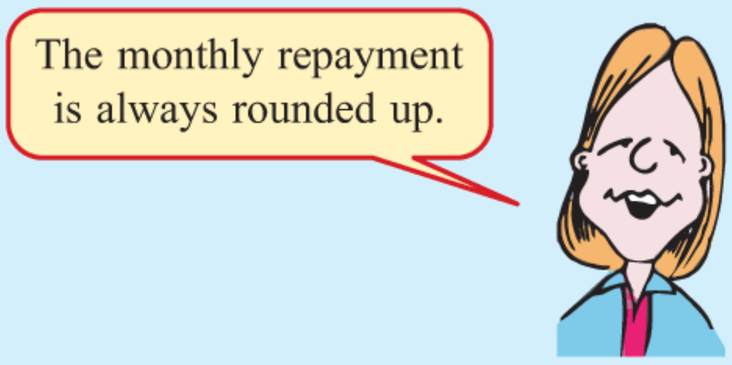
$\therefore$ repayments on \$16 500

$=$ \$23.2565 $\times$ 16.5 {16.5 lots of \$1000}

$=$ \$383.732

The repayments are \$383.74 per month.





b i Total repayment

$=$ monthly repayment $\times$ number of months

$=$ \$383.74 $\times$ 48

$=$ \$18 419.52

ii Interest

$=$ total repayment $-$ amount borrowed

$=$ \$18 419.52 $-$ \$16 500

$=$ \$1919.52

In the following Exercise, calculate monthly repayments using your calculator. Where appropriate, check your answers using the table.

EXERCISE 4A

1 Don takes out a personal loan for \$12 000 to fund his wedding. He will repay it over 5 years at 6% p.a. Calculate the:

a monthly repayments **b** total repayment **c** interest charged.

2 Jay and Penni need £9500 to fund house renovations. They take out a personal loan over 3 years at 4.5% p.a. Calculate the:

a monthly repayments **b** total repayment **c** interest charged.

3 Binh-vu needs \$15 000 to buy a boat. His bank offers him a personal loan at 6.5% p.a. Calculate the total interest he will pay if he repays the loan with monthly repayments over:

a 2 years **b** 5 years.

4 Mimi takes out a personal loan of €10 000. She will repay the loan over 4 years at 6% p.a. She is charged a €150 application fee and a €10 monthly service fee.
Calculate the total cost of taking out the loan.

5 The spreadsheet alongside shows the progress of a \$30 000 loan taken out over 5 years at 8% p.a. interest compounded monthly.

	A	B	C	D	E
1	**LOAN SPREADSHEET**				
2					
3	**Loan amount**	\$30,000.00			
4	**Number of years**	5			
5	**Rate p.a.**	8.00%			
6	**Periods p.a.**	12			
7	**Rate per period**	0.667%			
8	**Repayment**	\$608.30			
9					
10	**Month**	**Amount**	**Interest**	**Repayment**	**Balance**
11	1	\$30,000.00	\$200.00	\$608.30	\$29,591.70
12	2	\$29,591.70	\$197.28	\$608.30	\$29,180.68
13	3	\$29,180.68	\$194.54	\$608.30	\$28,766.92
14	4	\$28,766.92	\$191.78	\$608.30	\$28,350.40
15	5	\$28,350.40	\$189.00	\$608.30	\$27,931.10
16	6	\$27,931.10	\$186.21	\$608.30	\$27,509.01

a Check that the monthly repayment has been correctly calculated.

b What is the account balance after six months?

c How much interest is paid in:

i month 1 **ii** month 6?

Explain the differences between these amounts.

d Click on the spreadsheet icon.
Fill row 16 down to the end of the 60th month.

i How much interest is paid in the 60th month?

ii How much interest is paid in total?

SPREADSHEET

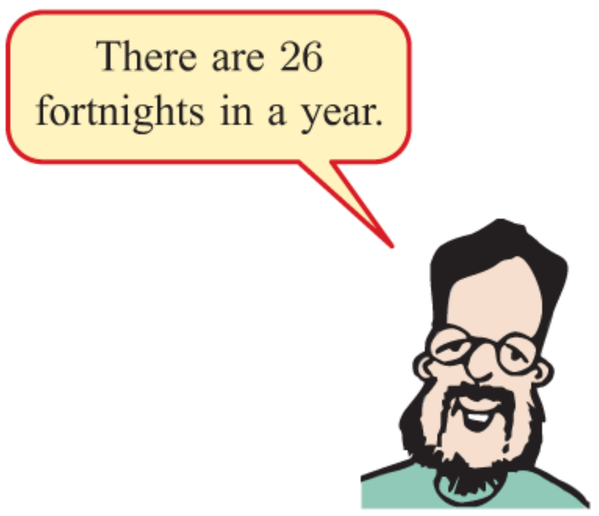


e Explain why the final payment is slightly less.

6 Grace takes out a loan of \$7000 to buy a jet ski. She will repay the loan over 2 years at 9.9% p.a. interest compounded fortnightly.

a Calculate Grace's fortnightly repayments.

b Find the total amount of interest charged on the loan.



Example 2 **Self Tutor**

Ryan takes out a \$10 000 loan to be repaid over three years at 9.6% p.a. interest compounded monthly.

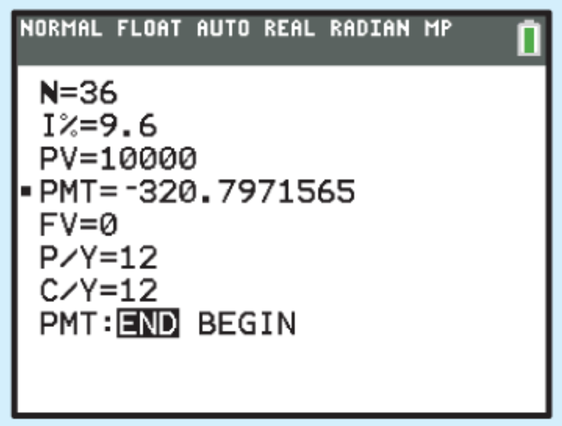
a Calculate the monthly repayments.

b Find the outstanding balance on the loan after 1 year of repayments.

a $N = 3 \times 12 = 36$, $I\% = 9.6$, $PV = 10\,000$, $FV = 0$, $P/Y = 12$, $C/Y = 12$

$\therefore \quad PMT \approx -320.80$

The monthly repayment is \$320.80 .

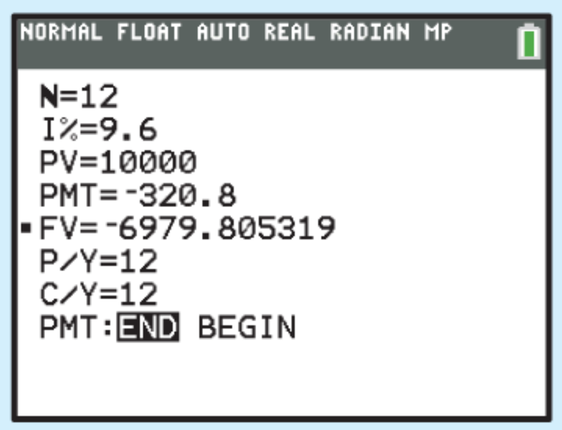


b $N = 1 \times 12 = 12$, $I\% = 9.6$, $PV = 10\,000$, $PMT = -320.80$, $P/Y = 12$, $C/Y = 12$

$\therefore \quad FV \approx -6979.81$

The outstanding balance on the loan after 1 year is \$6979.81 .

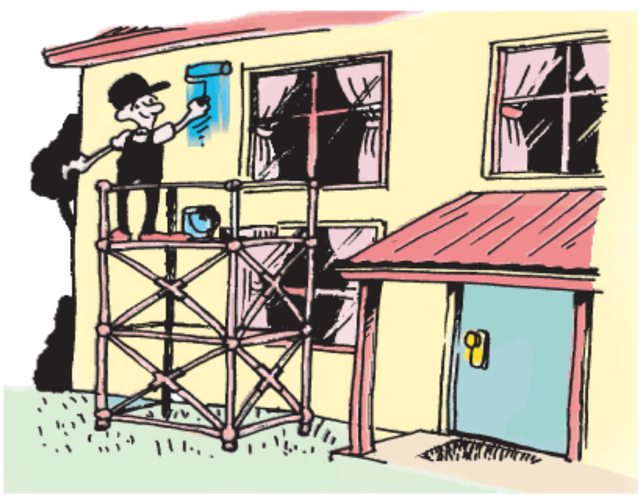




7 Pieter takes out a loan of £20 000 to renovate his home. He agrees to repay the loan over 4 years at 8.25% p.a. interest compounded monthly. Find:

a Pieter's monthly repayments

b the outstanding balance on the loan after 1 year of repayments.



8 Jacob wants to buy a piano valued at \$12 000. He has \$3500 in savings, and will borrow the remaining money.

a How much will Jacob borrow?

b Jacob will repay the loan over 5 years at 7.9% p.a. interest compounded quarterly.

i Calculate Jacob's quarterly repayments.

ii Calculate the total interest charged on the loan.

iii Find the outstanding balance on the loan after 2 years.

9 Simon took out a loan of \$7000 to pay for a holiday. He will repay the loan over 4 years with monthly repayments of \$165.36 .

a What annual interest rate, compounded monthly, is being charged?

b Calculate the total interest Simon will pay in the first year.

10 Consider a loan of \$25 000 at 8.5% p.a. interest compounded monthly.

a Calculate the monthly loan repayments if the loan is taken out for:

i 3 years **ii** 5 years **iii** 7 years.

b Which loan charges the least interest in total? Explain your answer.

11 Ally takes out an €18 000 car loan over 5 years at 10.5% p.a. interest compounded monthly.

a Calculate the monthly repayments.

b Calculate the total interest Ally will pay on the loan.

c Find the outstanding loan balance after $2\frac{1}{2}$ years of repayments.

d Explain why Ally still has more than half the loan to pay off when half the loan period has passed.

12 Shane and Julie have taken out a \$250 000 loan to purchase an apartment. They will repay the loan over 20 years at 6.25% p.a. fixed interest compounded monthly.



a Calculate the monthly repayments.

b Calculate the total interest charged.

c Find the outstanding balance on the loan after 10 years.

d After 10 years, the couple can afford to increase their monthly repayments. They aim to completely repay the loan over the next 5 years.

i Calculate their monthly repayments for the next 5 years.

ii Calculate the total interest that would be charged over the 15 years.

iii How much interest will the couple save by repaying their loan sooner?

13 A young couple borrows \$435 000 to buy their home at a starting rate of 6.25% p.a. interest compounded monthly over 30 years.

a Calculate the monthly repayments for this starting interest rate.

b Find the outstanding debt after 3 years of repayments.

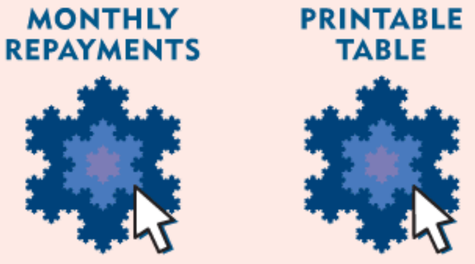
c After 3 years the interest rate drops to 5.5% p.a.

i Calculate the new monthly repayments.

ii If the couple is able to keep paying the original repayments, how much earlier will the loan be paid off? State any assumptions you make.

ACTIVITY 1

Interest rates have changed a great deal over time. They have a big impact on what homes a family can afford.



What to do:

1 Suppose you have saved \$50 000 as deposit for a new house. The house is valued at \$450 000 (including all fees), so you will need a loan for \$400 000.

Copy and complete the tables on the next page with the monthly repayments needed, and the total interest paid, for different interest rates and loan durations. You will need to click on the "Monthly repayments" icon and use the table of monthly repayments given. Consult the websites of some banks to find the current interest rates.

Monthly repayments

	4%	6%	8%	10%	12%	*Current interest rates*
5 years						
10 years						
15 years						
20 years						

Total interest paid

	4%	6%	8%	10%	12%	*Current interest rates*
5 years						
10 years						
15 years						
20 years						

2 Discuss the impact of higher interest rates on the total interest that needs to be paid over the course of a loan. Include a discussion of the benefit of paying off a loan over a shorter time period, and what effect interest rates have on a family's ability to do this.

ACTIVITY 2 THE LOAN REPAYMENTS FORMULA

In the previous Exercise we used technology and a table of values to find the necessary repayments for a loan. In this Activity we explore the calculations leading to these values. We will use our knowledge of geometric sequences and series.

Consider a loan for which PV is the amount originally borrowed
n is the number of repayment periods
i is the interest rate *per period*, and
p is the repayment per period.

After each time period, interest is charged on the loan by multiplying the balance by $(1+i)$. The repayment p is then subtracted from the balance.

So, the balance after:

- 0 repayment periods $= PV$
- 1 repayment period $= PV(1+i) - p$
- 2 repayment periods $= [PV(1+i) - p](1+i) - p = PV(1+i)^2 - p(1+i) - p$

What to do:

1 Show that the balance of the loan after:

a 3 repayment periods $= PV(1+i)^3 - p(1+i)^2 - p(1+i) - p$

b 4 repayment periods $= PV(1+i)^4 - p(1+i)^3 - p(1+i)^2 - p(1+i) - p$

2 Following the pattern in the formulae, show that the balance of the loan after n repayment periods will be

$$\underbrace{PV(1+i)^n}_{\text{compound interest formula for balance if no repayments were made at all.}} - \underbrace{\sum_{k=0}^{n-1} p(1+i)^k}_{\text{reduction due to regular repayments}} = PV(1+i)^n - p\,\frac{(1+i)^n - 1}{i}.$$

3 After n repayments, the loan must be completely repaid. Hence show that the regular repayment must be $p = \dfrac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$.

4 By setting $PV = 16\,500$, $i = \dfrac{0.055}{12}$, and $n = 4 \times 12 = 48$, verify the result in **Example 1**.

5 Use the formula $p = \dfrac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$ to check the repayments you calculated in **Exercise 4A**.

6 Jade takes a loan of \$20 000 to be repaid over 5 years at 5% p.a. interest compounded monthly. Use the formula to find her monthly repayments. Check your answer using technology.

7 Apply the formula in a spreadsheet to generate the table of values for loan repayments shown on page **79**.

8 Are there any other advantages to knowing a formula to find an answer, rather than relying on a calculator's built-in solver?

DISCUSSION

Do you think it is better to *borrow* money to purchase an item you want, or to *save* your money until you can afford to purchase the item?

In your discussion, consider:

- the costs associated with borrowing
- the benefits associated with having the item sooner rather than later
- the effect of inflation on the cost of the item.

B ANNUITIES

An **annuity** is an investment where an individual makes a lump-sum deposit, and then makes regular withdrawals over a fixed time period. The investment earns interest according to the balance of the annuity each time period.

In effect, an annuity is the reverse of a personal loan. The person loans the bank their money, and the bank makes regular repayments over time. As the balance in the annuity fund decreases, so does the interest it earns. However, the regular repayments remain the same.

Annuities are most commonly used as a source of regular income for people who have retired from work. Money saved during their working life is "rolled over" into an annuity fund. The regular withdrawals provide money for the person to live on.

DISCUSSION

In some countries, all workers receive **superannuation** payments as part of their salaries. They do not have access to this money until they reach retirement age, so it is a form of compulsory saving.

Why might a government make superannuation compulsory?

When performing calculations with annuities:

- The person is depositing a lump-sum with the bank, so PV is negative.
- The person receives regular repayments, so PMT is positive.

Example 3 **Self Tutor**

Heather has just retired at age 65. She has \$900 000 in her savings fund. She "rolls over" the money into an annuity fund which returns 4% p.a. compounded monthly.

a If Heather withdraws \$5400 per month to live on, how long will it take for the money in the fund to run out?

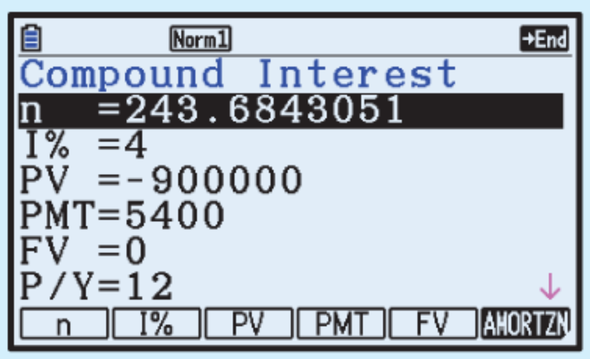
b If Heather wants the money to last until she is 90 years old, how much can she afford to withdraw each month?

a We need to find how long it will take for the future value to fall to \$0.

$I\% = 4$, $PV = -900\,000$, $PMT = 5400$, $FV = 0$,
$P/Y = 12$, $C/Y = 12$

$\therefore \quad N \approx 243.7$

It will take 244 months (or 20 years 4 months) for the money in the fund to run out.

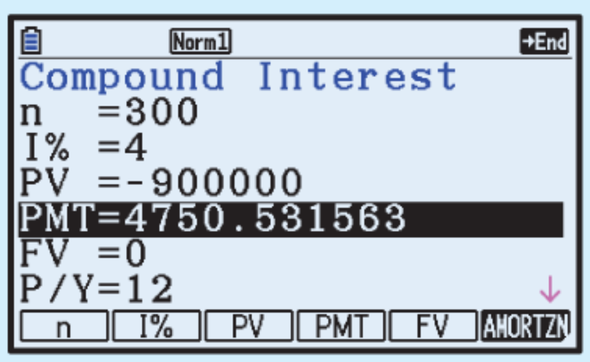


b Heather wants the money to last $90 - 65 = 25$ years.

$N = 25 \times 12 = 300$, $I\% = 4$, $PV = -900\,000$, $FV = 0$,
$P/Y = 12$, $C/Y = 12$

$\therefore \quad PMT \approx 4750.53$

Heather can afford to withdraw \$4750.53 each month.



EXERCISE 4B

1 Sue has achieved her target of \$700 000 in her savings fund for her retirement at age 55. She rolls the money into an annuity account earning 4.85% p.a. compounded monthly.

a How long will Sue's money last if she withdraws \$4000 per month?

b If Sue wants her money to last for 30 years, how much can she afford to withdraw per month?

2 Terrence has \$800 000 in his savings fund. He rolls his money into an annuity fund which earns 4.7% p.a. compounded monthly. He wants to withdraw \$7000 each month to live on.

a How long will his money last?

b How much *longer* would his money last if he only withdrew \$6000 each month?

3 Henry has retired at age 68 with £830 000 in his savings fund. He rolls the money into an annuity fund that earns 4% p.a. compounded monthly.

a If Henry wants the money to last until he is 85, how much can he withdraw each month?

b Henry currently lives on £6000 per month. Will Henry be able to maintain his current standard of living? Explain your answer.

4 Tamsyn invests \$750 000 in a savings account which pays 3.8% p.a. interest compounded quarterly.

a How much money will be in the account when she retires in 10 years' time?

b When she retires, Tamsyn rolls the money into an annuity fund which pays 4.9% p.a. interest compounded monthly. Tamsyn would like the money to last for 15 years.

i How much can Tamsyn withdraw each month?

ii How long will it take for the balance of the fund to fall to \$300 000?

5 After retiring at age 55, Danny rolls his £600 000 into an annuity account earning 5.9% p.a. compounded quarterly.

a If he wants his money to last for 25 years, how much can he withdraw each quarter?

b How much money will be left in Danny's annuity account when he is 68?

c How much *more* could Danny withdraw each quarter if his money only needed to last 20 years?

6 When Maggie retires at 70, she will deposit her savings in an annuity account which pays 6.2% p.a. interest compounded monthly. She wants to withdraw $4500 per month from the annuity account until she is 90.

a Show that Maggie will withdraw a total of $1 080 000 from her annuity account.

b Explain why Maggie does not need $1 080 000 in savings at the time when she retires.

c Calculate the amount Maggie will need in savings when she retires.



7 Abby has carefully saved $1 000 000 during her career. She rolls it into an annuity fund which pays 5% p.a. interest compounded monthly. If Abby withdraws $4000 each month, how long will Abby's money last?

Hint: Find the balance of the fund after 1 year.

8 Igor has saved €550 000. He wants to put the money in an annuity fund where he can withdraw €5000 per month for the next 15 years.

a What interest rate, compounded monthly, will Igor require?

b The best interest rate Igor can find is 4.9% p.a. compounded monthly.

i If he still wants to withdraw €5000 per month, how much *less* time will the money last?

ii If he still wants the money to last 15 years, how much *less* must his withdrawals be each month?

9 Luke rolls his $700 000 of savings into an annuity fund which earns 4.5% p.a. interest compounded monthly. He wants the money to last for 16 years.

a How much money can Luke withdraw each month?

b Find the outstanding balance of the fund after 10 years.

c After 10 years, Luke receives an inheritance of $100 000 which he adds to the annuity fund. How much is Luke now able to withdraw each month for the remaining 6 years?

10 Célia deposits €500 000 in an annuity fund which earns 4.5% p.a. interest compounded monthly. Célia wants the money to last for 20 years.

a How much can Célia afford to withdraw each month?

b Find the outstanding balance of her fund after 5 years.

c After 5 years, Célia transfers the balance of her fund to an account which earns 5.5% p.a. interest compounded monthly.

i How much can Célia afford to withdraw each month for the remaining 15 years?

ii How long will it take for the balance of the fund to fall to €200 000?

ACTIVITY 3 PERPETUITIES

A **perpetuity** is a special type of annuity in which the regular payments continue indefinitely. This is achieved by ensuring the interest earned by the investment matches the payments taken out.

The **price** we would expect to pay for a perpetuity is therefore the principal required to give interest equal to the regular repayment.

So, the price or **present value** of a perpetuity is given by

$PV = \dfrac{PMT}{r}$ where PMT is the payment received each year and r is the annual interest rate.

For example, for a perpetuity which pays \$2000 per year indefinitely, with an interest rate of 4% p.a., you would expect to pay $PV = \dfrac{PMT}{r} = \dfrac{\$2000}{0.04} = \$50\,000$.

What to do:

1 Find the price you would expect to pay for a perpetuity of:

a \$300 per year with an interest rate of 5% p.a.

b \$4000 per year with an interest rate of 3.2% p.a.

c \$18 000 per year with an interest rate of 4.5% p.a.

2 Find the annual payment you would expect to receive from a perpetuity which has:

a present value €1000 and interest rate 3.5% p.a.

b present value €25 000 and interest rate 4.8% p.a.

3 **a** Suppose you had \$500 000 in savings, and used it to purchase a perpetuity with interest rate 4% p.a. What payment would you expect to receive each year?

b Suppose \$500 000 is deposited in a pension fund which pays 4% p.a. interest compounded annually. How much can be withdrawn from the fund each year so that the money lasts for:

i 10 years **ii** 20 years **iii** 50 years **iv** 200 years **v** 1000 years?

c Comment on your answers to **a** and **b**.

ACTIVITY 4 GROWING ANNUITIES

In the previous Exercise, we have assumed that the amount withdrawn each time period does not change over time.

This is unrealistic for several reasons:

- The prices of goods and services increase over time with inflation, so the cost of living at the same standard increases.
- As we age, we inevitably need more health care and domestic services, so these expenses increase.

In a **growing annuity**, the amount withdrawn increases by a certain percentage, called the **growth rate**, each year.

What to do:

1 **a** Click on the icon to open a spreadsheet for a growing annuity fund. \$300 000 is deposited into the fund which earns 6% p.a. interest compounded monthly for 20 years. We assume a growth rate of 2% p.a.

The initial withdrawal is calculated as \$1819.92 .

SPREADSHEET



	A	B	C	D	E
1	**Growth Annuities**				
2					
3	**Amount**		\$300,000		
4	**Number of years**		20		
5	**Rate p.a.**		0.06		
6	**Periods p.a.**		12		
7	**Rate per period**		0.005		
8	**Growth rate p.a.**		0.02		
9	**Growth rate per period**		0.00167		
10					
11	**Initial withdrawal**		\$1,819.92		
12					
13	**Period**	**Amount**	**Interest**	**Withdrawal**	**Balance**
14	1	\$300,000.00	\$1,500.00	\$1,819.92	\$299,680.08
15	2	\$299,680.08	\$1,498.40	\$1,822.95	\$299,355.53
16	3	\$299,355.53	\$1,496.78	\$1,825.99	\$299,026.32
17	4	\$299,026.32	\$1,495.13	\$1,829.03	\$298,692.42
18	5	\$298,692.42	\$1,493.46	\$1,832.08	\$298,353.80

b Discuss what is happening to the withdrawals over time.

c Find the amount withdrawn for the final time period.

d Use your calculator to find the regular withdrawal you would make for this investment. Compare this value with the initial and final withdrawals in the growing annuity.

e How could you calculate the regular withdrawal found in **d** by adjusting the spreadsheet?

2 Suppose \$500 000 is deposited into a growing annuity fund which earns 4% p.a. interest compounded monthly for 25 years. Assume a growth rate of 3% p.a.

a Use the spreadsheet to find:

i the initial monthly withdrawal **ii** the total amount withdrawn over 25 years.

b Hence find the total interest earned by the annuity fund. Check your answer by finding the sum of the interest payments in the spreadsheet.

3 Predict the effect on the initial withdrawal of increasing:

a the amount originally deposited **b** the duration of the annuity

c the interest rate **d** the growth rate.

Use the spreadsheet to check your answers.

4 For a growing annuity over n periods with initial deposit PV, interest rate i per period, and growth rate g per period, the initial withdrawal w is given by $w = \dfrac{PV \times (i-g) \times (1+i)^n}{(1+i)^n - (1+g)^n}$.

a Verify that this formula gives the correct initial withdrawals for the scenarios in **1** and **2**.

b What happens to this formula when $g = 0$? Compare your answer with the formula found in **Activity 2** on page **83**.

REVIEW SET 4A

1 Alberto takes out a personal loan for \$23 000 at 7% p.a. over 5 years. Calculate:

a his monthly repayments **b** the total of the repayments

c the total interest charged.

2 Alexandra takes out a loan of €2000 to pay for an emergency vet bill. She will repay the loan over 6 months at 8.12% p.a. interest compounded fortnightly. Calculate:

a Alexandra's fortnightly repayments

b the outstanding balance on the loan after 6 fortnights.

3 Simone borrows $410 000 to purchase a house. She agrees to repay the loan over 25 years at 6.95% p.a. interest compounded monthly.

a Find Simone's minimum monthly repayments.

b Show that the total interest Simone will pay is greater than the amount she originally borrowed.



4 Yasmin retires at age 60 with $500 000 in her savings fund. She rolls the money into an annuity fund earning 5.25% p.a. interest compounded monthly.

a How long will Yasmin's money last if she withdraws $6000 per month?

b If Yasmin wants her money to last for 25 years, how much can she afford to withdraw per month?

5 When Scott retires at 68, he will deposit his savings in an annuity fund which pays 5.4% p.a. interest compounded monthly. He wants to be able to withdraw $6000 per month from the fund until he is 85.

a Calculate the amount Scott will need to have in savings when he retires.

b How much money will be left in Scott's account when he is 80?

6 After retiring at age 65, Vasili rolls €1 400 000 of savings into an annuity account earning 5.4% p.a. interest compounded monthly. He wants his money to last for 30 more years.

a How much can Vasili withdraw per month?

b How long will it take for the balance of the fund to fall below €1 000 000?

c How much of Vasili's annuity will be left after 20 years?

d After 20 years, Vasili must withdraw €20 000 from the account to pay for an operation.

i If Vasili continues to make the same monthly withdrawals he made for the first 20 years, how long will his money last?

ii How much is Vasili now able to withdraw each month if his money is to last for the remaining 10 years?

REVIEW SET 4B

1 Nicola and Hamish take out a personal loan of $12 000 to pay for their wedding. The loan is to be repaid over 4 years at 5.5% p.a. Calculate:

a their monthly repayments **b** the total interest charged on the loan.

2 A loan of 500 000 pesos is taken out at 6% p.a. interest compounded monthly.

a Calculate the monthly loan repayments if the loan is taken out for:

i 4 years **ii** 6 years.

b Which loan charges the least total interest? Explain your answer.

3 Peter has saved \$4500 towards buying a car. The car he wants to buy is valued at \$22 000, so he will borrow the remaining money. He is able to get a loan for 4 years at 6.9% p.a. interest, compounded quarterly.

a How much will Peter borrow?

b Calculate Peter's quarterly repayments.

c Find the outstanding balance after 2 years.

d After 2 years the interest rate for the loan increases to 7.5% p.a. compounded quarterly.

i Calculate the new quarterly repayments for the remaining 2 years.

ii How much *additional* interest must Peter pay as a result of the interest rate rise?

4 Answer the **Opening Problem** on page **78**.

5 Pia retires at age 62 with €350 000 in her savings fund. She rolls this money into an annuity fund earning 5.8% p.a. interest compounded monthly.

a How much will she be able to withdraw each month if her money is to last another 2 decades?

b How much *more* will she be able to withdraw each month if her money was to only last 15 years?

6 Harold rolls his £800 000 of savings into an annuity account. He wants the money to last for 15 years.

a Given that Harold can withdraw £6284.75 each month, find the annual interest rate, compounded monthly, of the account.

b How much longer would Harold's money last if he withdrew only £5000 each month?

Chapter 5

Modelling

Contents:

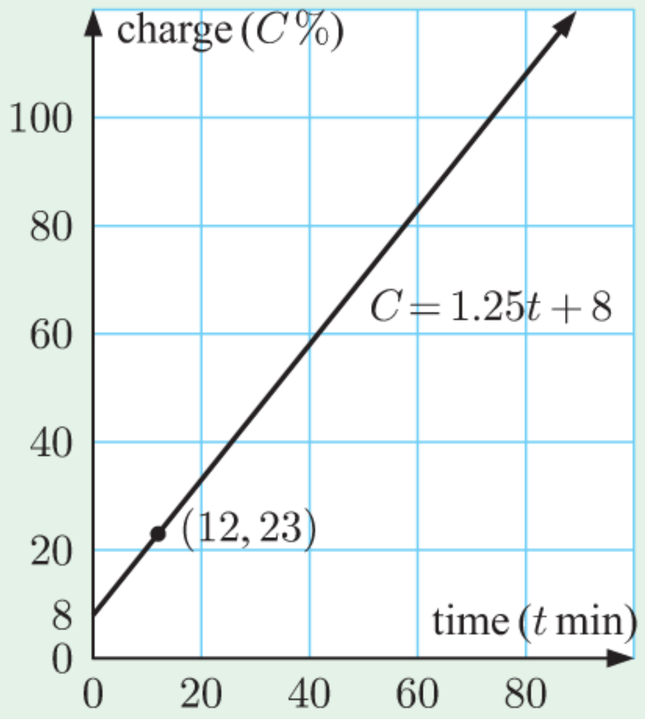
A The modelling cycle
B Linear models
C Piecewise models
D Systems of equations



OPENING PROBLEM

Briony's laptop had 8% charge left when she started to recharge it. 12 minutes later, the laptop had 23% charge.

Briony would like to know when the laptop will be fully charged. She constructs the model alongside to describe the laptop's charge after t minutes.



Things to think about:

a Can you explain how Briony constructed her model? What assumptions did Briony make? Do you think the assumptions are reasonable?

b What range of values can each variable take? Has Briony accounted for this correctly?

c According to Briony's model, how long will it take for the laptop to be fully charged?

d The laptop actually took 79 minutes to be fully charged.

i Why do you think the actual time differed from the time predicted by Briony's model?

ii Do you think Briony's model was accurate enough to be useful?

We can apply mathematics to real-world problems to help understand the world around us.

A **mathematical model** is a simplified description of a real system using mathematical concepts and language. The process of developing this model is called **mathematical modelling**.

A THE MODELLING CYCLE

Beyond basic problems of counting, most real-world problems are complicated. We generally cannot find an exact, perfect solution.

In mathematical modelling, we simplify the problem by making assumptions and removing less important details. This allows us to construct a mathematical description of the problem which is simple enough for us to work with, and which we can solve to approximate the real-world situation.

Mathematical models are developed using a **modelling cycle**:

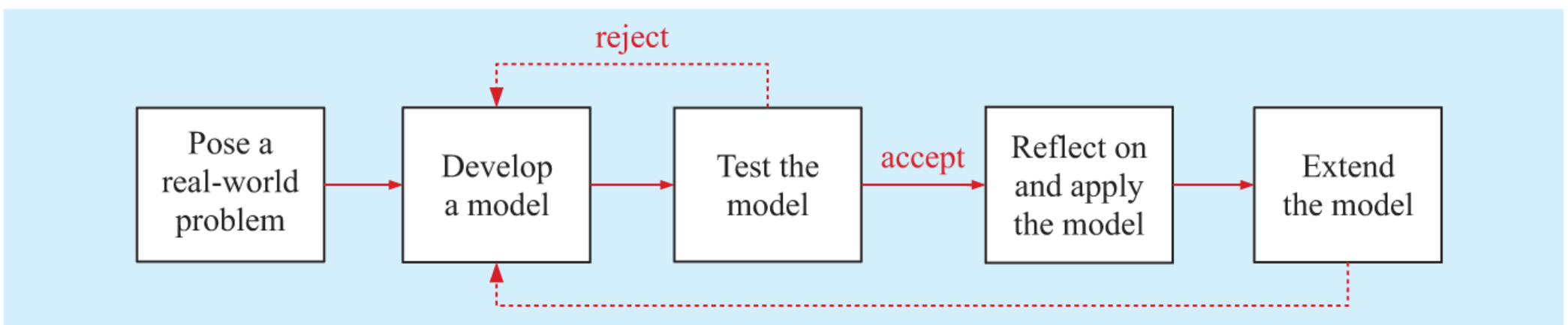
Step 1: **Pose** a real-world problem. Make **assumptions** which simplify the problem without missing key features.

Step 2: **Develop** a model which represents the problem with mathematics. This may involve a formula or an equation. The model should consider constraints such as the range of possible values each variable can take in the real world.

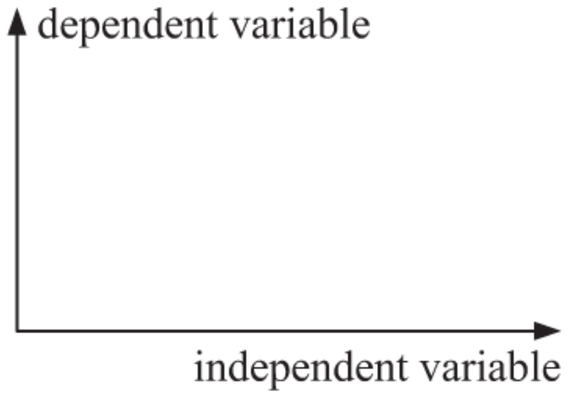
Step 3: **Test** the model by comparing its predictions with known data. If the model is unsatisfactory, return to *Step 2*.

Step 4: **Reflect** on your model and **apply** it to your original problem, interpreting the solution in its real-world context.

Step 5: If appropriate, **extend** your model to make it more general or accurate as needed.



In general, when we consider the relationship between two variables, the value of one of the variables is *dependent* on the value of the other *independent* variable. We place the independent variable on the horizontal axis, and the dependent variable on the vertical axis.



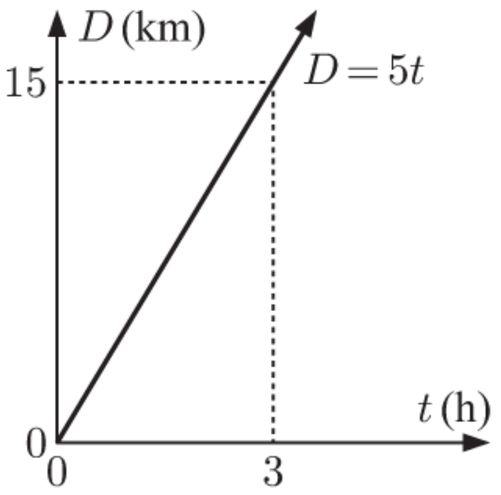
For example, suppose we are interested in the distance a person could walk in a certain time. *Time* is the independent variable and *distance* is the dependent variable. In reality, the person will not walk at the same speed the whole time, and they will not walk at *exactly* 5 km h^{-1}. However, we might *assume* that the person walks at this typical speed. This assumption allows us to construct a simplified model for the distance (D km) travelled in t hours:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$\therefore \text{ distance} = \text{speed} \times \text{time}$$

$$= 5\text{ km h}^{-1} \times t\text{ h}$$

$$\therefore D = 5t\text{ km}$$

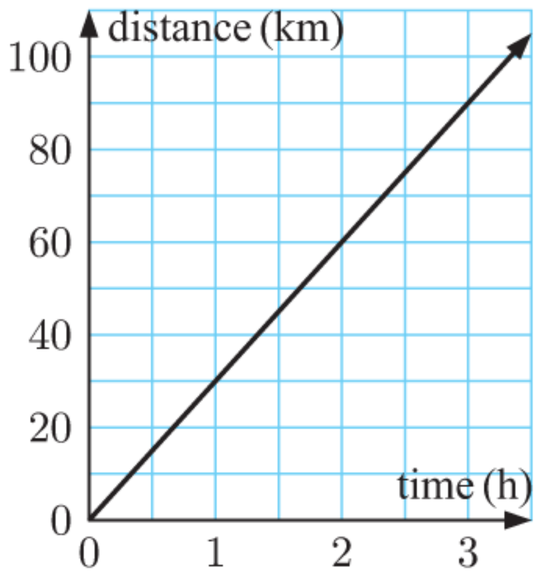


We can use the model to predict that, for example, in 3 hours the person will travel 15 km.

If this model is not sufficiently accurate, we might consider using different speeds for walking uphill, on flat ground, and downhill.

EXERCISE 5A

1 The graph alongside is a model for the distance travelled by a cyclist.



a What assumptions have been made in constructing the model? Do you think the assumptions are reasonable?

b Use the model to predict the distance travelled by the cyclist in 2 hours.

2 Answer the **Opening Problem** on page **92**.

3 Rick takes 15 seconds to run 100 metres.

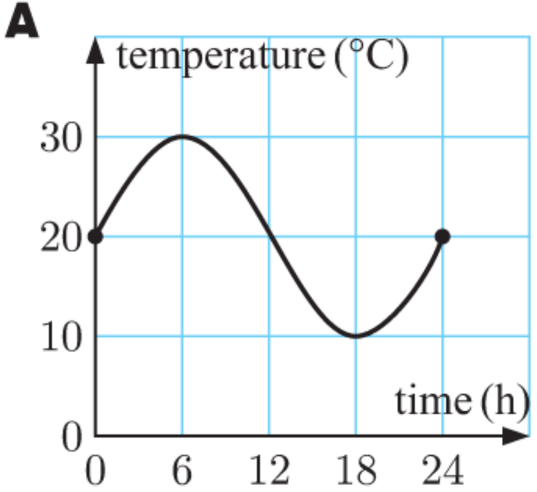
a Construct a model to describe how long Rick takes to run d metres.

b Hence predict how long Rick takes to run 500 metres.

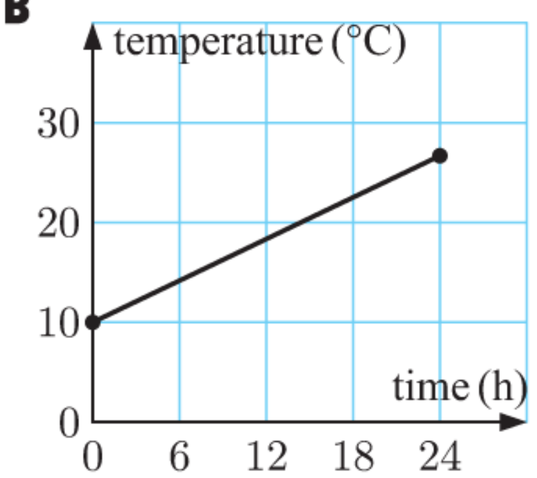
c Do you think the actual time Rick takes to run 500 metres will be longer or shorter than your prediction? Explain your answer.

4 **a** Which of these graphs is the most appropriate model for the temperature of a city on a particular day? Explain your reasoning. Time is measured in hours after midnight.

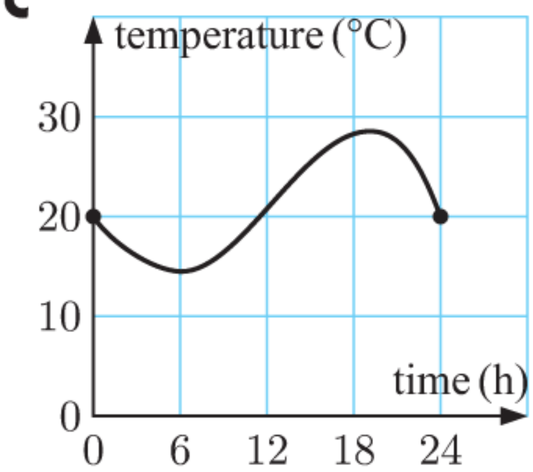
A



B



C

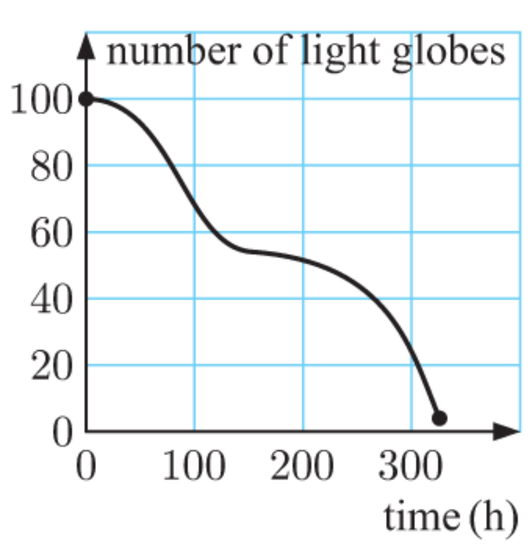


b Use the model you selected in **a** to predict the temperature of the city at 8 pm.

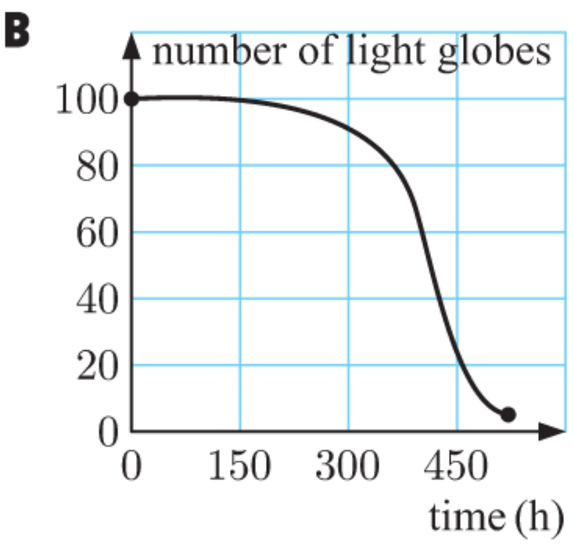
5 A particular type of light globe is claimed to last for 200 hours. The manufacturer simultaneously tests 100 of these globes.

a Which of these graphs do you think is the most appropriate model for the number of light globes still working at any given time? Explain your reasoning.

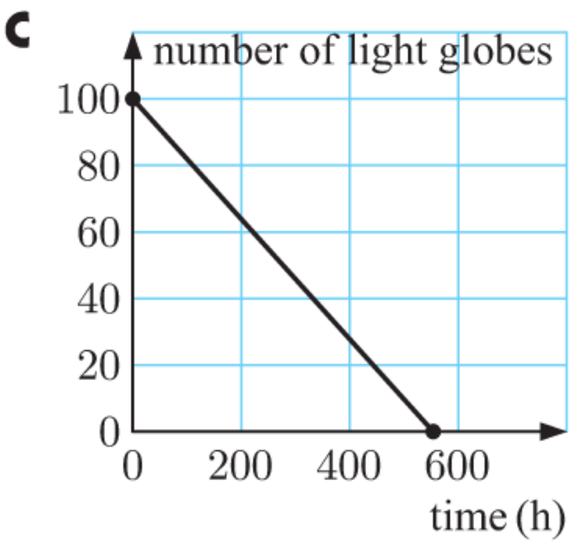
A



B

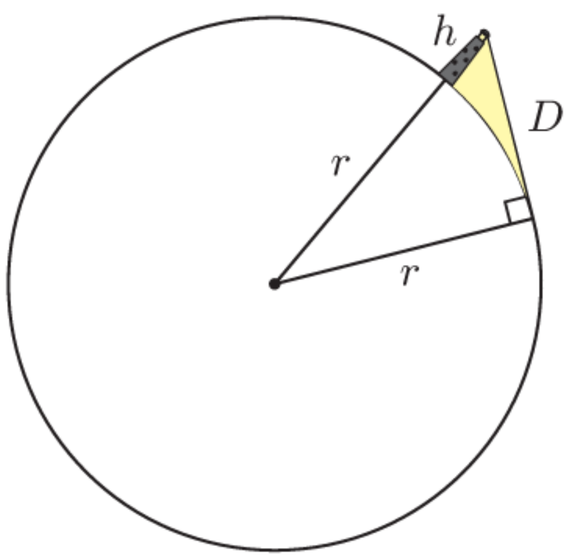


C



b Use the model you selected in **a** to predict the number of light globes still working after 250 hours.

6



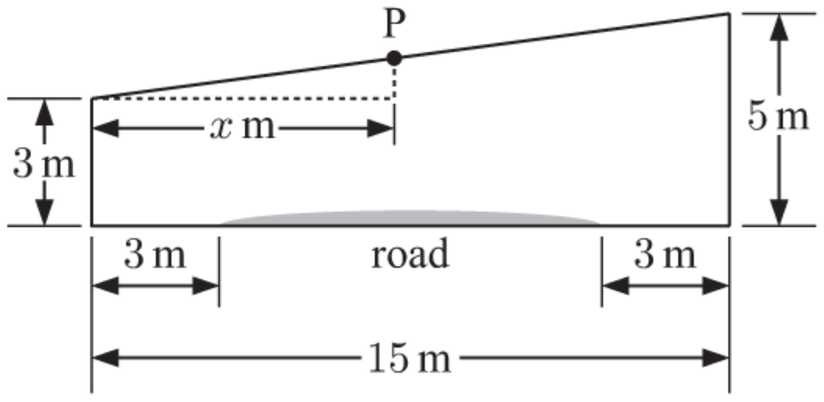
A lighthouse is 40 m tall. Darren wants to know how far the light at the top can be seen from.
He draws the diagram alongside to describe the situation, where r is the Earth's radius, h is the height of the lighthouse, and D is the required distance.

a What assumptions has Darren made in constructing this model? Do you think they are reasonable?

b Given that the Earth's radius is approximately 6370 km, use the model to find D.

7 A power line is connected between two poles as shown. Hashni needs to know the height of the power line above the road, as he needs to work out whether a truck can fit underneath.

a Use similar triangles to construct a model for the height of the power line above ground level at point P. Describe any assumptions you make when constructing your model.



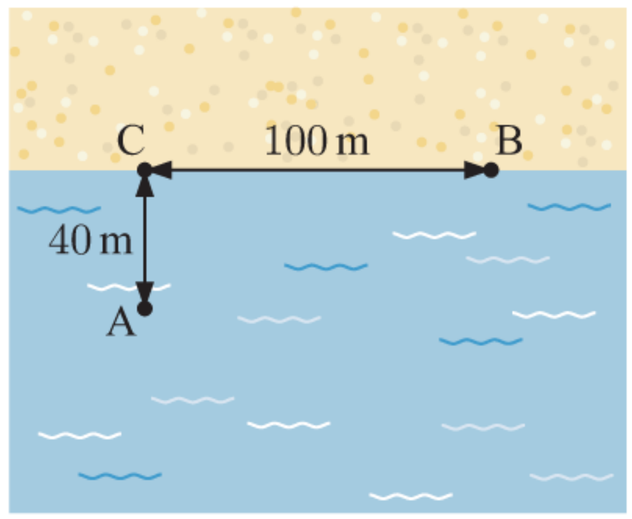
b For what values of x is it reasonable to apply the model?

c Use your model to predict the height of the power line above ground level as it passes over the middle of the road.

d Does your answer to **c** seem reasonable? Explain your answer.

e Do you think your model would be useful for Hashni's purposes? Explain your answer.

8 Antonio is swimming at A, and his belongings are at B. Feeling cold, he wonders how he can return to his belongings quickest.



Antonio can swim at 1.5 m s^{-1} and can jog at 4 m s^{-1}.

a Find the time Antonio would take to return to his belongings by:

i swimming directly to B

ii swimming to C, then jogging to B.

State any additional assumptions you have made in your calculations.

b Which option appears to be quicker?

c Can you find another option that is even quicker?

Example 1 **Self Tutor**

Henri takes 6 hours to clean the house. Marcia takes 5 hours to clean the house. How long would they take to clean the house working together?

Henri can clean $\frac{1}{6}$ of the house each hour, and Marcia can clean $\frac{1}{5}$ of the house each hour.

We assume Henri and Marcia can work without getting in each other's way when they work together. So, working together they will clean $\frac{1}{6} + \frac{1}{5} = \frac{11}{30}$ of the house each hour.

$\therefore$ it would take them $\frac{30}{11} \approx 2.73$ hours ≈ 2 hours 44 minutes to clean the house together.

This answer seems reasonable. With two people working we expect them to take about half the time, and 2 hours 44 minutes is between half of Henri's time and half of Marcia's time.



9 Kate takes 7 hours to dig a hole. Lenny takes 12 hours to dig a hole. How long would they take to dig a hole working together?

10 A tank holds 1200 litres of water. Pulling the plug would drain the tank in 15 minutes. A hole develops which would drain the tank in 25 minutes. How long would it take to drain the tank through the hole and the plug together?

11 Aaron can make 4 widgets per hour, Bonnie can make 3 widgets per hour, and Calum can make 2 widgets per hour. How long would they take to make 135 widgets working together?

12 Angus can paint a room in 3 hours. Angus and Beitidh would take 2 hours to paint the same room together. How long would Beitidh take to paint the room on her own?

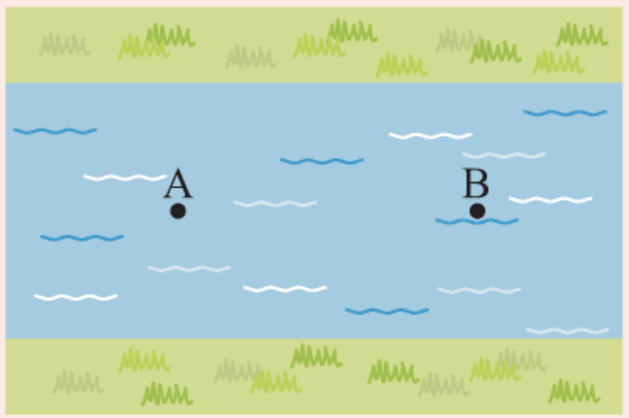
13 Huiliang and Lifen would take 2 hours to complete a jigsaw puzzle. Lifen and Suyin would take 3 hours, whereas Huiliang and Suyin would take 4 hours. How long would it take if all three of them worked together to complete the jigsaw puzzle?

ACTIVITY MATHEMATICAL MODELLING

What to do:

Working in small groups, try to solve the problems presented below. You will need to make some assumptions in order to simplify each problem to one that can be solved mathematically. Record your assumptions as well as your mathematical solution.

- How long would it take to count to 1 million out loud?
- Do people who take showers or baths use more water?
- A swimmer must swim up a river from A to B, and then back to A.
  Would a current in the river make the task take more time or less time?



- One tonne of sand is dumped into a pile on the ground. How high is the pile?

Compare your answers with those of other groups. If your answers differ significantly, compare the assumptions you have made.

INVESTIGATION CALENDARS

A **solar day** is the time it takes Earth to rotate about its axis. A **tropical year** is the time it takes the Earth to orbit around the Sun.

There are approximately 365.242 199 solar days in a tropical year. However, it is much more convenient to have a calendar with a whole number of days in the year. In this Investigation we will explore some models for achieving this.

What to do:

1 If we fixed a year as 365 days then quite quickly we would find the months drifting apart from the seasons. July would eventually move to winter in the Northern hemisphere!

a Find the difference between 365 days and a tropical year.

b How long would it take for the difference between 365 days and a tropical year to accumulate to one whole day?

c How many years would it take for the difference to accumulate to six months?

2 To fix this problem, we have leap years in which an extra day is added to the end of February. The **Julian calendar**, devised by **Julius Caesar** in 46 BC, adds an extra day to every fourth year, giving the average length of a year as 365.25 days.

a In days per year, how inaccurate is the Julian calendar?

b How long would it take for this inaccuracy to accumulate to:

i one whole day **ii** six months?

3 In the mid 1200s, **Roger Bacon** (1219 - 1292) noticed that the Julian calendar had already drifted by about nine days. Religious festivals such as Easter were being celebrated at the wrong time! However, it was only in 1582 that **Pope Gregory XIII** introduced the **Gregorian calendar** used in most of the world today.

In the Gregorian calendar, every 4th year is a leap year unless it is divisible by 100 and *not* by 400. For example:

- 1800 was divisible by 100 but not divisible by 400, so it was *not* a leap year.
- 2000 was divisible by 100 and by 400, so 2000 *was* a leap year.

When the Gregorian calendar was first introduced, people who went to bed on October 4th 1582 woke up the next day on October 15th. People did not like having so many days "stolen" from them! Few European countries used the new system immediately. For example, the Kingdom of Great Britain waited until 1752, Russia until 1917, and China until 1949. The Eastern Orthodox Church has still not adopted the Gregorian calendar.

a In each period of 400 years, how many leap years does the Gregorian calendar add?

b Find the length of an average year in the Gregorian calendar.

c How long will it take for the inaccuracy in the Gregorian calendar to accumulate to one whole day?

4 The Julian calendar has an approximate error of one day per 128 years. Noticing that $128 = 2^7$ gives rise to another possible model. We first write the year number in **binary**, which means that the number is written in base 2 rather than the traditional base 10.

a In base 2, a number will end in n 0s if it is divisible by 2^n.
Describe how the year number ends, if it is divisible by: **i** 4 **ii** 128.

b In the **binary calendar**, there is a leap year if the year number ends in "00", except if it ends in "0000000".

i Explain how the binary calendar improves the Julian calendar.

ii What advantages does this binary calendar have over the Gregorian calendar?

iii In each period of 128 years, how many leap years does the binary calendar add?

iv Find the length of an average year in the binary calendar.

v How long will it take for the inaccuracy in the binary calendar to accumulate to one whole day?

THEORY OF KNOWLEDGE — MOVEMENT OF THE PLANETS

People have always been fascinated by the Moon, the planets, and the stars. Many attempts have been made to accurately model their movement, given their importance in navigation and for calendars.

Early teaching in Europe put the Earth at the centre of the universe. However, **Nicolaus Copernicus** (1473 - 1543) created a new model of the solar system which placed the Sun at the centre. In this model, the orbits of the planets were circles centred at the Sun. The orbit of the Moon was also a circle, centred at the Earth. Putting the Sun at the centre made it much easier to describe the positions of the planets.



Nicolaus Copernicus

By carefully analysing astronomical observations, **Johannes Kepler** (1571 - 1630) realised that the positions of the planets did not exactly fit the theory of circular orbits. He proposed his *three laws of planetary motion*, the first two of which appeared in his book *Astronomia nova* in 1609:

- The planets move in *elliptical* orbits, with the Sun at one focus.
- The speed of the planet is proportional to the area swept out by the orbit.

In the Renaissance, clockwork mechanisms were developed to display the solar system. A mechanism like this is called an **orrery**.

In the late 17th century, **Isaac Newton** developed his law of gravitation and his laws of motion in which he suggested that forces result in *acceleration* and not velocity. His new theories now *predicted* the elliptical orbits suggested by Kepler.

However, Newton's elliptical orbits did not account for the gravitational pull of the planets on each other. A revised model was needed to more accurately account for the observed movement of the planets. Indeed, the existence of the planet Neptune was predicted by the mathematical model before it was directly observed. More recent mathematics has improved models of how bodies move under gravity, and the "three body problem" is the subject of ongoing mathematical research.

1. How did the different models for the solar system build on previous models?
2. Identify assumptions made in each model.
3. Can a mathematical model ever be "complete"?
4. The British statistician **George Box** stated that "All models are wrong, but some are useful".
   Discuss what he meant by this.

B LINEAR MODELS

Two variables are **linearly related** if the graph connecting them is a straight line.

Ted the taxi driver charges passengers an initial fee of £4, and then £2 for each kilometre travelled.



To study the relationship between the *distance travelled* (d km) and the *cost* (C pounds), we construct a table of values and draw a graph. The *cost* is dependent on the *distance travelled*, so we place d on the horizontal axis, and C on the vertical axis.

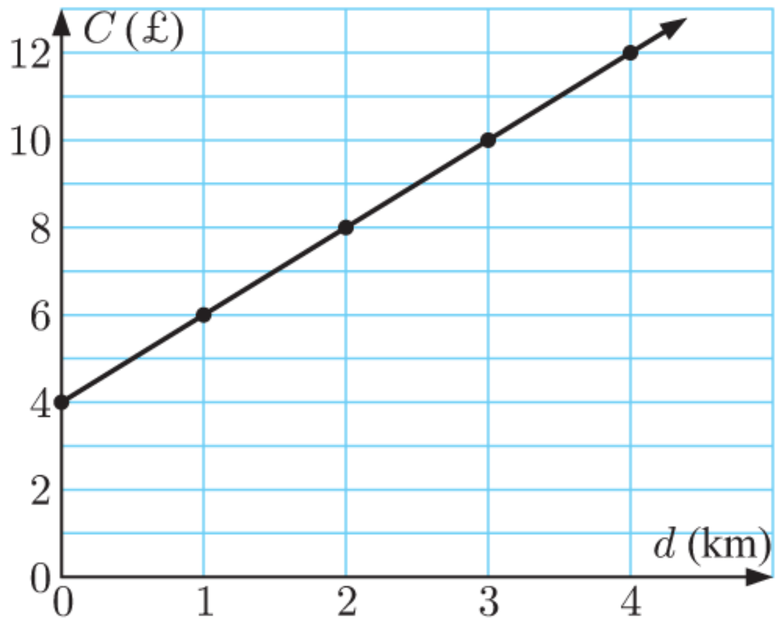
Distance travelled (d km)	0	1	2	3	4
Cost (£C)	4	6	8	10	12

+2 +2 +2 +2

The graph of C against d is a straight line, so C and d are **linearly related**.

Notice in the graph that:

- The C-intercept of the graph is 4. This is the initial fee, in pounds, of the taxi ride.
- The gradient of the graph is 2. This is the cost, in pounds, of each additional kilometre travelled.
- The variables are related by the **linear model** $C = 2d + 4$.



The model can be used to predict the value of one variable given the value of the other variable. For example, for a 10 km journey, $d = 10$, and $C = 2(10) + 4 = 24$. The cost of the 10 km journey is £24.

EXACT AND APPROXIMATE MODELS

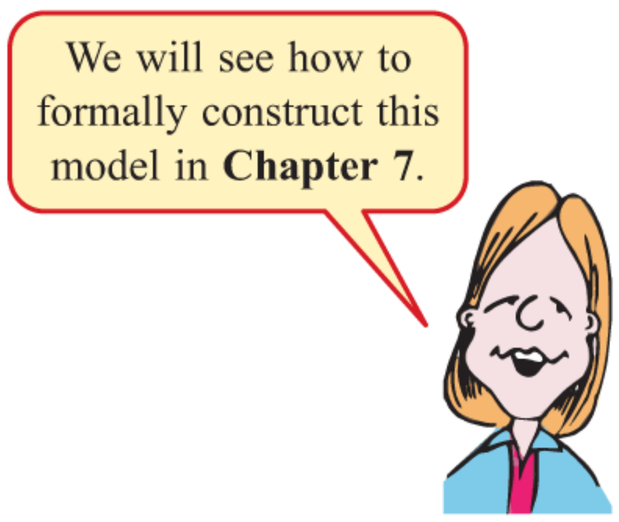
In the taxi situation on the previous page, the cost follows a specific rule, and the relationship between C and d is *exact*. We can use our linear model to find the exact cost C for any positive distance d.

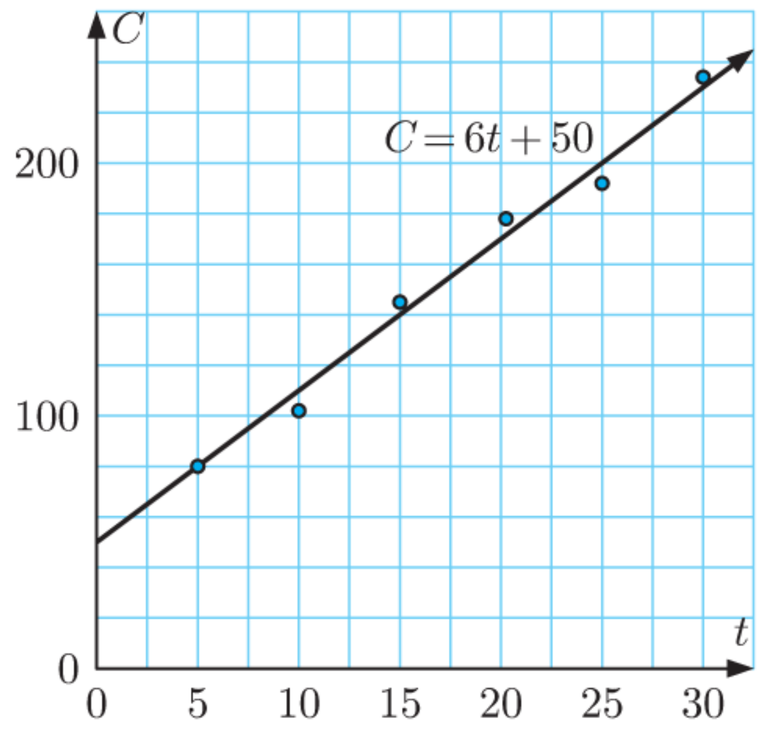
In many situations, however, the connection between the variables is not exact. When we plot a set of data points, we may observe a *linear trend*, which suggests a linear model will be a good *approximation* of the situation.

For example, the table below shows the number of customers at a restaurant every 5th day after it opens.

Time (t days)	5	10	15	20	25	30
Number of customers (C)	80	102	145	178	192	234

If we plot these points on a graph, we can see the variables are approximately linearly related. The line $C = 6t + 50$ is an *approximate* model for the data.



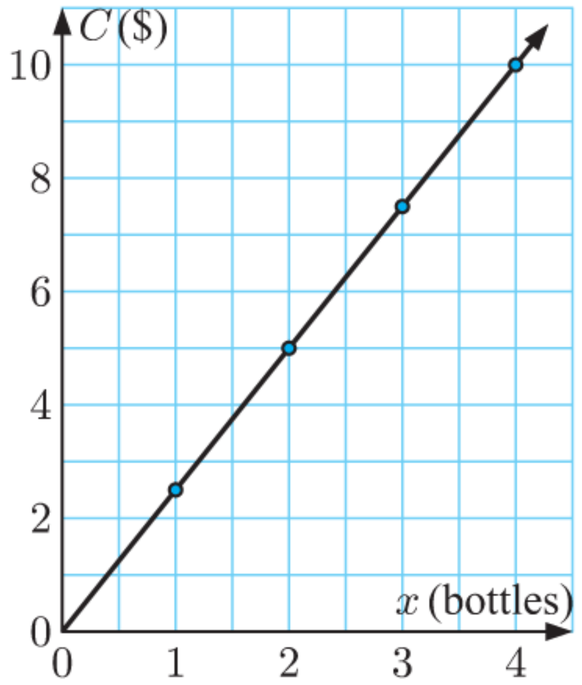


Care must be taken when using an approximate model to predict values beyond the range of the given data points. This process is called **extrapolation**.

For example, we could use the model to predict that, on the 100th day, the restaurant will receive $6(100) + 50 = 650$ customers. However, this assumes that the linear trend will continue far beyond the given data. This is unrealistic given the restaurant will have a maximum capacity.

EXERCISE 5B

1 This graph shows the cost $\$C$ of buying x bottles of juice.



a Find the model connecting C and x.

b Is the model exact or approximate? Explain your answer.

c Can the model be used to find the exact cost of 12 bottles of juice? If so, find the cost.

2 A tank contains 265 litres of water. The tap is left on and 11 litres of water escape per minute.

a Construct a table of values for the volume V litres left in the tank after t minutes, for $t = 0, 1, 2, 3, 4$, and 5.

b Draw the graph of V against t.

c Find the linear model connecting V and t.

d Determine:

i the amount of water left in the tank after 15 minutes

ii the time taken for the tank to empty.

3 Xuanyu planted a 30 cm high bamboo plant in her garden bed. She found that with consistent weather it grew 10 cm each day.

a Copy and complete this table of values which gives the height H of the bamboo after t days.

t (days)	0	1	2	3	4	5	6
H (cm)							

b Draw the graph of H against t.

c Discuss whether it is reasonable to continue the line for $t < 0$ and for $t > 6$ days. Hence state the domain of the function $H(t)$.

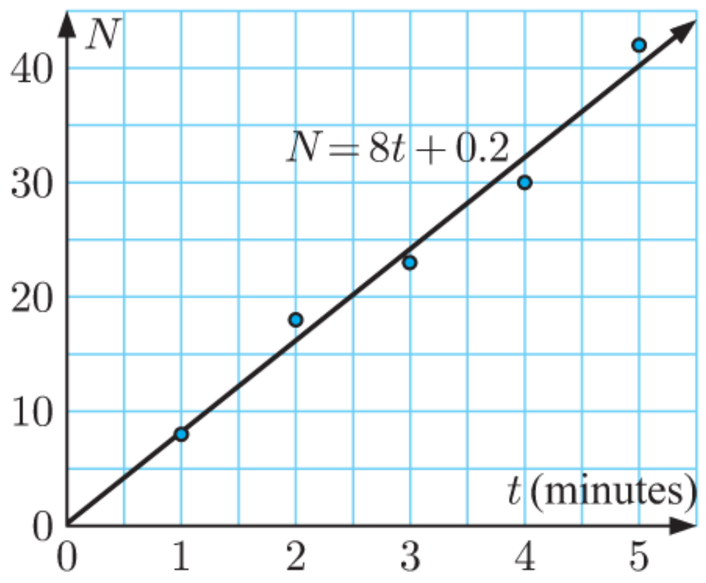
d Find the model connecting H and t.

e How long will it take for the bamboo to be 1 m high?

4 As punishment for misbehaving, Jack must pick up litter at lunchtime. This table shows how many pieces of litter he has picked up after t minutes.

Time (t minutes)	1	2	3	4	5
Number of pieces of litter (N)	8	18	23	30	42

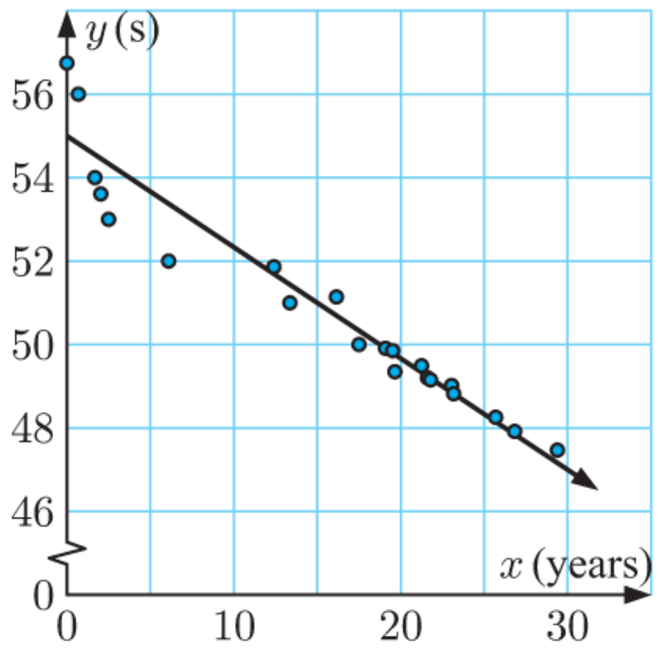
The graph alongside shows these data points, as well as the linear model $N = 8t + 0.2$.



a Is the linear model exact or approximate? Explain your answer.

b Use the model to predict how many pieces of litter Jack will pick up in 20 minutes. Discuss the accuracy of your prediction.

5 The graph alongside shows the world record times for the women's 400 m, between Marlene Mathews' record of 57.0 s on January 6, 1957 and Marita Koch's record of 47.60 s on October 6, 1985. The data is modelled with the straight line $y = 54.85 - 0.2660x$ where y is the time in seconds and x is the number of years since 1957.



a What does the model predict the time will be in 1957? What was the time, and why are these values different?

b Could the model be used to estimate world record times prior to 1957? Explain your answer.

c Florence Griffith-Joyner holds the women's 100 m record of 10.49 s. In what year does this model predict the women's 400 m will be completed in $10.49 \times 4 = 41.96$ s?

d Marita Koch's time of 47.60 s is still the world record in 2018. Discuss the reliability of this model for predicting future world record times.

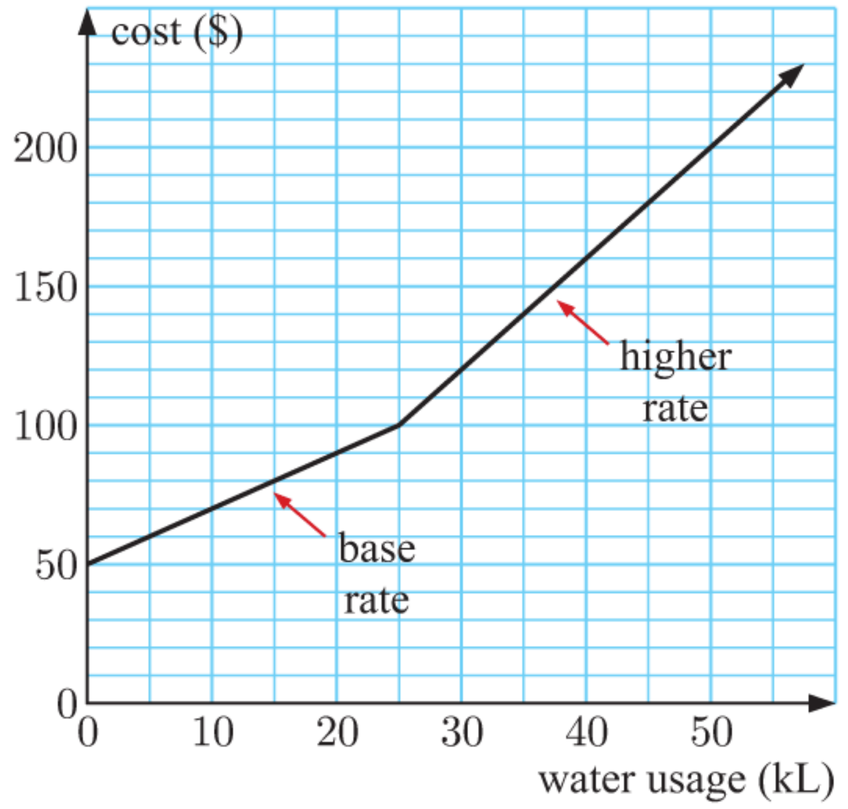
C PIECEWISE MODELS

In many real-world situations, a single formula is not sufficient to describe the behaviour of a variable over its entire domain. Instead, we can divide the domain into smaller subintervals, and build a function using separate formulae for each subinterval. This is known as a **piecewise model**.

PIECEWISE LINEAR MODELS

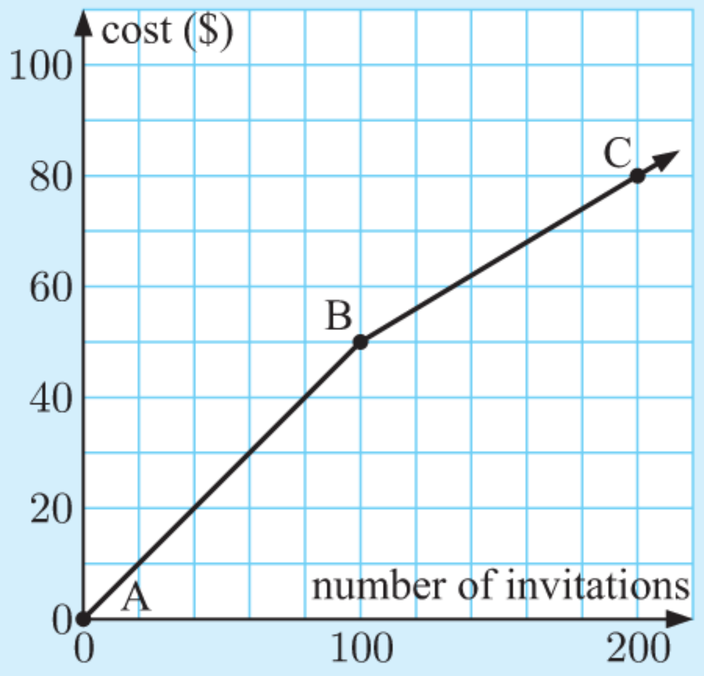
A **piecewise linear model** is a model made up of several straight line segments.

This piecewise linear graph shows the relationship between the *water used* by a household and the *cost* of the water. Water usage up to 25 kL is charged at a base rate, and any water usage above 25 kL is charged at a higher rate. We observe this in the graph because after 25 kL, the gradient of the graph suddenly increases.



Example 2 **Self Tutor**

The *cost* of printing invitations depends on the *number of invitations* printed, as shown in this piecewise linear model.



a Find the gradient of: **i** [AB] **ii** [BC]. Interpret your answer.

b Find the cost of printing 60 invitations.

c How many invitations can be printed for $74?

a **i** A is $(0, 0)$ and B is $(100, 50)$

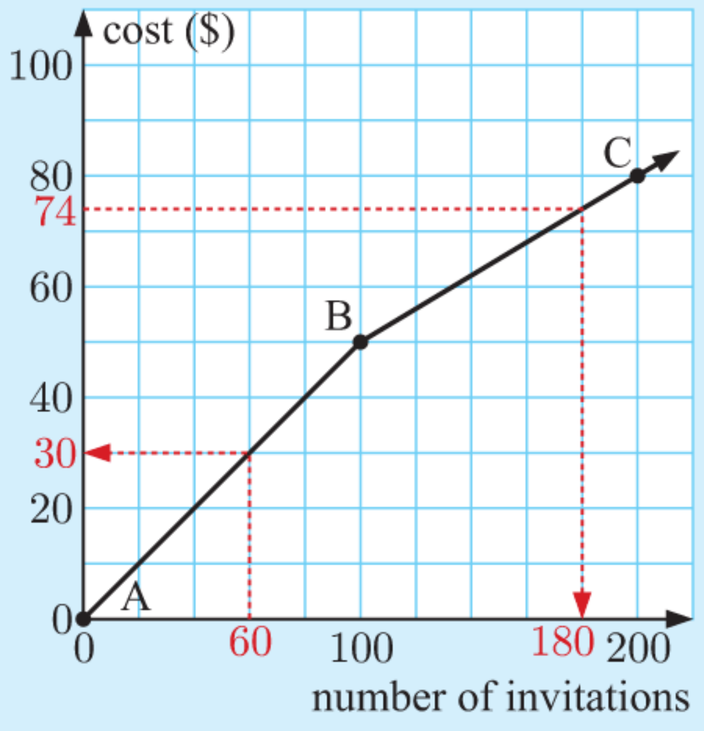
$\therefore$ gradient of [AB] $= \dfrac{50-0}{100-0} = 0.5$

The first 100 invitations cost 50 cents each to print.

ii B is $(100, 50)$ and C is $(200, 80)$

$\therefore$ gradient of [BC] $= \dfrac{80-50}{200-100} = 0.3$

Each invitation after the first 100 costs 30 cents to print.



b It costs $30 to print 60 invitations.

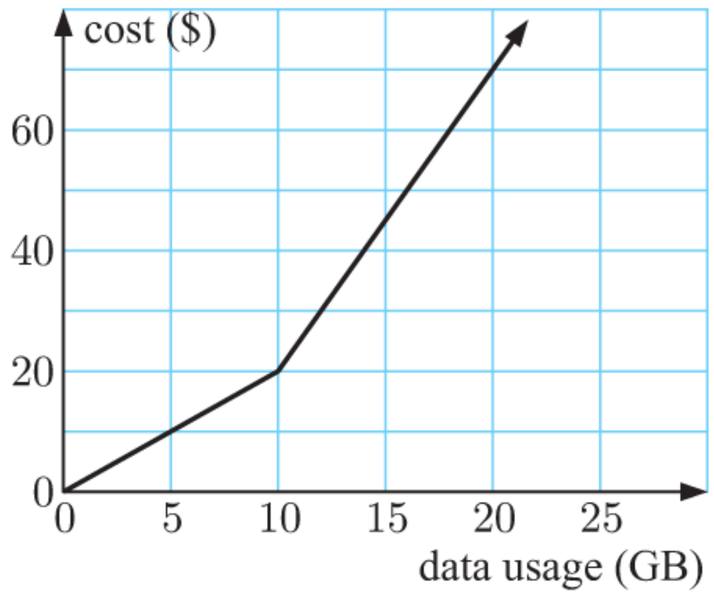
c 180 invitations can be printed for $74.

EXERCISE 5C.1

1 Consider the water costs graph on the previous page.

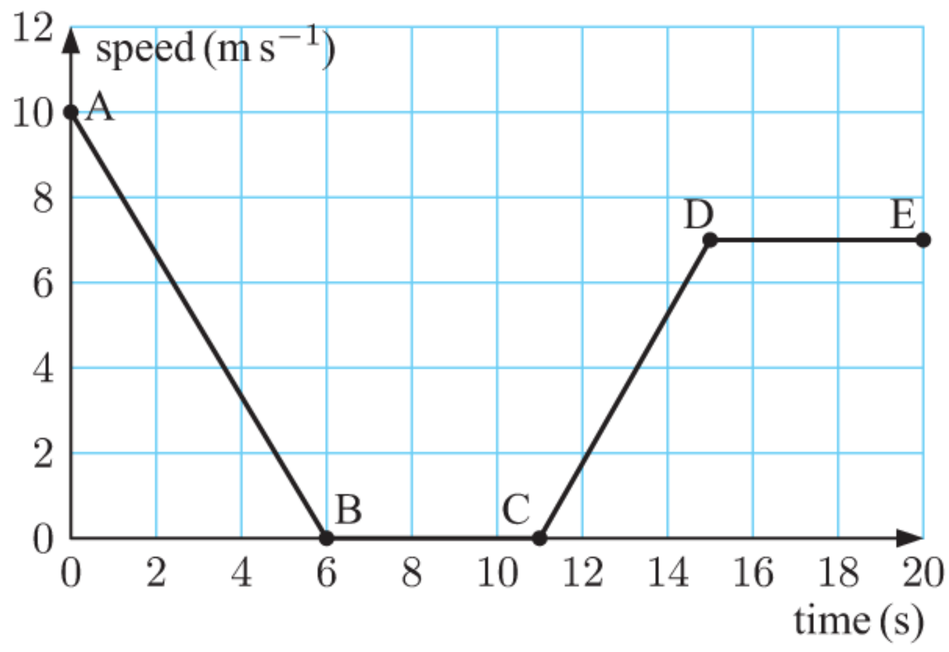
- **a** Find the gradient of each line segment. Interpret your answers.
- **b** Find the cost of using 40 kL of water.
- **c** Kelly's last water bill was \$80. How much water did she use?

2 Each month, Colin pays \$2 per gigabyte for mobile phone data up to his allowance, and then a higher rate for excess data.



- **a** What is Colin's data allowance?
- **b** At what rate is Colin charged for excess data?
- **c** How much does Colin need to pay for using 15 GB of data in a month?
- **d** What data usage would result in a bill of \$60?

3 This graph models the speed of a car over a 20 second period.

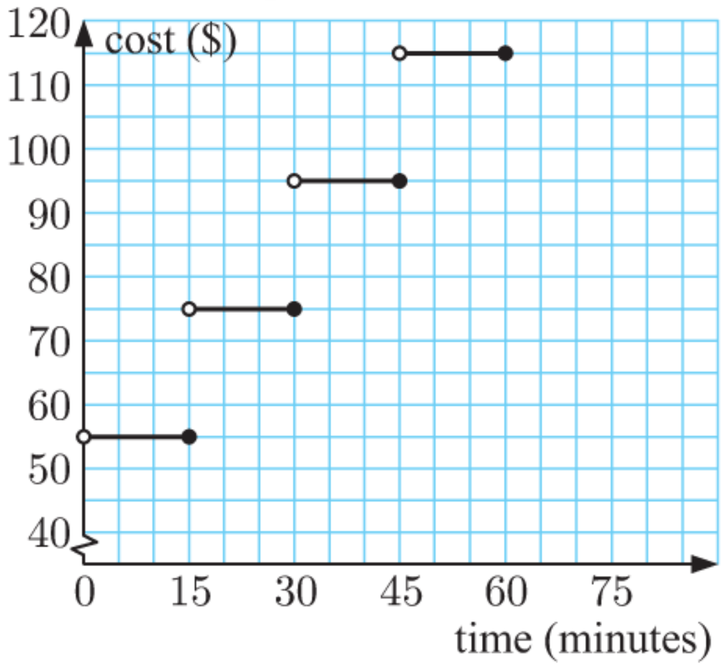


- **a** Describe the car's journey over this period.
- **b** What assumptions have been made when using this graph to model the car's speed? Are these assumptions reasonable?
- **c** i Find the *average* speed of the car between A and B.
  ii Hence find the distance travelled between A and B.
- **d** Find the total distance travelled by the car.

4 Jasper picks berries on a farm. He earns \$5 per kg for the first 12 kg of berries he picks each day, and \$8 for each additional kilogram.

- **a** Draw a graph to display the relationship between the *weight of berries* Jasper picks and his *daily wage*.
- **b** How much does Jasper earn for picking 9 kg of berries?
- **c** How many kilograms of berries must Jasper pick to earn \$100 in a day?

5 **Refrigerator service costs**



This graph shows the costs for refrigerator services.

- **a** Find the cost of a service which takes:
  i 20 minutes ii 45 minutes.
- **b** Find the maximum time for a service costing:
  i \$55 ii \$115
- **c** Predict the cost of an 80 minute service, stating any assumptions you are making.

6 The cost of sending a parcel overseas depends on the weight of the parcel.

Weight	*Cost*
Up to 500 g	\$13
Over 500 g up to 1 kg	\$23
Over 1 kg up to 2 kg	\$32
Over 2 kg up to 3 kg	\$40
Extra kg or part thereof	\$5

a Draw a graph to display the information given in the table, for weights up to 5 kg.

b Find the cost of sending a parcel weighing:

i 750 g **ii** 1.6 kg **iii** 3.4 kg

c Heather has a 1.7 kg parcel and a 2.8 kg parcel to send to her mother. How much money will she save by sending the parcels together as a single package, rather than separately?

7 A swimmer in a 200 m medley swims all 4 strokes (butterfly, backstroke, breaststroke, and freestyle) in one race. Laura's times for each 50 m leg are shown in the table.

Stroke	*Time* (s)
Butterfly	28.20
Backstroke	29.28
Breaststroke	33.91
Freestyle	27.43

a Sketch a distance-time graph to model Laura's position during the race.

b Describe any assumptions you made when constructing your graph.

c Use your graph to estimate how far Laura had swum after 1 minute.

Example 3 — Self Tutor

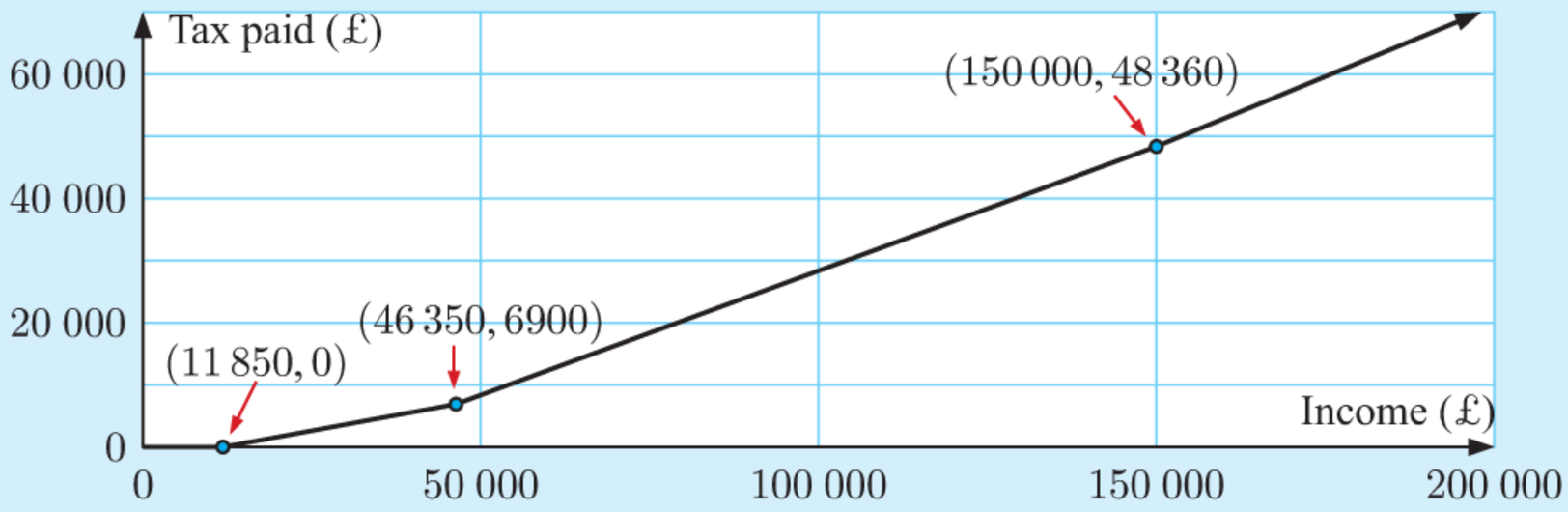
This table shows the income tax rates in the United Kingdom in 2018.

Taxable income (per year)	*Tax paid*
Up to £11 850	Nil
£11 851 - £46 350	£0.20 for each £1 above £11 850
£46 351 - £150 000	£6900 + £0.40 for each £1 above £46 350
Over £150 000	£48 360 + £0.45 for each £1 above £150 000

a Draw a graph of tax paid against income.

b Find the amount of tax paid by a person who earns £50 000 per year.

a



b £50 000 = £46 350 + £3650

$\therefore$ the tax paid = £6900 + £0.40 × 3650

= £8360

8 Use the income tax rates in **Example 3** to answer the following problems:

a The minimum wage in the United Kingdom is £7.83 per hour. Find the annual tax paid by a person working on minimum wage for 36 hours per week, 50 weeks of the year.

b Harold is a train driver earning £53 172 per year.

i How much income tax must Harold pay?

ii Find Harold's salary after income tax has been deducted.

c On April 1, 2018 the United Kingdom Prime Minister's total salary was £153 907.

i How much income tax did the Prime Minister pay?

ii What *percentage* of the Prime Minister's income was paid as income tax?

9 For 2018 - 2019, the Australian government set the following income tax rates for residents:

Taxable income (per year)	*Tax paid*
Up to \$18 200	Nil
\$18 201 - \$37 000	\$0.19 for each \$1 over \$18 200
\$37 001 - \$90 000	\$3572 plus \$0.325 for each \$1 over \$37 000
\$90 001 - \$180 000	\$20 797 plus \$0.37 for each \$1 over \$90 000
\$180 001 and over	\$54 097 plus \$0.45 for each \$1 over \$180 000

a Draw a graph of tax paid against income for incomes between \$0 and \$200 000.

b On a separate set of axes, draw a graph showing the *rate* of tax against income for incomes between \$0 and \$200 000.

c Calculate the income tax for the following annual incomes:

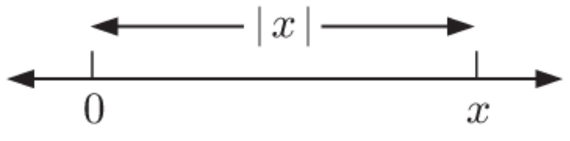
i \$28 000 **ii** \$48 300 **iii** \$96 150

ABSOLUTE VALUE FUNCTIONS

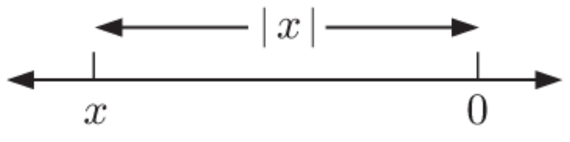
The **absolute value** or **modulus** of a real number x is its distance from 0 on the number line. We write the absolute value of x as $|x|$.

Because the absolute value is a distance, it cannot be negative.

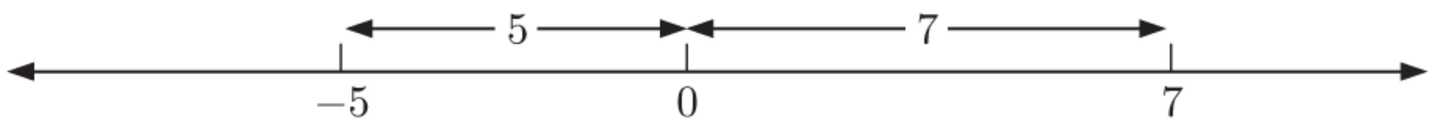
- If $x > 0$, $|x| = x$.



- If $x < 0$, $|x| = -x$.



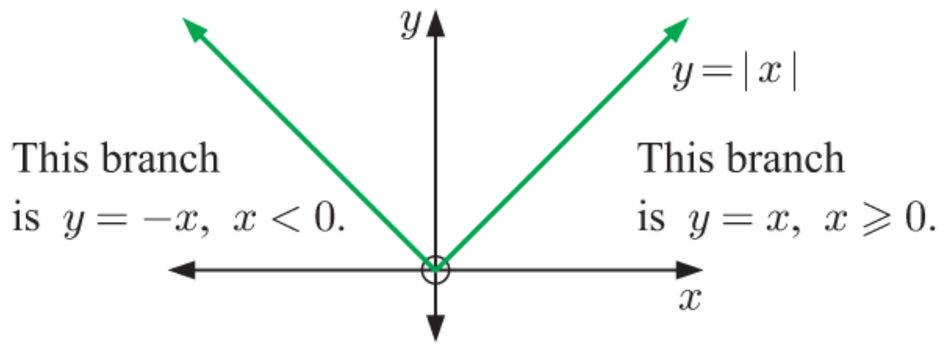
For example: $|7| = 7$ and $|-5| = 5$.



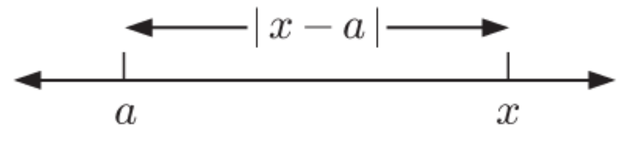
This leads us to the **algebraic definition**:

The **absolute value of x** is $|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$.

The relation $y = |x|$ is a piecewise linear function. We call it the **absolute value function**, and it has the graph shown.

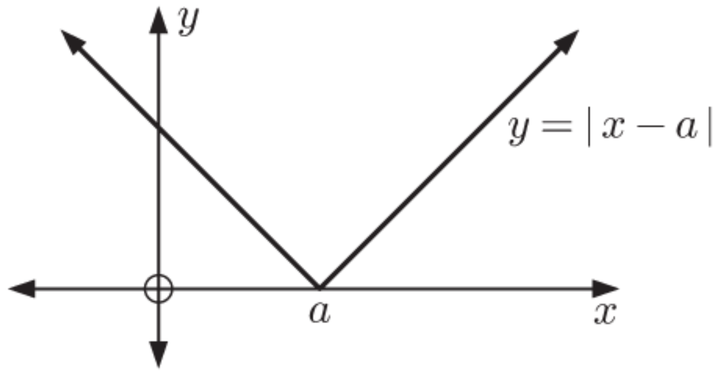


More generally, we can define $|x - a|$ as the distance between x and a on the number line.



Algebraically, we define:

$$|x-a| = \begin{cases} x-a & \text{if } x \geqslant a \\ a-x & \text{if } x < a \end{cases}.$$

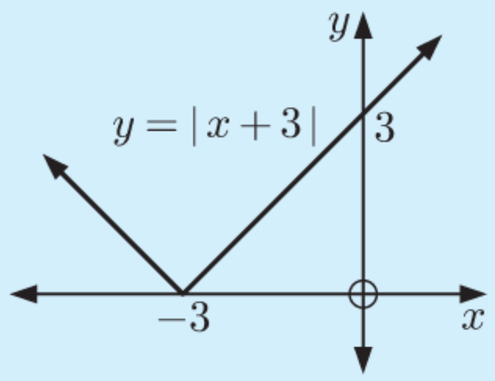


Example 4 **Self Tutor**

Sketch the graph of:

a $y = |x+3|$ **b** $y = |x| + |x-4|$

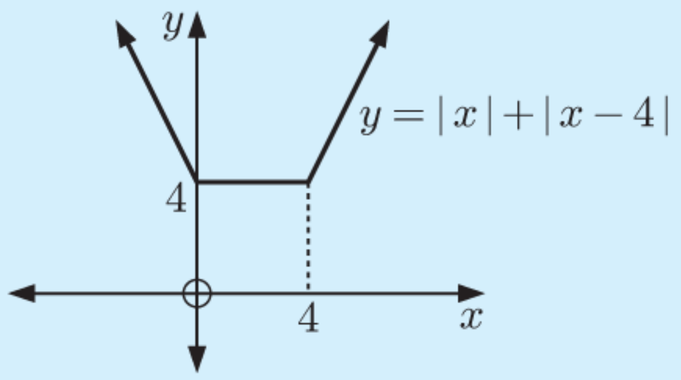
a $|x+3| = \begin{cases} x+3 & \text{if } x \geqslant -3 \\ -x-3 & \text{if } x < -3 \end{cases}$

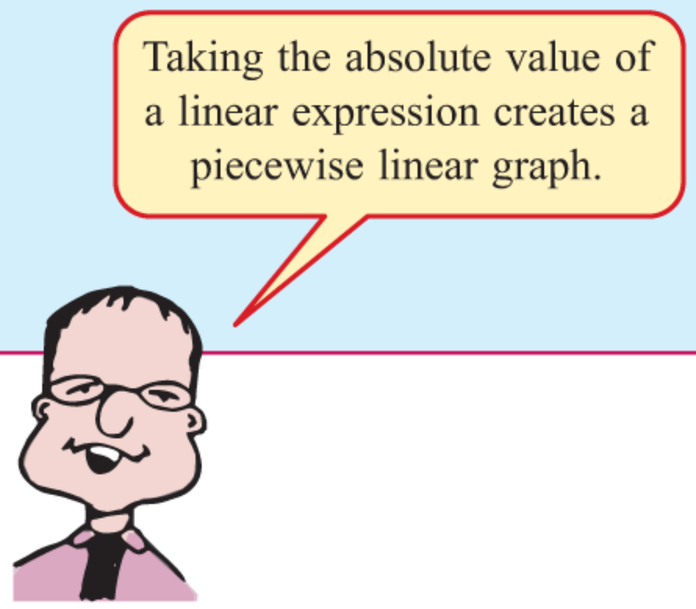


b $|x| + |x-4|$

$$= \begin{cases} x + (x-4) & \text{if } x \geqslant 4 \\ x + (4-x) & \text{if } 0 \leqslant x < 4 \\ -x + (4-x) & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 2x-4 & \text{if } x \geqslant 4 \\ 4 & \text{if } 0 \leqslant x < 4 \\ 4-2x & \text{if } x < 0 \end{cases}$$





EXERCISE 5C.2

1 Sketch the graph of:

a $y = |x - 1|$ **b** $y = |x + 2|$

c $y = |x| + |x + 5|$ **d** $y = |x + 1| + |x - 3|$

e $y = |x - 2| - |x + 1|$ **f** $y = 2|x + 3| - |x|$

2 The *absolute error* of a weather forecast is the difference between the forecast maximum temperature and the actual maximum temperature.

Let $x°$C be the forecast maximum temperature on a day with maximum temperature 15°C.

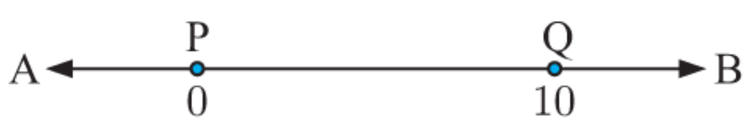
a Use an absolute value function to construct a model for the absolute error E of the forecast.

b Find E when:

i $x = 18$ **ii** $x = 15$ **iii** $x = 13$.

c Sketch the graph of E against x.

3 Schools P and Q lie 10 km apart on the road (AB). Susie and her sister Sarah live on road (AB). Susie attends school P, and Sarah attends school Q. Both sisters cycle to school each morning.



Suppose x is the position of their house relative to school P.

a Construct a model for D, the total distance the sisters cycle each morning.

b Find D if:

i $x = 7$ **ii** $x = -3$.

c Sketch the graph of D against x.

d Find the value(s) of x for which D is minimised.

4 **a** Draw the graph of $y = |x + 5| + |x + 2| + |x| + |x - 3|$.

b



P, Q, and R are swimming beaches which are respectively 5, 2, and 3 km from beach O.

An ambulance service wants to locate its premises along the coast such that in an emergency its average distance to any of the four beaches is minimised.

i Suppose the ambulance service is located at position x relative to O. Explain why the average distance to any of the beaches is given by $\frac{1}{4}(|x + 5| + |x + 2| + |x| + |x - 3|)$ km.

ii Where should the ambulance service locate its premises? Explain your answer.

iii Funding for the new ambulance service premises is approved, but it is conditional to the service supporting an extra beach at S, 7 km to the right of O. Where should the ambulance service locate its premises now? Explain your answer.

NON-LINEAR PIECEWISE MODELS

In the world around us, there are many situations where piecewise models involve curves.

Example 5 **Self Tutor**

The weight, in kilograms, of a cat at age t months is given by

$$W(t) = \begin{cases} \frac{t^2}{40} + 0.15, & 0 \leqslant t < 12 \\ 4.5 - \frac{9}{t}, & 12 \leqslant t < 18 \\ 4, & t \geqslant 18 \end{cases}.$$



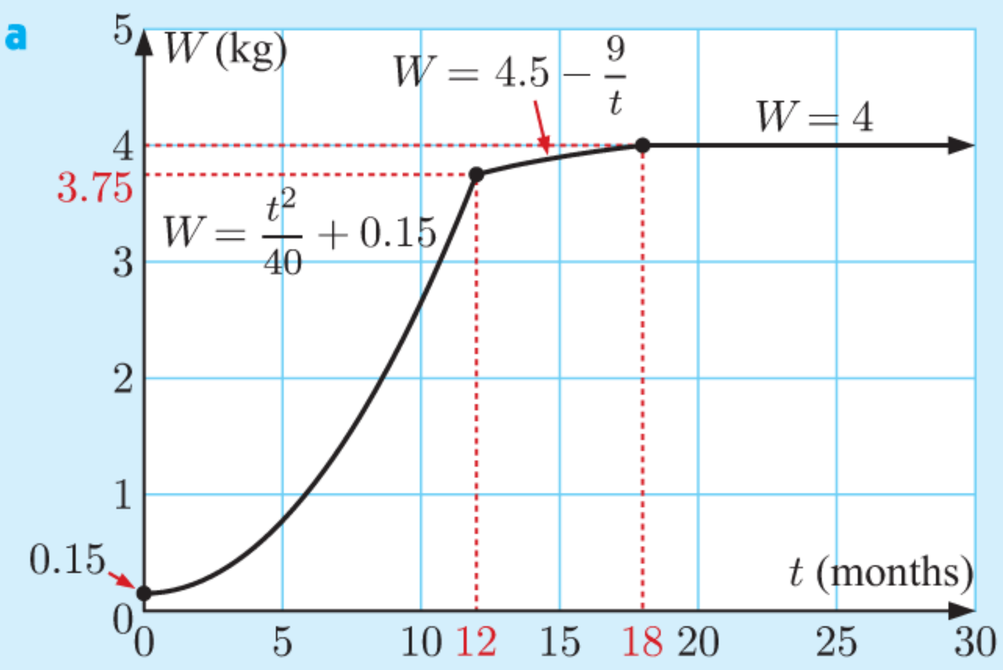
a Sketch the graph of W against t.

b Find the weight of the cat at age:

i 4 months **ii** 15 months.

c How long did it take for the cat to be fully grown?

a



c The cat was fully grown after 18 months.

b **i** $W(4) = \frac{4^2}{40} + 0.15$

$\{\text{as } 0 \leqslant t < 12\}$

$= 0.55$

The cat weighed 0.55 kg at age 4 months.

ii $W(15) = 4.5 - \frac{9}{15}$

$\{\text{as } 12 \leqslant t < 18\}$

$= 3.9$

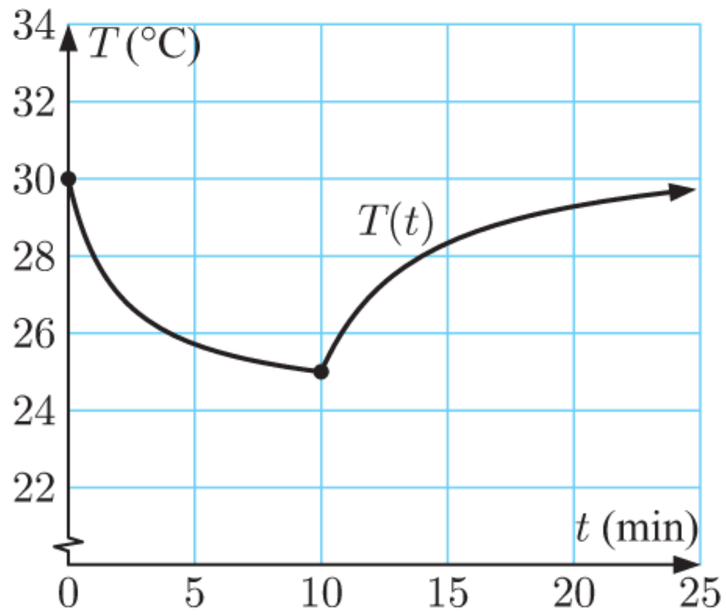
The cat weighed 3.9 kg at age 15 months.

EXERCISE 5C.3

1 The temperature $T°C$ in a room, t minutes after the air conditioning is turned on, is

$$T(t) = \begin{cases} 24 + \frac{12}{t+2}, & 0 \leqslant t < 10 \\ 31 - \frac{24}{t-6}, & t \geqslant 10 \end{cases}.$$

The graph of T against t is shown alongside.



a Find the initial temperature of the room.

b How long do you think the air conditioner was turned on for?

c Find the temperature of the room at the time when the air conditioner was turned off.

d How long did it take for the room to return to its initial temperature?

2 The speed S of a car in km h^{-1} is given by $S(t) = \begin{cases} 2t^2, & 0 \leqslant t < 5 \\ 2t + 40, & 5 \leqslant t < 10 \\ 60, & t \geqslant 10 \end{cases}$

where t is the time in seconds.

a Sketch the graph of $S(t)$.

b What do you think is the speed limit where the car is driving?

c Find the speed of the car after: **i** 3 seconds **ii** 8 seconds **iii** 12 seconds.

d Find the time at which the speed of the car was 32 km h^{-1}.

3 A restaurant orders tomatoes from either Supplier A or Supplier B. The costs of purchasing x kg of tomatoes from each supplier, including delivery, are given by the functions $C_A(x) = 9 + 3x$ and $C_B(x) = 12 + 2x + 2\sqrt{x}$ dollars, respectively.

Each week the restaurant chooses the cheaper supplier, based on the size of their order.

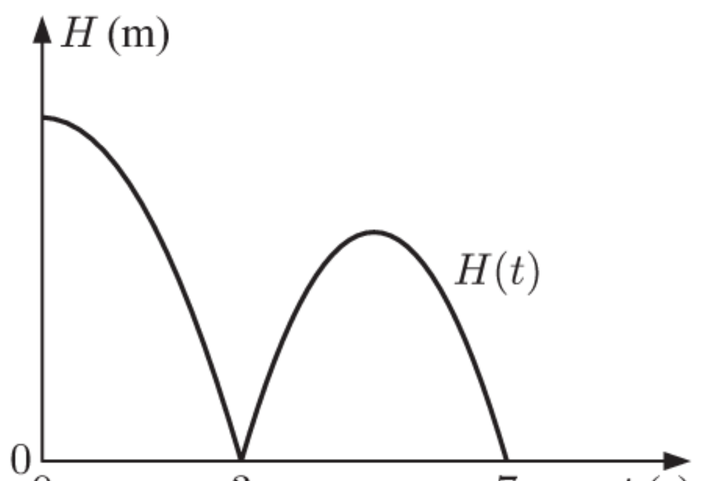
a Sketch $C_A(x)$ and $C_B(x)$ on the same set of axes.

b Construct a piecewise model for the cost C of x kg of tomatoes.

c Hence find the cost of: **i** 6 kg of tomatoes **ii** 9 kg of tomatoes.

4 When a ball is dropped from the top of a tower, it hits the ground after 3 seconds. The ball then bounces before hitting the ground again after a further 4 seconds. The height in metres of the ball above the ground after t seconds follows the piecewise model

$$H(t) = \begin{cases} -5t^2 + c, & 0 \leqslant t < 3 \\ -5t^2 + dt + e, & 3 \leqslant t \leqslant 7 \end{cases}.$$



a Find all of the unknowns in the model.

b How high is the tower from which the ball was dropped?

c Find the maximum height reached by the ball after the first bounce.

d Find the times at which the ball was 15 m above ground level.

DISCUSSION

Suppose a piecewise model has the form $f(x) = \begin{cases} f_1(x), & x < a \\ f_2(x), & x \geqslant a \end{cases}$.

- Is it always a requirement that $f_1(a) = f_2(a)$?
- List some situations where:
  - $f_1(a) = f_2(a)$ is required
  - $f_1(a) \neq f_2(a)$

D SYSTEMS OF EQUATIONS

When we construct a mathematical model, we can often use our knowledge of mathematical functions to guess the *form* that the model should take.

For example, when a ball is thrown into the air, we can predict that its height above the ground will be modelled by a function of the form $y = at^2 + bt + c$, where t is the time in seconds, and a, b, and c are unknown **coefficients** or **parameters**.

DISCUSSION

- What do we mean by coefficients, parameters, and constants?
- Can we use these mathematical terms interchangeably without causing confusion?

If we have some data values that the model needs to fit, we can substitute the points into the function to construct a **system of equations** for the unknown parameters. By solving the system of equations simultaneously, we obtain the parameters which complete the model.


GRAPHICS CALCULATOR INSTRUCTIONS

Example 6

Self Tutor

The table alongside shows the quantity V of air affected by smoke t seconds after a fire is started.

t (seconds)	1	2	3
V (L)	2	4	12

V and t are connected by a cubic model of the form $V = at^3 + bt^2 + ct$.

Find the coefficients a, b, and c.

Substituting $(1, 2)$ into the model gives $2 = a(1)^3 + b(1)^2 + c(1)$

$$\therefore \quad a + b + c = 2$$

Substituting $(2, 4)$ into the model gives $4 = a(2)^3 + b(2)^2 + c(2)$

$$\therefore \quad 8a + 4b + 2c = 4$$

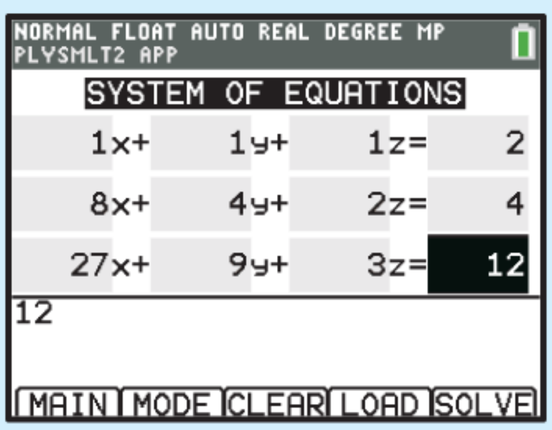
Substituting $(3, 12)$ into the model gives $12 = a(3)^3 + b(3)^2 + c(3)$

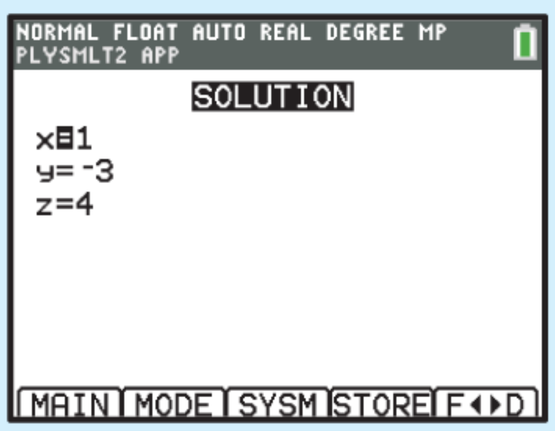
$$\therefore \quad 27a + 9b + 3c = 12$$

So, we have the system of equations $\begin{cases} a + b + c = 2 \\ 8a + 4b + 2c = 4 \\ 27a + 9b + 3c = 12 \end{cases}$.

Solving these equations simultaneously using technology, we find that $a = 1$, $b = -3$, and $c = 4$.

So, the model is $V = t^3 - 3t^2 + 4t$.

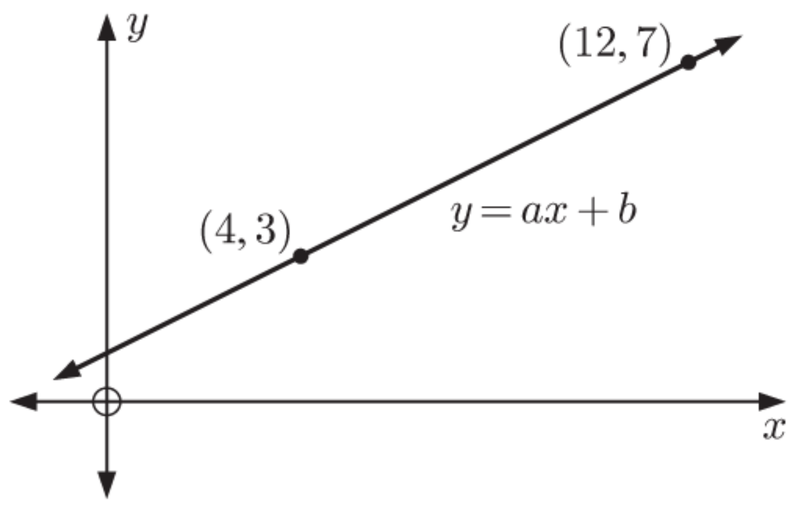




EXERCISE 5D

1 **a** Use technology to find this linear model.

b Check your answer algebraically.

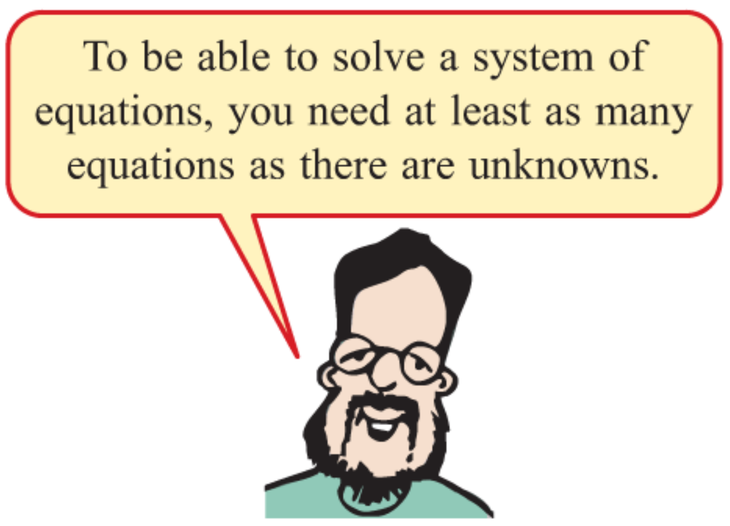


2 The points $(1, 7)$, $(2, 10)$, and $(3, 11)$ lie on a model of the form $y = ax^2 + bx + c$.

a Construct three equations in terms of a, b, and c.

b Use technology to find a, b, and c.

c Hence determine the model.



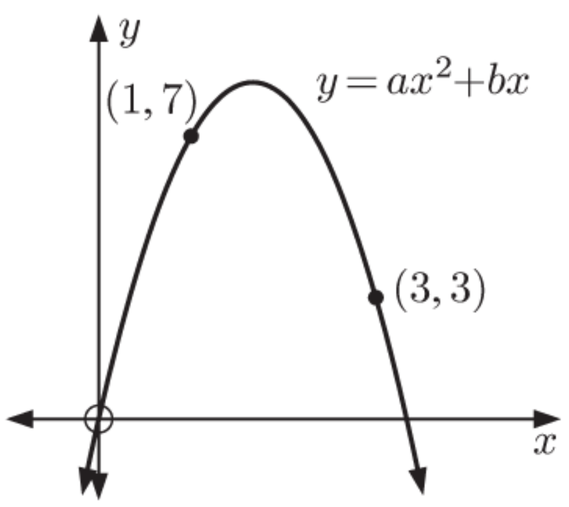
3 Find the equation of the quadratic function which passes through the points:

a $(1, -2)$, $(2, 4)$, and $(3, 12)$ **b** $(-1, 3)$, $(2, 9)$, and $(4, -7)$.

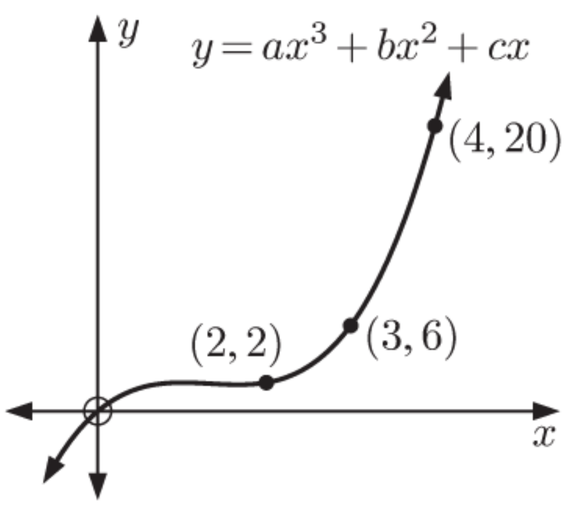
4 Try to find the equation of the quadratic which passes through the points $(-1, 7)$, $(2, 1)$, and $(3, -1)$. Explain your answer.

5 Determine the model for each graph:

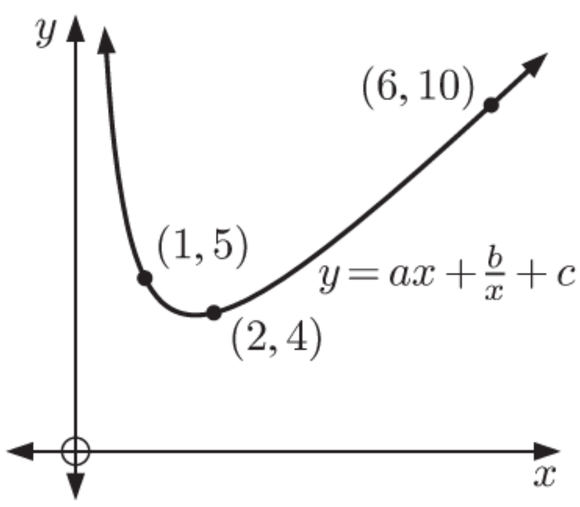
a



b



c



6 The table alongside shows the distance D m a scooter has travelled t seconds after it starts moving.

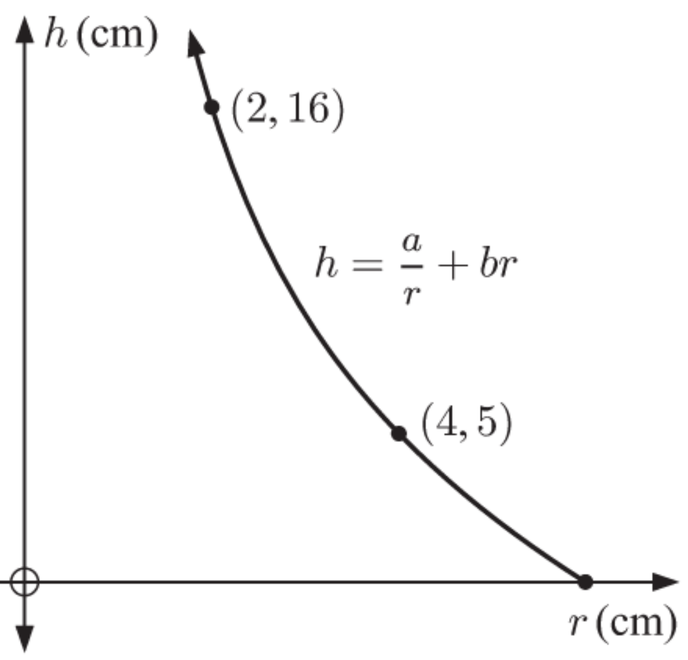
The model connecting D and t has the form $D = at^2 + bt + c\sqrt{t}$.

Find the parameters a, b, and c.

t (seconds)	1	4	9
D (metres)	1.7	7.2	23.7

7 This graph shows the relationship between the height and radius of cylinders which have a constant surface area.

The model connecting h and r has the form $h = \frac{a}{r} + br$, where a and b are constants.



a Use the given data points to determine the model connecting h and r.

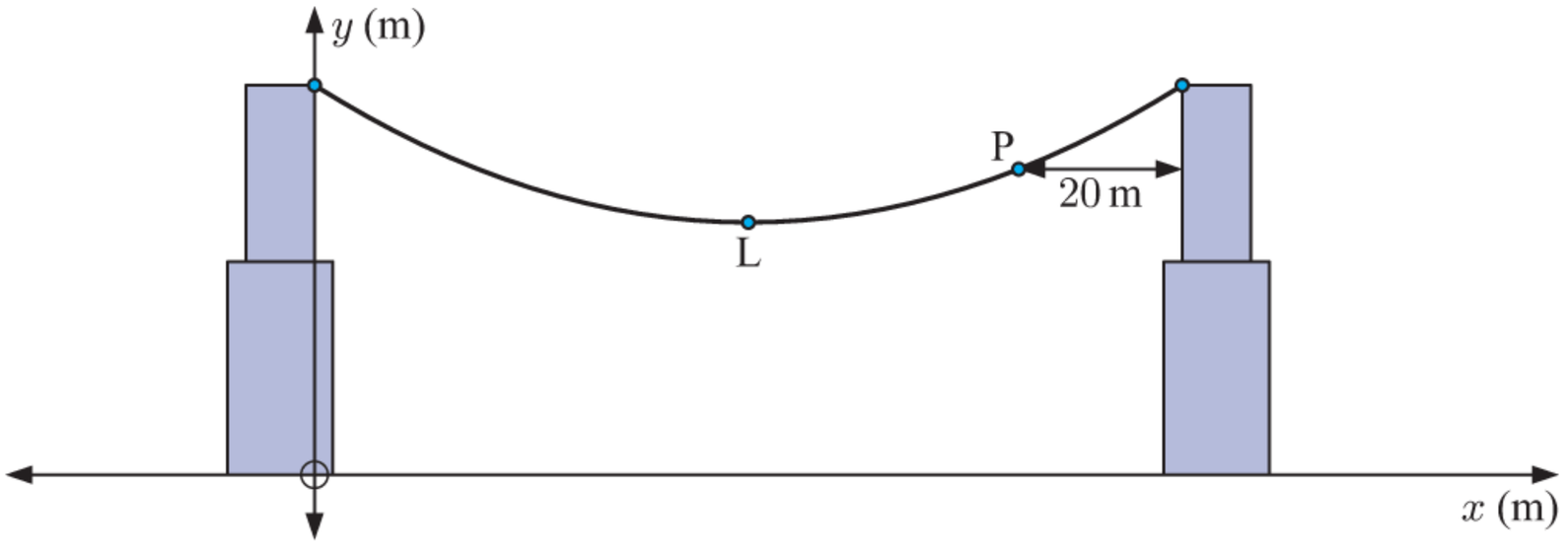
b Use the formula for the surface area of a cylinder to explain why a model of this form is reasonable.

c Find the constant surface area of these particular cylinders.

d Use your model to find the value of h when $r = 9$. Is your answer reasonable?

e For what values of r can this model be correctly applied?

8



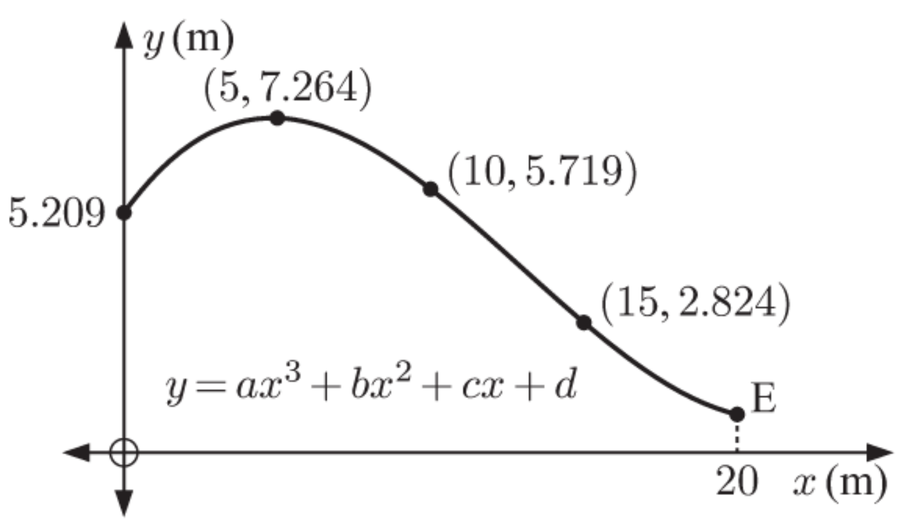
A tightrope connects two elevated platforms which are 50 m high and 100 m apart. Julian wants to know the height of the tightrope above ground level as a function of the distance along it. He knows that the tightrope is 30 m high at its lowest point L, and he assumes the height follows a quadratic function.

a Find the quadratic model connecting y and x.

b Hence estimate the height of the tightrope above ground level at point P.

c For what values of x is the quadratic model valid?

9 The cross-section of a mound on a bike track is shown alongside. It can be modelled by the function $y = ax^3 + bx^2 + cx + d$ for $0 \leqslant x \leqslant 20$.
Units are in metres.



a State the value of d.

b Use technology to find a, b, and c.

c Find the height of the mound at point E.

RESEARCH

The height of a free-hanging rope, such as the tightrope in the previous Exercise, does not in fact follow a quadratic model.

- Research the correct model for the height of a free-hanging rope.
- Do you think the quadratic model gives a good approximation for the height of the tightrope? What advantages are there to using the quadratic model as opposed to the correct model?

REVIEW SET 5A

1 Ben loves to kayak. He can paddle 100 metres in 40 seconds.

a Construct a model to describe how far Ben can kayak in t seconds.

b Hence predict how far Ben can kayak in 10 minutes.

c Do you think the actual distance Ben can kayak in 10 minutes will be more or less than your prediction? Explain your answer.

2 Amy and Bernard are electricians. It would take each of them 3 days to wire a particular house.

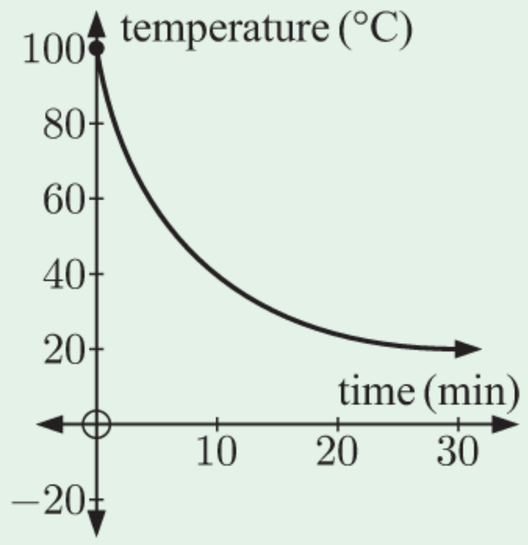
a Assuming they were able to work without getting in each other's way, how long would it take for the two of them to do the job?

b Do you think the assumption in **a** is reasonable? If not, do you think it would take more or less time than your answer in **a**?

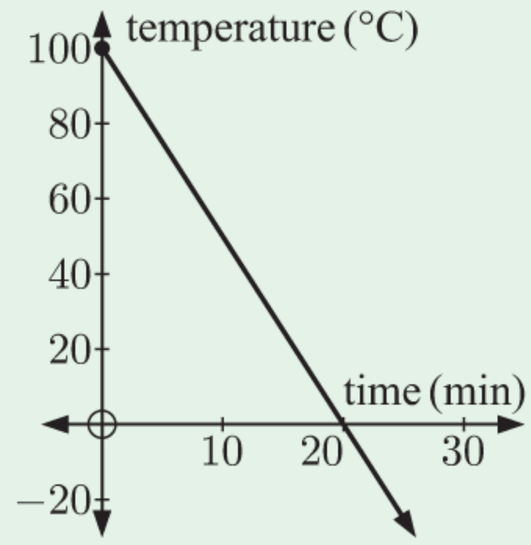
3 Some water is boiled, then poured into a cup.

a Which of these graphs do you think is the most appropriate model for the temperature of the water?

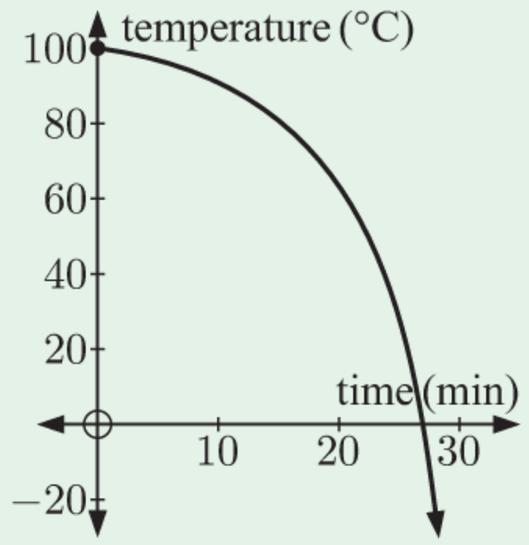
A



B

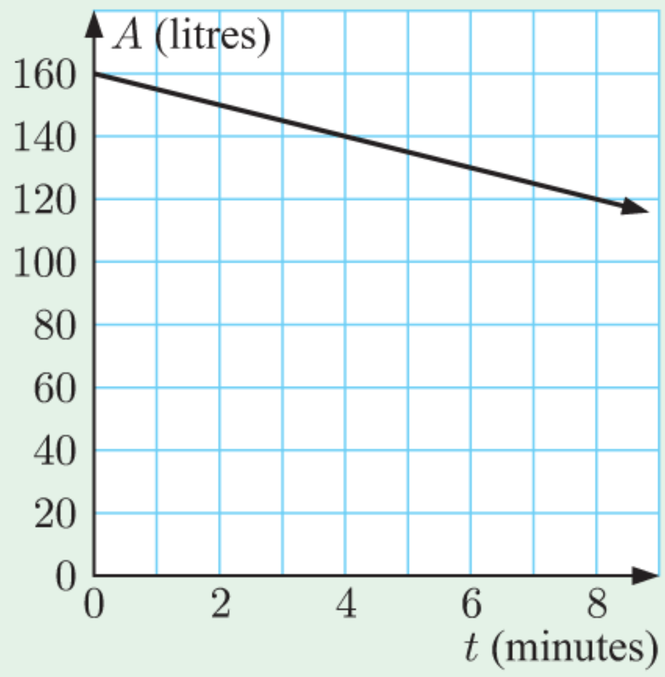


C



b Use the model you selected to predict the temperature of the water after 10 minutes.

4



The amount of oil A left in a leaky barrel after t minutes is shown on the graph alongside.

a Find the gradient and A-intercept of the line. Interpret your answers.

b Find the model connecting A and t.

c How much oil will be left after 15 minutes?

d For what values of t is it reasonable to apply this model?

5 A plumber charges customers a £50 call-out fee, and then £80 for each hour he spends on the job.

a Copy and complete this table of values:

Time (t hours)	0	1	2	3	4
Cost (£C)					

b Draw the graph of C against t.

c Find the linear model connecting C and t.

d Hence find the cost of a job which takes 6 hours.

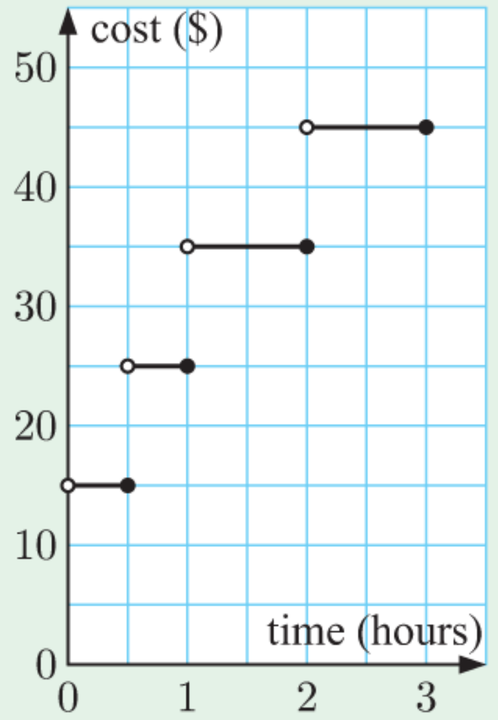


6 This graph shows the cost of hiring a squash court.

a Find the cost of hiring the court for:

i 45 minutes **ii** 2 hours.

b Kate and Peggy can spend no more than $15 each to hire a court. What is the longest time they can hire the court for?



7 A triathlon consists of a 1.5 km swim, a 40 km bicycle ride, and a 10 km run. The table alongside shows the times Alana obtained for each leg of the triathlon.

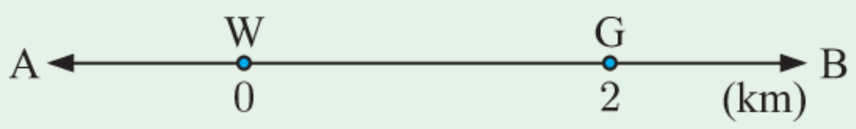
Leg	*Time* (min)
Swim	36
Bicycle ride	81
Run	54

a Draw a distance-time graph to model Alana's position during the triathlon.

b Describe any assumptions you made when constructing your graph.

c Use your graph to estimate how far Alana had travelled after 1 hour.

8 A house is to be built along an isolated road (AB). The house must be connected to the water access point W with a water pipe, and to the gas access point G with a gas pipe.



Let x be the position of the house along (AB).

a Construct a model for L, the total length of pipe required.

b Find L when:

i $x = 1.5$ **ii** $x = -1$.

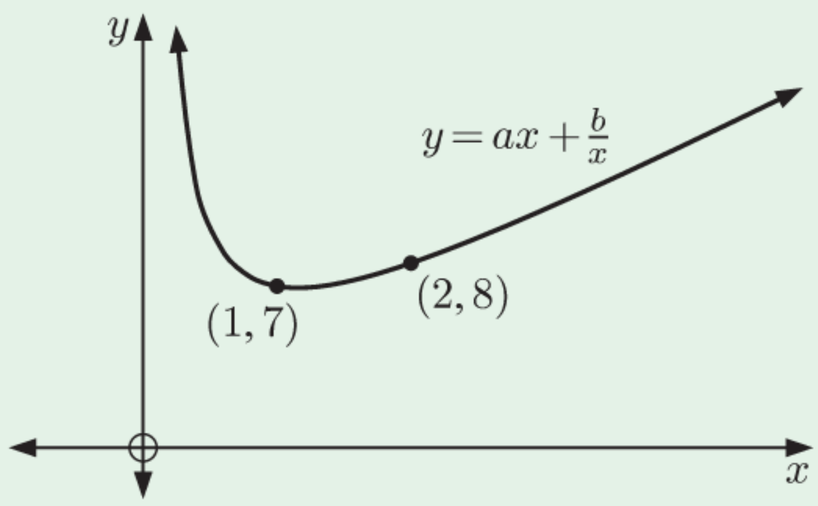
c Find the value(s) of x for which L is minimised.

d Suppose each metre of gas pipe costs twice as much as each metre of water pipe. Where should the house be built to minimise the total cost of the pipes?

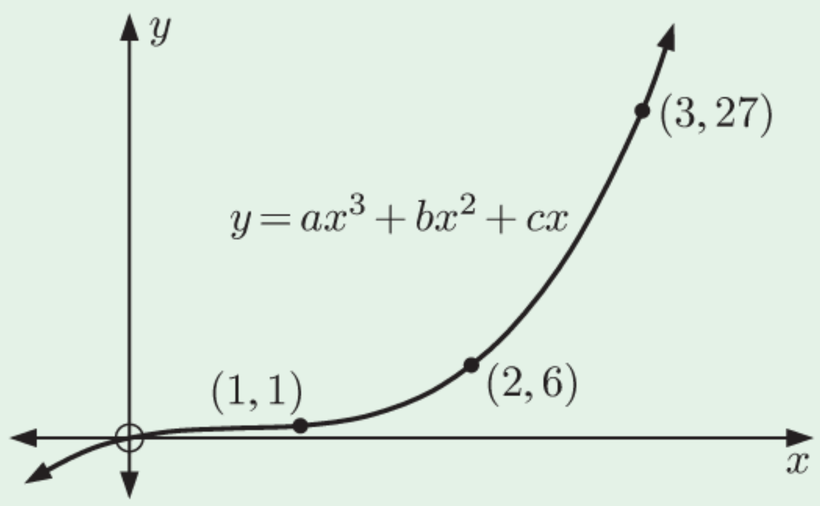
9 Find the equation of the quadratic function which passes through the points $(-8, 4)$, $(-4, -8)$, and $(6, 32)$.

10 Determine the model for each graph:

a

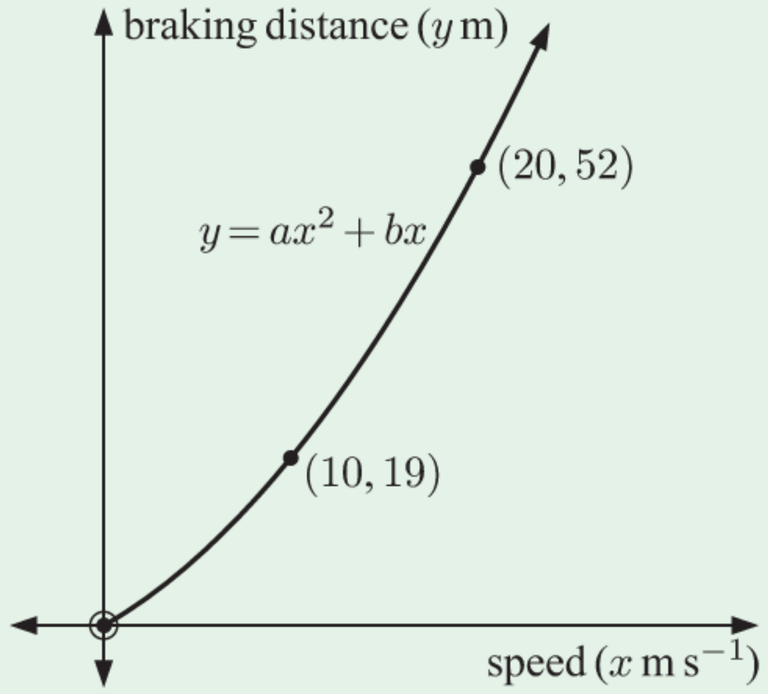


b



11 This graph shows the relationship between the *speed* of Daniel's car and his *braking distance*, which is the distance required to bring the car to a complete stop in an emergency.

The variables are related by the model $y = ax^2 + bx$.



a Find a and b.

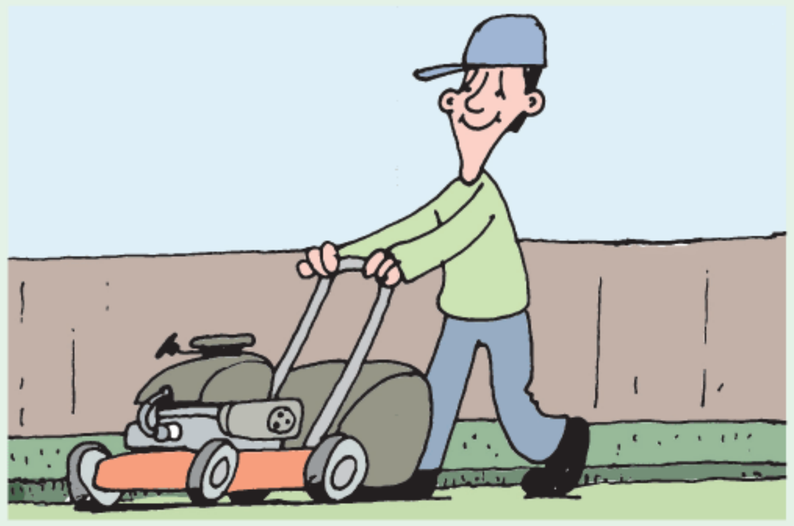
b Use the model to predict Daniel's braking distance when he is travelling at 30 m s^{-1}.

c Can we use this model to predict the braking distance for a different person? Explain your answer.

REVIEW SET 5B

1 Todd takes 3 hours to weed the garden. Sophie takes 3.5 hours to weed the garden. How long would they take to weed the garden working together? State any assumptions you make.

2 Rohan's lawn mower cuts grass to a height of 17 mm. The grass grows 3 mm each day.



a Construct a table of values for the height H of the lawn d days after it is cut, for $d = 0, 1, 2, 3, 4$.

b Draw the graph of H against d.

c Determine the linear model connecting H and d.

d Find the height of the lawn after 12 days.

e Rohan mows the lawn again when it is 8 cm high. How often does he mow the lawn?

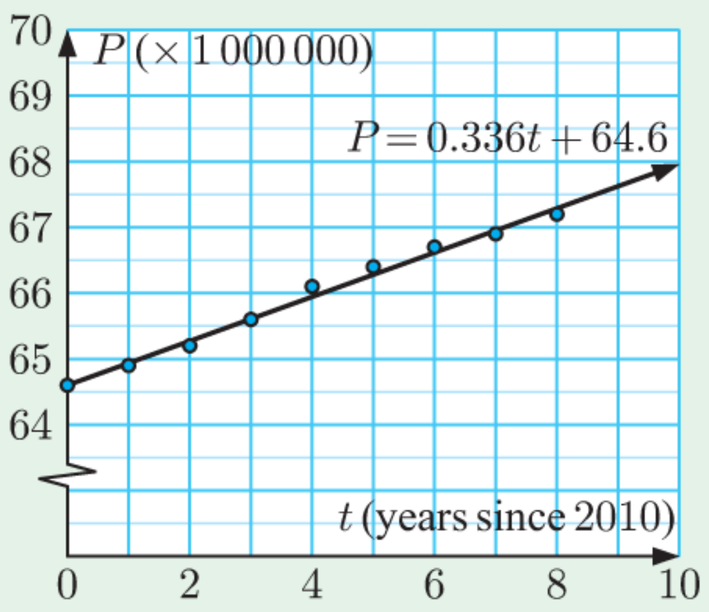
3 Elizabeth needs to estimate the volume of usable wood in the trees growing in a plantation. For her calculations, she assumes that the trees are perfectly cylindrical and that 80% of the wood in each tree is usable. She measures a sample of trees and finds the average trunk diameter is 45 cm.

a Do you think these assumptions are reasonable? Explain your answer.

b Based on these assumptions, construct a model for the volume V of usable wood in a tree that is h metres high.

c Hence predict the volume of usable wood in a 15 m high tree.

4 The graph alongside shows the population of France, in millions, since the start of 2010. The linear model $P = 0.336t + 64.6$.



a Is the linear model exact or approximate? Explain your answer.

b Use the model to predict when France's population will reach 75 million. Do you think your prediction is reasonable?

5 Shayne is looking for somewhere to stay during his vacation. He is choosing between the following places:

Bob's Beach House: \$150 per night for the first 2 nights, \$120 for each additional night
Claire's Cottage: \$180 per night for the first 3 nights, \$100 for each additional night

A "night" is a 24-hour period starting after 10 am.

a On the same set of axes, graph *cost* against *number of nights* for each place.

b Find the cost of staying for 4 nights at:

i Bob's Beach House **ii** Claire's Cottage.

c For what length stay is the cost the same at each place?

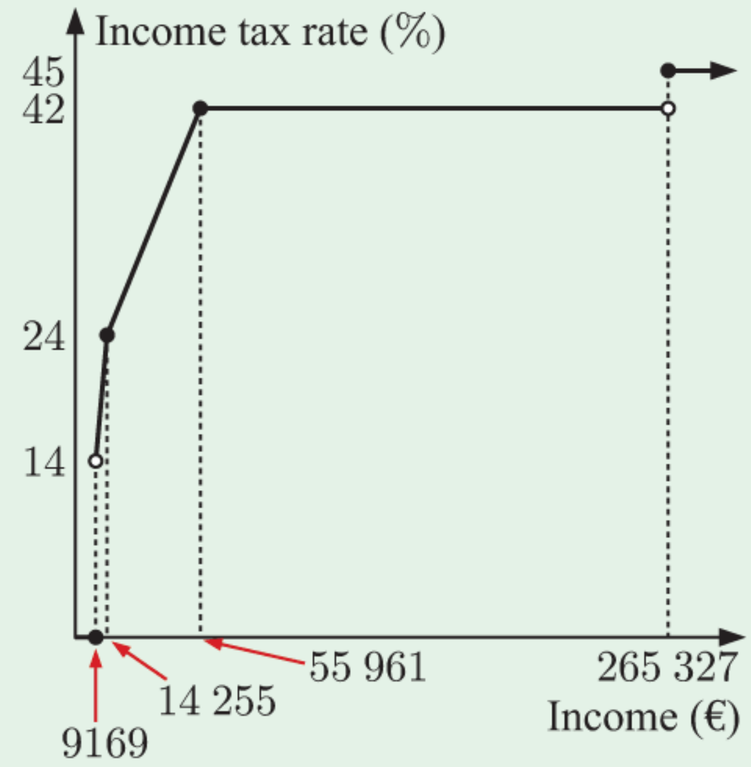
d Shayne wants to stay for 8 nights. Which place will be cheaper, and by how much?

6 For 2019, the income tax rates for a single person resident in Germany are shown in the graph alongside.

a What type of model is this?

b Estimate the *rate* of income tax payable by Gert-Jan, who has an annual income of €34 500.

c The tax payable on the first €55 961 is €14 729.32. Calculate the income tax payable by Nikolas, who earns €108 609 in 2019.



7 In a show jumping competition, a competitor is given one time penalty for every commenced four seconds by which they exceed the time allowed.

For example, if the time allowed is 75 seconds, and the competitor takes 79.4 seconds, they are 4.4 seconds over the time allowed, and so are given 2 time penalties.



a Draw a graph to display the time penalties for different times over the time allowed.

b If the time allowed is 82 seconds, find the time penalties given to a rider who takes:

i 83.1 seconds **ii** 87.9 seconds **iii** 81.5 seconds **iv** 96.3 seconds

8 The downwards speed S m s^{-1} of a skydiver t seconds after jumping from a plane is approximated by the piecewise model $S(t) = \begin{cases} 60 - \dfrac{k}{t+1}, & 0 \leqslant t < T \\ 10 + \dfrac{45}{t-10}, & t \geqslant T \end{cases}$.

a Find the value of k.

b Sketch the graphs of $y = 60 - \dfrac{k}{t+1}$ and $y = 10 + \dfrac{45}{t-10}$ on the same set of axes.

c Find the time T at which the skydiver opened her parachute.

d What would the terminal velocity of the skydiver have been if she had not opened her parachute?

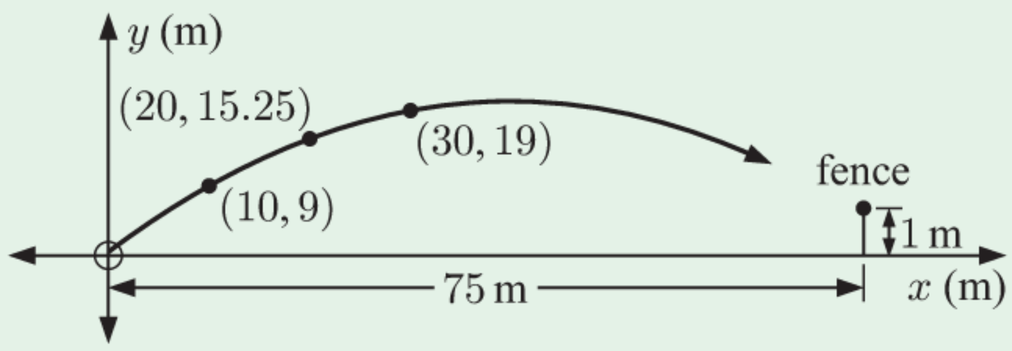
e Find the speed of the skydiver after:

i 5 seconds **ii** 15 seconds.

9 A cricketer hits the ball in the air with the path shown alongside.

a Use technology to find the quadratic model connecting y and x.

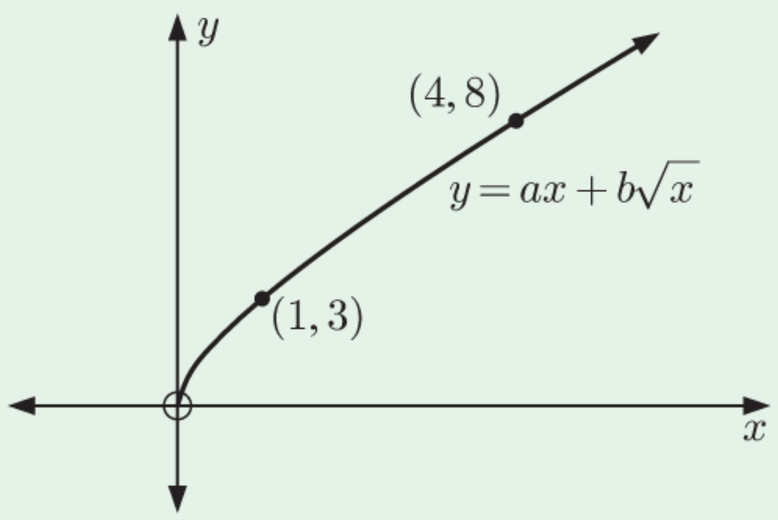
b Find the maximum height reached by the ball.



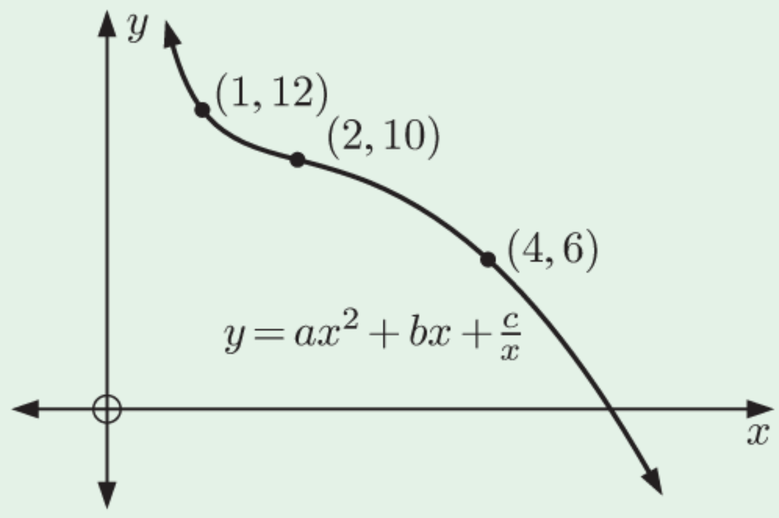
c The 1 m high boundary fence is 75 m away. Will the ball clear the boundary fence?

10 Determine the model for each graph:

a

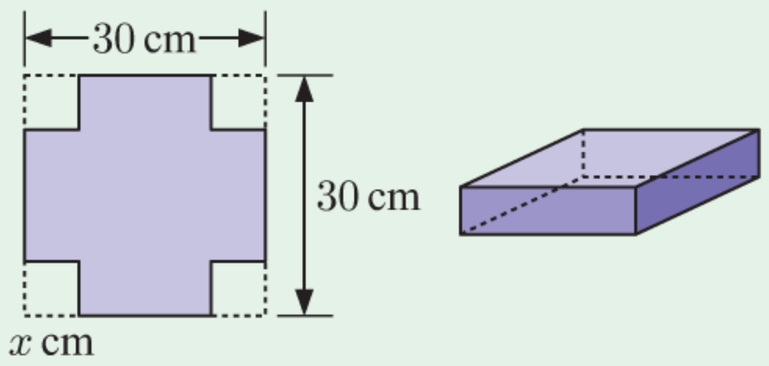


b



11 A tray is formed by cutting squares from each corner of a 30 cm $\times$ 30 cm sheet of metal, then folding the remainder.

If the squares cut out are x cm $\times$ x cm, the volume of the container can be written in the form $V = ax^3 + bx^2 + cx + d$ cm^3.



a Explain why $d = 0$.

b Use the data points alongside to find the values of a, b, and c.

x (cm)	2	5	10
V (cm^3)	1352	2000	1000

c Use the formula for the volume of a rectangular prism to explain why a model of this form is reasonable.

d For what values of x can this model be correctly applied?

Chapter 6

Direct and inverse variation

Contents:

A Direct variation
B Powers in direct variation
C Inverse variation
D Powers in inverse variation
E Determining the variation model
F Using technology to find variation models

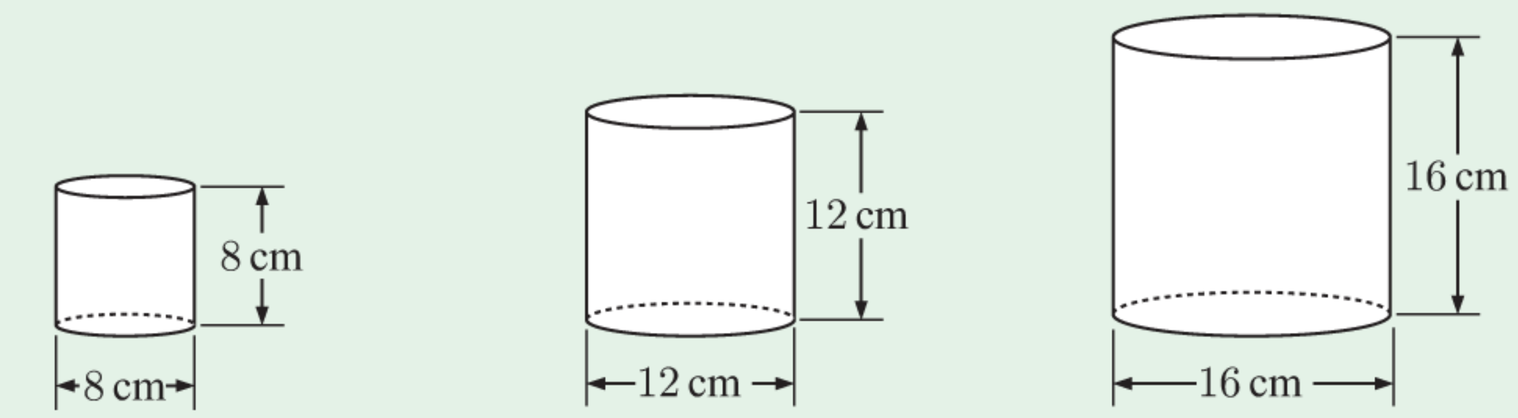


OPENING PROBLEM

Mrs Cornwall has set a challenge for the students in her Mathematics class. Each student must try to guess the number of jelly beans in a large cylindrical jar.

Jill measures the jar, and finds that its height and width are each 20 cm. She cannot find another jar that size, but she finds three other jars with equal height and width at home. She fills them with jelly beans and counts how many they can contain:





95 jelly beans | 321 jelly beans | 763 jelly beans

Things to think about:

a What does the graph of the *number of jelly beans* against the *jar height* look like?

b Does the *number of jelly beans* increase *in proportion* with the *jar height*?

c Explain why we should expect the *number of jelly beans* to increase in proportion with the *cube* of the *jar height*.

d Can you find an equation which connects the *number of jelly beans* and the *jar height*?

e What guess do you think Jill should make in the challenge?

A DIRECT VARIATION

Suppose petrol costs \$3 per gallon.

The table alongside shows how the *cost* y of the petrol varies with the *amount* x of petrol bought.

Amount of petrol (x gallons)	0	1	2	3	4	5	6
Cost (\$$y$)	0	3	6	9	12	15	18

$\times 2$ $\times 3$ (above the table) and $\times 2$ $\times 3$ (below the table)

Notice that:

- If x is doubled from 1 to 2, y is also doubled (from 3 to 6).
- If x is tripled from 2 to 6, y is also tripled (from 6 to 18).

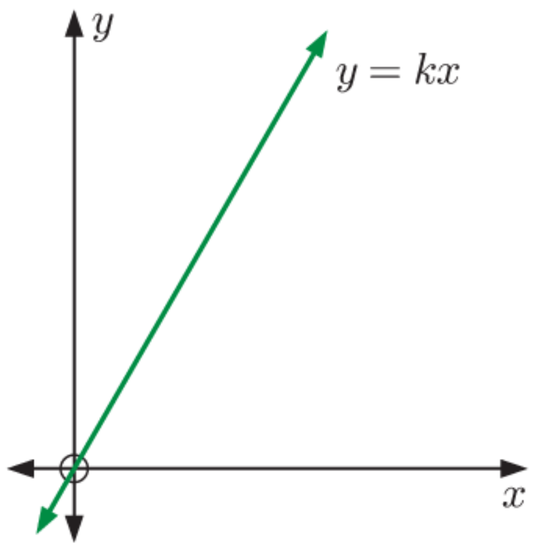
The variables x and y vary *in proportion* to each other. We say that x and y are **directly proportional**.

Two variables are **directly proportional** or **vary directly** if multiplying one of them by a number results in the other one being multiplied by the same number.

If the variables x and y are directly proportional, we write $y \propto x$.

If $y \propto x$ then $y = kx$ where k is a constant called the **proportionality constant**.

When y is graphed against x, the graph is a straight line with **gradient** k, which passes through the **origin**.





In the petrol scenario on the previous page, the proportionality constant $k = 3$, and the variables are connected by the formula $y = 3x$.

Example 1 **Self Tutor**

The table alongside shows the relationship between the *side length* of a square and its *perimeter*.

Side length (x cm)	1	2	3	5
Perimeter (P cm)	4			

a Copy and complete the table.

b Use a graph to show that P is directly proportional to x.

c Find a formula connecting P and x.

a

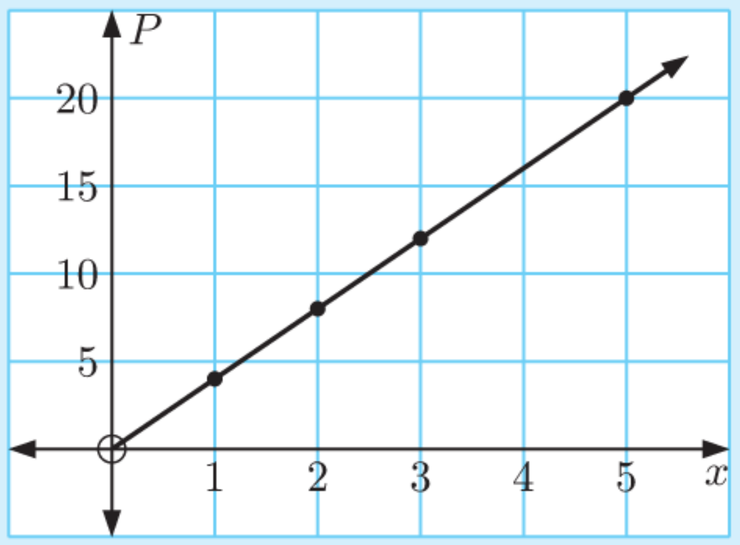
Side length (x cm)	1	2	3	5
Perimeter (P cm)	4	8	12	20

b The graph is a straight line which passes through the origin, so $P \propto x$.

c The gradient of the line $= \dfrac{8-4}{2-1}$

$= 4$

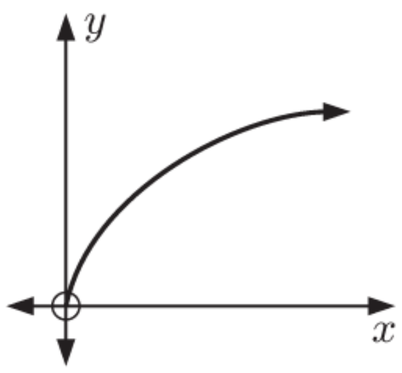
$\therefore$ the proportionality constant $k = 4$, and the formula connecting P and x is $P = 4x$.



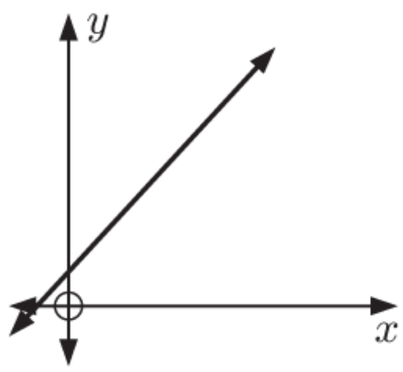
EXERCISE 6A

1 Which graph indicates that y is directly proportional to x?

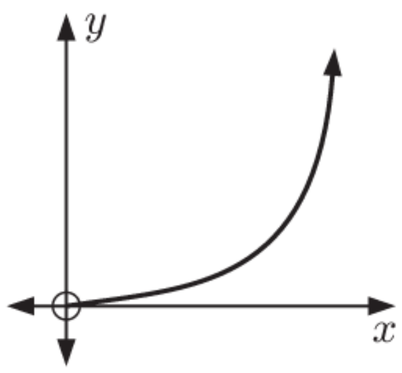
A



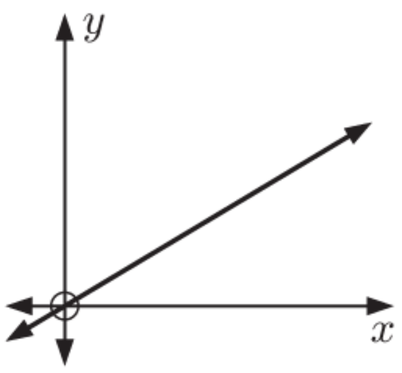
B



C



D



2 For each table of values:

i Plot the graph of y against x.

ii Determine whether y is directly proportional to x. If it is, find the proportionality constant.

a

x	0	1	2	3	4
y	0	5	10	15	20

b

x	0	1	2	3	4
y	2	4	6	8	10

c

x	2	3	4	6
y	3	4.5	6	7.5

d

x	2	4	5	7
y	10	20	30	35

3 This table shows the cost of parking in a parking lot.

Time (t hours)	*Cost* (€C)
$0 < t \leqslant 1$	6
$1 < t \leqslant 2$	12
$2 < t \leqslant 3$	18
$3 < t \leqslant 4$	24
$4 < t \leqslant 5$	30

a Find the cost of parking for:

i 1 hour **ii** 2 hours

iii 2.5 hours **iv** 4 hours.

b Are C and t directly proportional? Explain your answer.

4 Suppose y is directly proportional to x. State what happens to:

a y if x is doubled **b** x if y is multiplied by 6

c y if x is divided by 3 **d** x if y is increased by 20%

e y if x is decreased by 70% **f** x if y is increased by 10.

5 Water freezes at 0°C, which is approximately 32°F. Explain why the variables *temperature in* °C and *temperature in* °F cannot be directly proportional.

6 Water is poured into this drinking glass at a constant rate. Determine whether the following pairs of variables are directly proportional:

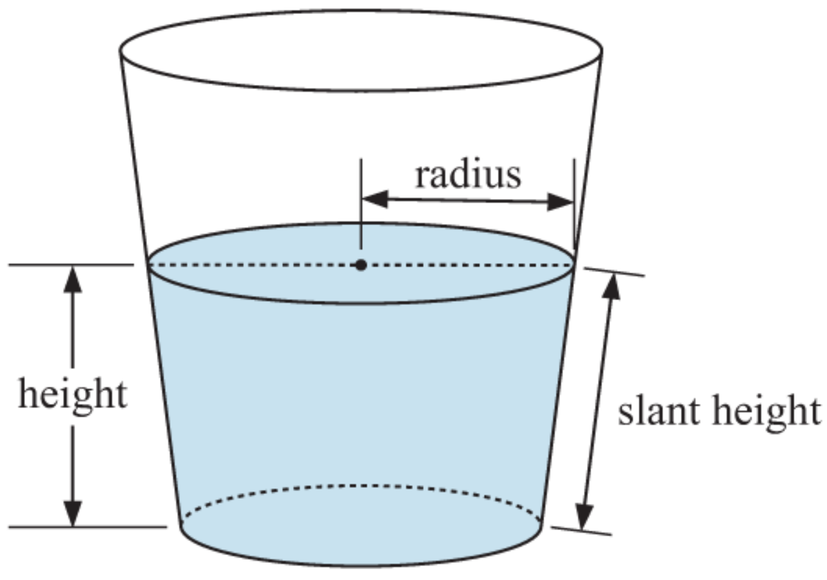
a *volume of water* and *time*

b *volume of water* and *height*

c *height* and *slant height*

d *height* and *radius*

e *slant height* and *weight of water*.



Example 2 **Self Tutor**

Suppose $p \propto t$ and that $p = 17$ when $t = 5$. Find t when $p = 60$.

t	5	?
p	17	60

$\times \frac{60}{17}$

To change p from 17 to 60, we multiply by $\frac{60}{17}$.

Since $p \propto t$, we must also multiply t by $\frac{60}{17}$

$\therefore \quad t = 5 \times \frac{60}{17} \approx 17.6$

7 Suppose $y \propto x$ and that $y = 24$ when $x = 7$. Find:

a y when $x = 21$ **b** x when $y = 30$.

8 The amount of energy used by a heater is directly proportional to the time it is on. When it is on for 2 hours, the heater uses 3.6 kWh of energy. Find:

a the proportionality constant and explain what it means

b the energy used by the heater if it is on for 1.5 hours

c the time it would take for the heater to use 6 kWh of energy.

9 Isabella has drawn a scale diagram of her school. An actual length of 60 m is represented by 3 cm on her diagram.

a Find the proportionality constant. Explain how it relates to the scale of the diagram.

b The football pitch is 110 m long. How long is the pitch on Isabella's diagram?

c The front fence is 15 cm long on Isabella's diagram. Find the actual length of the front fence.

10 The momentum of a gridiron player is directly proportional to his velocity. When he moves at $5\ \text{m s}^{-1}$, his momentum is $610\ \text{kg m s}^{-1}$. Find:

a the player's momentum when he moves at $3\ \text{m s}^{-1}$

b the velocity required for his momentum to be $420\ \text{kg m s}^{-1}$.

11 A plumber charges a £30 call-out fee plus a constant rate per hour of labour.
Suppose the total charge for a job requiring t hours labour is £C.

a Explain why:

i C and t are not directly proportional

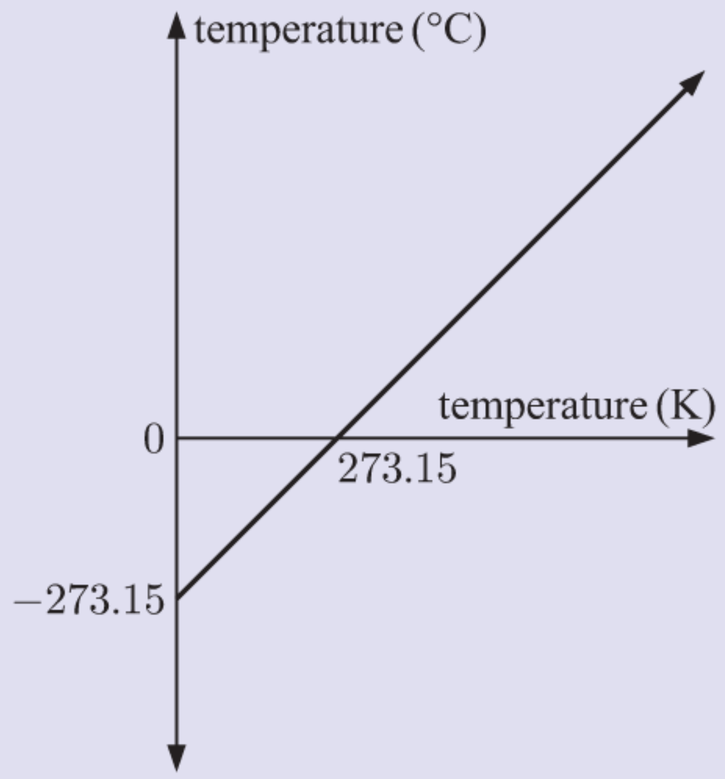
ii $(C - 30)$ and t are directly proportional.

b The total charge for a 1.5 hour job is £120. Find the total charge for a 5 hour job.

THEORY OF KNOWLEDGE

In 1848, **William Thomson** (1824 - 1907), also known as **Lord Kelvin**, proposed the need for a temperature scale starting at *absolute zero*. His idea stemmed from research showing a proportional relationship between the kinetic energy of a system and its temperature. By extrapolating his results to a point where the kinetic energy of a system was zero, Thomson was able to predict *absolute zero* as about $-273°\text{C}$.

The SI unit for temperature is the kelvin (K), named in Thomson's honour. *Absolute zero* is regarded as 0 kelvin, and is defined as $-273.15°\text{C}$. An increase of 1 kelvin corresponds to an increase of $1°\text{C}$, so $0°\text{C}$ is equivalent to 273.15 K, and $100°\text{C}$ is equivalent to 373.15 K.



1 In what ways is it useful to use variables in direct proportion?

2 Which measure of temperature is most convenient?

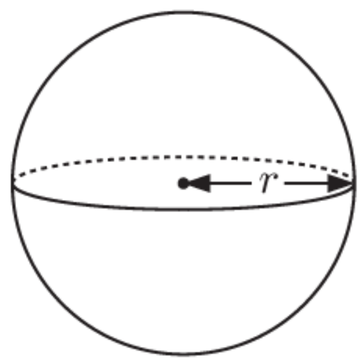
3 What is the most *natural* measure of temperature?

B POWERS IN DIRECT VARIATION

In many circumstances, the variables we consider are not directly proportional, but there is direct variation between their *powers*.

For example, the surface area of a sphere with radius r is $A = 4\pi r^2$.

A is not directly proportional to r. However, $A \propto r^2$ with proportionality constant 4π.



Example 3 — Self Tutor

Suppose $y = \dfrac{2x^3}{7}$. State two variables which are directly proportional, and determine the proportionality constant k.

$y = \frac{2}{7}(x^3)$, so $y \propto x^3$ and $k = \frac{2}{7}$.

y is directly proportional to the cube of x, and the proportionality constant $k = \frac{2}{7}$.

EXERCISE 6B

1 State which two variables are directly proportional, and determine the proportionality constant k:

a $A = \pi r^2$ **b** $V = \frac{4}{3}\pi r^3$ **c** $T = \dfrac{3n^4}{4}$

2 Suppose $y \propto x^3$. Describe what happens to:

a y if x is doubled **b** y if x is divided by 10

c y if x is increased by 20% **d** x if y is multiplied by 2.5

e x if y is decreased by 30% **f** x if y is increased by 100%.

Example 4 — Self Tutor

Suppose D is directly proportional to z^2, and that $D = 14$ when $z = 5$. Find:

a D when $z = 15$ **b** z when $D = 140$, given $z > 0$.

a

z	5	15
D	14	

($\times 3$)

z is multiplied by 3

$\therefore$ z^2 is multiplied by $3^2 = 9$

$\therefore$ D is multiplied by 9 $\{$as $D \propto z^2\}$

$\therefore$ $D = 14 \times 9 = 126$

b

z	5	
D	14	140

($\times 10$)

D is multiplied by 10

$\therefore$ z^2 is multiplied by 10 $\{$as $D \propto z^2\}$

$\therefore$ z is multiplied by $\sqrt{10}$ $\{$as $z > 0\}$

$\therefore$ $z = 5\sqrt{10} \approx 15.8$

3 Suppose M is directly proportional to t^2, where $t > 0$. When $t = 6$, $M = 40$. Find:

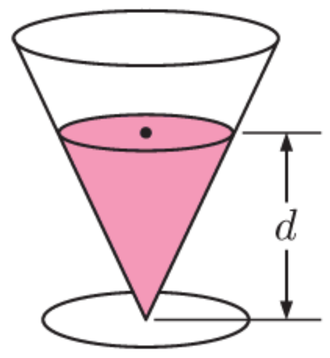
a M when $t = 9$ **b** t when $M = 120$.

4 Suppose V is directly proportional to the cube of y, and that when $y = 3$, $V = 30$. Find:

a V when $y = 12$ **b** y when $V = 180$.

5 The mass of a square sheet of glass is directly proportional to the square of its length. Given that a 30 cm square sheet has mass 900 g, find the mass of a 50 cm square sheet.

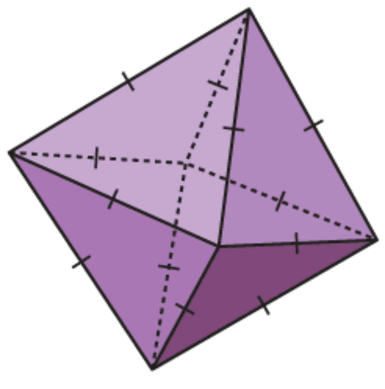
6 The amount of medicine in this glass is directly proportional to the cube of its depth d. The glass is 6 cm high, and has capacity 40 mL.



a Find the amount of medicine needed to fill the glass to a depth of 4 cm.

b Jimmy needs 30 mL of medicine. To what depth should the glass be filled?

7 The volume of a regular octahedron is directly proportional to the cube of its side lengths.



a If the side lengths increase by 5%, by what percentage does the volume change?

b What percentage change in side length is required to double the volume?

8 The *kinetic energy* of an object with mass m and speed v is given by $E = \frac{1}{2}mv^2$.

a For objects travelling at a particular speed, what proportionality exists between E and m?

b For an object with constant mass, what proportionality exists between E and v?

c What happens to the kinetic energy of an object if its speed decreases by 10%?

d The brakes of a car turn kinetic energy into heat at a constant rate. Explain why the stopping distance of a car is proportional to the square of the speed.

9 Suppose y is directly proportional to $\sqrt{x}$, and that z is directly proportional to x^2. What power of z is y directly proportional to?

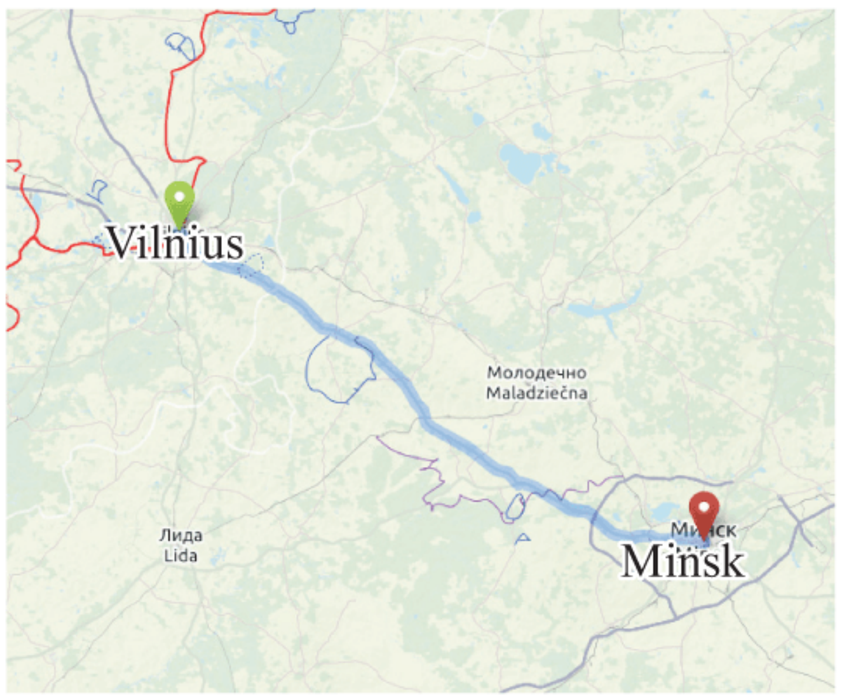
C INVERSE VARIATION

Emilija is driving 200 km from Vilnius to Minsk.

Travelling at an average speed of 50 km h^{-1}, the trip will take $\frac{200}{50} = 4$ hours. Travelling at an average speed of 100 km h^{-1}, the trip will only take $\frac{200}{100} = 2$ hours.

Notice that doubling the *average speed* will halve the *time taken*. In other words, when one variable was *multiplied* by 2, the other was *divided* by 2.

In a case like this, we have an **inverse proportion**.



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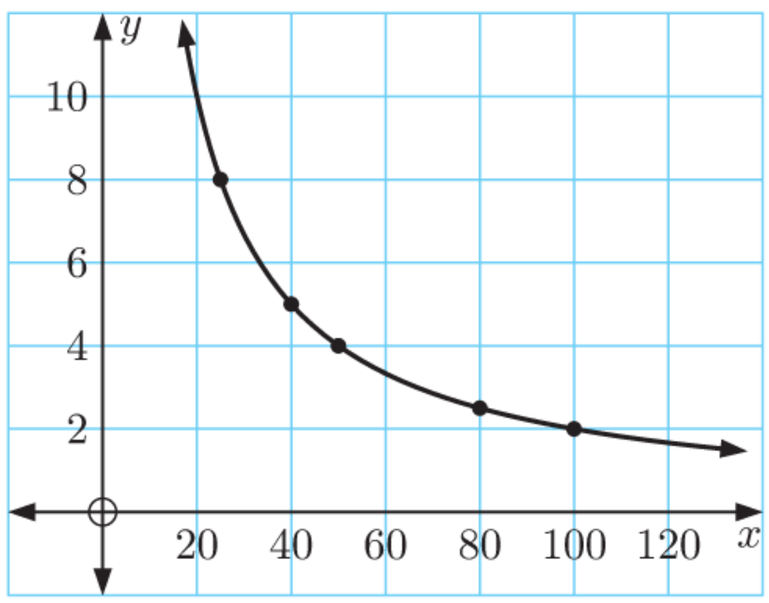
Two variables are **inversely proportional** or **vary inversely** if, when one is *multiplied* by a constant, the other is *divided* by the same constant.

Suppose Emilija travels with average speed x km h^{-1}, and the trip takes y hours.

The table below shows some possible combinations of x and y.

x	25	40	50	80	100
y	8	5	4	2.5	2

When these points are plotted, they form part of a **hyperbola**.



Notice that:

- As x gets closer and closer to 0, y gets infinitely large. The curve approaches but never reaches the vertical asymptote $x = 0$.
- As x gets infinitely large, y gets closer and closer to 0. The curve approaches but never reaches the horizontal asymptote $y = 0$.
- $xy = 200$ for all points in the table.

If y is **inversely proportional** to x, then y is **directly proportional** to $\frac{1}{x}$.

We write this as $y \propto \frac{1}{x}$.

Consequently, $\boldsymbol{y = \frac{k}{x}}$ or $\boldsymbol{xy = k}$.

Example 5 **Self Tutor**

Suppose y is inversely proportional to x, and that $y = 12$ when $x = 10$. Find:

a y when $x = 40$ **b** x when $y = 19$

Since y is inversely proportional to x, $y \propto \frac{1}{x}$.

a

x	10	40
y	12	

$\times 4$

x is multiplied by 4

$\therefore$ y is multiplied by $\frac{1}{4}$ $\{$as $y \propto \frac{1}{x}\}$

$\therefore$ $y = 12 \times \frac{1}{4} = 3$

b

x	10	
y	12	19

$\times \frac{19}{12}$

y is multiplied by $\frac{19}{12}$

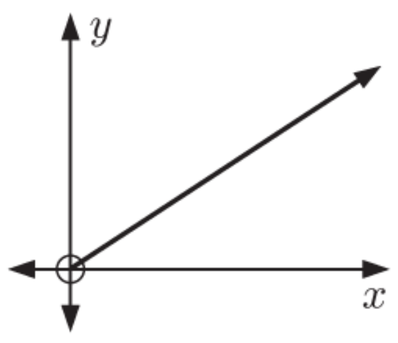
$\therefore$ x is multiplied by $\frac{12}{19}$ $\{$as $y \propto \frac{1}{x}\}$

$\therefore$ $x = 10 \times \frac{12}{19} \approx 6.32$

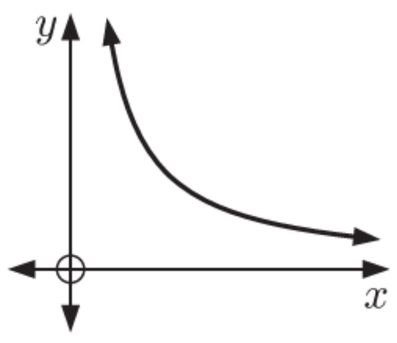
EXERCISE 6C

1 Which graph could indicate that y is inversely proportional to x?

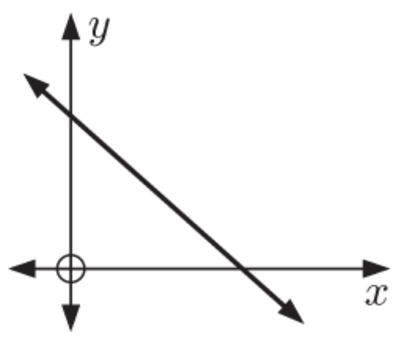
A



B



C



2 For each of the following tables, calculate the value of xy for each point. Hence determine whether x and y are inversely proportional. If an inverse proportionality exists, determine the law connecting the variables, and draw the graph of y against x.

a

x	2	3	4	6
y	12	8	6	4

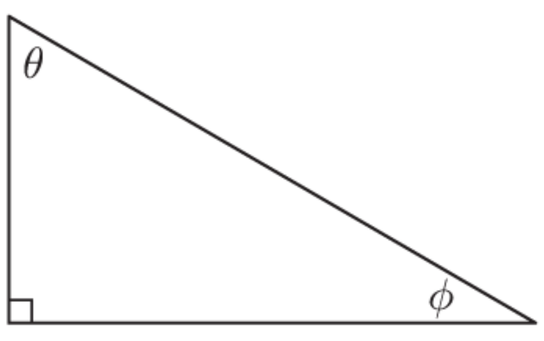
b

x	1	2	4	5
y	20	10	6	4

c

x	3	5	10	12
y	20	12	6	5

3



θ and ϕ are the two smaller angles of a right angled triangle.

a Are θ and ϕ inversely proportional?

b Are $\tan\theta$ and $\tan\phi$ inversely proportional?

Explain your answers.

4 Suppose y is inversely proportional to x. Explain what happens to y if:

a x is doubled

b x is divided by 7

c x is multiplied by $\frac{9}{5}$

d x is increased by 30%.

5 Suppose C is inversely proportional to t, and that $C = 15$ when $t = 6$. Find:

a C when $t = 18$

b t when $C = 20$.

6 The time taken to landscape a garden is inversely proportional to the number of gardeners working on the task. If 5 gardeners could do the task in 6 hours, how long would it take 3 gardeners to do the task?

7 When a constant force is applied to an object, the object's *acceleration* is inversely proportional to its *mass*.

When the force is applied to a 5 kg object, the resulting acceleration is $1.5\ \text{m s}^{-2}$.

a Find the acceleration which would result if the same force acted on a 2 kg object.

b Find the mass of an object whose acceleration under the force is $10\ \text{m s}^{-2}$.

8 Wendy wants to use the money in her savings account to buy shares in her favourite company.

a Explain why the *number of shares* she can buy is inversely proportional to the *share price*.

b The share price is currently £6.25, and Wendy can buy n shares at that price.

i If the share price drops by £0.15, how many shares can Wendy now buy?

ii By what percentage must the share price fall so that Wendy can buy $1.5n$ shares?

DISCUSSION

Ohm's law states that in an electrical circuit, the potential difference V across a resistor is equal to the current I times the resistance R.

- Assuming the potential difference is fixed, are the current and resistance inversely proportional?
- In general, is it reasonable to say that the current and resistance are inversely proportional?

D POWERS IN INVERSE VARIATION

In this Section we consider the inverse variation between one variable and the power of another. We use the same techniques as for direct variation.

Example 6 **Self Tutor**

Suppose P is inversely proportional to the square of n, and that $P = 40$ when $n = 6$.

a Find P when $n = 12$.

b Given $n > 0$, find n when $P = 90$.

Since P is inversely proportional to n^2, $P \propto \frac{1}{n^2}$

a

n	6	12
P	40	

($\times 2$ from 6 to 12)

n is multiplied by 2

$\therefore$ n^2 is multiplied by $2^2 = 4$

$\therefore$ $\frac{1}{n^2}$ is multiplied by $\frac{1}{4}$

$\therefore$ P is multiplied by $\frac{1}{4}$ $\{$as $P \propto \frac{1}{n^2}\}$

$\therefore$ $P = 40 \times \frac{1}{4} = 10$

b

n	6	
P	40	90

($\times \frac{90}{40}$ from 40 to 90)

P is multiplied by $\frac{90}{40} = \frac{9}{4}$

$\therefore$ $\frac{1}{n^2}$ is multiplied by $\frac{9}{4}$ $\{$as $P \propto \frac{1}{n^2}\}$

$\therefore$ n^2 is multiplied by $\frac{4}{9}$

$\therefore$ n is multiplied by $\sqrt{\frac{4}{9}} = \frac{2}{3}$ $\{$as $n > 0\}$

$\therefore$ $n = 6 \times \frac{2}{3} = 4$

EXERCISE 6D

1 Suppose $y \propto \frac{1}{x^3}$. Explain what happens to:

a y if x is doubled

b y if x is multiplied by $\frac{3}{5}$

c x if y is multiplied by 64.

2 Suppose y is inversely proportional to the square of x, and that $y = 27$ when $x = 8$.

a Find y when $x = 24$.

b Given $x > 0$, find x when $y = 75$.

3 Suppose M is inversely proportional to the cube of c, and that $M = 64$ when $c = 12$.

a Find M when $c = 8$.

b Find c when $M = 1$.

4 A drink company wants to adjust the dimensions of their cylindrical soft drink cans, but keep the same volume. The current can is 12.9 cm high with radius 3.04 cm.

a Explain why the height of the can is inversely proportional to the square of its radius.

b Find the height of the can if the radius chosen is 3.39 cm.

c Find the radius of the can if the height chosen is 15.3 cm.

d The production manager restricts the possible radius of the can to values between 2.7 cm and 3.8 cm. Can you suggest why this was done?

5 The *tidal acceleration* acting between two bodies is inversely proportional to the cube of the *distance* between them. Find:

a the percentage change in the tidal acceleration if the distance between the objects increases by 10%

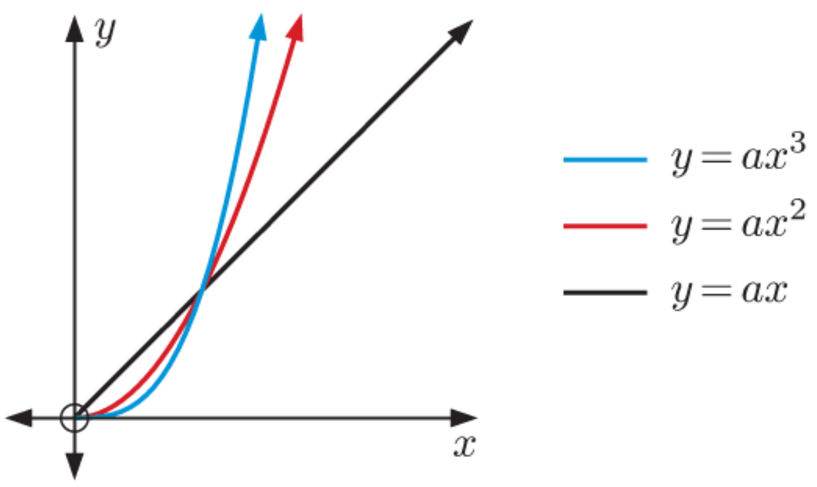
b the percentage change in the distance between two objects such that the tidal acceleration triples.

6 Suppose $y \propto \frac{1}{x}$, and $z \propto \frac{1}{y^2}$. What variation exists between x and z?

E DETERMINING THE VARIATION MODEL

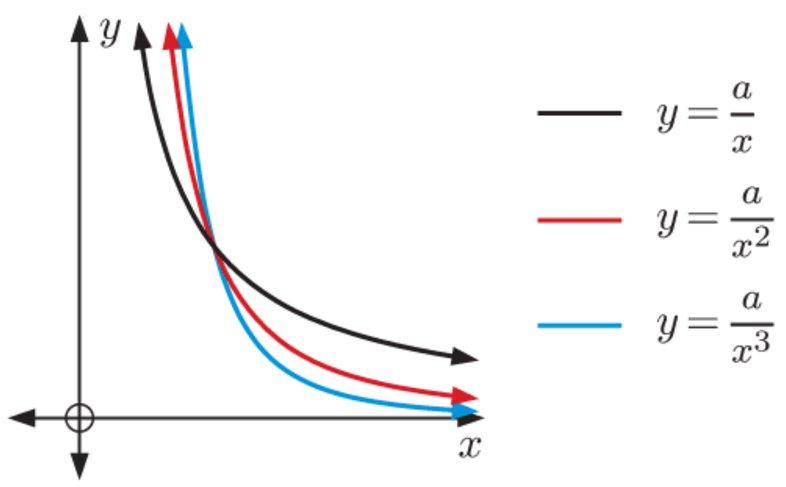
The direct and inverse variations we have studied have equations of the form $y = ax^n$ where $n \in \mathbb{Z}$, $n \neq 0$. These equations are called **variation models**.

- If $n > 0$ we have **direct variation**.



The graph passes through the origin $(0, 0)$.

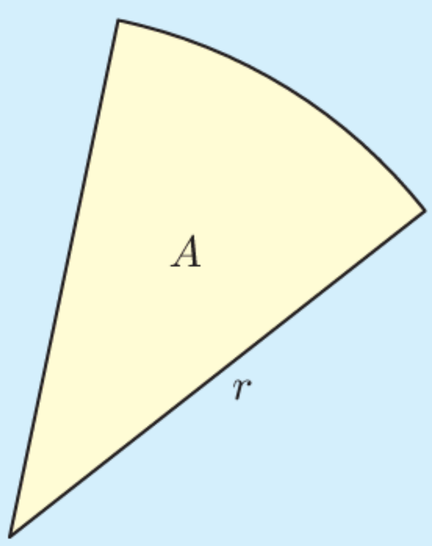
- If $n < 0$ we have **inverse variation**.



The graph is asymptotic to both the x and y axes.

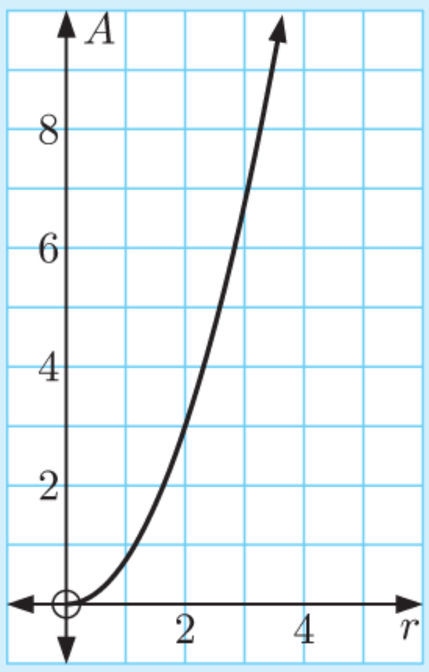
In some cases, we know what type of variation exists between the variables. Given this knowledge, we can use a point which lies on the graph to find the exact equation of the variation model.

Example 7 — Self Tutor



The area A of a sector of given angle is directly proportional to the square of its radius r. The graph of A against r is shown alongside.

Find the equation of the variation model.

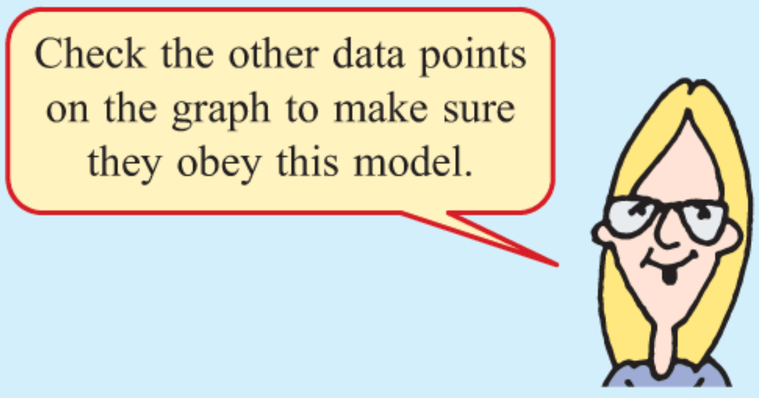


Since $A \propto r^2$, $A = ar^2$ where a is a constant.

From the graph we see that when $r = 2$, $A = 3$

$\therefore \quad 3 = a \times 4$ and so $a = \frac{3}{4}$.

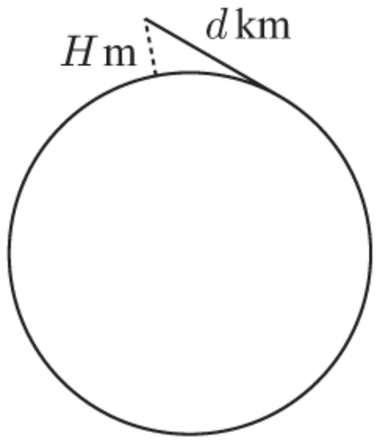
The equation of the variation model is $A = \frac{3}{4}r^2$.

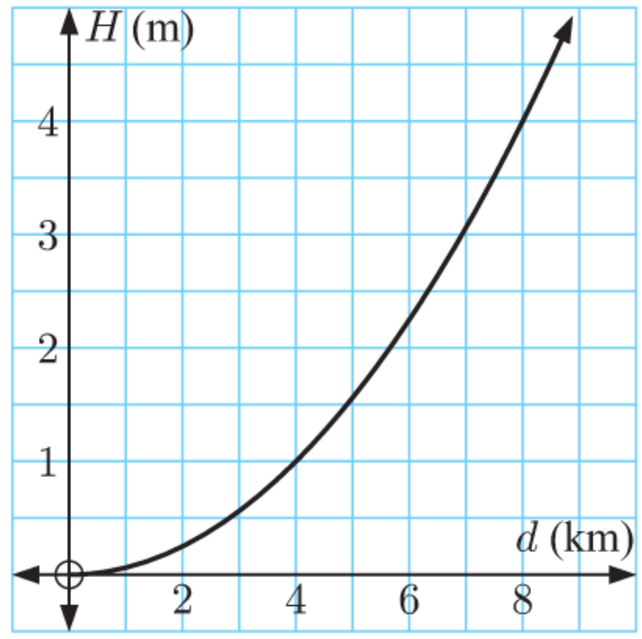


EXERCISE 6E

1 The height (H m) of a person above sea level is directly proportional to the square of the distance (d km) to the horizon.

The graph of H against d is shown alongside.

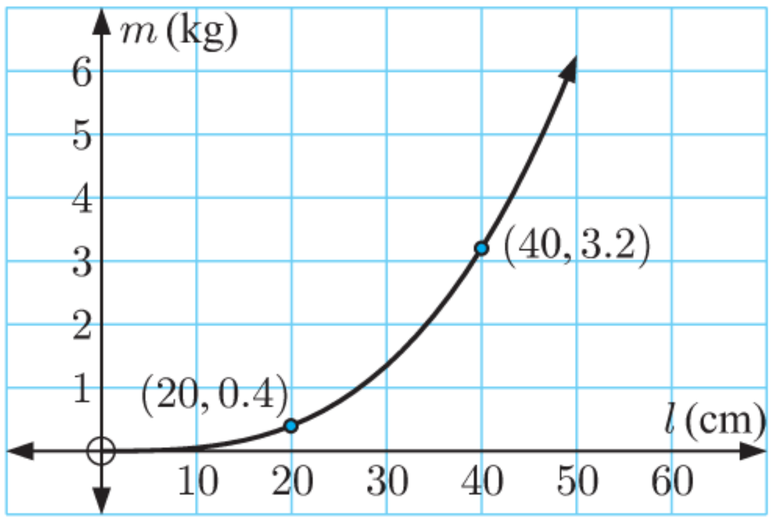




a Find the equation of the variation model.

b Teresa is 16 km from the horizon. Find her height above sea level.

2



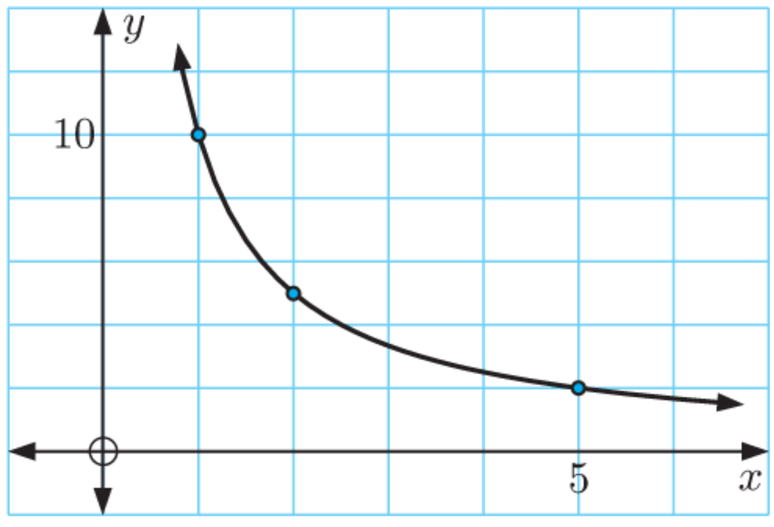
Scale models of a car are made in different sizes.
The mass of a model car (m kg) is directly proportional to the cube of its length (l cm).
The graph of m against l is shown alongside.

a Find a model connecting m and l.

b Find the mass of a model car which is 50 cm long.

c Find the length of a model car with mass 1 kg.

3 It is suspected that variable y varies inversely with variable x.



a Do the three marked points confirm this relationship? Explain your answer.

b State the equation of the model connecting y and x.

c Find the value of y when $x = 8$.

4 The table opposite contains data from an experiment.

x	0.25	0.5	1	2
y	80	20	5	1.25

a Show that the model relating x to y has the form $y = \dfrac{k}{x^2}$ and find the value of k.

b Find the value of $x > 0$ when $y = 0.5$.

5 A car designer wants to find the relationship between the air resistance R and the velocity v km h^{-1} of the car. He performs a wind tunnel experiment and records the results in the table alongside:

v	10	20	30	40
R	0.5	4	13.5	32

a The designer initially suspects that $R \propto v^2$.

i Assuming $R \propto v^2$, use the first data point to find the model for R in terms of v.

ii Use the other data points to show that this model is incorrect.

b Show that a model of the form $R = kv^3$ fits all data points.

c Hence find the air resistance when the car is travelling at 50 km h^{-1}.

d What reduction in velocity is required to reduce air resistance by 20%?

F USING TECHNOLOGY TO FIND VARIATION MODELS

So far we have only considered data which a variation model fits exactly. This is unlikely to be the case when we collect data from real-world situations, due to experimental error, inaccuracies in measurement, and rounding. In these circumstances, we can use technology to find the variation model which best fits the data.

Variation models are often referred to as **power models**, because the variable is raised to a power. We use the **power regression** function on our calculator to find variation models.


GRAPHICS CALCULATOR INSTRUCTIONS

Example 8 **Self Tutor**

A company manufactures ball bearings in different sizes. The table alongside shows the radius and mass of each type of ball bearing.

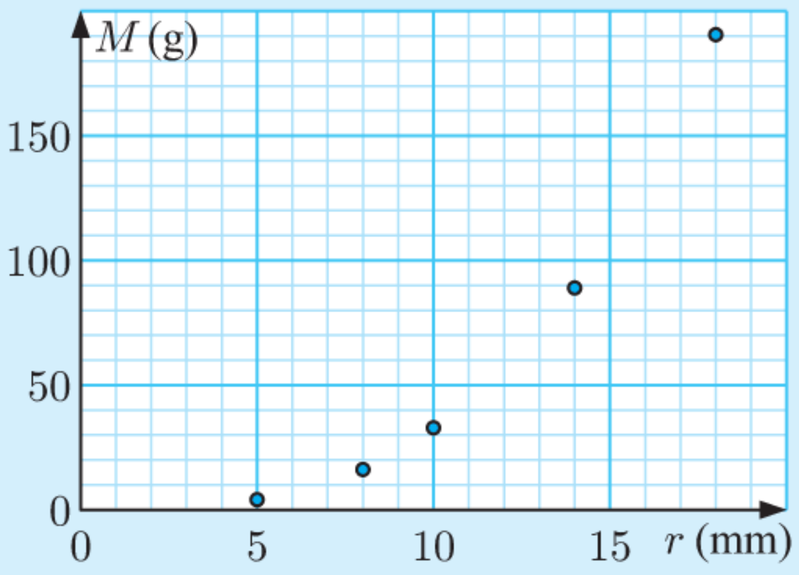
Radius (r mm)	5	8	10	14	18
Mass (M g)	4.08	16.73	32.67	89.65	190.54

a Draw a scatter diagram of the data. Discuss the shape of the graph.

b Obtain the power model which best fits the data.

c Estimate the mass of a ball bearing with radius 22 mm.

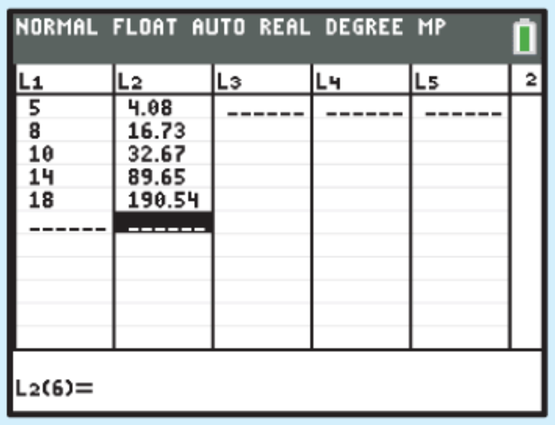
a

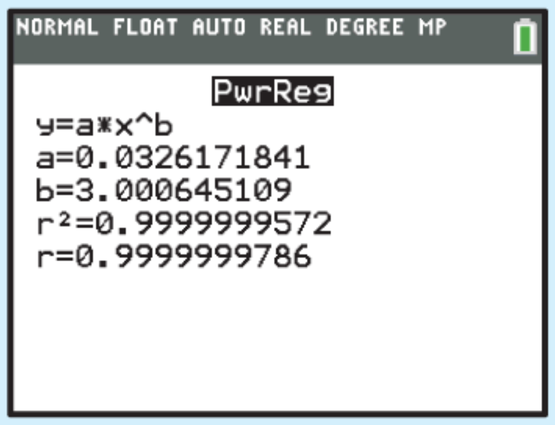


The data points appear to lie on a curve passing through the origin. This suggests that a variation model is appropriate.

b The power is very close to 3, so it is reasonable to conclude that M is directly proportional to r^3.

The model is $M \approx 0.0326r^3$.





c When $r = 22$, $M \approx 0.0326 \times 22^3 \approx 347$

So, a ball bearing with radius 22 mm has mass ≈ 347 g.

DISCUSSION

In the Example above, why should we expect that mass $\propto$ radius^3?

Discuss other situations where you might expect variation models.

EXERCISE 6F

1 For each data set, obtain the power model which best fits the data:

a

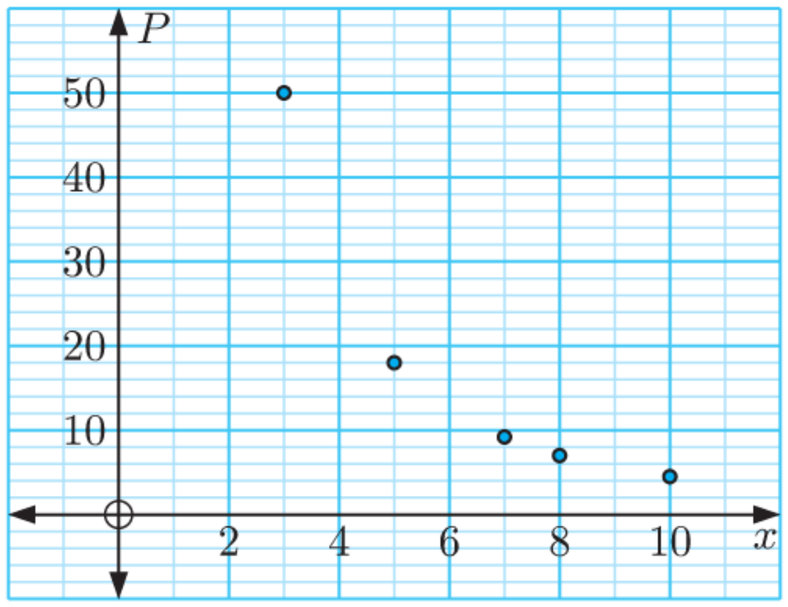
x	1	2	3	4
y	0.6	9.7	48.8	153.5

b

x	2	3	6	9
y	100	29.6	3.7	1.1

2 The data below is displayed on the scatter diagram.

x	3	5	7	8	10
P	50	18	9.2	7	4.5



a Do you think there is direct variation or inverse variation between the variables? Explain your answer.

b Obtain the variation model which best fits the data.

c Estimate the value of P when $x = 4$.

3 Zach never remembers to charge his phone until the battery has completely run out. This table shows the percentage charge his phone receives when it is charged for different times:

Time (t minutes)	8	25	32	45
Percentage charge ($C\%$)	10	31	40	56

a What type of variation would you expect between C and t? Explain your answer.

b Find the power model which best fits the data.

c Estimate the percentage charge of the phone after 56 minutes.

d For what values of t is it reasonable to apply this model? Explain your answer.

4 The land blocks in a new housing development have different dimensions, but they all have approximately the same area. Some of the land block dimensions are shown in the table below.

Width (w metres)	15	18	20	22	24
Length (l metres)	40	33	30	27	25

a What type of variation do you expect between l and w? Explain your answer.

b Draw a scatter diagram of the data. Is the diagram consistent with your answer to **a**?

c Obtain the power model which best fits the data.

d Estimate the length of a land block which is 23 m wide.

e For what range of values of w can this model sensibly be applied? Explain your answer.

5 When a car turns a corner, its *turning radius* depends on its *speed*.

Speed (s m s^{-1})	5.56	6.00	6.41	6.80	7.17	7.52
Turning radius (R m)	1.0	1.2	1.4	1.6	1.8	2.0

a Obtain the variation model which best fits the data.

b Estimate the turning radius for a car travelling at 10 m s^{-1}.

c Estimate the speed of a car if its turning radius is 4 m.

6 Answer the **Opening Problem** on page **118**.

7 The force between two positively charged spheres was measured at different distances. The results are recorded below.

Distance (d m)	0.1	0.25	0.5	0.75	1	1.5
Force (F N)	563	90.0	22.5	10.0	5.63	2.50

a Obtain a power model which best fits the data.

b Estimate the strength of the force when the spheres are 0.4 m apart.

c Find the distance between the spheres if the force between them is 650 N.

8 Donna had heard that the pressure of a constant volume of gas is directly proportional to its *temperature*.

To test this for herself, Donna observed how the pressure in a tube changed as the gas inside was heated. Her results are shown in the table:

Temperature (T °C)	5	10	15	20	25	30	35
Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39	3.45	3.51	3.57

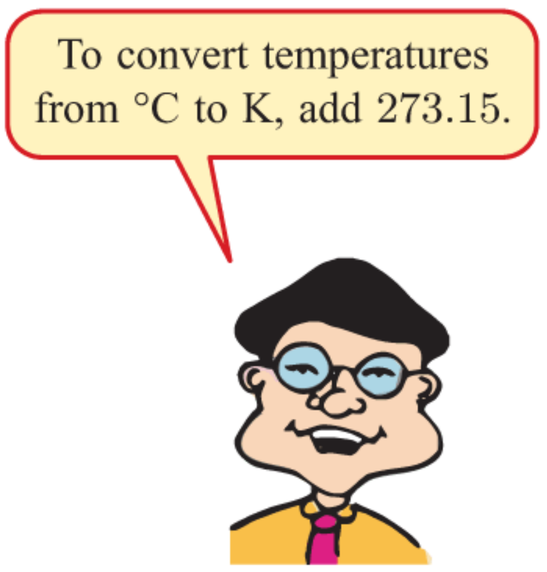
a Obtain the power model which best fits the data. Explain why P is *not* directly proportional to T.

b Donna's friend Penny suggests that the temperatures should be recorded in kelvin (K) rather than degrees Celsius.

i Redraw the table with the temperatures recorded in kelvin.

ii Obtain the power model which best fits the data.

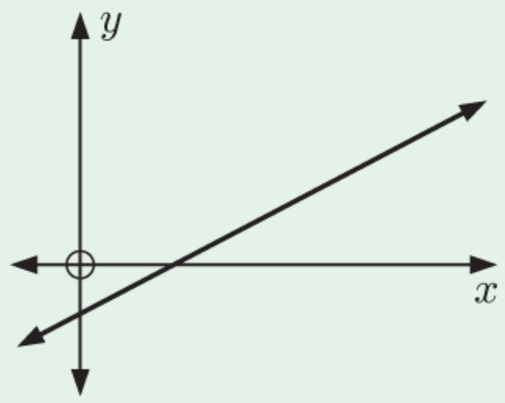
iii Is it now reasonable to conclude that pressure and temperature are directly proportional? Explain your answer.



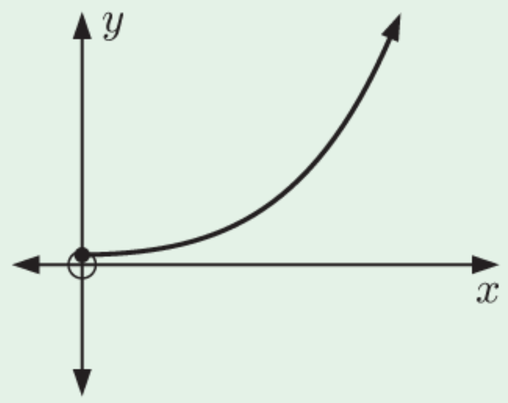
REVIEW SET 6A

1 Which graph indicates that y is directly proportional to x?

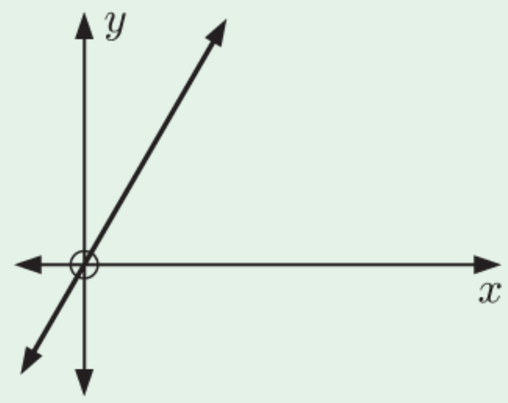
A



B



C



2 Suppose A is directly proportional to t. Explain what happens to:

a A when t is multiplied by 4 **b** t when A is increased by 5%.

3 A person's weight on the Moon is directly proportional to their weight on Earth. A person weighing 750 N on Earth weighs only 124 N on the Moon.
Given that John weighs 640 N on Earth, find his weight on the Moon.

4 State which two variables are directly proportional, and determine the proportionality constant k:

a $y = 5x^2$ **b** $P = \frac{2n^4}{3}$ **c** $V = \frac{\sqrt{5}}{4}a^3$

5 Suppose y is directly proportional to x^2, where $x > 0$. When $x = 8$, $y = 30$.

a Find a model connecting x and y.

b Hence find: **i** y when $x = 4$ **ii** x when $y = 150$.

6 For each of the following data sets, use values of xy to determine whether x and y are inversely proportional. If an inverse proportionality exists, determine the law connecting the variables, and draw the graph of y against x.

a

x	2	4	5	8
y	20	10	8	5

b

x	3	5	8	10
y	20	12	8	6

7 The *frequency* of a light wave is inversely proportional to its *wavelength*. Light waves that are seen as orange have wavelength 600 nm and frequency 500 THz. Light waves that are seen as blue have wavelength 480 nm. Find the frequency of this blue light wave.

8 The resistance to the flow of electricity in a wire varies inversely to the square of the diameter of the wire. When the diameter is 0.44 cm, the resistance is 0.24 ohms. Find:

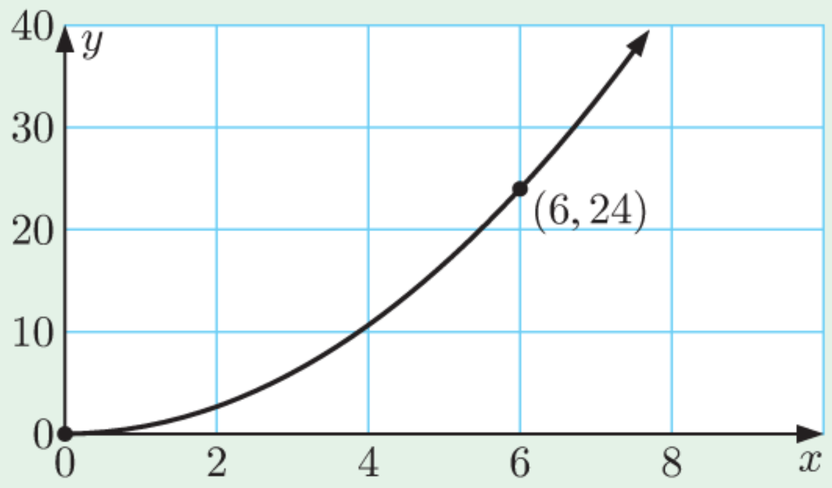
a the resistance when the diameter is 0.3 cm

b the diameter when the resistance is 0.45 ohms.

9 In the graph alongside, y varies directly with the square of x.

a Find the equation of the model connecting the variables.

b Find the value of y when $x = 11$.



10 Kelly makes glass regular pyramids of height h cm. She suspects that the volume of glass V cm^3 she uses is directly proportional to a power of h, so $V \propto h^n$. A table of volumes for various heights is shown below.

h	2	4	6
V	3.2	25.6	86.4

a Explain why we should expect that $n = 3$.

b Use the first piece of data to find the variation model connecting V and h.

c Check that the model you have found is satisfied by the remaining data points.

d Hence find: **i** V when $h = 8$ **ii** h when $V = 50$.

11 For each data set, use technology to find the power model which best fits the data.

a

x	2	4	5	7
y	12	96	188	514

b

x	0.9	1.4	2.2	2.5
y	762	130	21.3	12.8

12 The current was measured across a range of resistors for a constant potential difference:

Resistance (R ohms)	6.8	9.1	15	30	68	100
Current (I amperes)	17.6	13.2	8.00	4.00	1.76	1.20

a Draw a graph of I against R.

b Do you think there is a direct variation relationship or inverse variation between the variables? Explain your answer.

c Obtain the variation model which best fits the data.

d Estimate the current for a 250 ohm resistor given the same potential difference.

REVIEW SET 6B

1 Consider the table of values alongside.

t	2	5	8	10	15
P	5	12.5	20	25	37.5

a Draw a graph of P against t.

b Explain why P and t are directly proportional.

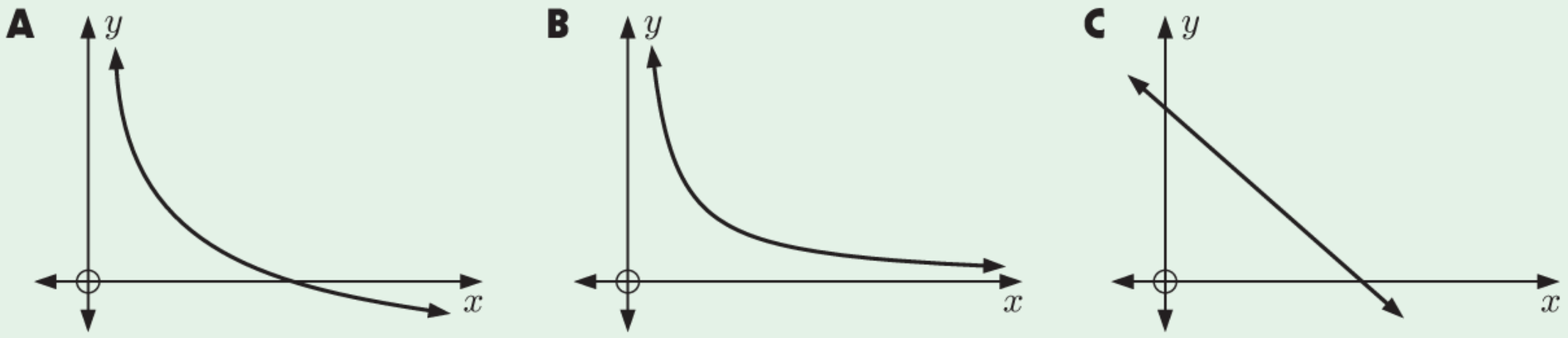
c Find a formula connecting P and t.

2 The force on a length of wire in a constant magnetic field is directly proportional to the current flowing through the wire. When the current is 1.09 A, the force on the wire is 2.18 N. Find:

a the force when the current is 1.45 A

b the current when the force is 3.6 N.

3 Which graph could indicate that y is inversely proportional to x?



4 Tamzin works at the botanic gardens. She has been told by the head gardener that 250 kg of compost is needed to cover a circular garden bed of radius 3 m. However, she has new plants to add to the collection, so she has decided to increase the size of the garden bed at the same time.

a Explain why the amount of compost needed is directly proportional to the square of the radius of the garden bed.

b How much *extra* compost will Tamzin need to extend the radius of the garden bed by 15%?

c By how far can Tamzin extend the radius of the garden bed with an extra 40 kg of compost?

5 Suppose y is inversely proportional to the cube of x, and that $y = 16$ when $x = 6$.

a Find a model connecting x and y.

b Hence find: **i** y when $x = 4$ **ii** x when $y = -2$.

6 If 3 people could paint a grain silo in 18 days, how long would it take 8 people to paint the silo working at the same rate?

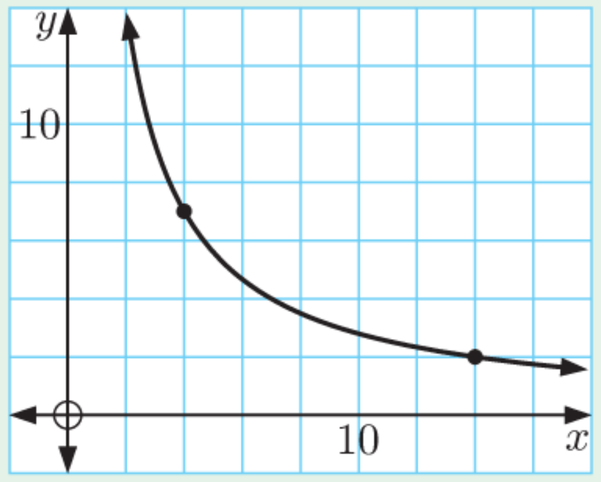
7 For an object moving in a circular orbit under gravity, the radius of the orbit is inversely proportional to the square of the object's orbital speed.

If the radius of the orbit increases by 20%, what happens to the orbital speed of the object?

8 In the graph alongside, y varies inversely with x.

a Find the equation of the variation model.

b Find the value of y when $x = 0.1$.



9 The power P required by a cyclist to maintain constant speed on flat ground is directly proportional to the cube of their speed, v.

To maintain a speed of 35 km h^{-1}, the power required is 262 W.

a Find the model connecting P and v.

b Hence find:

i the power needed to maintain a speed of 45 km h^{-1}

ii the speed of a cyclist exerting 205 W on flat ground.

10 It is suspected that two variables D and p are related by a law of the form $D = \dfrac{k}{p^2}$ where k is a constant.

An experiment was conducted to find D for various values of p, and the results are given in the table.

p	1.5	2	2.5	3	4
D	40	22.5	14.4	10	5.625

a Graph D against p for these data values.

b What features of the graph suggest that $D = \dfrac{k}{p^2}$ may be an appropriate model?

c Determine the value of k.

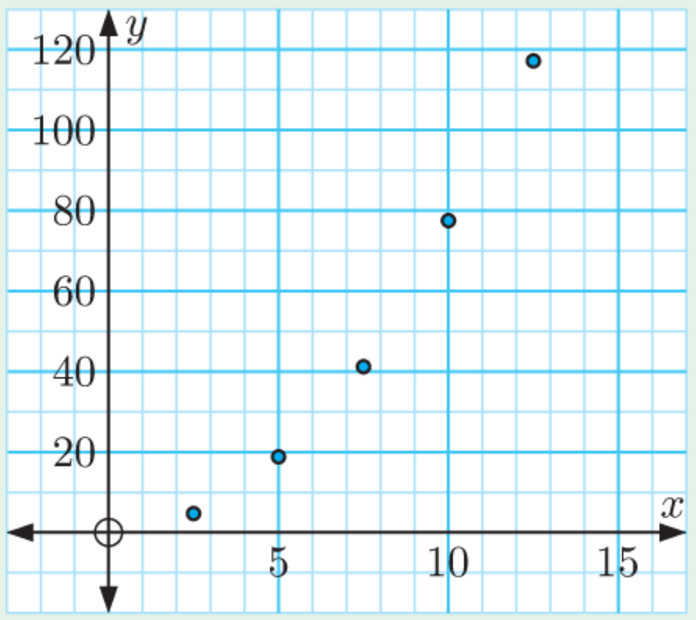
d Find the value of D when $p = 5$.

11 The data below is displayed on the scatter diagram.

x	2.5	5	7.5	10	12.5
y	4.7	18.8	41.2	75	117.2

a Do you think there is a direct variation relationship or inverse variation between the variables? Explain your answer.

b Obtain the variation model which best fits the data.



12 Abbas wanted to test the sound intensity of the speakers in the local hall. He set the output to a constant power, and took measurements at different distances. The results are given below.

Distance (d m)	1	5	10	15	20
Sound intensity (I W m^{-2})	63.7	2.55	0.637	0.283	0.159

a Abbas thinks that $I \propto \dfrac{1}{d}$. By calculating the value of $I \times d$ for each data point, show that he is incorrect.

b Find the power model which best fits the data.

c Hence find the percentage change in intensity if the distance is increased by 40%.

Chapter 7

Bivariate statistics

Contents:

A Association between numerical variables
B Pearson's product-moment correlation coefficient
C The coefficient of determination
D Line of best fit by eye
E The least squares regression line
F Statistical reliability and validity
G Spearman's rank correlation coefficient



OPENING PROBLEM

At a junior tournament, some young athletes each throw a discus. The *age* and *distance thrown* are recorded for each athlete.

Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

Things to think about:

a Do you think the distance an athlete can throw is related to the person's age?

b What happens to the distance thrown as the age of the athlete increases?

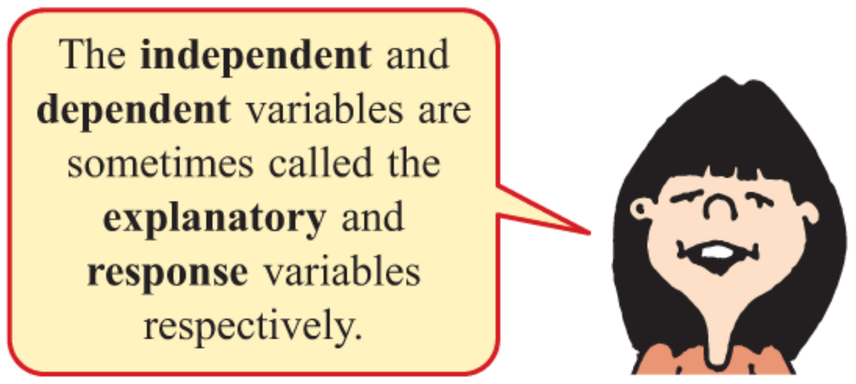
c How could you graph the data to more clearly see the relationship between the variables?

d How can we *measure* the relationship between the variables?

In the previous Chapter we saw how the relationship between two variables can be described using a function. In particular, we considered linear relationships between variables, and discussed how they could be *exact* or *approximations*.

In this Chapter we consider **bivariate data**, which means data which has two variables recorded for each individual. In most real-world situations, there will not be an exact relationship between these variables. Our task is to find the model which *best fits* the data, and measure how strong the relationship between the variables is.

For example, each athlete in the **Opening Problem** has had the *two* variables *age* and *distance thrown* recorded about them. We expect the *distance thrown* will *depend* on the athlete's *age*, so *age* is the independent variable and *distance thrown* is the dependent variable.



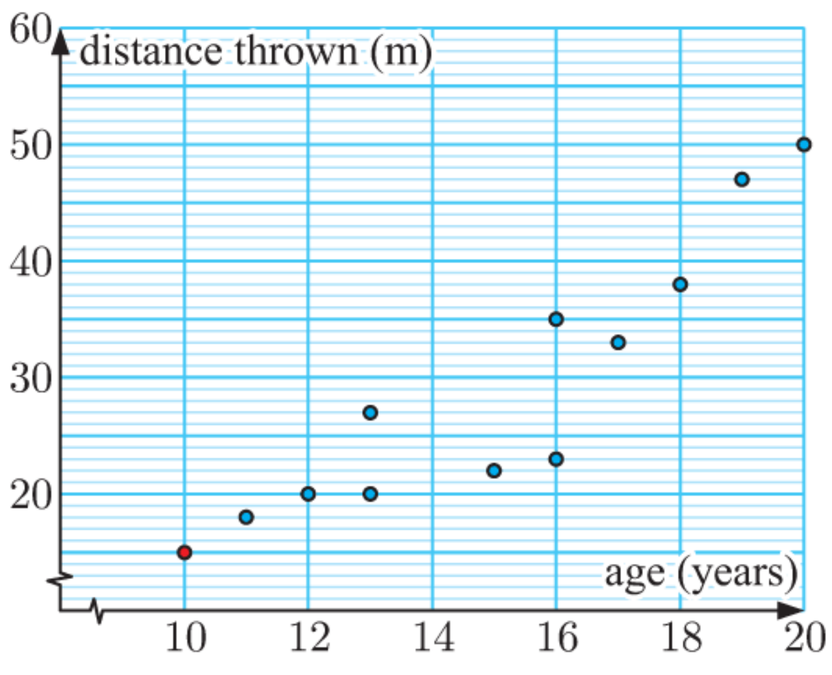
A ASSOCIATION BETWEEN NUMERICAL VARIABLES

We can observe the relationship between two numerical variables using a **scatter diagram**. We usually place the independent variable on the horizontal axis, and the dependent variable on the vertical axis.

In the **Opening Problem**, the independent variable *age* is placed on the horizontal axis, and the dependent variable *distance thrown* is placed on the vertical axis.

We then graph each data value as a point on the scatter diagram. For example, the red point represents athlete H, who is 10 years old and threw the discus 15 metres.

From the general shape formed by the dots, we can see that as the *age* increases, so does the *distance thrown*.

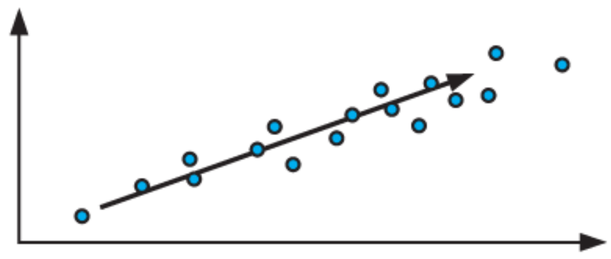


CORRELATION

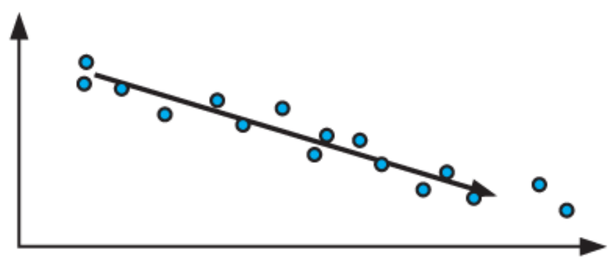
Correlation refers to the relationship or association between two numerical variables.

There are several characteristics we consider when describing the correlation between two variables: direction, linearity, strength, outliers, and causation.

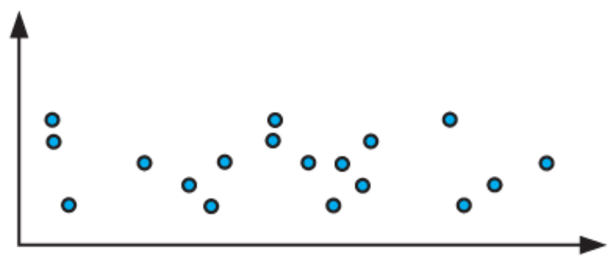
DIRECTION



For a generally *upward* trend, we say that the correlation is **positive**. An increase in the independent variable generally results in an increase in the dependent variable.



For a generally *downward* trend, we say that the correlation is **negative**. An increase in the independent variable generally results in a decrease in the dependent variable.

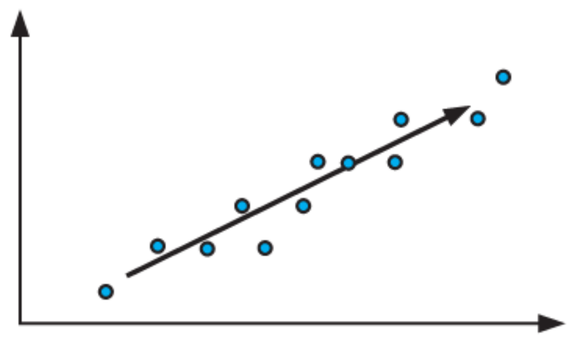


For *randomly scattered* points, with no upward or downward trend, we say there is **no correlation**.

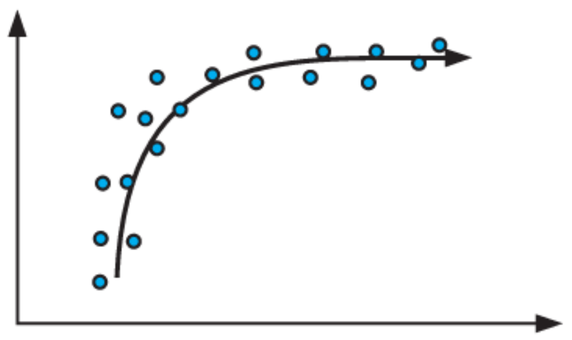
LINEARITY

When a trend exists, and the points approximately form a straight line, we say the trend is **linear**.

These points are roughly linear.

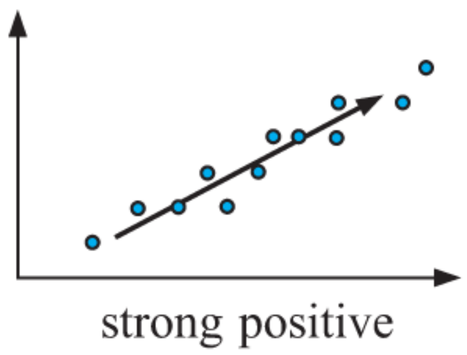
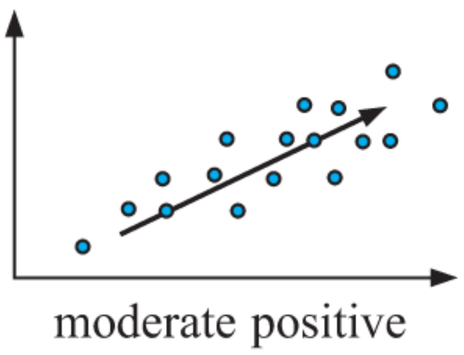
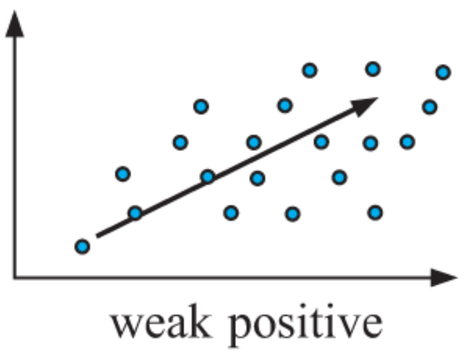
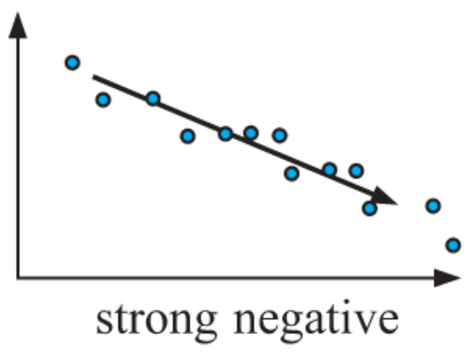
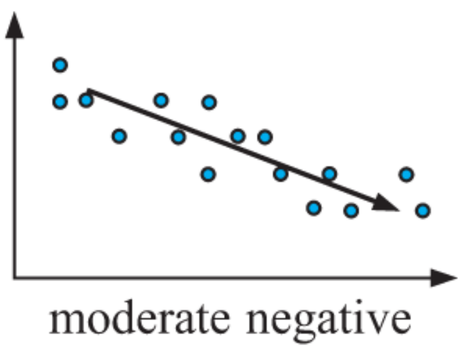
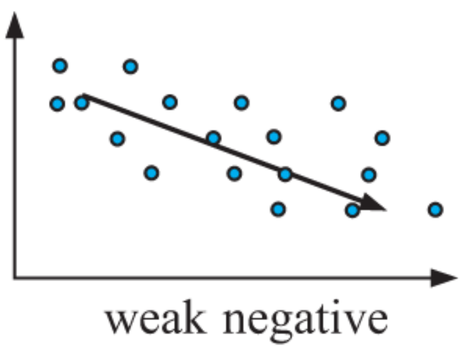


These points do not follow a linear trend.



STRENGTH

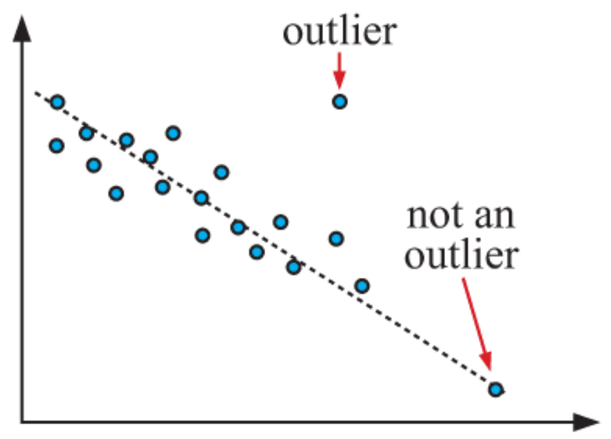
To describe how closely the data follows a pattern or trend, we talk about the **strength** of correlation. It is usually described as either **strong**, **moderate**, or **weak**.

strong	**moderate**	**weak**
 strong positive	 moderate positive	 weak positive
 strong negative	 moderate negative	 weak negative

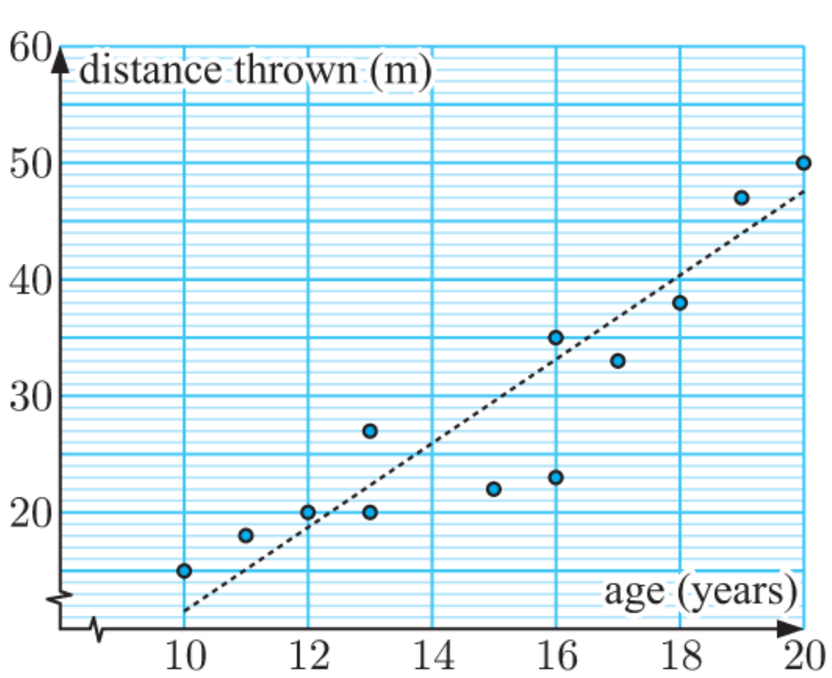
OUTLIERS

Outliers are isolated points which do not follow the trend formed by the main body of data.

If an outlier is the result of a recording or graphing error, it should be discarded. However, if the outlier is a genuine piece of data, it should be kept.



For the scatter diagram of the data in the **Opening Problem**, we can say that there is a strong, positive correlation between *age* and *distance thrown*. The relationship appears to be linear, with no outliers.



CAUSALITY

Correlation between two variables does not necessarily mean that one variable *causes* the other.

For example:

- The *arm length* and *running speed* of a sample of young children were measured, and a strong, positive correlation was found between the variables.

  This does *not* mean that short arms cause a reduction in running speed, or that a high running speed causes your arms to grow long.

  Rather, there is a strong, positive correlation between the variables because both *arm length* and *running speed* are closely related to a third variable, *age*. Up to a certain age, both *arm length* and *running speed* increase with *age*.



- The number of television sets sold in London and the number of stray dogs collected in Boston were recorded over several years. A strong, positive correlation was found between the variables.

  Obviously the number of television sets sold in London was not influencing the number of stray dogs collected in Boston. It is coincidental that both these variables increased over this period of time.



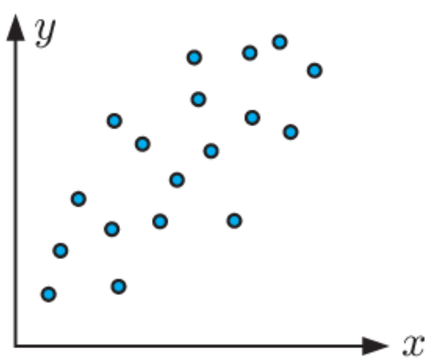
If a change in one variable *causes* a change in the other variable then we say that a **causal relationship** exists between them. In these cases, we can say that the independent variable *explains* the dependent variable. It may be more natural to use the terminology **explanatory variable** and **response variable**.

In cases where a causal relationship is not apparent, we cannot conclude that a causal relationship exists based on high correlation alone.

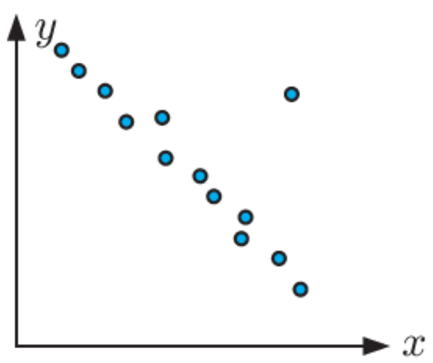
EXERCISE 7A

1 For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.

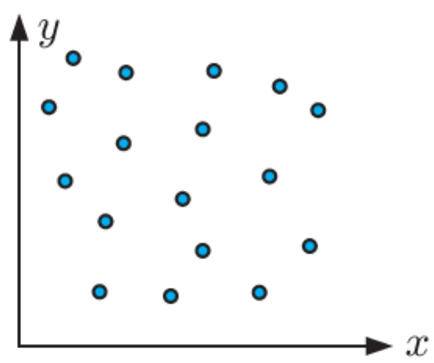
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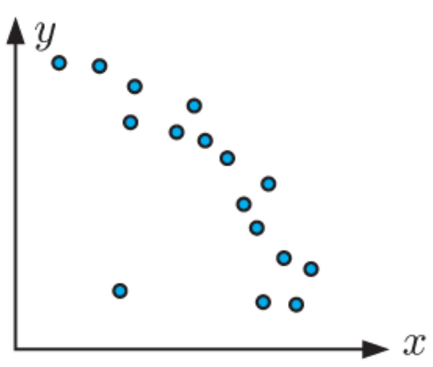
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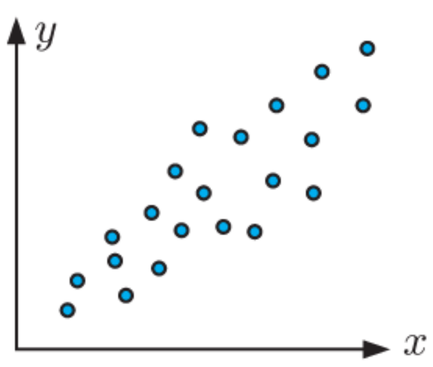
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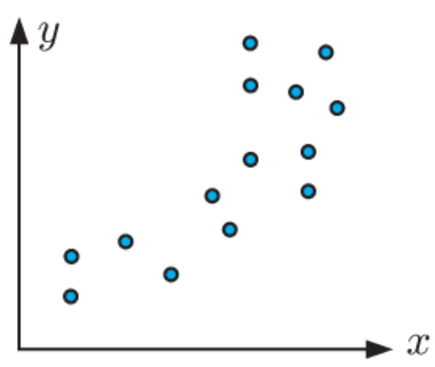
d



e



f



2 Tiffany is a hairdresser. The table below shows the number of hours she worked each day last week, and the number of customers she had.

Day	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Hours worked	8	4	5	10	8	3	6
Number of customers	9	6	5	12	7	4	5

a Which is the explanatory variable, and which is the response variable?

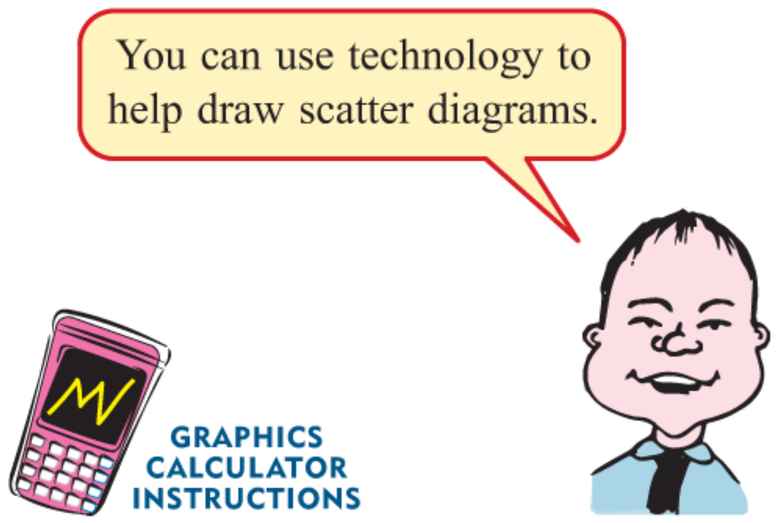
b Draw a scatter diagram of the data.

c On which two days did Tiffany:

i work the same number of hours

ii have the same number of customers?

d Explain why you would expect a positive correlation between the variables.



3 The scores awarded by two judges at an ice skating competition are shown in the table.

Competitor	P	Q	R	S	T	U	V	W	X	Y
Judge A	5	6.5	8	9	4	2.5	7	5	6	3
Judge B	6	7	8.5	9	5	4	7.5	5	7	4.5

a Construct a scatter diagram for the data, with Judge A's scores on the horizontal axis and Judge B's scores on the vertical axis.

b Copy and complete the following comments about the scatter diagram:

There appears to be ....., ..., ... correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores .....

c Would it be reasonable to conclude that an increase in Judge A's scores *causes* an increase in Judge B's scores? Explain your answer.

4 Paul owns a company which installs industrial air conditioners. The table below shows the number of workers at the company's last 10 jobs, and the time it took to complete the job.

Job	A	B	C	D	E	F	G	H	I	J
Number of workers	5	3	8	2	5	6	1	4	2	7
Time (hours)	4	6	2.5	9	3	4	10	4	7.5	3

a Which job: **i** took the longest **ii** involved the most workers?

b Draw a scatter diagram to display the data.

c Describe the relationship between the variables *number of workers* and *time*.

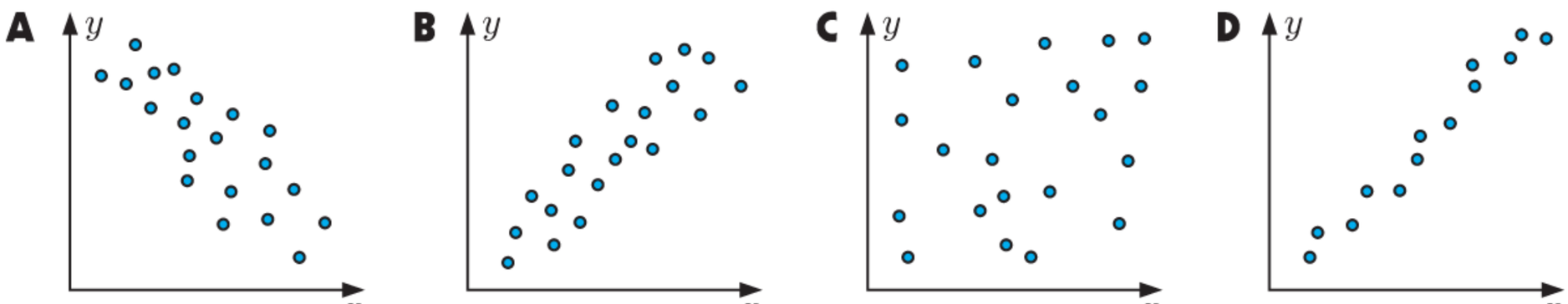
5 Choose the scatter diagram which would best illustrate the relationship between the variables x and y.

a $x =$ the number of apples bought by customers, $y =$ the total cost of apples bought

b $x =$ the number of pushups a student can perform in one minute,
$y =$ the time taken for the student to run 100 metres

c $x =$ the height of a person, $y =$ the weight of the person

d $x =$ the distance a student travels to school, $y =$ the height of the student's uncle

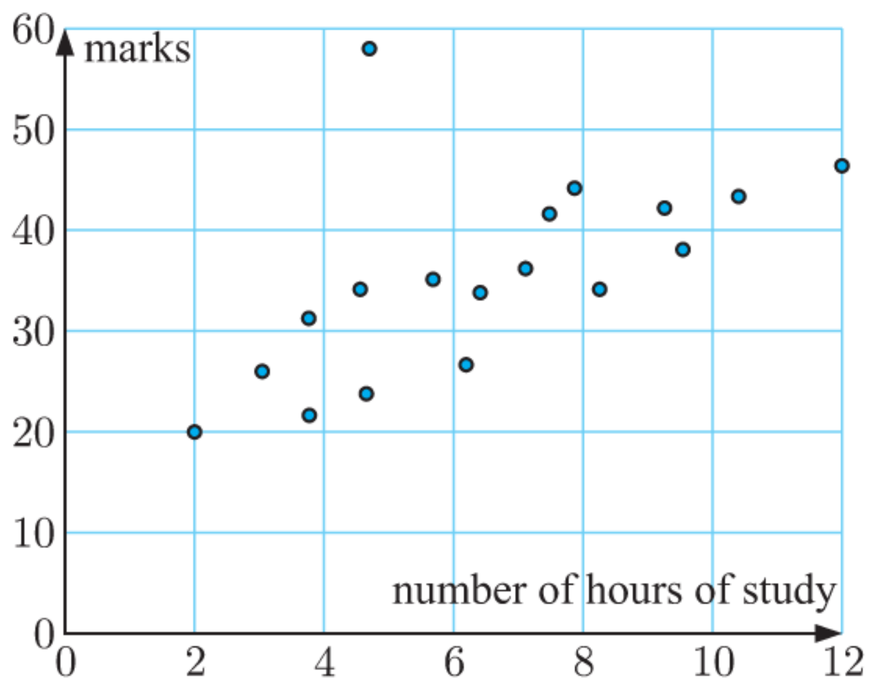


6 The scatter diagram shows the marks obtained by students in a test out of 50 marks, plotted against the number of hours each student studied for the test.

a Describe the correlation between the variables.

b How should the outlier be treated? Explain your answer.

c Do you think there is a causal relationship between the variables? Explain your answer.



7 When the following pairs of variables were measured, a strong, positive correlation was found between each pair. Discuss whether a causal relationship exists between the variables. If not, suggest a third variable to which they may both be related.

a The lengths of one's left and right feet.

b The damage caused by a fire and the number of firefighters who attend it.

c A company's expenditure on advertising, and the sales they make the following year.

d The heights of parents and the heights of their adult children.

e The number of hotels and the number of service stations in rural towns.

B PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT

In the previous Section, we classified the strength of the correlation between two variables as either strong, moderate, or weak. We observed the points on a scatter diagram, and judged how clearly the points formed a linear relationship.

Since this method is *subjective* and relies on the observer's opinion, it is important to get a more precise measure of the strength of linear correlation between the variables. We achieve this using **Pearson's product-moment correlation coefficient** r.

For a set of n data given as ordered pairs (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n),

Pearson's product-moment correlation coefficient is $$r = \frac{\sum(x - \overline{x})(y - \overline{y})}{\sqrt{\sum(x - \overline{x})^2 \sum(y - \overline{y})^2}}$$

where $\overline{x}$ and $\overline{y}$ are the means of the x and y data respectively, and $\sum$ denotes the sum over all the data values.

You are not required to learn this formula, but you should be able to calculate the value of r using technology.



GRAPHICS CALCULATOR INSTRUCTIONS

HISTORICAL NOTE

Karl Pearson (1857 - 1936) was an English statistician who developed the product-moment correlation coefficient together with his academic advisor **Sir Francis Galton**.

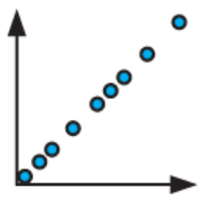
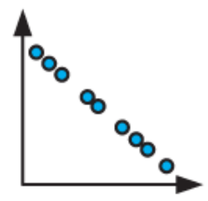
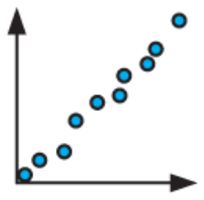
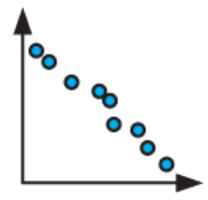
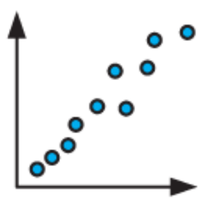
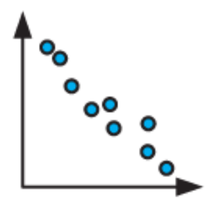
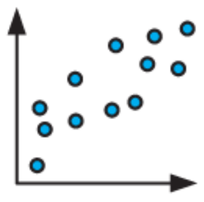
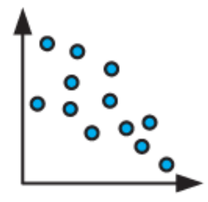
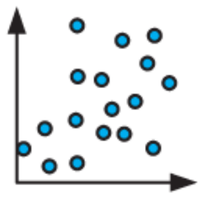
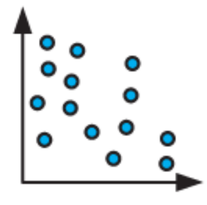
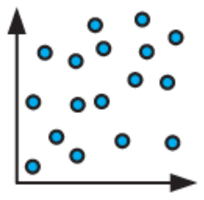
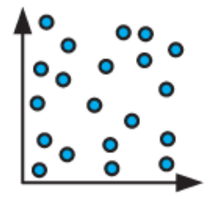
Pearson made many other contributions to statistics including the use of histograms in exploratory data analysis, parameter estimation, and hypothesis testing.

He is considered a key figure in the development of mathematical statistics.

PROPERTIES OF PEARSON'S PRODUCT-MOMENT CORRELATION COEFFICIENT

- The values of r range from -1 to $+1$.
- The **sign** of r indicates the **direction** of the correlation.
  - A positive value for r indicates the variables are **positively correlated**.
    An increase in one variable results in an increase in the other.
  - A negative value for r indicates the variables are **negatively correlated**.
    An increase in one variable results in a decrease in the other.
  - If $r = 0$ then there is **no correlation** between the variables.
- The **size** of r indicates the **strength** of the correlation.
  - A value of r close to -1 or $+1$ indicates strong correlation between the variables.
  - A value of r close to zero indicates weak correlation between the variables.

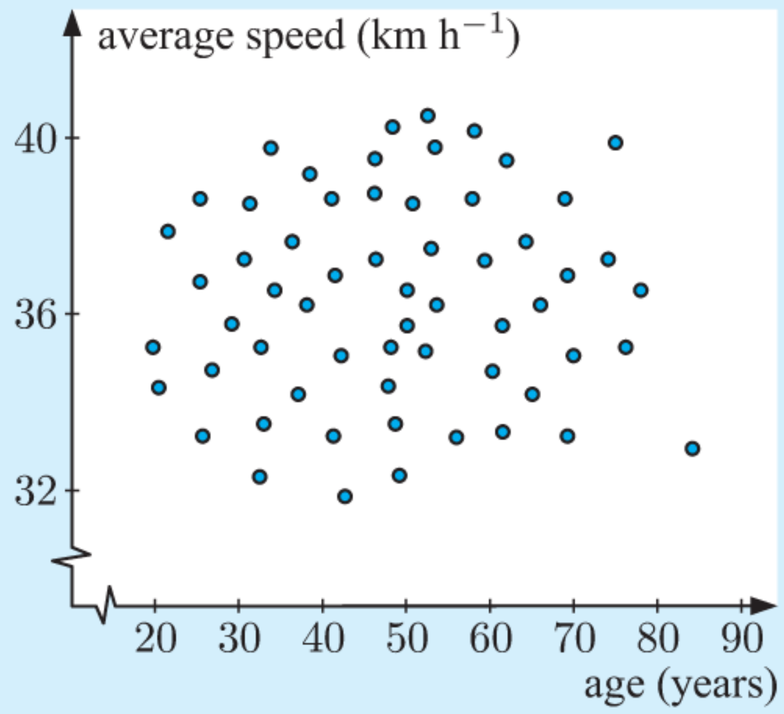
The following table is a guide for describing the strength of linear correlation using r.

Positive correlation		Negative correlation	
$r = 1$	perfect positive correlation 	$r = -1$	perfect negative correlation 
$0.95 \leqslant r < 1$	very strong positive correlation 	$-1 < r \leqslant -0.95$	very strong negative correlation 
$0.87 \leqslant r < 0.95$	strong positive correlation 	$-0.95 < r \leqslant -0.87$	strong negative correlation 
$0.7 \leqslant r < 0.87$	moderate positive correlation 	$-0.87 < r \leqslant -0.7$	moderate negative correlation 
$0.5 \leqslant r < 0.7$	weak positive correlation 	$-0.7 < r \leqslant -0.5$	weak negative correlation 
$0 < r < 0.5$	very weak positive correlation 	$-0.5 < r < 0$	very weak negative correlation 

Example 1 **Self Tutor**

The Department of Road Safety wants to know if there is any association between *average speed* in the metropolitan area and the *age of drivers*. They commission a device to be fitted in the cars of drivers of different ages.

The results are shown in the scatter diagram.
The r-value for this association is $+0.027$.
Describe the association.



Since $0 < r < 0.5$, there is a very weak positive correlation between the two variables.
We observe this in the graph as the points are randomly scattered.

Example 2 **Self Tutor**

The botanical gardens have been trying a new chemical to control the number of beetles infesting their plants. The results of one of their tests are shown in the table.

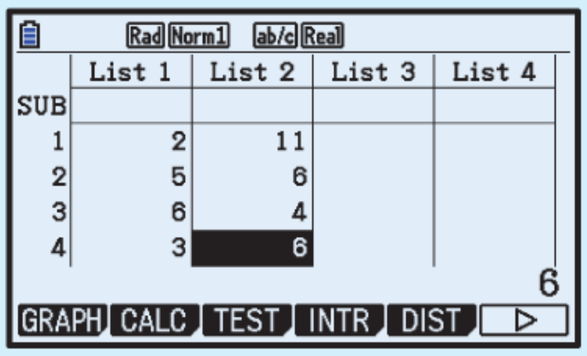
Sample	*Quantity of chemical* (g)	*Number of surviving beetles*
A	2	11
B	5	6
C	6	4
D	3	6
E	9	3

a Draw a scatter diagram for the data. **b** Determine the correlation coefficient r.

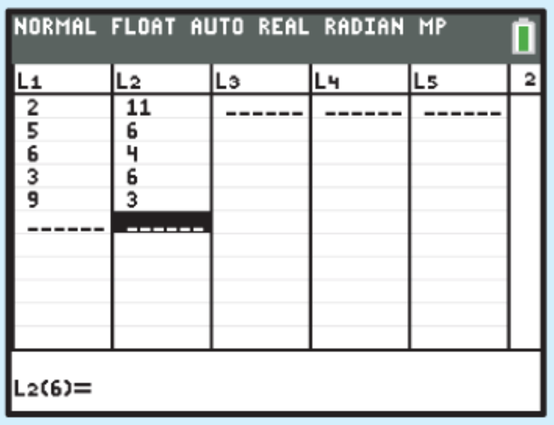
c Describe the correlation between the *quantity of chemical* and the *number of surviving beetles*.

We first enter the data into separate lists:

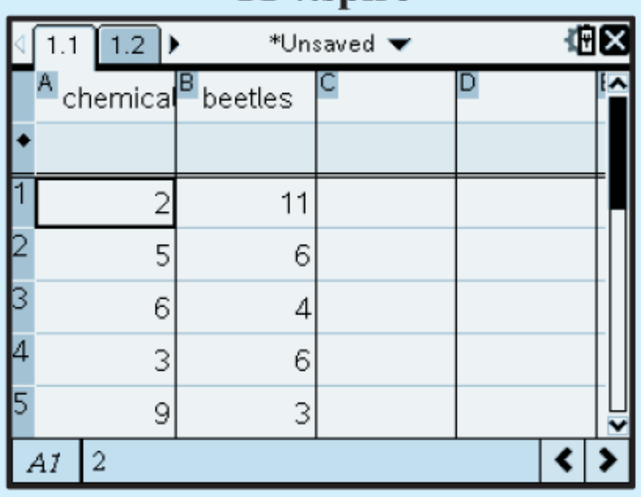
Casio fx-CG50



TI-84 Plus CE

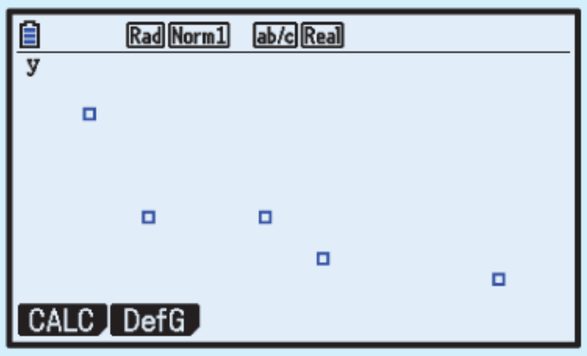


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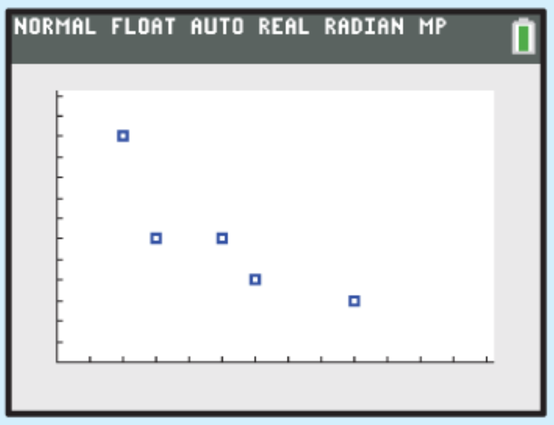


a

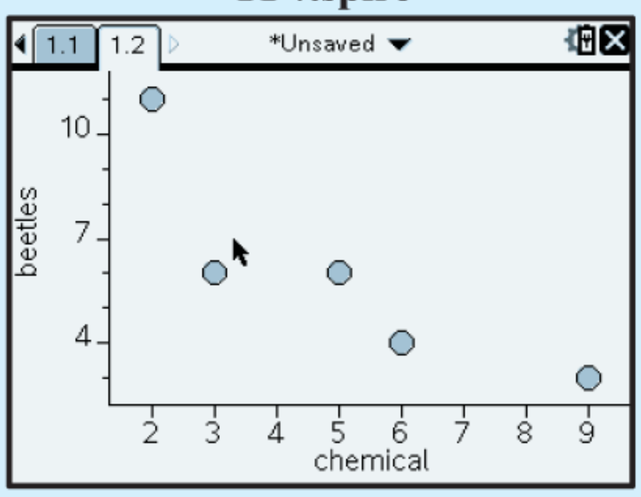
Casio fx-CG50



TI-84 Plus CE

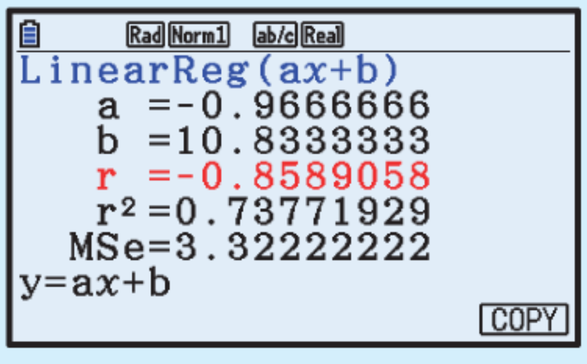


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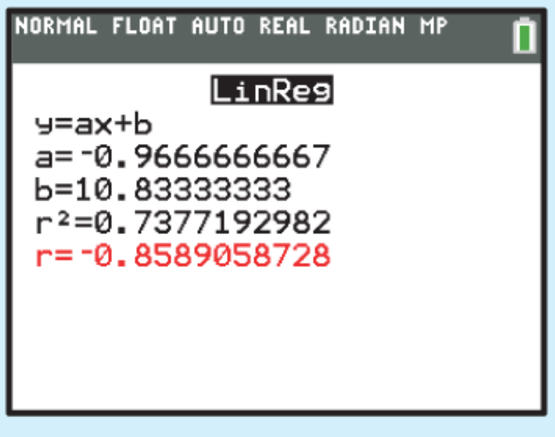


b

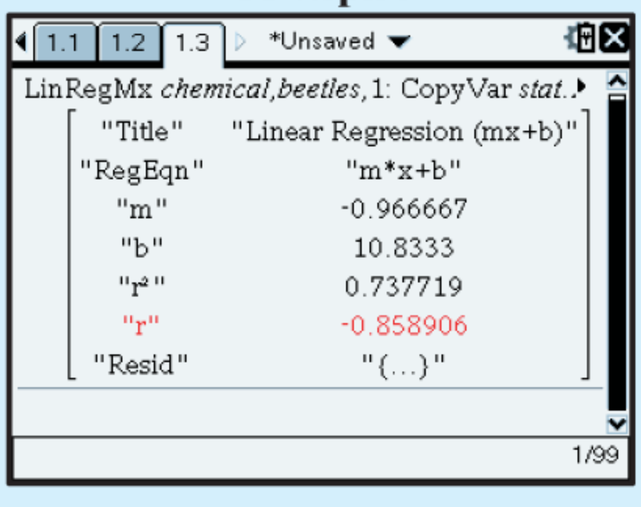
Casio fx-CG50



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So, $r \approx -0.859$.

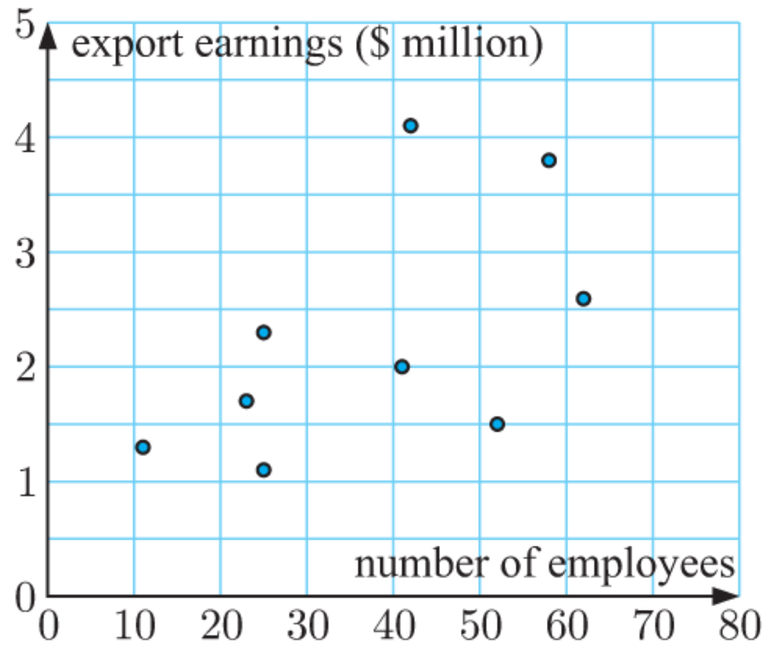
c There is a moderate negative correlation between the *quantity of chemical used* and the *number of surviving beetles*.

In general, the more chemical that is used, the fewer beetles that survive.

EXERCISE 7B

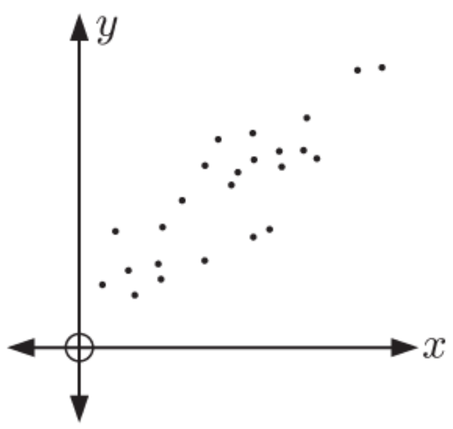
1 In a recent survey, the Department of International Commerce compared the *number of employees of a company* with its *export earnings*. A scatter diagram of their data is shown alongside. The corresponding value of r is 0.556.

Describe the association between the variables.

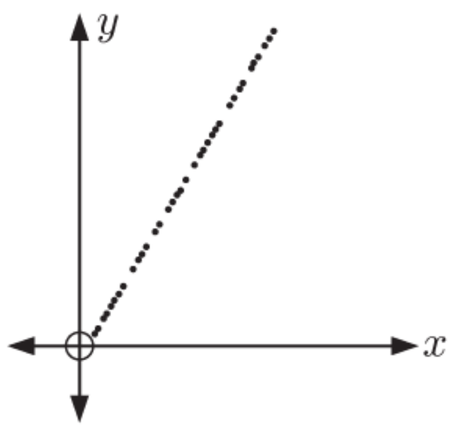


2 Match each scatter diagram with the correct value of r.

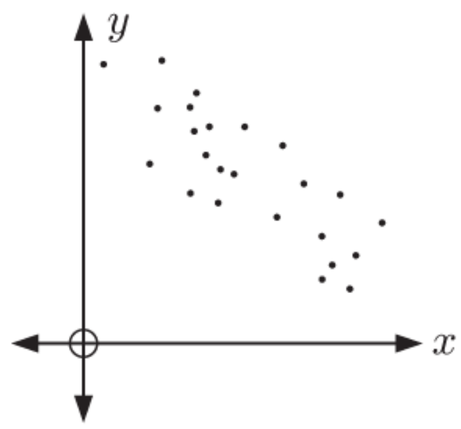
a



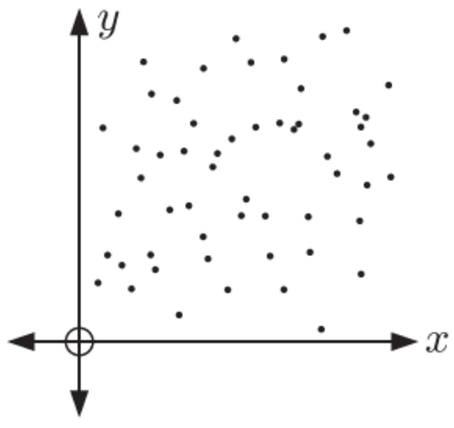
b



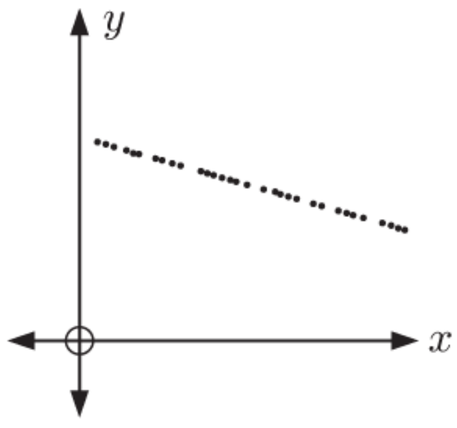
c



d



e



A $r = 1$ **B** $r = 0.6$ **C** $r = 0$ **D** $r = -0.7$ **E** $r = -1$

3 For each of the following data sets:

i Draw a scatter diagram for the data.

ii Calculate Pearson's product-moment correlation coefficient r.

iii Describe the linear correlation between x and y.

a

x	1	2	3	4	5	6
y	3	2	5	5	9	6

b

x	3	8	5	14	19	10	16
y	17	12	15	6	1	10	4

c

x	3	6	11	7	5	6	8	10	4
y	2	8	8	4	7	9	11	1	5

4 A selection of students were asked how many phone calls and text messages they received the previous day. The results are shown alongside.

Student	A	B	C	D	E	F	G	H
Phone calls received	4	7	1	0	3	2	2	4
Text messages received	6	9	2	2	5	8	4	7

a Draw a scatter diagram for the data. **b** Calculate r.

c Describe the linear correlation between *phone calls received* and *text messages received*.

d Give a reason why this correlation may occur.

5 Consider the **Opening Problem** on page **136**.

a Calculate r for the data.

b Hence describe the association between the variables.

6 Jill does her washing every Saturday and hangs her clothes out to dry. She notices that the clothes dry faster some days than others. She investigates the relationship between the temperature and the time her clothes take to dry:

Temperature (x °C)	25	32	27	39	35	24	30	36	29	35
Drying time (y minutes)	100	70	95	25	38	105	70	35	75	40

a Draw a scatter diagram for the data.

b Calculate r.

c Describe the correlation between *temperature* and *drying time*.

7 This table shows the number of supermarkets in 10 towns, and the number of car accidents that have occurred in these towns in the last month.

Number of supermarkets	5	8	12	7	6	2	15	10	7	3
Number of car accidents	10	13	27	19	10	6	40	30	22	37

a Draw a scatter diagram for the data.

b Calculate r.

c Identify the outlier in the data.

d It was found that the outlier was due to an error in the data collection process.

i Recalculate r with the outlier removed.

ii Describe the relationship between the variables.

iii Discuss the effect of removing the outlier on the value of r.

e Do you think there is a causal relationship between the variables? Explain your answer.

8 A health researcher notices that the incidence of Multiple Sclerosis (MS) is higher in some parts of the world than in others.

To investigate further, she records the *latitude* and *incidence of MS per 100 000 people* of 20 countries.

Latitude (degrees)	55	25	41	22	47	37	56	14	34	25
MS incidence per 100 000	165	95	75	20	180	140	230	15	45	65

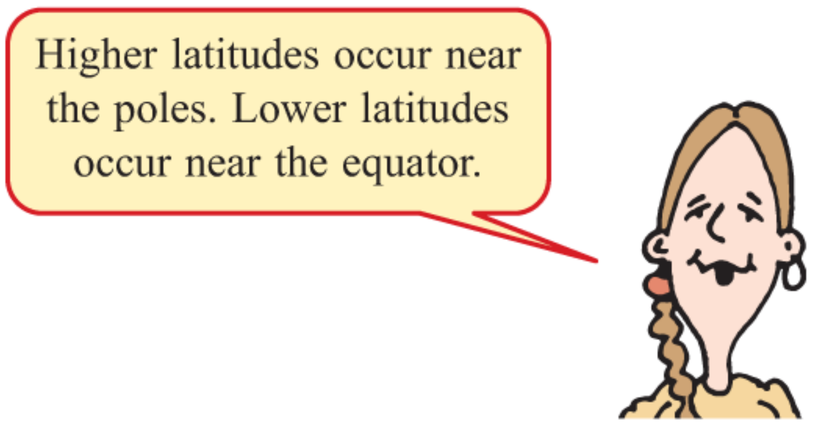
Latitude (degrees)	27	65	10	24	4	56	46	8	50	40
MS incidence per 100 000	30	140	5	15	2	290	95	8	160	105

a Draw a scatter diagram for the data.

b Calculate the value of r.

c Describe the relationship between the variables.

d Is the incidence of MS higher near the equator, or near the poles?



ACTIVITY 1 COMPARING HEIGHT AND FOOT LENGTH

In this Activity, you will explore the relationship between the *height* and *foot length* of the students in your class.

You will need: ruler, tape measure

What to do:

1. Predict whether there will be positive correlation, no correlation, or negative correlation between the *height* and *foot length* of the students in your class.
2. Measure the height and foot length of each student in your class. Record your measurements in a table like the one alongside.
3. Use technology to draw a scatter diagram for the data.

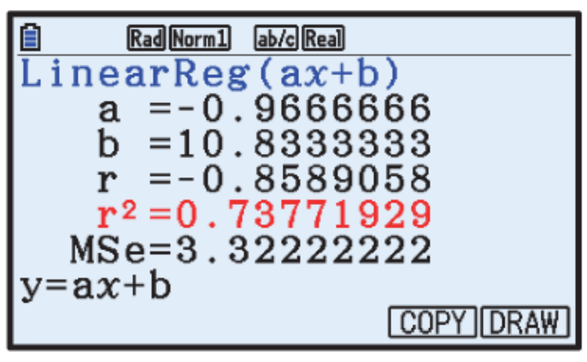
Student	*Height* (cm)	*Foot length* (cm)

4. Calculate Pearson's product-moment correlation coefficient r for the data.
5. Describe the relationship between *height* and *foot length*. Was your prediction correct?
6. Do you think that a high value of r indicates a causal relationship in this case?

C THE COEFFICIENT OF DETERMINATION

To help describe the correlation between two variables, we can also calculate the **coefficient of determination**, r^2. This is simply the square of Pearson's product-moment correlation coefficient r, so the direction of correlation is eliminated.

We can find r^2 using technology, or if r is already known, we can simply square this value.



INTERPRETATION OF THE COEFFICIENT OF DETERMINATION

If there is a causal relationship, then r^2 indicates the degree to which change in the independent variable explains change in the dependent variable.

For example, an investigation into many different brands of muesli found that there is strong positive correlation between the variables *fat content* and *kilojoule content*. It was found that $r \approx 0.862$ and $r^2 \approx 0.743$.

An interpretation of this r^2 value is:

dependent or response variable — independent or explanatory variable

74.3% of the variation in *kilojoule content* of muesli can be explained by the variation in *fat content* in the muesli.

In this case, we assume that the other $100\% - 74.3\% = 25.7\%$ of the variation in *kilojoule content* in the muesli can be explained by other factors.

Example 3 **Self Tutor**

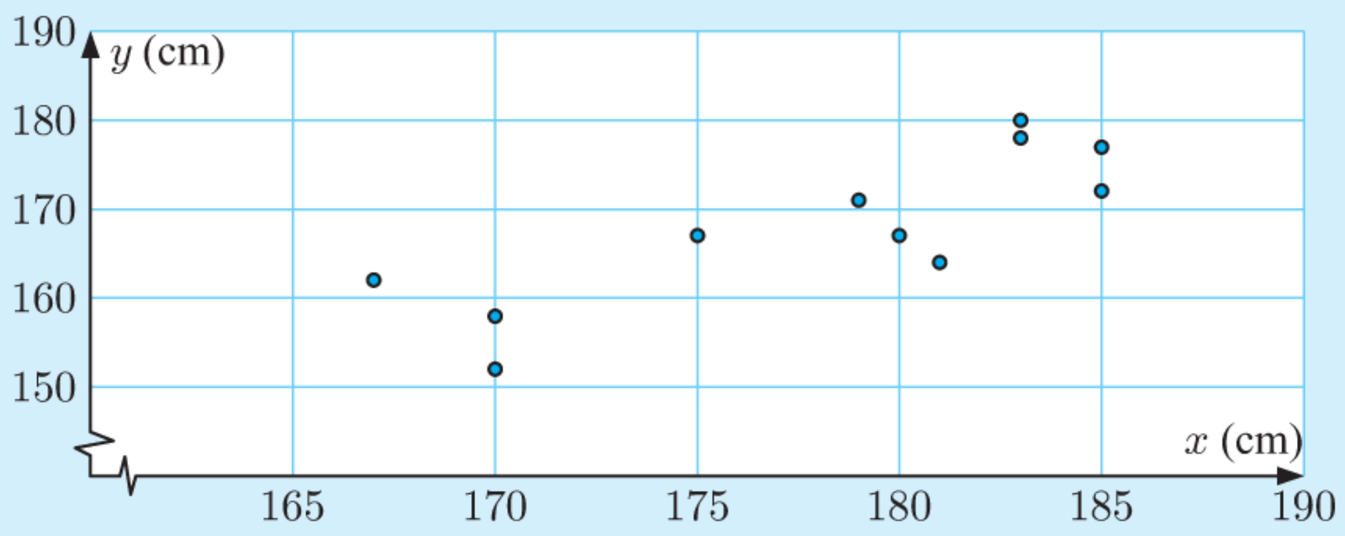
At a father-son camp, the heights of the fathers and their sons were measured.

Father's height (x cm)	175	183	170	167	179	180	183	185	170	181	185
Son's height (y cm)	167	178	158	162	171	167	180	177	152	164	172

a Draw a scatter diagram for the data.

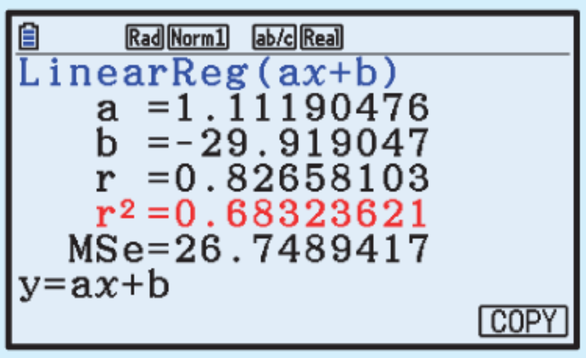
b Calculate r^2 for the data, and interpret its value.

a

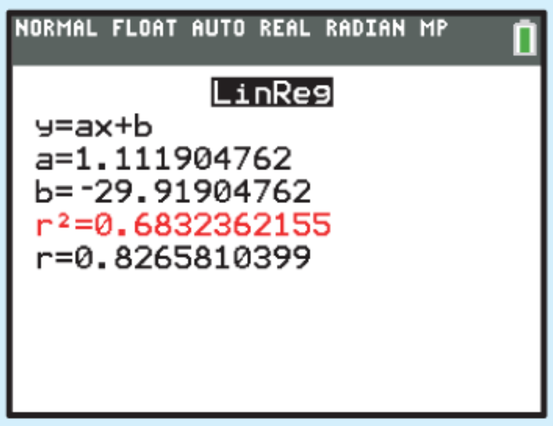


b

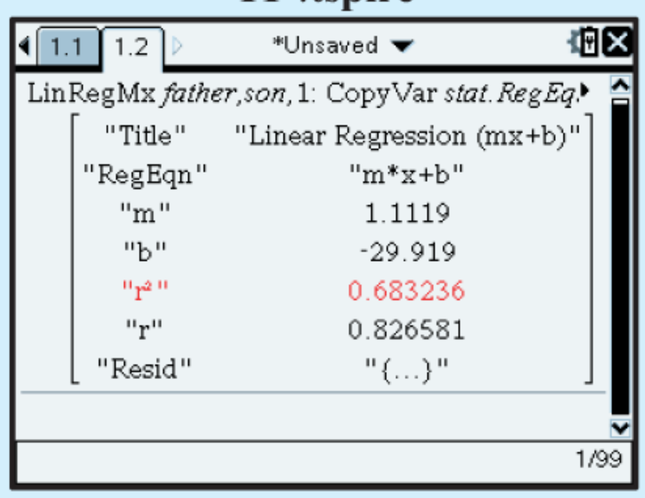
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So, $r^2 \approx 0.683$.

$\therefore$ 68.3% of the variation in the son's height can be explained by variation in the father's height.

EXERCISE 7C

1 From an investigation at an aquatic centre, the coefficient of determination between the variables *number of visitors* and *maximum temperature* is found to be 0.578. Complete the following interpretation of the coefficient of determination:

..... % of the variation in the ..... can be explained by the variation in .....

2 An investigation has found the association between the variables *time spent gambling* and *money lost* has correlation coefficient 0.7732. Find the coefficient of determination and interpret its meaning.

3 For a group of children, the correlation coefficient -0.365 is found between the variables *heart rate* and *age*. Find the coefficient of determination and interpret its meaning.

4 For a particular species of bear, the variables *weight* and *running speed* are found to be negatively correlated with coefficient of determination $r^2 = 0.821$. Find the correlation coefficient r.

5 Joanne is a real estate agent. This table shows the *distance from the city* and the *selling price* of the last 12 houses she has sold.

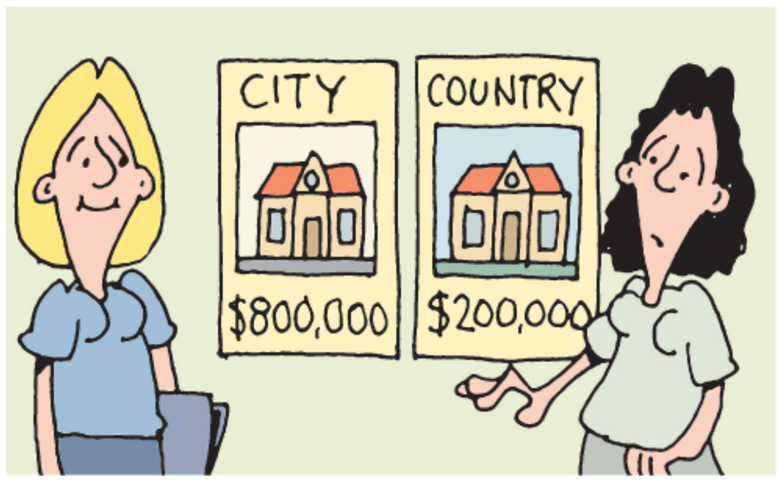
Distance from city (km)	10	4	23	16	3	35	8	7	12	24	14	12
Selling price ($\$ \times 1000$)	380	495	350	420	540	260	480	340	350	310	470	350

a Draw a scatter diagram for the data.

b Calculate the coefficient of determination r^2.

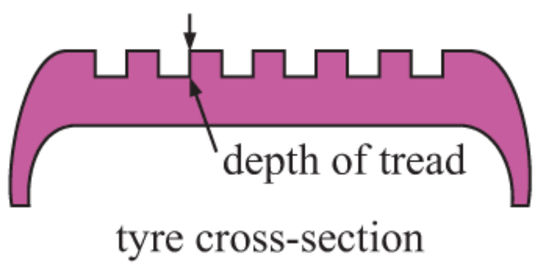
c What percentage of the variation in selling price can be explained by the variation in the distance from the city?

d What other factors could explain the variation in selling price?



6 A sample of 8 tyres was taken to examine the association between the *tread depth* and the *distance travelled*.

Distance (x km $\times$ 1000)	14	17	24	34	35	37	38	39
Tread depth (y mm)	5.7	6.5	4.0	3.0	1.9	2.7	1.9	2.3



tyre cross-section

a Draw a scatter diagram for the data.

b Calculate r^2 for the data, and interpret its meaning.

D LINE OF BEST FIT BY EYE

If there is a sufficiently strong linear correlation between two variables, we can draw a line of best fit to illustrate their relationship. In general, it is only worth drawing a line of best fit if the coefficient of determination $r^2 \geqslant 0.7$.

If we draw the line just by observing the points, we call it a **line of best fit by eye**. This line will vary from person to person.

We draw a line of best fit connecting variables x and y as follows:

Step 1: Calculate the mean $\overline{x}$ of the x-values, and the mean $\overline{y}$ of the y-values.

Step 2: Mark the **mean point** $(\overline{x}, \overline{y})$ on the scatter diagram.

Step 3: Draw a line through the mean point which fits the trend of the data, and so that about the same number of data points are above the line as below it.

Consider again the data from the **Opening Problem**:

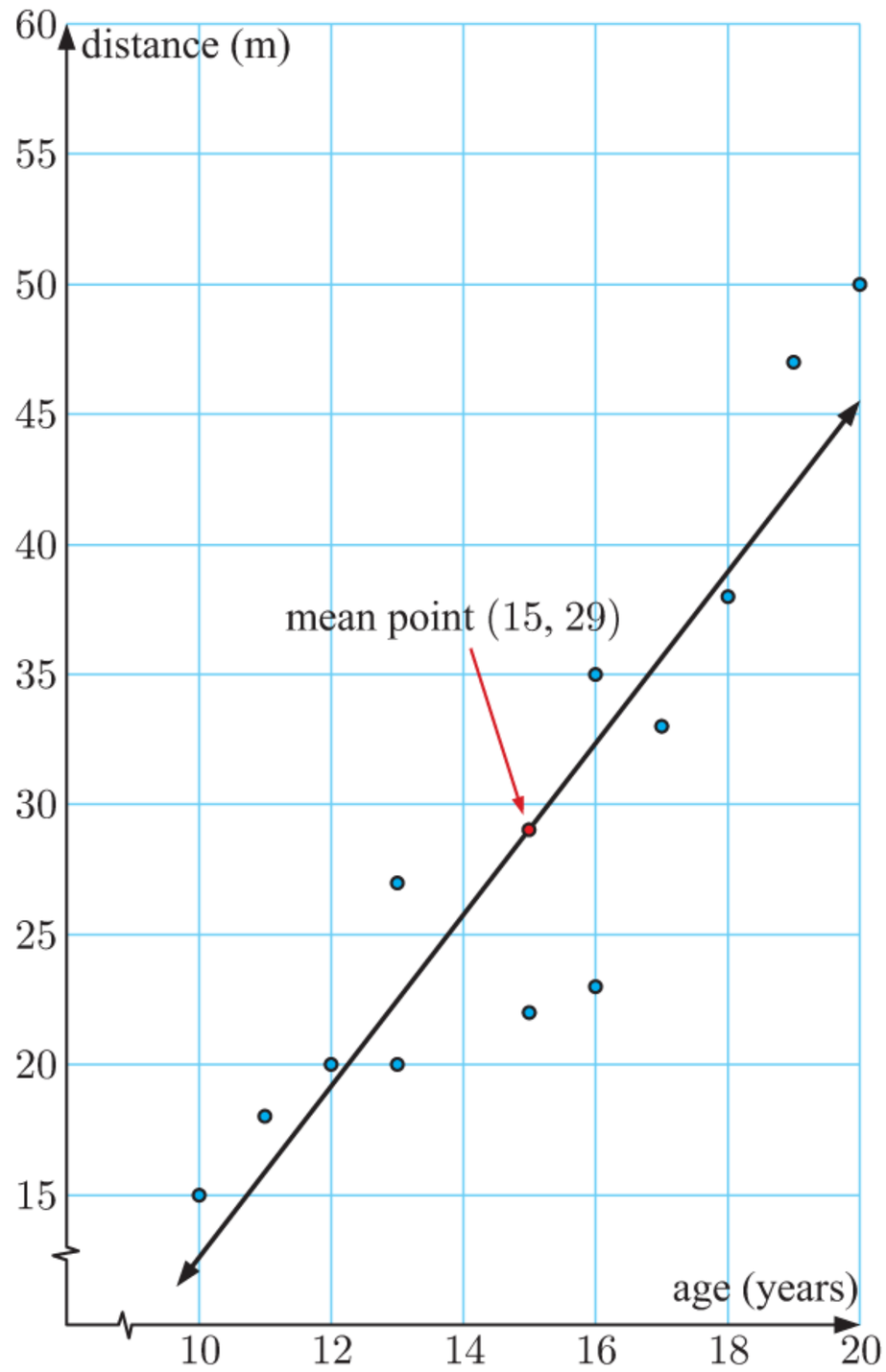
Athlete	A	B	C	D	E	F	G	H	I	J	K	L
Age (years)	12	16	16	18	13	19	11	10	20	17	15	13
Distance thrown (m)	20	35	23	38	27	47	18	15	50	33	22	20

We have seen that there is a strong, positive linear correlation between *age* and *distance thrown*.

We can therefore model the data using a line of best fit.

The mean age is 15 years and the mean distance thrown is 29 m. We therefore draw our line of best fit through the mean point $(15, 29)$.

We can use the line of best fit to estimate the value of y for any given value of x, and vice versa.



We draw the line through the mean point so it follows the trend of the data and there are about the same number of points above the line as below the line.

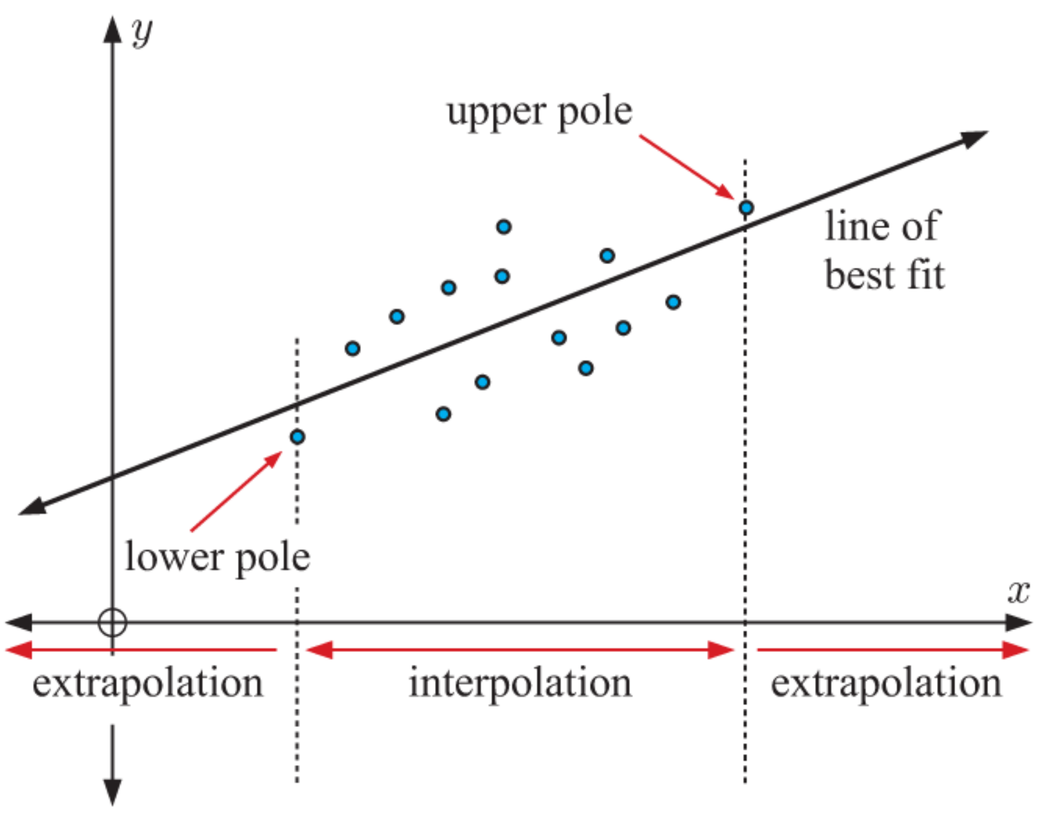


INTERPOLATION AND EXTRAPOLATION

Consider the data in the scatter diagram alongside. The data with the highest and lowest x-values are called the **poles**.

The line of best fit for the data is also drawn on the scatter diagram. We can use this line to predict the value of one variable for a given value of the other.

- If we predict a y-value for an x-value **in between** the poles, we say we are **interpolating** in between the poles.
- If we predict a y-value for an x-value **outside** the poles, we say we are **extrapolating** outside the poles.



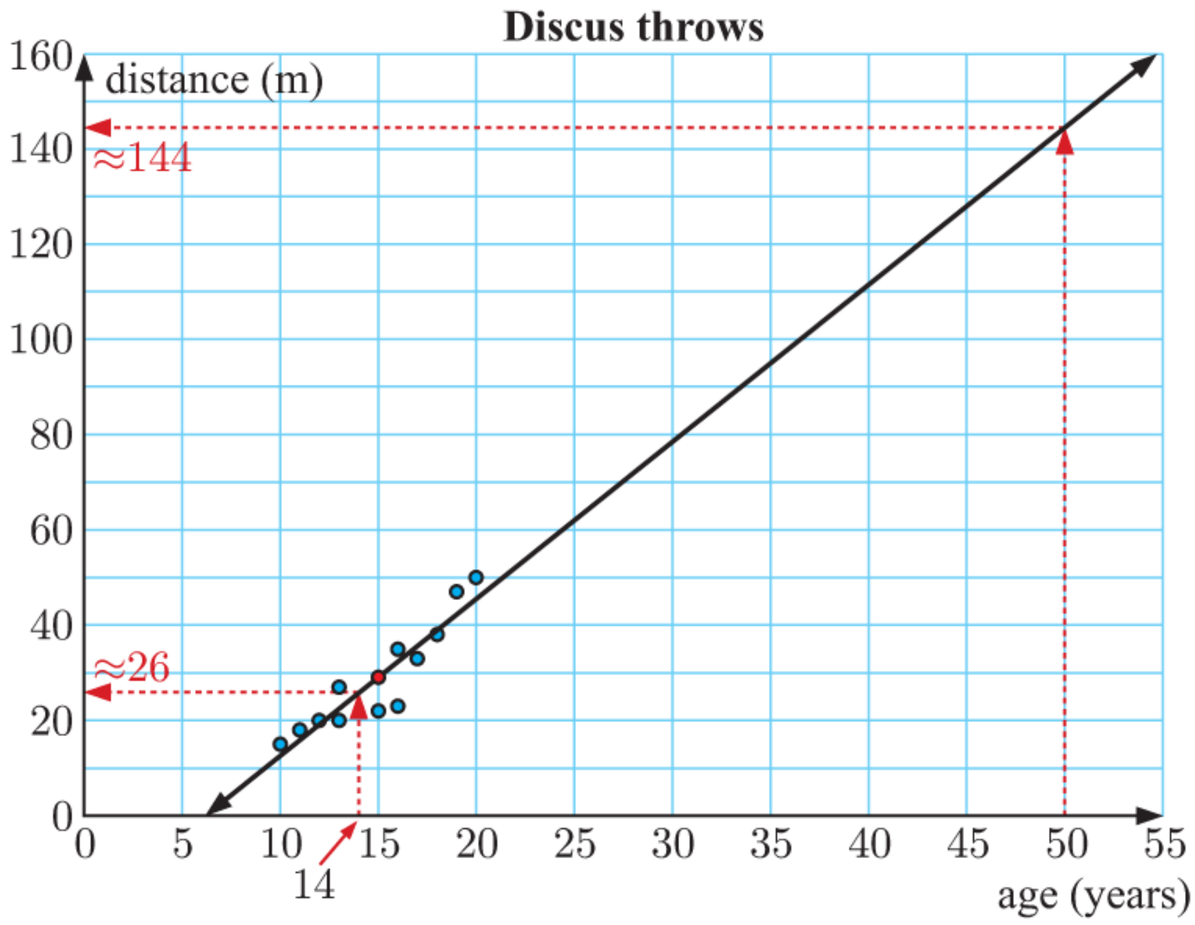
The accuracy of an interpolation depends on how well the linear model fits the data. This can be gauged by the correlation coefficient and by ensuring that the data is randomly scattered around the line of best fit.

The accuracy of an extrapolation depends not only on how well the model fits, but also on the assumption that the linear trend will continue past the poles. The validity of this assumption depends greatly on the situation we are looking at.

For example, consider the line of best fit for the data in the **Opening Problem**. It can be used to predict the distance a discus will be thrown by an athlete of a particular age.

The age 14 is within the range of ages in the original data, so it is reasonable to predict that a 14 year old will be able to throw the discus 26 m.

However, it is unlikely that the linear trend shown in the data will continue far beyond the poles. For example, according to the model, a 50 year old might throw the discus 144 m. This is almost twice the current world record of 76.8 m, so it would clearly be an unreasonable prediction.



Example 4

Self Tutor

On a hot day, six cars were left in the sun in a car park. The length of time each car was left in the sun was recorded, as well as the temperature inside the car at the end of the period.

Car	A	B	C	D	E	F
Time (x minutes)	50	5	25	40	15	45
Temperature (y °C)	47	28	36	42	34	41

a Calculate $\overline{x}$ and $\overline{y}$.

b Draw a scatter diagram for the data.

c Locate the mean point $(\overline{x}, \overline{y})$ on the scatter diagram, then draw a line of best fit through this point.

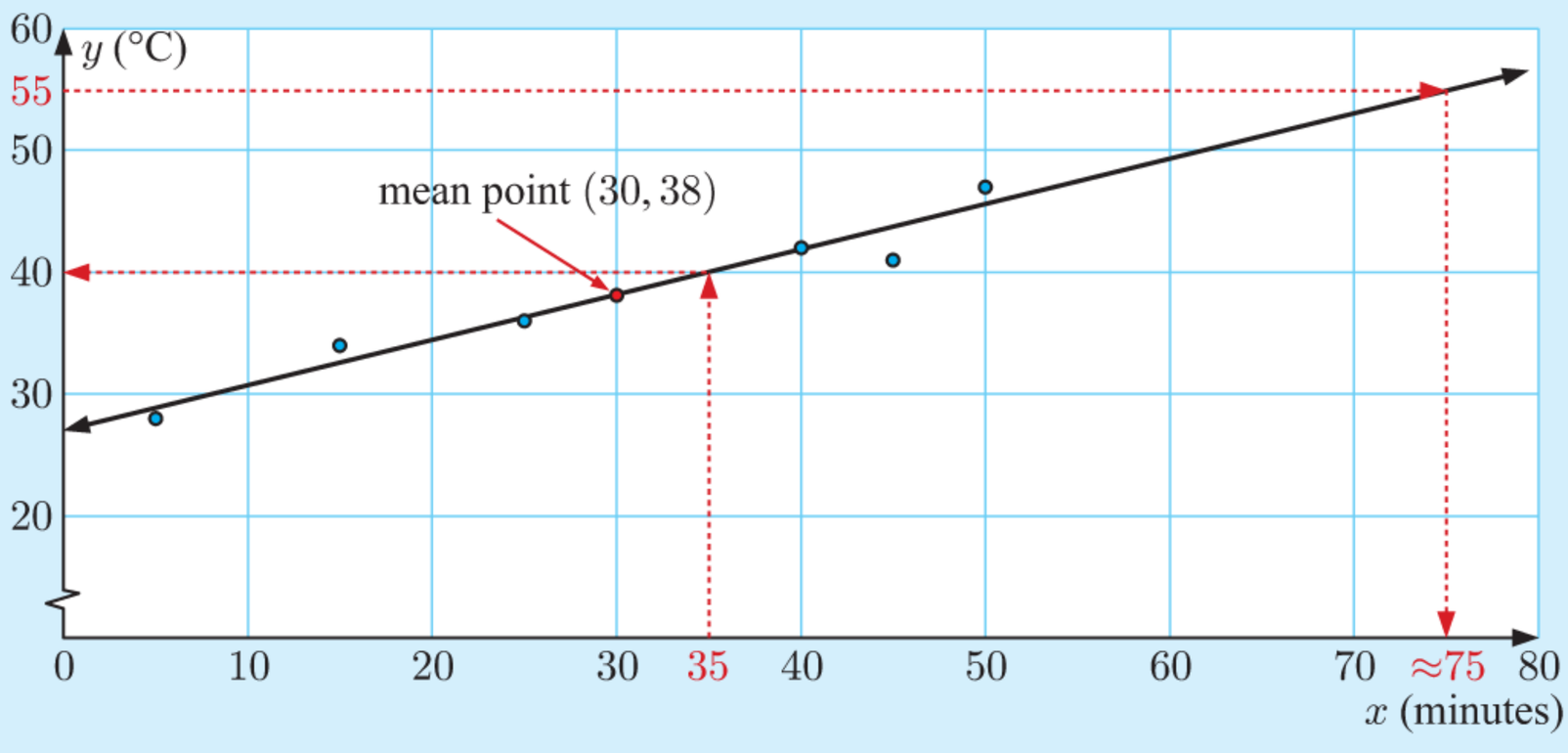
d Predict the temperature of a car which has been left in the sun for 35 minutes.

e Predict how long it would take for a car's temperature to reach 55°C.

f Comment on the reliability of your predictions in **d** and **e**.

a $\overline{x} = \dfrac{50+5+25+40+15+45}{6} = 30, \qquad \overline{y} = \dfrac{47+28+36+42+34+41}{6} = 38$

b, c



d When $x = 35$, $y \approx 40$.
The temperature of a car left in the sun for 35 minutes will be approximately $40°$C.

e When $y = 55$, $x \approx 75$.
It would take approximately 75 minutes for a car's temperature to reach $55°$C.

f The prediction in **d** is reliable, as the data appears linear, and this is an interpolation.
The prediction in **e** may be unreliable, as it is an extrapolation and the linear trend displayed by the data may not continue beyond the 50 minute mark.

EXERCISE 7D

1 Consider the data set alongside.

x	5	12	20	17	10	8	25	15
y	28	19	4	18	22	20	7	10

a Draw a scatter diagram for the data.
b Does the data appear to be positively or negatively correlated?
c Calculate r for the data.
d Describe the strength of the relationship between x and y.
e Calculate the mean point $(\overline{x}, \overline{y})$.
f Locate the mean point, then use it in drawing a line of best fit.
g Estimate the value of y when $x = 22$.

2 Fifteen students were weighed and their pulse rates were measured:

Weight (x kg)	46	37	32	57	47	64	42	30	52	56	65	43	36	28	40
Pulse rate (y beats per min)	65	59	54	74	69	87	61	59	70	69	75	60	56	53	58

a Draw a scatter diagram for the data. **b** Calculate r.
c Describe the relationship between *weight* and *pulse rate*.
d Calculate the mean point $(\overline{x}, \overline{y})$.
e Locate the mean point on the scatter diagram, then use it in drawing a line of best fit.
f Estimate the pulse rate of a 50 kg student. Comment on the reliability of your estimate.

3 The trunk widths and heights of the trees in a garden are given below:

Trunk width (x cm)	35	47	72	40	15	87	20	66	57	24	32
Height (y m)	11	18	24	12	3	30	22	21	17	5	10

a Draw a scatter diagram for the data.
b Which of the points is an outlier?
c How would you describe the tree represented by the outlier?
d Calculate the mean point $(\overline{x}, \overline{y})$.
e Locate the mean point on the scatter diagram, then draw a line of best fit through the mean point.
f Predict the height of a tree with trunk width 120 cm. Comment on the reliability of your prediction.
g Predict the trunk width of a tree with height 10 m. Comment on the reliability of your prediction.



E THE LEAST SQUARES REGRESSION LINE

The problem with drawing a line of best fit by eye is that the line drawn will vary from one person to another. For consistency, we use a method known as **linear regression** to find the equation of the line which best fits the data. The most common method is the method of "**least squares**".

RESIDUALS

When we fitted a line of best fit by eye, we tried to minimise the distances between the line and each data point. We can formally define each of these distances as a *residual*:

The **residual** of the ith data point (x_i, y_i) is

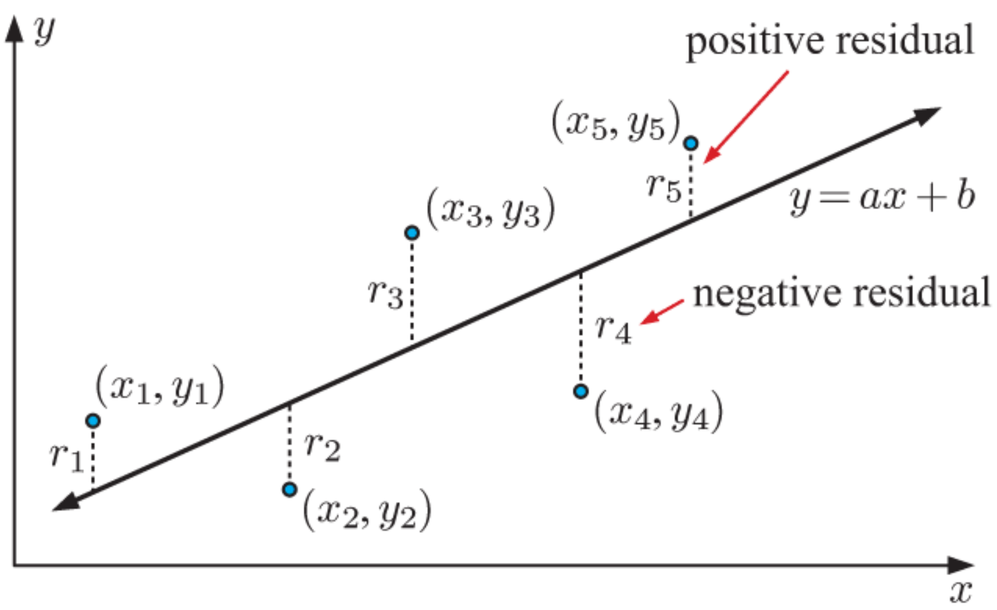
$$r_i = y_i - \hat{y}_i$$

where $\hat{y}_i$ is the predicted value of y at $x = x_i$.

THE LEAST SQUARES REGRESSION LINE

The goal of least squares linear regression is to minimise the **sum of the squared residuals**.

$$SS_{\text{res}} = \sum_{i=1}^{n} r_i^2$$
$$= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$



If our least squares regression line is $y = ax + b$, we need to choose a and b to minimise $SS_{\text{res}} = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$.

The required values of a and b are:

$$a = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sum(x_i - \overline{x})^2} \quad \text{and} \quad b = \overline{y} - a\overline{x}$$

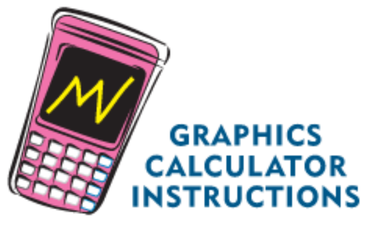
Click on the icon to see how these formulae are derived.



In this course you will not be required to find the equation of the least squares regression line by hand.

Instead, you can use your **graphics calculator** or the **statistics package**.





Example 5

Self Tutor

The annual income and average weekly grocery bill for a selection of families is shown below:

Income (x thousand pounds)	55	36	25	47	60	64	42	50
Grocery bill (y pounds)	120	90	60	160	190	250	110	150

a Construct a scatter diagram to illustrate the data.

b Use technology to find the equation of the regression line.

c State and interpret the gradient of the regression line.

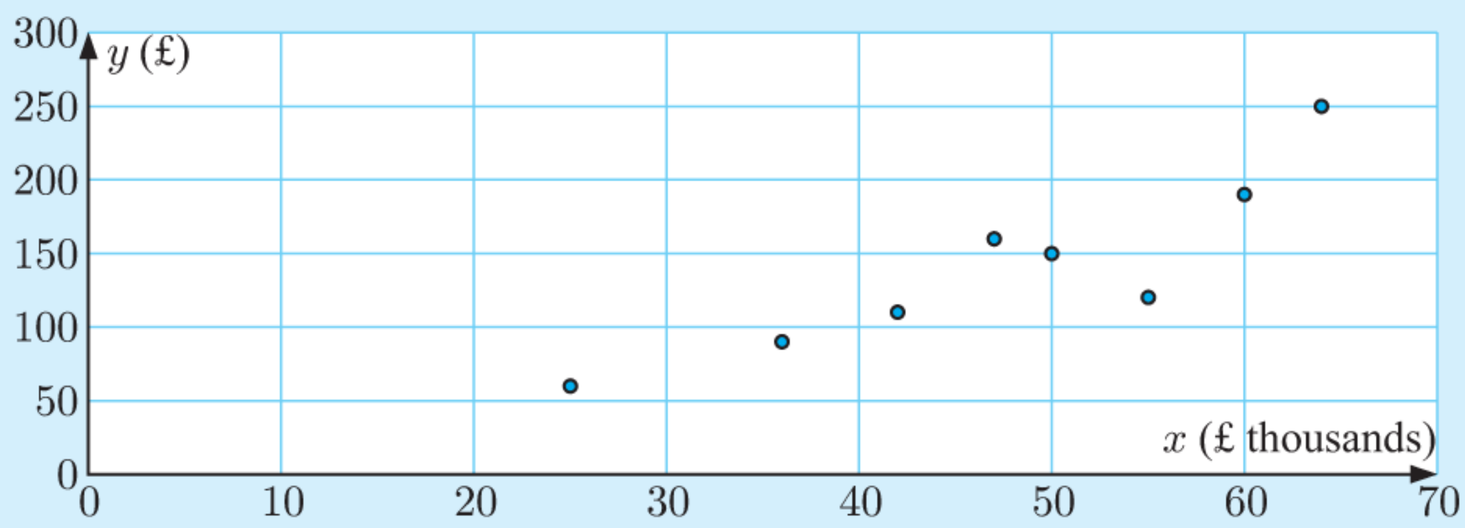
d Estimate the weekly grocery bill for a family with an annual income of £95 000.

e Estimate the annual income of a family whose weekly grocery bill is £100.

f Comment on whether the estimates in **d** and **e** are likely to be reliable.

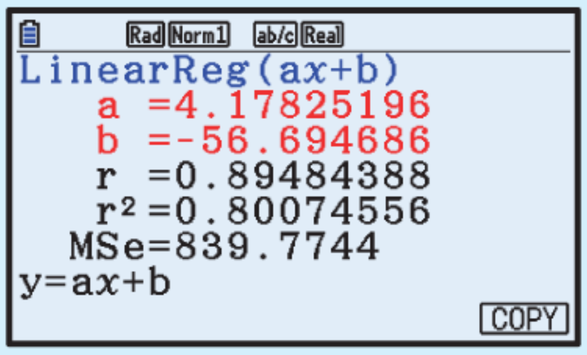
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a

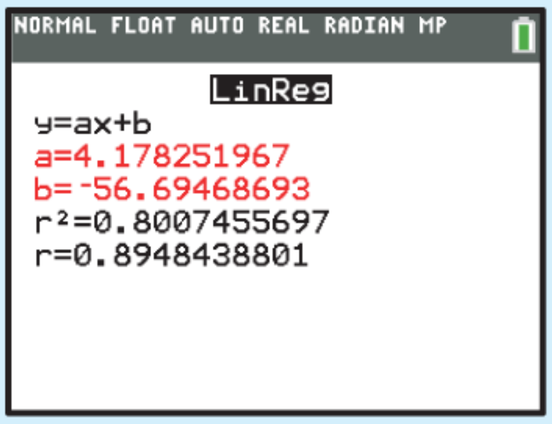


b

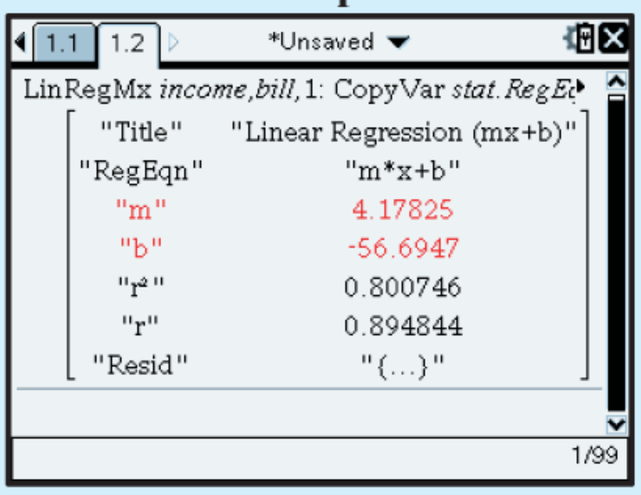
Casio fx-CG50



TI-84 Plus CE



TI-*n*spire



Using technology, the regression line is $y \approx 4.18x - 56.7$

c The gradient of the regression line ≈ 4.18. This means that for every additional £1000 of income, a family's weekly grocery bill will increase by an average of £4.18.

d When $x = 95$, $y \approx 4.18(95) - 56.7 \approx 340$

So, we expect a family with an income of £95 000 to have a weekly grocery bill of approximately £340.

e When $y = 100$, $100 \approx 4.18x - 56.7$

$\therefore \quad 156.7 \approx 4.18x$ {adding 56.7 to both sides}

$\therefore \quad x \approx 37.5$ {dividing both sides by 4.18}

So, we expect a family with a weekly grocery bill of £100 to have an annual income of approximately £37 500.

f The estimate in **d** is an extrapolation, so the estimate may not be reliable.
The estimate in **e** is an interpolation and there is strong linear correlation between the variables. We therefore expect this estimate to be reliable.

THE COEFFICIENT OF DETERMINATION

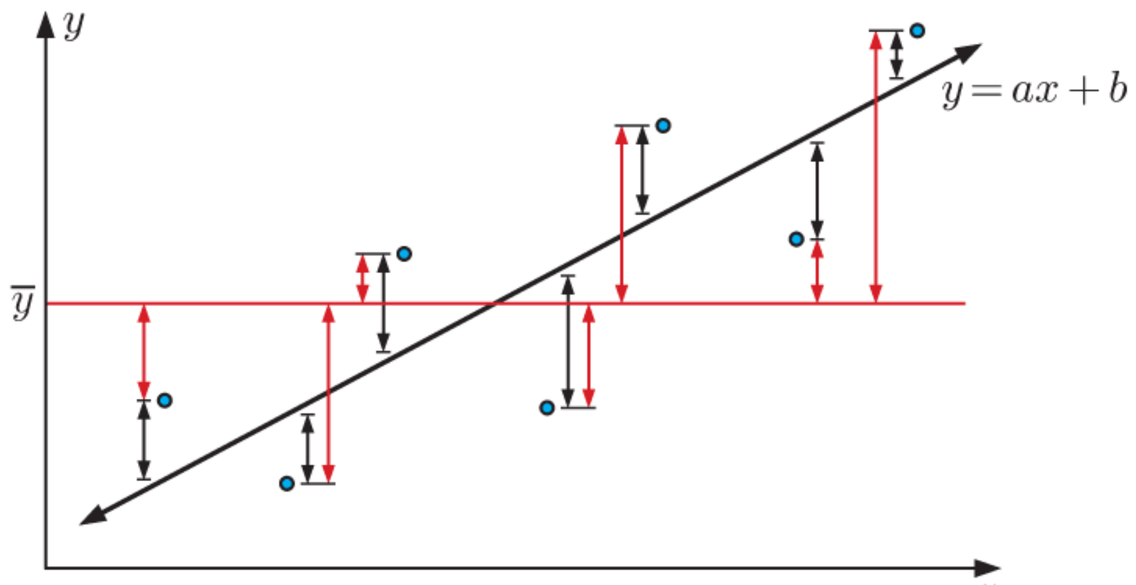
The graph alongside illustrates:

- the **residual sum of squares** (black)

$$SS_{\text{res}} = \sum_{i=1}^{n} (y_i - \widehat{y}_i)^2$$

- the **total sum of squares** (red)

$$SS_{\text{tot}} = \sum_{i=1}^{n} (y_i - \overline{y})^2$$



Notice that SS_{tot} is also the *variance* of the y-values, so SS_{tot} is the *total* variation in y.

SS_{res} is the variation in y that is not explained by the model.

So, the fraction $\dfrac{SS_{\text{res}}}{SS_{\text{tot}}}$ gives us the *proportion* of variation that is **not** explained by the model.

Thus, $1 - \dfrac{SS_{\text{res}}}{SS_{\text{tot}}}$ is the proportion of variation that **is** explained by the model.

In **Section C** we claimed that the coefficient of determination is the proportion of variation that is explained by the model. So, coefficient of determination $= 1 - \dfrac{SS_{\text{res}}}{SS_{\text{tot}}}$.

Click on the icon to see how this definiton of the coefficient of determination is equivalent to the value r^2, the square of Pearson's product-moment correlation coefficient.

COEFFICIENT OF DETERMINATION

The **coefficient of determination** is therefore defined as $\boldsymbol{r^2 = 1 - \dfrac{SS_{\text{res}}}{SS_{\text{tot}}}}$.

You should be able to use both r and r^2 to assess how well a linear model fits data.

DISCUSSION

Is it always true that $SS_{\text{res}} \leqslant SS_{\text{tot}}$?

EXERCISE 7E

1 Consider the data set below.

x	10	4	6	8	9	5	7	1	2	3
y	20	6	8	13	20	12	13	4	2	7

GRAPHICS CALCULATOR INSTRUCTIONS

a Draw a scatter diagram for the data.

b Use technology to find the equation of the regression line, and plot the line on your calculator.

c Use **b** to draw the regression line on your scatter diagram.

2 Steve wanted to see whether there was any relationship between the temperature when he leaves for work in the morning, and the time it takes for him to get to work.
He collected data over a 14 day period:

Temperature (x °C)	25	19	23	27	32	35	29	27	21	18	16	17	28	34
Time (y minutes)	35	42	49	31	37	33	31	47	42	36	45	33	48	39

a Draw a scatter diagram for the data.

b Calculate r and r^2.

c Describe the relationship between the variables.

d Is it reasonable to fit a linear model to this data? Explain your answer.

3 The table below shows the price of petrol and the number of customers per hour for sixteen petrol stations.

Petrol price (x cents per litre)	105.9	106.9	109.9	104.5	104.9	111.9	110.5	112.9
Number of customers (y)	45	42	25	48	43	15	19	10

Petrol price (x cents per litre)	107.5	108.0	104.9	102.9	110.9	106.9	105.5	109.5
Number of customers (y)	30	23	42	50	12	24	32	17

a Calculate Pearson's product-moment correlation coefficient and the coefficient of determination for the data.

b Describe the relationship between the *petrol price* and the *number of customers*.

c Use technology to find the equation of the regression line.

d Interpret r^2 and hence describe how well the linear model fits the data.

e State and interpret the gradient of the regression line.

f Estimate the number of customers per hour for a petrol station which sells petrol at 115.9 cents per litre.

g Estimate the petrol price at a petrol station which has 40 customers per hour.

h Comment on the reliability of your estimates in **f** and **g**.

4 To investigate whether speed cameras have an impact on road safety, data was collected from several cities. The number of speed cameras in operation was recorded for each city, as well as the number of accidents over a 7 day period.

Number of speed cameras (x)	7	15	20	3	16	17	28	17	24	25	20	5	16	25	15	19
Number of car accidents (y)	48	35	31	52	40	35	28	30	34	19	29	42	31	21	37	32

a Construct a scatter diagram to display the data.

b Calculate r and r^2 for the data.

c Describe the relationship between the *number of speed cameras* and the *number of car accidents*.

d Find the equation of the regression line.

e Interpret r^2 and hence describe how well the linear model fits the data.

f State and interpret the gradient and y-intercept of the regression line.

g Estimate the number of car accidents in a city with 10 speed cameras.

5 The table below contains information about the *maximum speed* and *ceiling* (maximum altitude obtainable) for nineteen World War II fighter planes. The maximum speed is given in km h^{-1}, and the ceiling is given in km.



Maximum speed	*Ceiling*
460	8.84
420	10.06
530	10.97
530	9.906
490	9.448
530	10.36
680	11.73

Maximum speed	*Ceiling*
680	10.66
720	11.27
710	12.64
660	11.12
780	12.80
730	11.88

Maximum speed	*Ceiling*
670	12.49
570	10.66
440	10.51
670	11.58
700	11.73
520	10.36

a Draw a scatter diagram for the data.

b Calculate r and r^2.

c Describe the association between *maximum speed* (x) and *ceiling* (y).

d Use technology to find the regression line, and draw the line on your scatter diagram.

e Describe how well the linear model fits the data.

f State and interpret the gradient of the regression line.

g Estimate the ceiling for a fighter plane with a maximum speed of 600 km h^{-1}.

h Estimate the maximum speed for a fighter plane with a ceiling of 11 km.

6 A group of children was asked the numbers of hours they spent exercising and watching television each week.

Exercise (x hours per week)	4	1	8	7	10	3	3	2
Television (y hours per week)	12	24	5	9	1	18	11	16

a Draw a scatter diagram for the data.

b Calculate r and r^2.

c Describe the correlation between *time exercising* and *time watching television*.

d Find the equation of the regression line, and draw the line on your scatter diagram.

e Describe how well the linear model fits the data.

f State and interpret the gradient and y-intercept of the regression line.

g **i** One of the children in the group exercised for 7 hours each week. How much television does this child watch weekly?

ii Use the regression line to predict the amount of television watched each week by a child who exercises for 7 hours each week.

iii Compare your answers to **i** and **ii**.

7 The yield of pumpkins on a farm depends on the quantity of fertiliser used.

Fertiliser (x g per m^2)	4	13	20	26	30	35	50
Yield (y kg)	1.8	2.9	3.8	4.2	4.7	5.7	4.4

a Draw a scatter diagram for the data, and identify the outlier.

b What effect do you think the outlier has on:

i the strength of correlation of the data **ii** the gradient of the regression line?

c Calculate the correlation coefficient and coefficient of determination:

i with the outlier included **ii** without the outlier.

d Calculate the equation of the regression line:

i with the outlier included **ii** without the outlier.

e If you wish to estimate the yield when 15 g per m^2 of fertiliser is used, which regression line from **d** should be used? Explain your answer.

f Can you explain what may have caused the outlier? Do you think the outlier should be kept when analysing the data?

ACTIVITY 2 FITTING A LINE THROUGH THE ORIGIN

Consider a bivariate data set $(x_1, y_1), ..., (x_n, y_n)$ where we *know* the model connecting the variables should pass through the origin. The model therefore has the form $y = \alpha x$.

What to do:

1 Show that the sum of the squared residuals for this model is

$$SS_{\text{res}} = \alpha^2 \sum_{i=1}^{n} (x_i{}^2) - 2\alpha \sum_{i=1}^{n} (x_i y_i) + \sum_{i=1}^{n} (y_i{}^2)$$

2 Hence show that SS_{res} is minimised when $\alpha = \dfrac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i{}^2}$.

3 Consider the data in the table below:

x	1	2	3	4	5	6	7	8	9	10
y	2.6	9.6	11.0	22.8	23.5	24.5	33.0	38.2	42.2	44.1

a Fit the zero-intercept model $y = \alpha x$ to the data.

b Fit the regular linear regression model $y = ax + b$ to the data.

c Compare the zero-intercept model with the regular linear regression model.

4 Explain how you might fit the following linear models to a data set:

a $y = ax + c$ where c is a known constant

b $y = ax + b + c$ where c is a known constant.

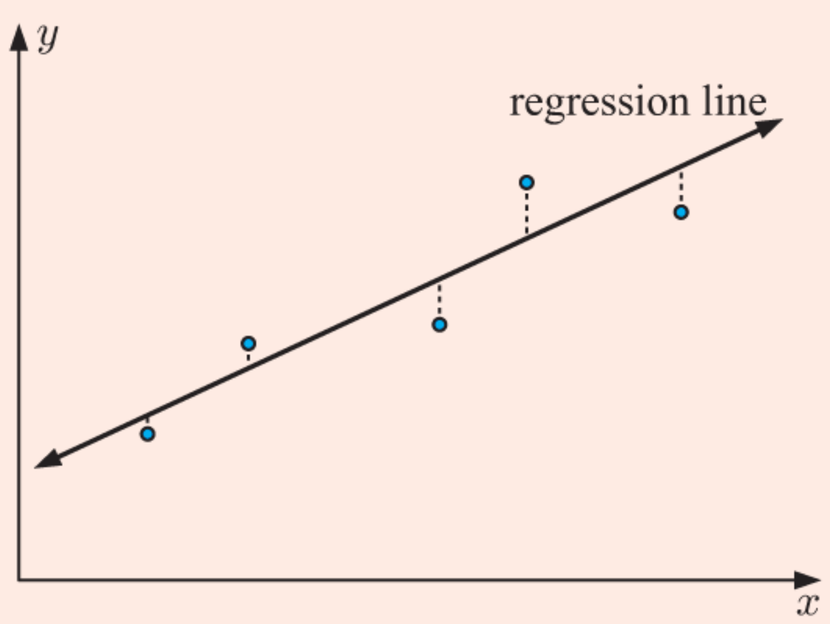
ACTIVITY 3 RESIDUAL PLOTS

In addition to the *correlation coefficient* and the *linearity* of a scatter diagram, we can use a **residual plot** to decide whether a linear model is appropriate. Click on the icon to explore these graphs.

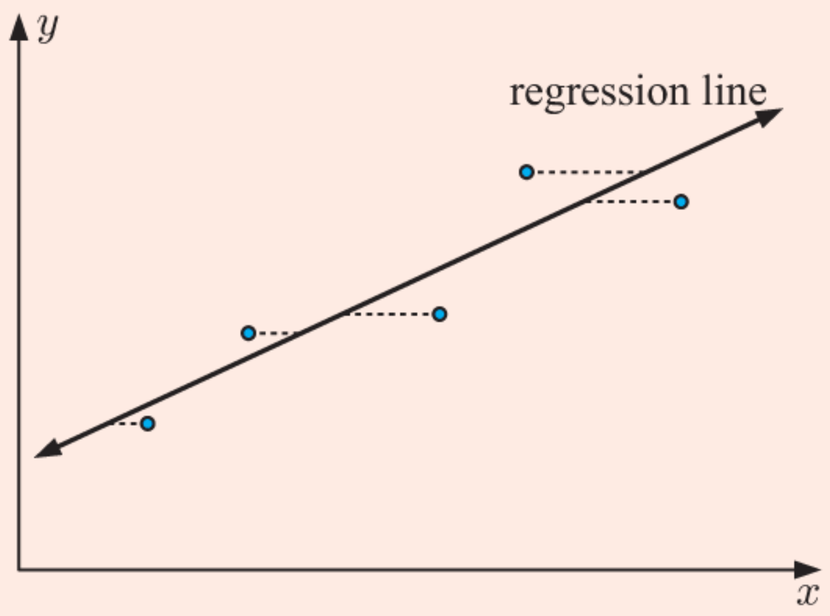


DISCUSSION

When we generated our least squares regression line, our aim was to minimise the *vertical* distances between the data points and the line. This means that when we use the regression line to estimate a value of y given a value of x, we can be sure that the error in our estimate is minimised.



The graph alongside shows the same data points and regression line. It also shows the *horizontal* distance each point is from the line.



1 Do you think that this regression line also minimises the horizontal distances between the data points and the line?
How would the regression line be different if we tried to minimise the horizontal distances instead?

2 Suppose we use a regression line to estimate x given a value of y.

a Can we be sure that the error in this estimate has been minimised?

b Will this estimate be as reliable as an estimate of y given a value of x?

THEORY OF KNOWLEDGE

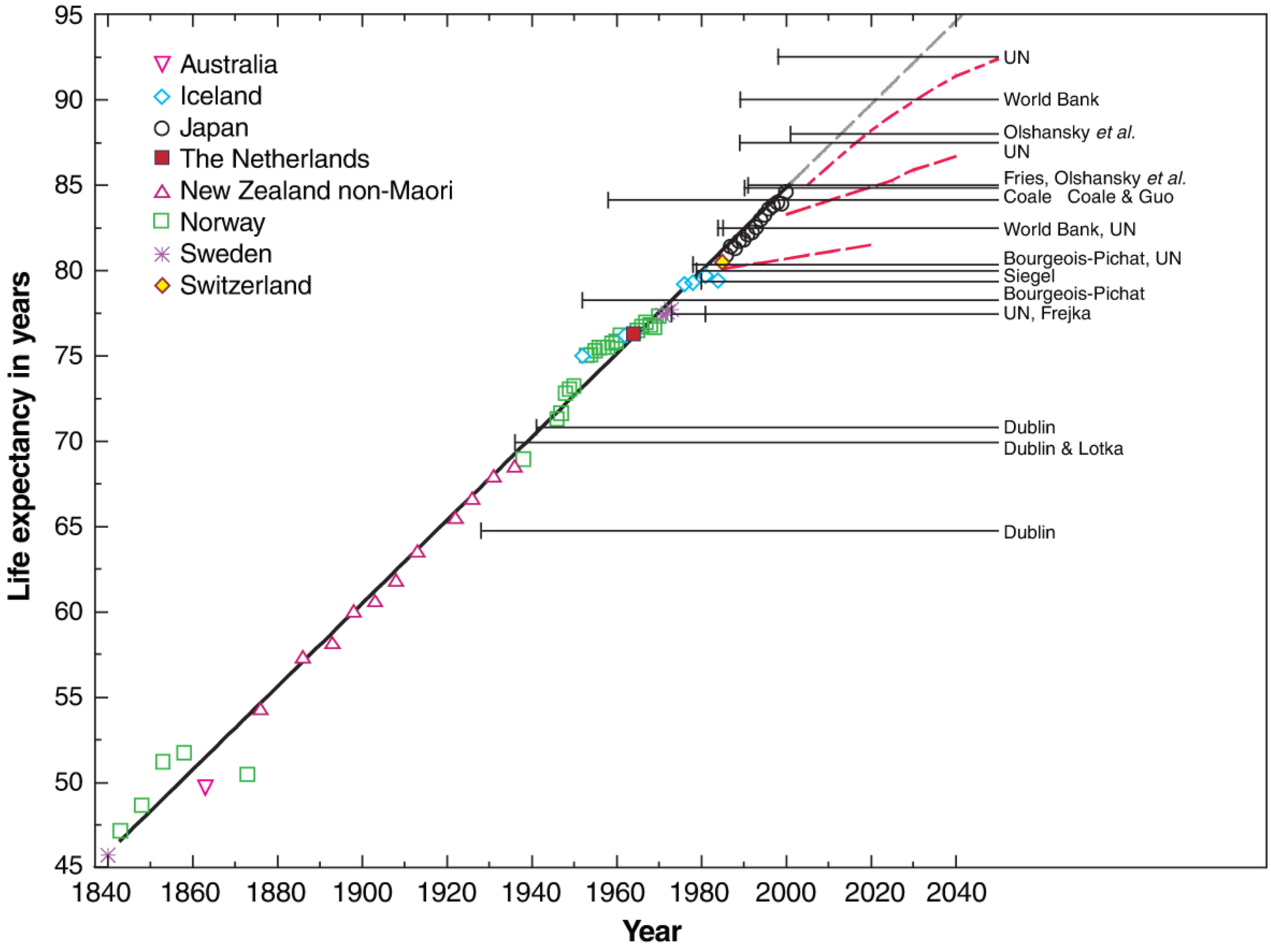
The use of extrapolation for predicting the future leads to debate on many global issues. Even when data shows a strong linear correlation, we need to consider whether it is reasonable for the trend to continue in the long term.

For example, the graph on the next page is based on the article by Oeppen and Vaupel (2002)[1]. It shows female life expectancy from 1840 to the early 2000s, and the country with the highest female life expectancy at each point in time.

Notice that:

- The linear regression trend line is drawn in black, and extrapolated in grey.
- The horizontal black lines show asserted "ceilings" on life expectancy. The vertical line at the left end shows the year of publication.
- The dashed red lines denote projections of female life expectancy in Japan published by the United Nations (UN) in 1986, 1999, and 2001.

Record female life expectancy from 1840 to the present - Oeppen and Vaupel (2002)



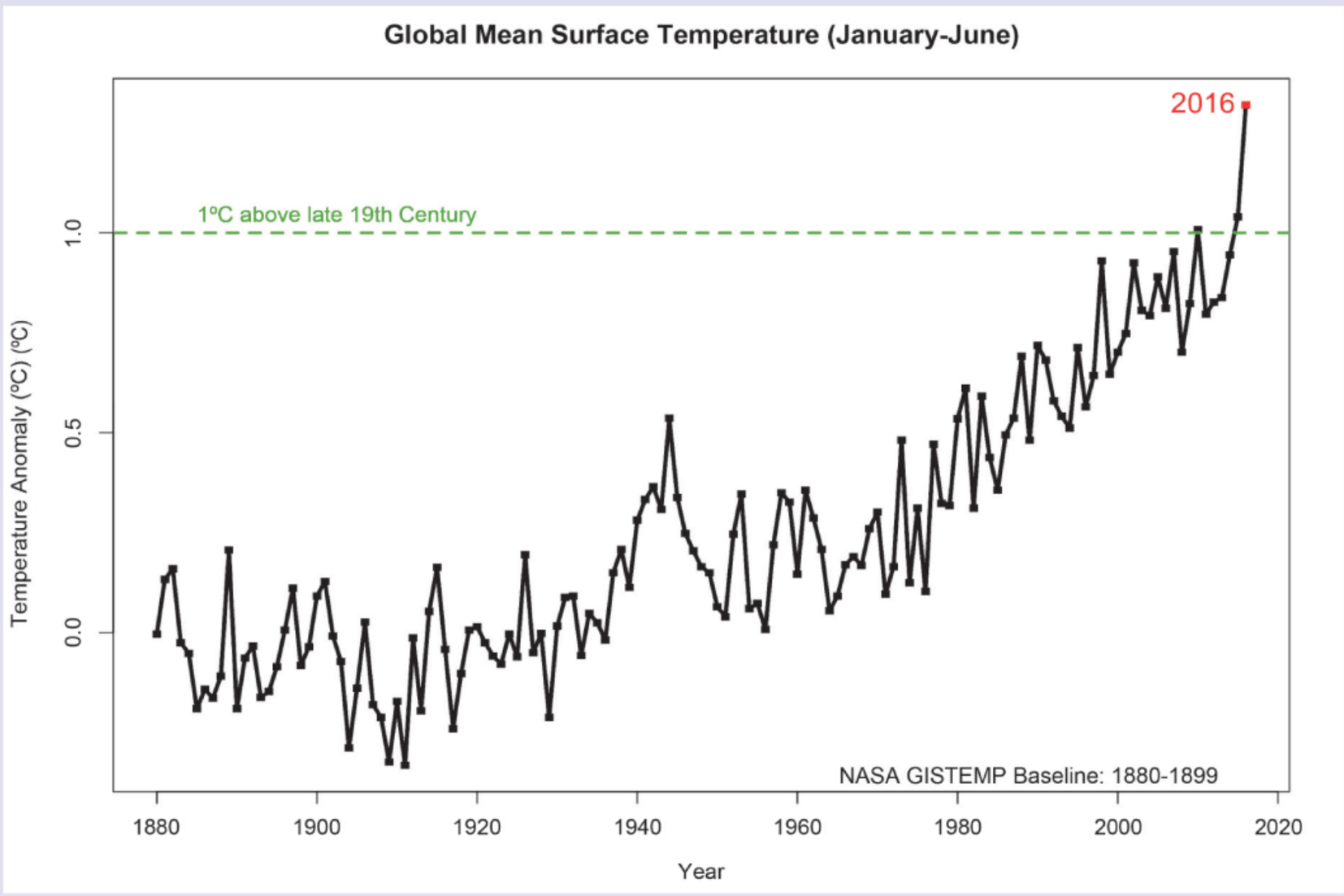
1 Discuss the relationship between the variables.

2 Use the regression line to predict female life expectancy in the year 2100. Do you think this is realistic?

3 Discuss the "ceilings" suggested by publishers over time. Is there evidence to suggest that human life expectancy will approach a limiting "ceiling"?

4 Discuss the accuracy of the UN projections for females in Japan from 1986 to 1999. Is there reason to expect the latest projection will be more reliable?

The graph below shows data from the NASA Goddard Institute for Space Studies[2]. The data for each point is for the first six months of the corresponding year.



5 Discuss the relationship between the variables. Is it reasonable to use a linear model to describe the mean surface temperature of the Earth over time? Is it reasonable to even conclude that the mean surface temperature of the Earth is increasing?

6 How can we predict the mean surface temperature of the Earth in the future?

7 Is mathematical extrapolation valid evidence for dictating environmental policy?

References:

[1] Oeppen and Vaupel, *Broken limits to life expectancy*, Science, **296**, 5570, 1029-1031, 2002.

[2] www.nasa.gov/feature/goddard/2016/climate-trends-continue-to-break-records

F STATISTICAL RELIABILITY AND VALIDITY

We have already studied the various errors that can arise in sampling. We have seen how they can be minimised by choosing both an appropriate:

- *sampling method* so the sample is representative of the whole population, and an appropriate
- *data collection method* so the data obtained is useful for answering the question we are investigating.

In this Section we will see how Pearson's product-moment correlation coefficient can be used to assess the **reliability** and **validity** of an experiment's design.

RELIABILITY

Statistical **reliability** is a measure of how *consistently* a variable can be measured.

To assess the reliability of a measurement, we want to know if we would get the same results if we repeated the experiment with the *same sample* and the *same conditions*.

You can contrast this with sampling error, where measurements taken from different samples will vary due to the natural differences between individuals.

Suppose a researcher gives a reading comprehension test to a group of adults. There are several ways for the researcher to consider reliability:

- **Test-retest reliability**
  The researcher could give the same test to the same group of adults under the same conditions a week later. In this case, the researcher is considering how consistent the measurement is over *time*.
  The main disadvantage of test-retest reliability in this scenario is that sampled individuals are likely to perform better on the retest, because they are now familiar with the test and can probably remember some of the answers.
- **Parallel forms reliability**
  The researcher could write similar sets of questions for similar reading passages, and give the sample of adults each passage as a separate test. In this case, the researcher is considering how consistent the measurement is across different *versions*.

For either form of reliability test, the measurement of a variable is considered **reliable** if the repeated measurements give similar results. In other words, there should be a *strong positive correlation* between the measurements.

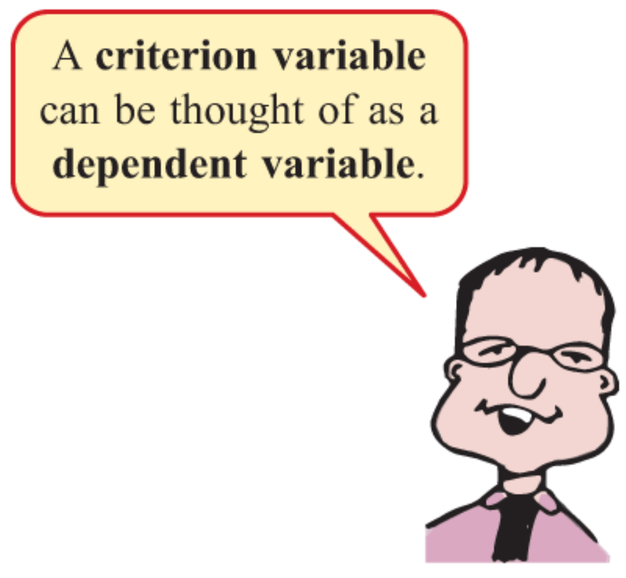
VALIDITY

Statistical **validity** considers how *accurately* a variable measures a particular aspect of a population.

There are several types of validity we can consider:

- **Content validity** considers how well the field of study or *content domain* is covered.
  For example:
  - A final IB examination needs to include questions covering all aspects of the course. Questions of varying difficulty are also needed to test the skills that students should have acquired throughout the course.
  - An experiment testing a person's cognitive skills should consider many areas including memory, problem solving, comprehension, language skills, and spatial awareness.

  This type of validity is not necessarily statistical as it requires knowledge of the particular field of study.
- **Criterion validity** considers how well a variable predicts another valid variable, called the **criterion variable**. A variable has high criterion validity if it can accurately predict the value of the criterion variable.
  For example, if a university entrance exam can accurately predict a student's academic performance at the university, then the entrance exam has high criterion validity.
  To measure the criterion validity of a predictor variable, we consider the strength of its *correlation* with the criterion variable.



DISCUSSION

Suppose the correlation coefficient between a predictor variable and a criterion variable is r. Does high criterion validity correspond to:

- r close to $+1$
- r close to -1
- $|r|$ close to 1?

Example 6 **Self Tutor**

Mr Pettigrew is a primary school teacher. Each week, he gives his students a test to help them practise their times tables. Each test has 20 randomly chosen times tables which the students are given 10 minutes to complete. The table below shows the results for consecutive weeks:

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Week 1	8	11	8	19	9	16	20	14	16	4	7	2	16	8	17
Week 2	8	11	13	20	9	18	18	17	13	4	9	8	14	10	16

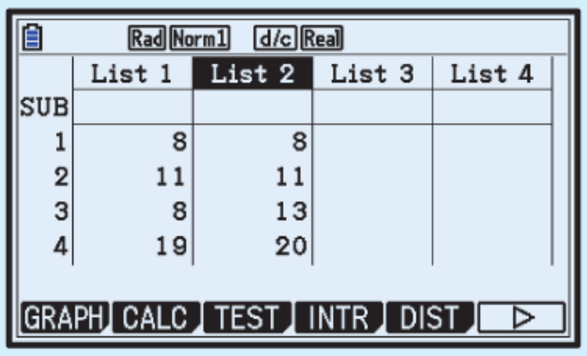
a Calculate Pearson's product-moment correlation between the *Week 1* and *Week 2* results.

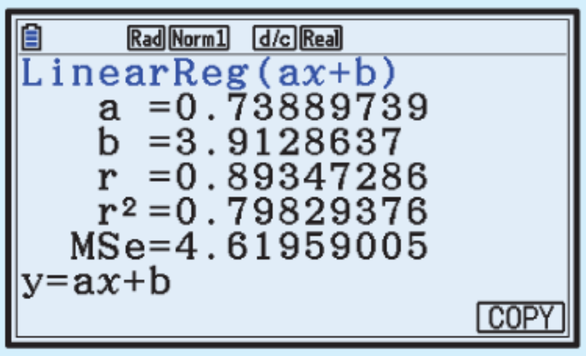
b What type of reliability is being considered here?

c Comment on the reliability of Mr Pettigrew's times tables test.

d Does this test necessarily have content validity for assessing the student's arithmetic skills? Explain your answer.

a





Using technology, $r \approx 0.893$.

b Each test has the same number of questions and is done under the same conditions by the same set of students each week. However, the tests are not identical as the 20 times tables are randomly selected each week. Parallel forms reliability is being considered here.

c From **a**, there is a strong positive correlation between the students' *Week 1* and *Week 2* results. Mr Pettigrew's times tables test is therefore reliable.

d This test only considers multiplication. A valid test for assessing arithmetic skills should include questions covering all arithmetic operations including addition, subtraction, multiplication, and division.

EXERCISE 7F

1 For each of the following scenarios, determine the type of reliability being considered:

a A language teaching app randomly selects 10 Spanish sentences for Frederick to translate into English every evening.

b Lucien measures his resting blood pressure every morning as soon as he wakes up.

c The longest distance thrown by each javelin thrower in a track and field team is recorded at the end of each training session.

2 Jo is a PhD candidate studying the impact of improper footwear while running.

Jo has designed an experiment in which volunteers run on a treadmill. To satisfy the research requirements outlined by the School of Human Movement and Sport, she must obtain at least 3 repeated measurements from each volunteer.

a List the factors that Jo should control to help ensure that her measurements are *reliable*.

b Would test-retest reliability or parallel forms reliability be more relevant in Jo's experiment? Give reasons for your answer.

3 The Association of Tennis Professionals (ATP) assigns ranks to professional tennis players based on how they perform in tournaments.

The following table shows the number of grand slams won by the top 15 male singles players as of January 2019:

ATP rank	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of grand slams	14	17	20	0	1	0	1	0	0	0	0	0	0	0	0

a Calculate the Pearson's product-moment correlation coefficient between the player's *ATP rank* and *number of grand slams won*.

b Explain why criterion validity is being considered here, and identify the criterion variable.

c Comment on the criterion validity of the predictor variable.

4 To test the accuracy of a set of scales used to weigh parcels, a post office worker records the readings on the scales when weighing a set of standard weights:

Standard weight (grams)	50	100	150	200	250	300
Measured weight (grams)	50.4	100.5	150.4	200.7	250.5	300.4

a Calculate the Pearson's product-moment correlation coefficient between the *standard weights* and *measured weights*.

b Explain why the post office worker is concerned about the *validity* of the scales' measurements rather than their *reliability*.

c Is this an example of content or criterion validity? Explain your answer.

d Explain why using the correlation coefficient to measure the validity of the scales' measurements is not necessarily appropriate in this case.

5 A psychologist asks a group of 20 school children to play a card matching game. She randomly places 6 pairs of cards in a 4×3 grid, and records the time it takes for each child to match all of the cards correctly.

After a short break, the psychologist asks the children to play the game again with the same set of cards, again randomly placed.

The results for the two games are shown below:

Child	1	2	3	4	5	6	7	8	9	10
1st game (minutes)	4.4	8.3	5.8	4.0	2.9	5.2	5.2	5.1	3.8	4.3
2nd game (minutes)	3.1	4.7	8.3	6.3	2.8	5.8	5.4	8.4	3.7	3.7

Child	11	12	13	14	15	16	17	18	19	20
1st game (minutes)	4.9	5.3	4.3	4.5	5.7	7.6	6.8	5.4	4.1	1.9
2nd game (minutes)	4.2	4.9	4.6	2.0	4.5	6.1	6.1	4.8	4.8	2.0

a Calculate Pearson's product-moment correlation between the *1st game* and *2nd game* times.

b Explain why parallel forms reliability is being considered in this case.

c Comment on the reliability of the psychologist's test.

d The main disadvantage of test-retest reliability is the child remembering the answers. Is this a problem in this case? Explain your answer.

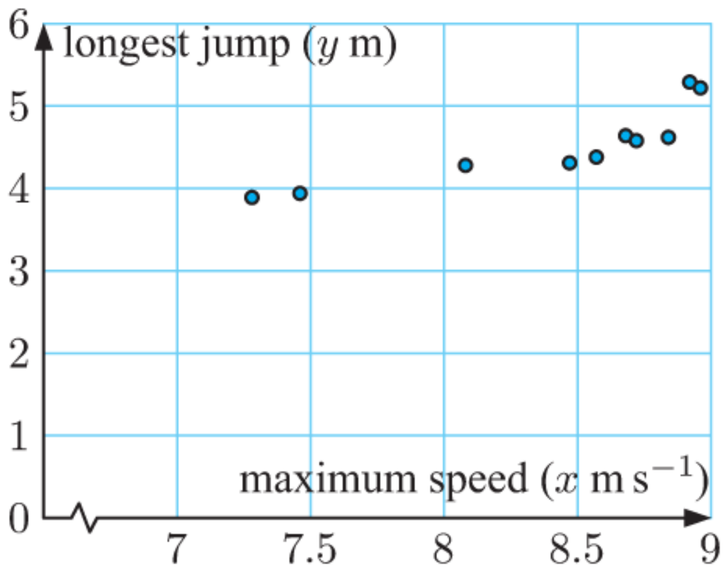
G SPEARMAN'S RANK CORRELATION COEFFICIENT

The table below shows data from 10 young athletes competing in a track and field competition. It includes the length of their longest jump, and the maximum speed attained on their approach.

Athlete	A	B	C	D	E	F	G	H	I	J
Maximum speed (x m s^{-1})	8.68	8.92	8.57	8.47	8.96	8.84	8.72	7.46	7.28	8.08
Longest jump (y m)	4.64	5.29	4.38	4.31	5.22	4.62	4.58	3.94	3.89	4.28

The scatter diagram of the data is shown alongside.

Pearson's product-moment correlation coefficient for the data ≈ 0.864, which suggests a strong (but not perfect) positive correlation between the variables.

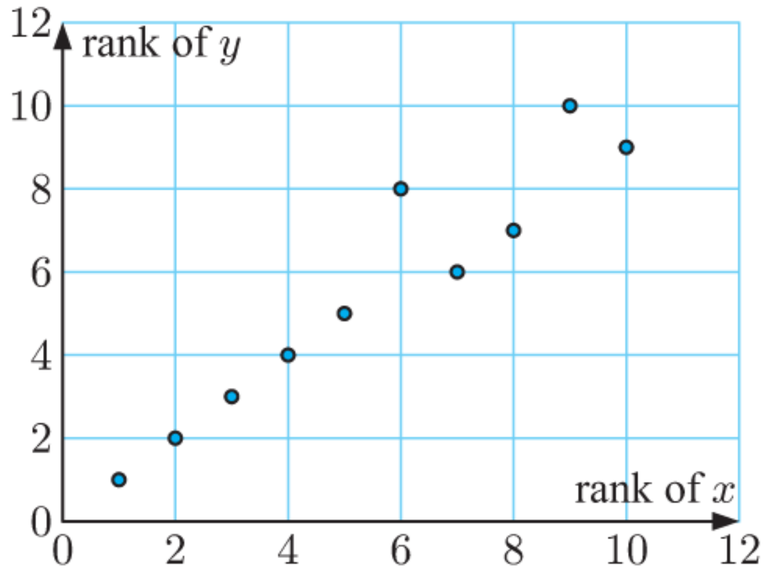


In a case like this, we may not be convinced that a linear model is appropriate for the data. We may therefore wish to focus just on the upward or downward *trend* of the data. One way to do this is to consider the relationship between each variable's **ranks** instead of the raw values.

For the above example, we can assign each x-value and each y-value a rank. A rank of 1 corresponds to the smallest value for the variable, a rank of 2 corresponds to the second smallest, and so on.

Athlete	A	B	C	D	E	F	G	H	I	J
Maximum speed (x m s^{-1})	8.68	8.92	8.57	8.47	8.96	8.84	8.72	7.46	7.28	8.08
rank of x	6	9	5	4	10	8	7	2	1	3
Longest jump (y m)	4.64	5.29	4.38	4.31	5.22	4.62	4.58	3.94	3.89	4.28
rank of y	8	10	5	4	9	7	6	2	1	3

The scatter diagram for the ranked data is shown alongside. For this data we observe a more clear linear trend with Pearson's product-moment correlation coefficient ≈ 0.952. This indicates a strong upward *but not necessarily linear* trend in the original data.



Spearman's rank correlation coefficient of a bivariate data set is defined as the Pearson product-moment correlation coefficient of the variables' **ranks**.

To distinguish between the two correlation coefficients, it is common to use r_p and r_s for Pearson's product-moment correlation and Spearman's rank correlation respectively.

Since Spearman's correlation only considers the relationship between the ranks and not the data itself, r_s is often used instead of r_p when the data is clearly non-linear, but has an upward or downward trend.

Example 7 **Self Tutor**

The population of black bears on a particular island has been recorded every 5 years since 1978:

Years since 1978 (t)	0	5	10	15	20	25	30	35	40
Population (P)	1030	1836	2678	3077	3394	3501	3479	3418	3511

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient r_p for this data.

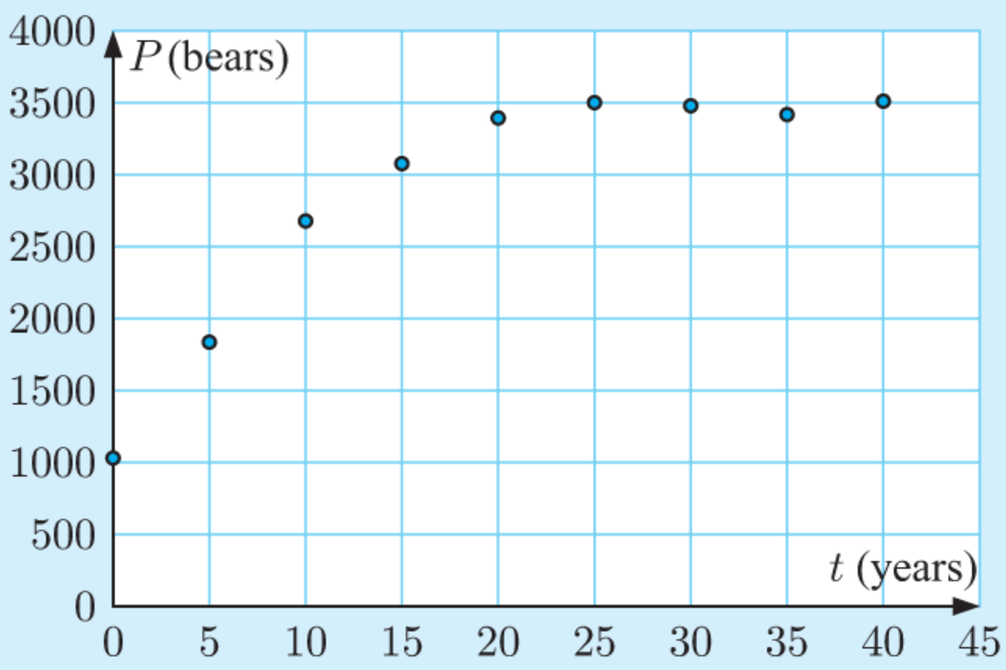
c Find the ranks for each variable and draw a scatter diagram of the ranks.

d Calculate Spearman's rank correlation coefficient, r_s.

e Which correlation coefficient is more appropriate to use for this data? Explain your answer.

f Describe the correlation between the variables.

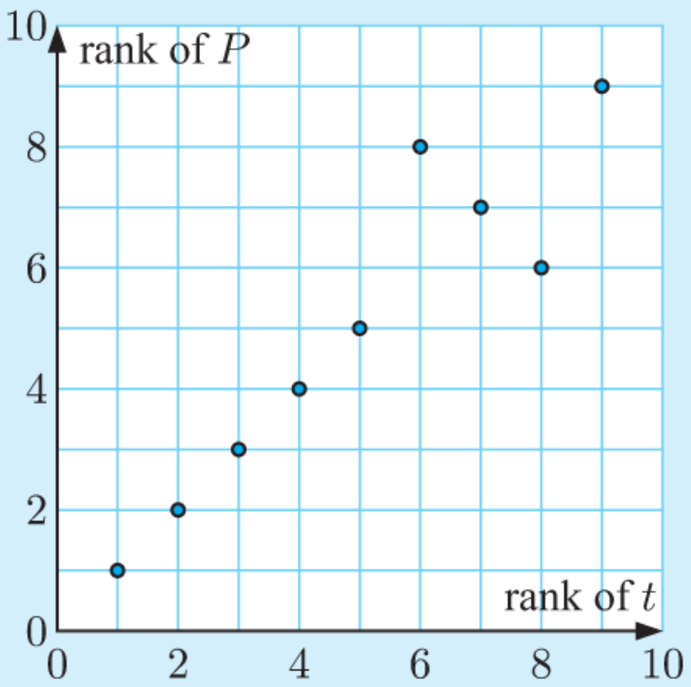
a



b Using technology, $r_p \approx 0.860$.

c

t	0	5	10	15	20	25	30	35	40
rank of t	1	2	3	4	5	6	7	8	9
P	1030	1836	2678	3077	3394	3501	3479	3418	3511
rank of P	1	2	3	4	5	8	7	6	9



d Using technology, $r_s \approx 0.933$.

e Looking at the scatter diagram of the raw data, the relationship between the variables is clearly *non-linear*. Spearman's rank correlation coefficient is therefore more appropriate.

f Based on the value of r_s, there is a strong, positive non-linear correlation between the variables.

RANKS OF TIED VALUES

In some cases, we have x-values or y-values that are equal. These are called **ties**. For example, in the data alongside we have one tie in the y-values.

x	3	2	5	4	8	6	7
y	3	4	6	8	7	7	1

The values in a tie must be given the same rank. To do this, the unique values are assigned ranks which leave out values for the ties.

x	3	2	5	4	8	6	7
rank of x	2	1	4	3	7	5	6
y	3	4	6	8	7	7	1
rank of y	2	3	4	7			1

ranks 5 and 6

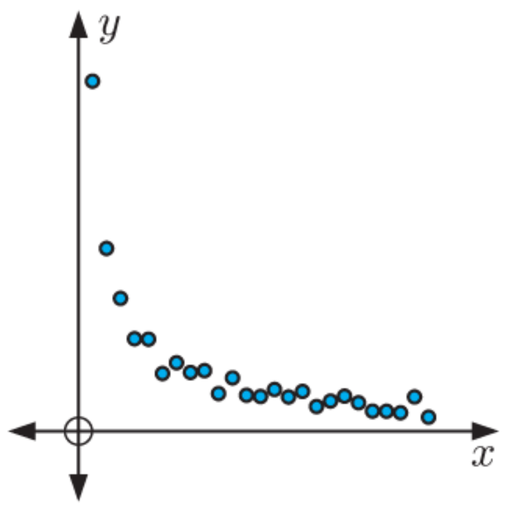
We then assign the ties the **average** of the ranks that were left out for them. In this case the ranks left out were 5 and 6, so we give the two shaded 7s the rank $= \dfrac{5+6}{2} = 5.5$.

x	3	2	5	4	8	6	7
rank of x	2	1	4	3	7	5	6
y	3	4	6	8	7	7	1
rank of y	2	3	4	7	5.5	5.5	1

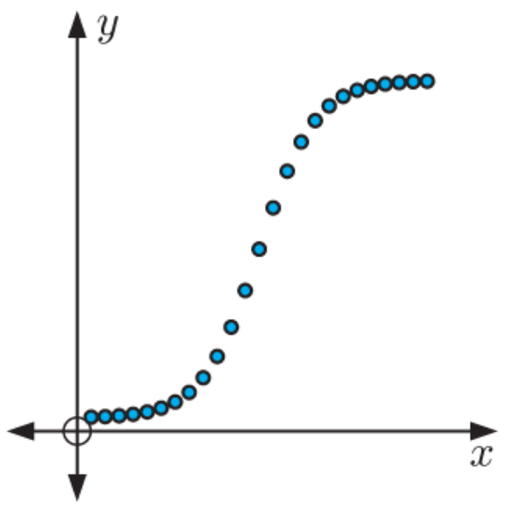
EXERCISE 7G

1 Match each scatter diagram with the correct rank scatter diagram:

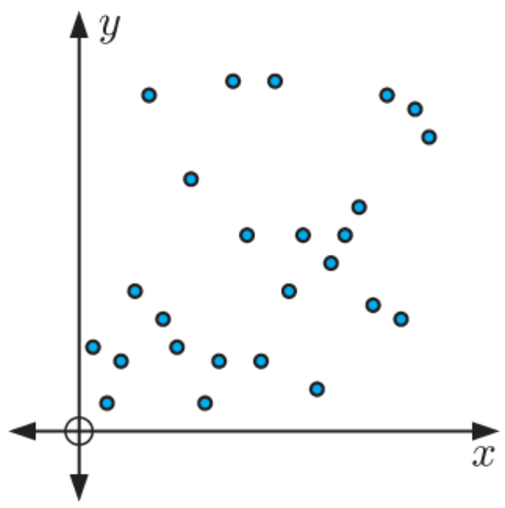
a



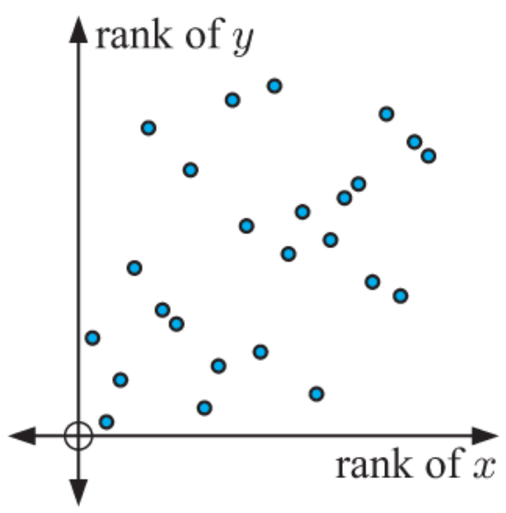
b



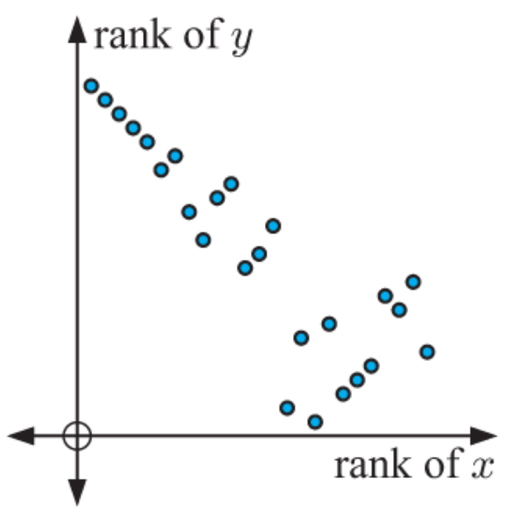
c



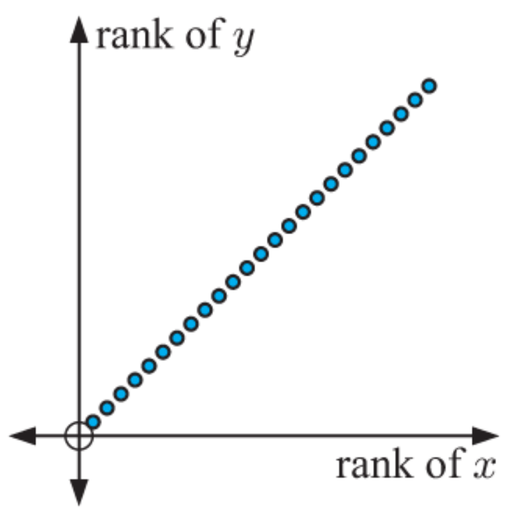
A



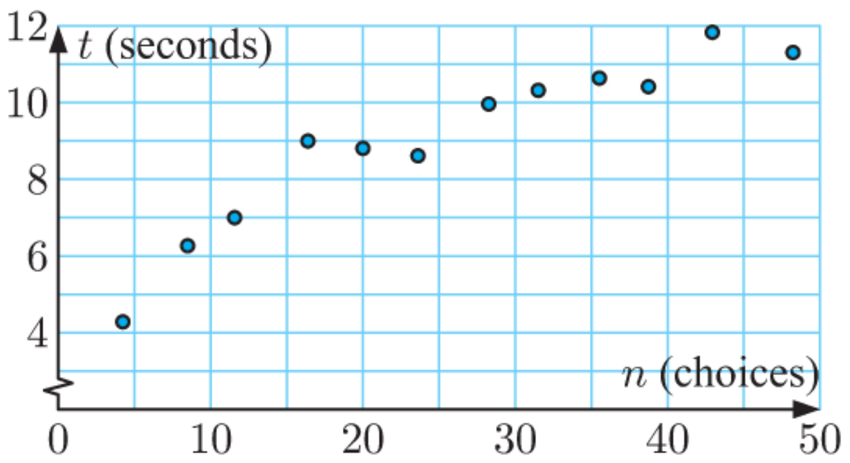
B



C

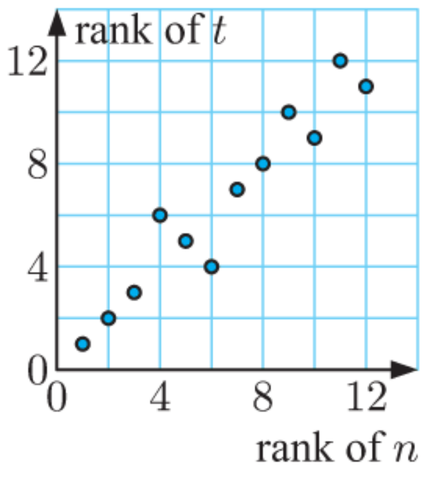


2 The scatter diagram alongside shows the time t that a person takes to make a selection given a set of n choices.

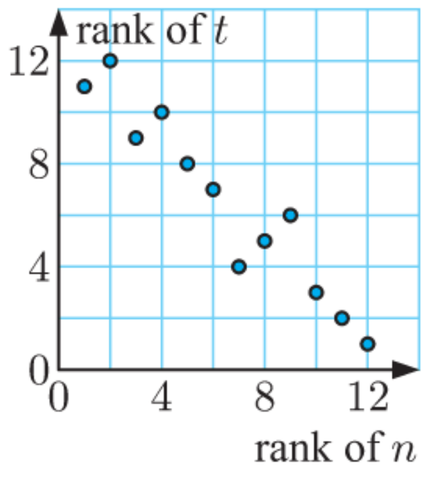


a Which of the following scatter diagrams is the scatter diagram of the ranks?

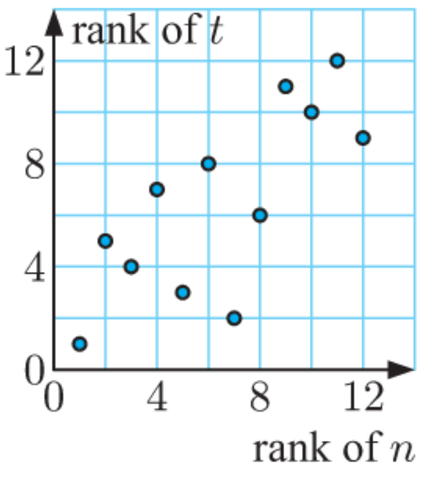
A



B



C



b *Hence* identify the correct value of Spearman's rank correlation coefficient for this data:

A $r_s \approx 0.958$ **B** $r_s \approx -0.958$ **C** $r_s \approx 0.755$

3 For a particular data set, Spearman's rank correlation coefficient $r_s \approx -0.7$.

a Is the trend in the data positive or negative?

b Can we determine the *linearity* of the data from the value of r_s alone?

4 In a 60 minute Art lesson, students were asked to make as many paper cranes as possible. The table shows how long it took each student to make a paper crane, and how many cranes they made during the lesson:

Time taken (t min)	6	9.5	4	5	8	7.5	11	6.5
Cranes made (C)	9	5	16	11	6	7	4	8

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient.

c Find the ranks for each of the variables and draw a scatter diagram for the ranks.

d Calculate Spearman's rank correlation coefficient.

e Which correlation coefficient is more appropriate to use for this data? Explain your answer.

f Describe the correlation between the variables.



5 A local council collected data for the *area* of its parks and the *number of maple trees* each contains:

Area (A hectares)	2.8	6.9	7.4	4.3	2.3	9.4	5.2	8.0	4.9	6.2	3.3	4.5
Number of maple trees (n)	18	31	33	24	17	40	32	37	30	32	25	28

a Draw a scatter diagram for this data.

b Use the scatter diagram to explain why you would expect Pearson's correlation and Spearman's correlation coefficients to have similar values.

c Calculate r_p and r_s to verify your answer to **b**.



6 Ten students in a typing contest were given one minute to type as many words as possible. The table below shows how many words each student typed, and how many errors they made:

Number of words (x)	40	53	20	65	35	60	85	49	35	76
Number of errors (y)	11	15	2	20	4	22	30	16	27	25

a Calculate r_p and r_s.

b Draw a scatter diagram for this data and hence identify the outlier.

c Recalculate r_p and r_s without the outlier.

d Which correlation coefficient is most affected by the presence of the outlier?

7 Consider the long jump data from the start of this Section.

a Use the longest jumps to determine the *placing* of each athlete in the competition.

b Out of *longest jump* or *placing*, which is the independent variable?

c Explain why:

i the variable *placing* is already ranked

ii Spearman's correlation coefficient for the variables *longest jump* and *placing* must be exactly -1.

ACTIVITY 4 ANSCOMBE'S QUARTET

Anscombe's quartet is a collection of four bivariate data sets which have interesting statistical properties.

It was first described in 1973 by the English statistician **Francis Anscombe** (1918 - 2001). At the time, computers were becoming increasingly popular in statistics, as they allowed for more large scale and complex computations to be done within a reasonable amount of time. However, many common statistical packages primarily performed numerical calculations rather than produced graphs. Such output was often limited to those with advanced programming skills.



Francis Anscombe
Photo courtesy of Yale University.

In his 1973 article, Anscombe stressed that:

> "*A computer should make* both *calculations* and *graphs. Both sorts of output should be studied; each will contribute to understanding.*"

The data values for Anscombe's quartet are given in the tables below:

Data set A:

x	10	8	13	9	11	14	6	4	12	7	5
y	8.04	6.95	7.58	8.81	8.33	9.96	7.24	4.26	10.84	4.82	5.68

Data set B:

x	10	8	13	9	11	14	6	4	12	7	5
y	9.14	8.14	8.74	8.77	9.26	8.1	6.13	3.1	9.13	7.26	4.74

Data set C:

x	10	8	13	9	11	14	6	4	12	7	5
y	7.46	6.77	12.74	7.11	7.81	8.84	6.08	5.39	8.15	6.42	5.73

Data set D:

x	8	8	8	8	8	8	8	19	8	8	8
y	6.58	5.76	7.71	8.84	8.47	7.04	5.25	12.5	5.56	7.91	6.89

Enter the data into your **graphics calculator** or click on the icon to access the data in the **statistics package**.

STATISTICS PACKAGE

What to do:

1 For each data set, use technology to calculate:

a the mean of each variable **b** the population variance of each variable.

Comment on your answers.

2 Find the regression line and Pearson's product-moment correlation coefficient for each data set. What do you notice?

3 Construct a scatter diagram for each data set, and plot the corresponding regression line on the same set of axes.

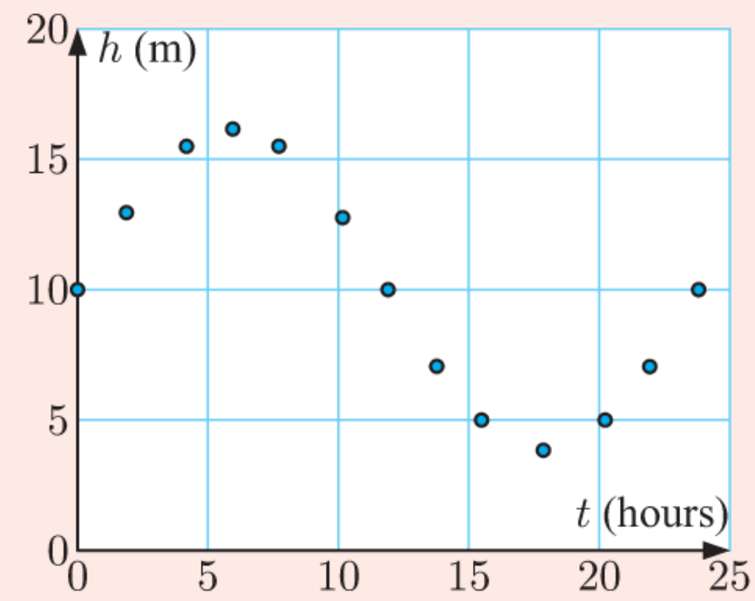
4 How do your calculations in **1** and **2** compare to your graphs in **3**? Is a linear model necessarily appropriate for each data set?

5 Calculate Spearman's rank correlation coefficient for each data set and comment on your results.

6 Discuss why it is important to consider both graphs *and* descriptive statistics when analysing data.

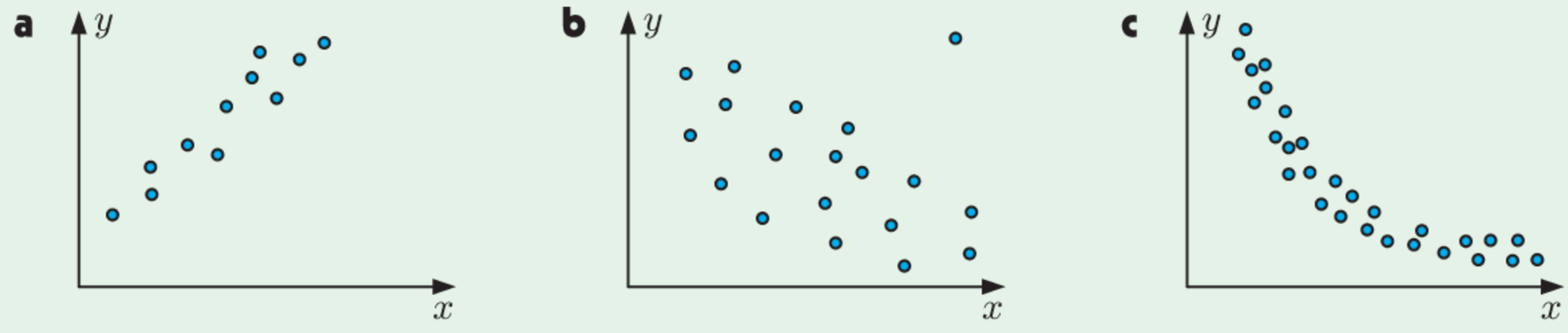
For example, consider the data in the scatter diagram alongside.

Is either Pearson's product-moment correlation coefficient or Spearman's rank correlation coefficient appropriate for measuring the strength of the relationship between these variables?



REVIEW SET 7A

1 For each scatter diagram, describe the relationship between the variables. Consider the direction, linearity, and strength of the relationship, as well as the presence of any outliers.



2 Kerry wants to investigate the relationship between the *water bill* and the *electricity bill* for the houses in her neighbourhood.

a Do you think the correlation between the variables is likely to be positive or negative? Explain your answer.

b Is there a causal relationship between the variables? Justify your answer.

3 The table below shows the ticket and beverage sales for each day of a 12 day music festival:

Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8

a Draw a scatter diagram for the data.

b Calculate Pearson's product-moment correlation coefficient r_p.

c Find the ranks for each variable and draw a scatter diagram for the ranks.

d Calculate Spearman's rank correlation coefficient r_s for these variables.

e Describe the correlation between *ticket sales* and *beverage sales*.

4 Jamie is a used car salesman. This table shows the age and selling price of the last 10 cars he has sold.

Age (years)	2	10	5	4	3	4	7	12	15	8
Selling price ($\$ \times 1000$)	25	10	12	7	17	15	4	3	2	7

a Draw a scatter diagram for the data.

b Describe the correlation between *age* and *selling price*.

c Calculate the coefficient of determination r^2.

d What percentage of the variation in selling price can be explained by variation in age?

e What other factors could explain the variation in selling price?

5 A garden centre manager believes that during March, the number of customers is related to the temperature at noon. Over a fortnight, the number of customers and the noon temperature were recorded.

Temperature (x°C)	23	25	28	30	30	27	25	28	32	31	33	29	27
Number of customers (y)	57	64	62	75	69	58	61	78	80	35	84	73	76

a Draw a scatter diagram of the data.

b Calculate Pearson's product-moment correlation coefficient r_p.

c Calculate Spearman's rank correlation coefficient r_s.

d Identify the outlier.

e Remove the outlier, and recalculate r_p and r_s.

f Which correlation coefficient is most affected by the presence of the outlier?

g Describe the association between the *number of customers* and the *noon temperature* at the garden centre.

6 A clothing store recorded the length of time customers were in the store and the amount they spent.

Time (min)	8	18	5	10	17	11	2	13	18	4	11	20	23	22	17
Money (€)	40	78	0	46	72	86	0	59	33	0	0	122	90	137	93

a Find the mean for each variable.

b Draw a scatter diagram for the data. Plot the mean point, and draw a line of best fit by eye.

c Describe the relationship between *time in the store* and the *money spent*.

7 Thien is a postgraduate Chemistry student. He wants to improve the reliability of his measurements when performing titrations. Should Thien consider the test-retest reliability or parallel forms reliability of his measurements? Give reasons for your answer.

8 The table below shows the marks an English teacher gave his students for an internally assessed short response assignment, and the marks the students obtained for their mid-year examinations:

Student	A	B	C	D	E	F	G
Short response (marks)	13.0	17.0	14.5	19.0	16.5	15.5	16.0
Mid-year exam (%)	70	81	71	94	90	72	83

Student	H	I	J	K	L	M	N
Short response (marks)	17.0	16.5	17.0	18.5	17.0	13.5	11.0
Mid-year exam (%)	86	86	72	96	86	80	59

a Calculate the Pearson's product-moment correlation coefficient between the *short response* marks and the *mid-year exam* results.

b Explain why criterion validity is being considered here. Which variable is the criterion variable?

c Comment on the criterion validity of the predictor variable.

9 The ages and heights of children at a playground are given below:

Age (x years)	3	9	7	4	4	12	8	6	5	10	13
Height (y cm)	94	132	123	102	109	150	127	110	115	145	157

a Draw a scatter diagram for the data.

b Use technology to find the regression line.

c State and interpret the gradient of the regression line.

d Use the regression line to predict the height of a 5 year old child.

e Based on the given data, at what age would you expect a child to reach 140 cm?

10 Tomatoes are sprayed with a pesticide-fertiliser mix. The table below shows the *yield of tomatoes* per bush for various *spray concentrations.*

Spray concentration (x mL per L)	3	5	6	8	9	11	15
Yield of tomatoes (y per bush)	67	90	103	120	124	150	82

a Draw a scatter diagram to display the data.

b Determine the value of r^2 and interpret your answer.

c Is there an outlier present that is affecting the correlation?

d The outlier was found to be a recording error. Remove the outlier from the data set, and recalculate r^2. Is it reasonable to now fit a linear model?

e Determine the equation of the regression line.

f State and interpret the gradient and y-intercept of the regression line.

g Use your line to estimate:

i the yield if the spray concentration is 7 mL per L

ii the spray concentration if the yield is 200 tomatoes per bush.

h Comment on the reliability of your estimates in **g**.

REVIEW SET 7B

1 For each pair of variables, discuss whether the correlation between the variables is likely to be positive or negative, and whether a causal relationship exists between the variables:

a *price of tickets* and *number of tickets sold*

b *ice cream sales* and *number of shark attacks*.

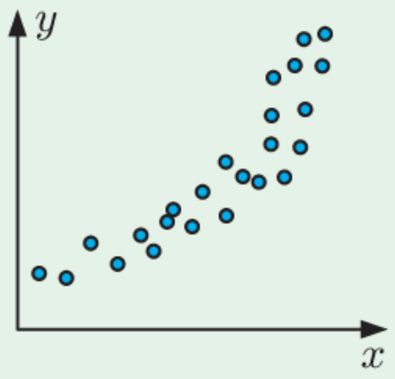
2 For a group of foods, a Pearson's product-moment correlation coefficient of 0.787 is found between the *fat content* and *energy*.

a Describe the relationship between the variables.

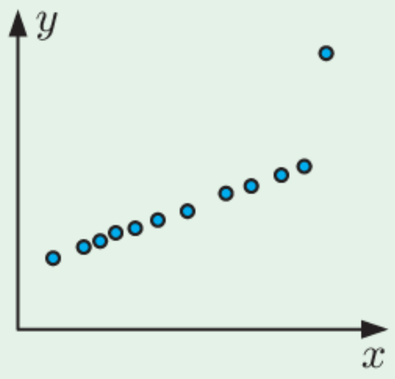
b Calculate the coefficient of determination, and interpret your answer.

3 Match each scatter diagram to the correct Spearman's rank correlation coefficient:

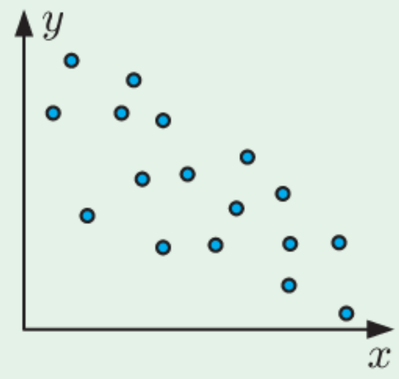
a



b



c



A $r_s = 1$ **B** $r_s = -0.4$ **C** $r_s = 0.7$

4 A group of students is comparing their results for a Mathematics test and an Art project:

Student	A	B	C	D	E	F	G	H	I	J
Mathematics test	64	67	69	70	73	74	77	82	84	85
Art project	85	82	80	82	72	71	70	71	62	66

a Construct a scatter diagram for the data.

b Describe the relationship between the Mathematics and Art marks.

c Given that the coefficient of determination $r^2 \approx 0.864$, find Pearson's product-moment correlation coefficient r.

5 Safety authorities advise drivers to travel three seconds behind the car in front of them. This gives the driver a greater chance of avoiding a collision if the car in front has to brake quickly or is itself involved in an accident.

A test was carried out to find out how long it would take a driver to bring a car to rest from the time a red light was flashed. The following results were recorded for a particular driver in the same car under the same test conditions.

Speed (v km h^{-1})	10	20	30	40	50	60	70	80	90
Stopping time (t s)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a Find the mean point $(\overline{v}, \overline{t})$.

b Draw a scatter diagram of the data. Add the mean point and draw a line of best fit by eye.

c Hence estimate the stopping time for a speed of:

i 55 km h^{-1} **ii** 110 km h^{-1}

d Which of your estimates in **c** is more likely to be reliable?

6 A craft shop sells canvasses in a variety of sizes. The table below shows the area and price of each canvas type.

Area (x cm^2)	100	225	300	625	850	900
Price (£y)	6	12	13	24	30	35

a Construct a scatter diagram for the data.

b Calculate r and r^2.

c Describe the correlation between *area* and *price*.

d Find the regression line and draw it on your scatter diagram.

e Estimate the price of a canvas with area 1200 cm^2. Discuss whether your estimate is likely to be reliable.

7 Eight identical flower beds contain petunias. The different beds were watered different numbers of times each week, and the number of flowers each bed produced was recorded in the table below:

Number of waterings (n)	0	1	2	3	4	5	6	7
Flowers produced (f)	18	52	86	123	158	191	228	250

a Draw a scatter diagram for the data, and describe the correlation between the variables.

b Find the equation of the regression line.

c Calculate the coefficient of determination. Hence comment on how well the linear model fits the data.

d Is it likely that a causal relationship exists between these two variables? Explain your answer.

e Plot the regression line on the scatter diagram.

f Violet has two beds of petunias. She waters one of the beds 5 times a fortnight and the other 10 times a week.

i How many flowers can she expect from each bed?

ii Discuss which of your estimates is likely to be more reliable.



8 A drinks vendor varies the price of Supa-fizz on a daily basis. He records the number of sales of the drink as shown:

Price ($p)	2.50	1.90	1.60	2.10	2.20	1.40	1.70	1.85
Sales (s)	389	450	448	386	381	458	597	431

a Produce a scatter diagram for the data.

b Are there any outliers? If so, should they be included in the analysis?

c Calculate the equation of the regression line.

d State and interpret the gradient of the regression line.

e Calculate r^2 and hence comment on how well the linear model fits the data.

f Do you think the regression line would give a reliable prediction of sales if Supa-fizz was priced at 50 cents? Explain your answer.

9 Mrs Wang teaches a class of Year 2 students in Nanjing. Each week she randomly selects 20 Chinese characters from the 800 her students should know, and asks the students to write them down. The table below shows the students' results for two consecutive weeks:

Student	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
Week 1	18	14	17	20	19	16	18	15	20	20	17	19	12	16	17	20	18	15	16	19	20	17
Week 2	16	13	15	20	17	16	17	12	18	20	18	16	11	15	16	20	14	15	14	17	19	16

a Calculate the Pearson's product-moment correlation coefficient between the students' *Week 1* and *Week 2* results.

b What type of reliability is being considered here?

c Comment on the reliability of Mrs Wang's tests.

d Does this test have content validity for assessing the students' language skills? Explain your answer.

10 Rumours can spread quickly in small towns. The table below shows the number of people who have heard a rumour in a particular town:

Time (t days)	3	6	9	12	15	18	21	24
Number of people (N)	7	13	103	167	397	450	496	497

a Draw a scatter diagram of N against t.

b Calculate Pearson's product-moment correlation coefficient r_p for the data.

c By inspecting the data and your scatter diagram, state the value of Spearman's rank correlation coefficient r_s.

d Which correlation coefficient is more appropriate for this data? Explain your answer.

Chapter 8

Non-linear modelling

Contents:

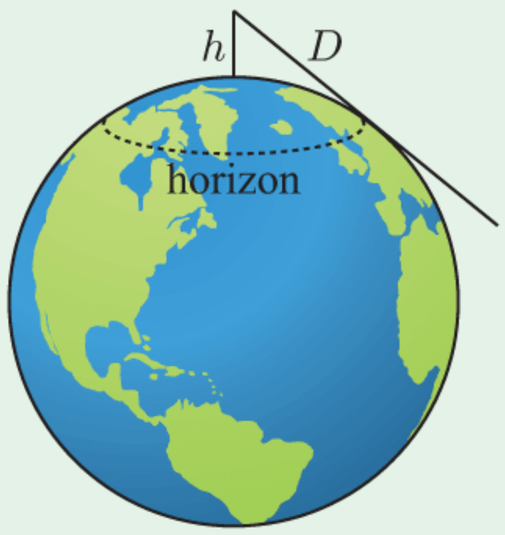
A Logarithmic models
B Exponential models
C Power models
D Problem solving
E Non-linear regression



OPENING PROBLEM

The higher a person is above the Earth, the further they can see. The table below shows the distance D, in kilometres, a person can see from a height h metres above the Earth's surface.

Height (h m)	2	10	20	50	100	200
Distance (D km)	5.3	11.5	17	27	35.6	52.2



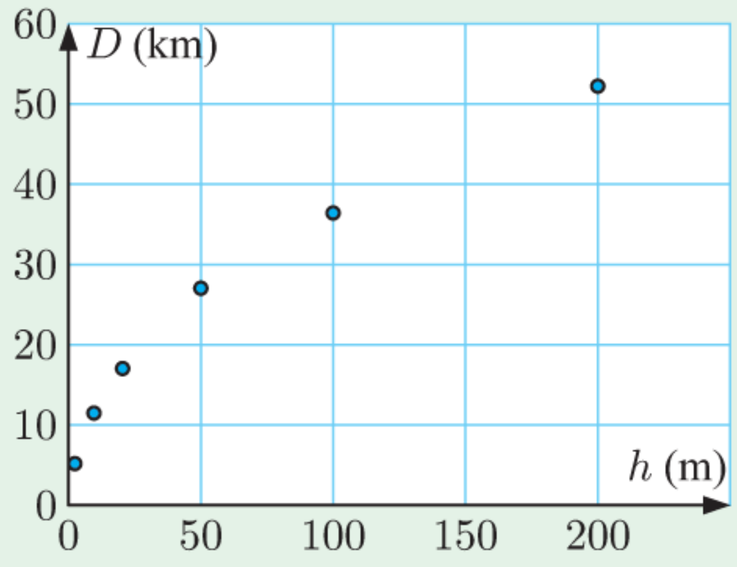
Things to think about:

a A scatter diagram of D against h is shown alongside. Which of the following model types might be appropriate for modelling the relationship between the variables?

A linear **B** logarithmic

C exponential **D** power

b How can we determine which model *best* describes the relationship between the variables?



For data which is clearly **non-linear**, it is not sensible to fit a linear model directly to the data. However, for some relationships between variables we can **transform** the data into data which *is* linearly related. We can then use linear regression to find a model connecting the transformed variables.

In this Chapter we will use **logarithmic transformations** to fit **logarithmic models**, **exponential models**, and **power models** to data.

A LOGARITHMIC MODELS

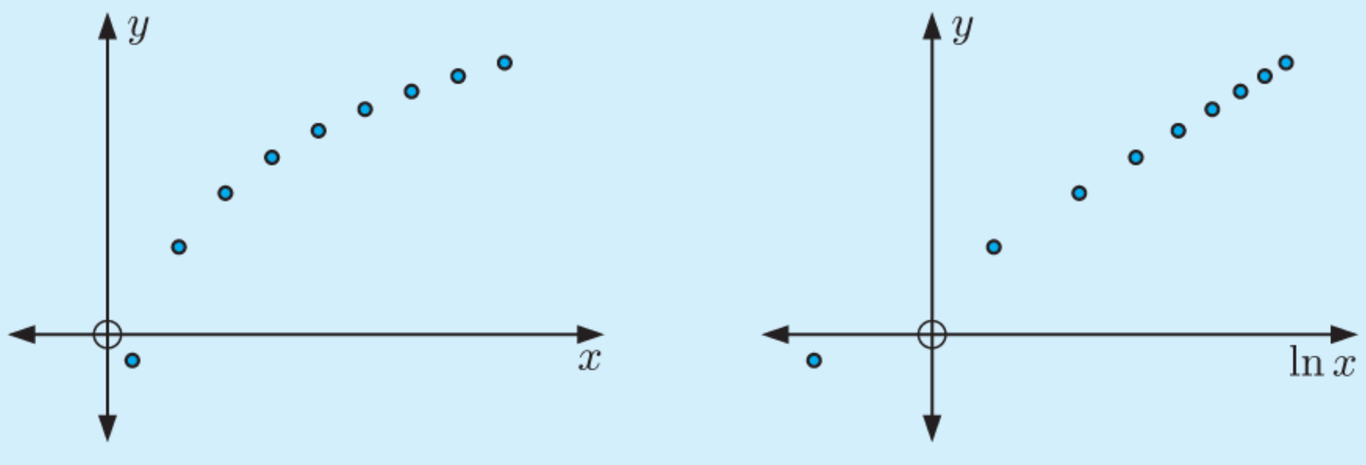
A **logarithmic model** has the form $y = a + b\ln x$.

The logarithmic function $y = a + b\ln x$ is clearly not linear.

However, if we consider $\ln x$ as a variable, there is a linear relationship between y and $\ln x$.

We can therefore use linear regression to find an equation connecting y and $\ln x$.

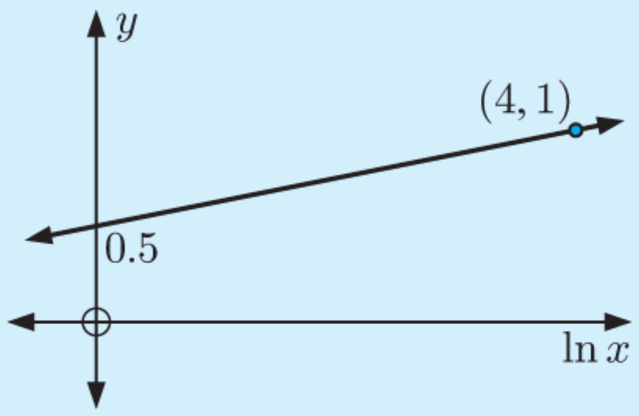
If the variables x and y are connected by a logarithmic model, the graph of y against $\ln x$ is linear.



Example 1

Self Tutor

Find the logarithmic model connecting x and y.



The graph of y against $\ln x$ is linear with gradient $m = \dfrac{1-0.5}{4-0} = 0.125$ and y-intercept $c = 0.5$.

$\therefore \quad y = 0.5 + 0.125 \ln x.$

Example 2

Self Tutor

Consider the data:

x	1	4	9	16	25
y	0.6	8.6	11.4	14.9	18.0



GRAPHICS CALCULATOR INSTRUCTIONS

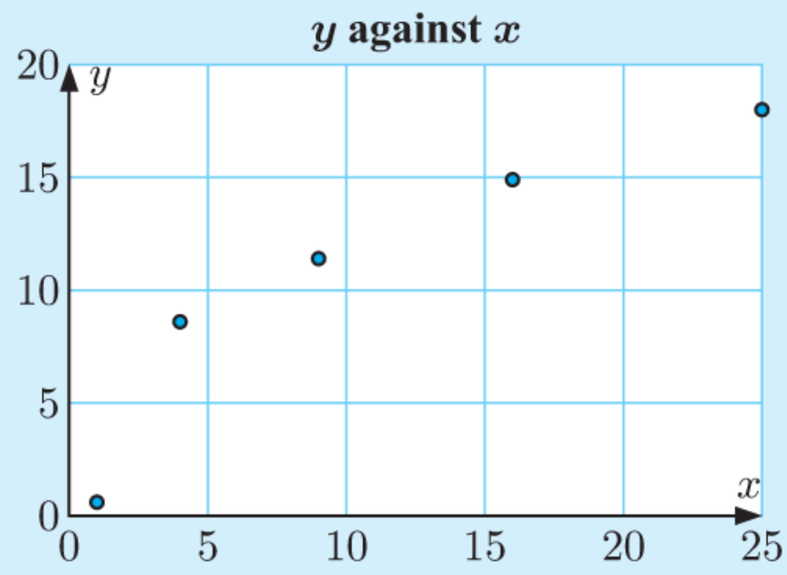
a Draw scatter diagrams of y against x and y against $\ln x$.

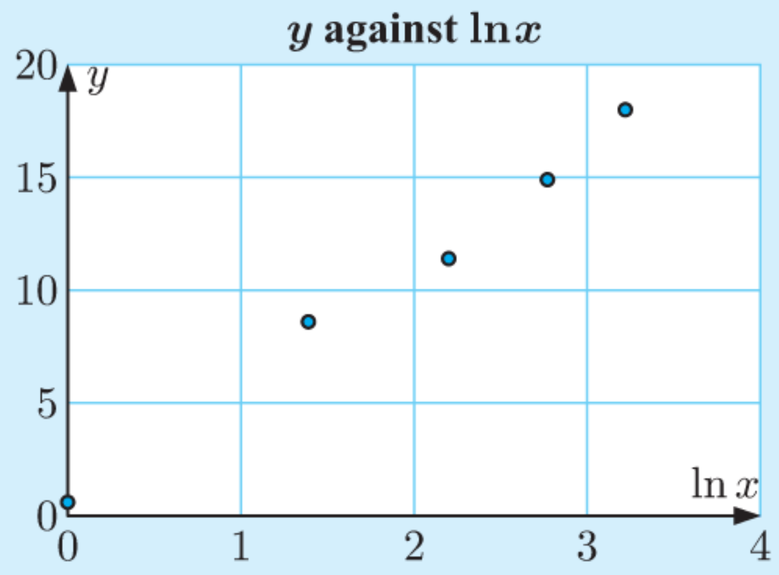
b Explain why y and x are related by a logarithmic model.

c Find the equation connecting y and x.

a

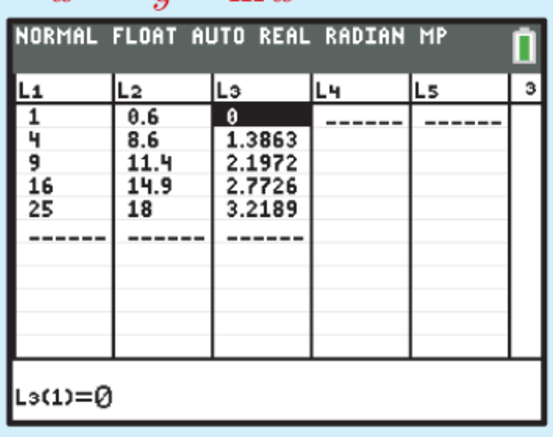
x	1	4	9	16	25
y	0.6	8.6	11.4	14.9	18.0
$\ln x$	0	1.39	2.20	2.77	3.22

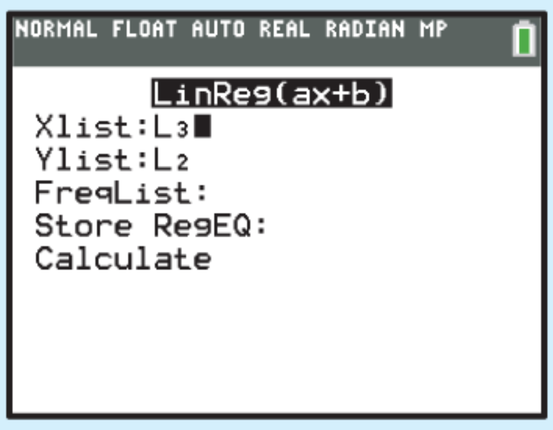


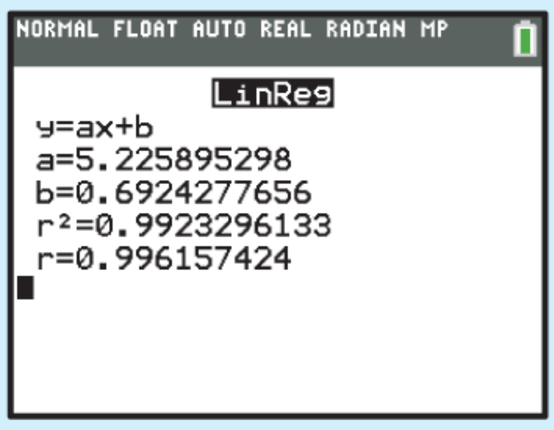


b The graph of y against $\ln x$ appears linear, so a logarithmic model is appropriate.

c $x \quad y \quad \ln x$





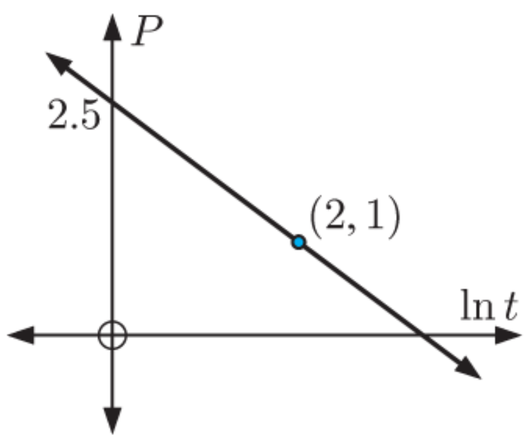


Using technology, the linear model connecting y and $\ln x$ is $y \approx 0.692 + 5.23 \ln x$.

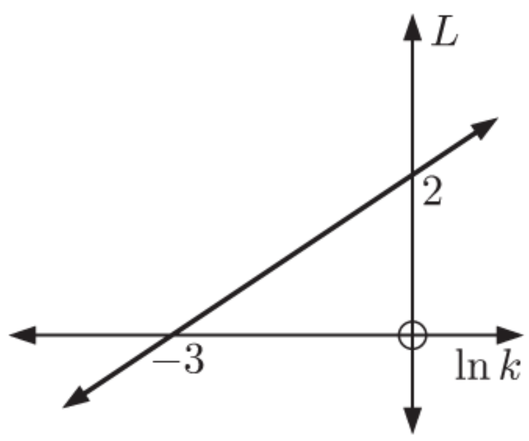
EXERCISE 8A

1 For each of the following graphs, find the logarithmic model connecting the variables:

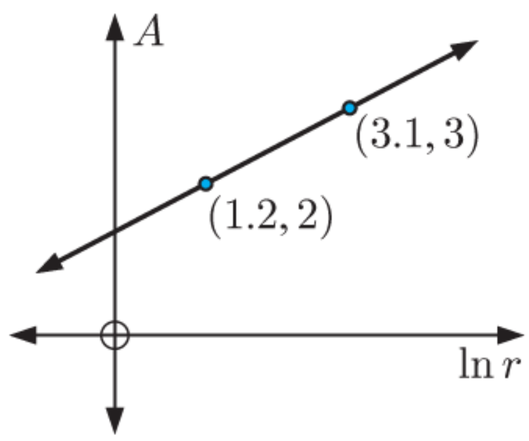
a



b



c



2 Which of the following models is equivalent to the form $y = a + b\ln x$? Explain your answers.

a $y = p + q\ln(rx)$ **b** $y = \alpha + \beta\ln(x - \gamma)$

3 The following table shows the sum of the first n terms of the **harmonic series**, $S_n = \sum_{i=1}^{n} \frac{1}{i}$, $n \geqslant 1$, correct to 3 decimal places where applicable.

n	1	2	3	4	5	6	7	8	9	10
S_n	1	1.5	≈ 1.833	≈ 2.083	≈ 2.283	2.45	≈ 2.593	≈ 2.718	≈ 2.829	≈ 2.929

a Draw scatter diagrams of S_n against n and S_n against $\ln n$.

b Explain why the model connecting S_n and n is logarithmic.

c Find the logarithmic model connecting S_n and n.

d Do you think the infinite series $S = \sum_{i=1}^{\infty} \frac{1}{i}$ is convergent? Explain your answer.

B EXPONENTIAL MODELS

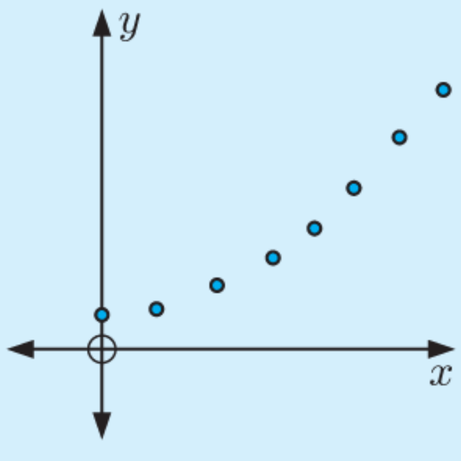
An **exponential model** has the form $y = kb^x$ where $k > 0$ and $b > 0$, $b \neq 1$ are constants.

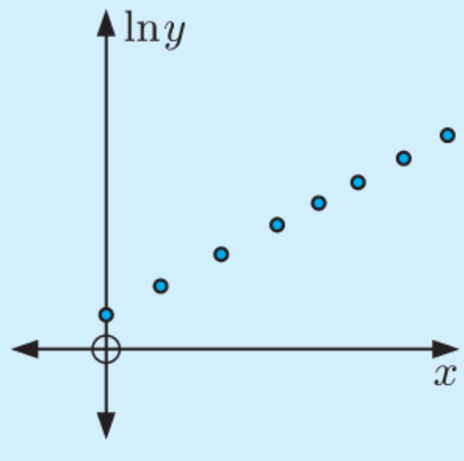
Using the laws of logarithms, notice that:

If $y = kb^x$, then $\ln y = \ln(kb^x)$ {taking the logarithm of both sides}
$= \ln k + \ln(b^x)$
$= \ln k + x\ln b$

Since $\ln k$ and $\ln b$ are constants, there is a linear relationship between the variables $\ln y$ and x.

If the variables x and y are connected by an exponential model, the graph of $\ln y$ against x is linear.

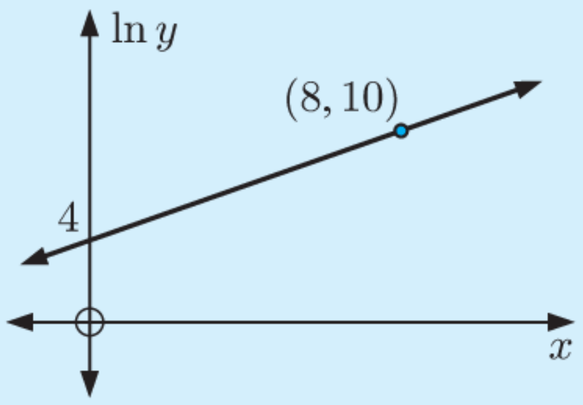




Once we have used linear regression to find an equation connecting $\ln y$ and x, we can rearrange the equation to make y the subject.

Example 3 **Self Tutor**

Find the exponential model connecting x and y.



The graph of $\ln y$ against x is linear with gradient $m = \dfrac{10-4}{8-0} = 0.75$ and y-intercept $c = 4$.

$\therefore \ \ln y = 0.75x + 4$

$\therefore \ y = e^{0.75x+4}$

$\therefore \ y = e^{0.75x} \times e^4$

$\therefore \ y = e^4 \times (e^{0.75})^x$

$\therefore \ y \approx 54.6 \times 2.12^x$

Example 4 **Self Tutor**

The mass of bacteria in a culture is measured each day for five days.

t (days)	1	2	3	4	5
M (grams)	3.6	5.7	9.1	14.6	23.3

a Draw scatter diagrams of M against t and $\ln M$ against t.

b Explain why M and t are related by an exponential model.

c Find the equation connecting M and t.

d Estimate the original mass of the bacteria.

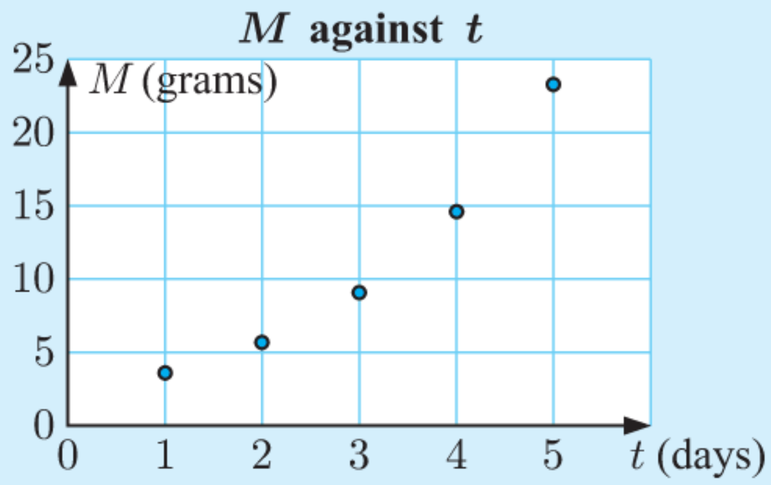


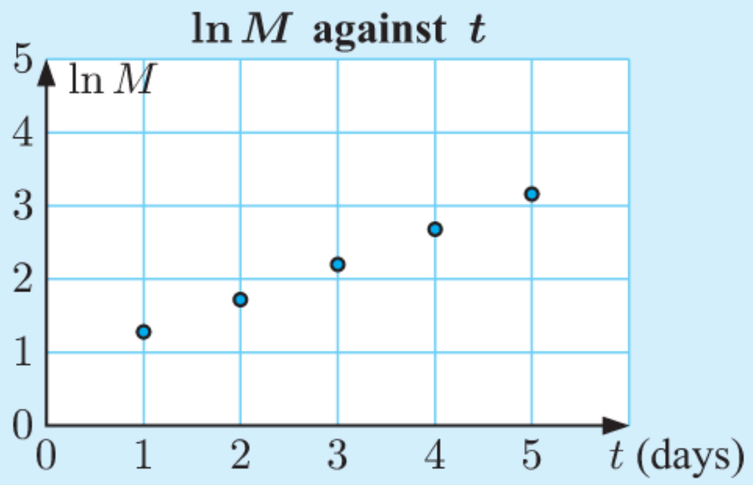
GRAPHICS CALCULATOR INSTRUCTIONS

a

t (days)	1	2	3	4	5
M (grams)	3.6	5.7	9.1	14.6	23.3
$\ln M$	1.28	1.74	2.21	2.68	3.15

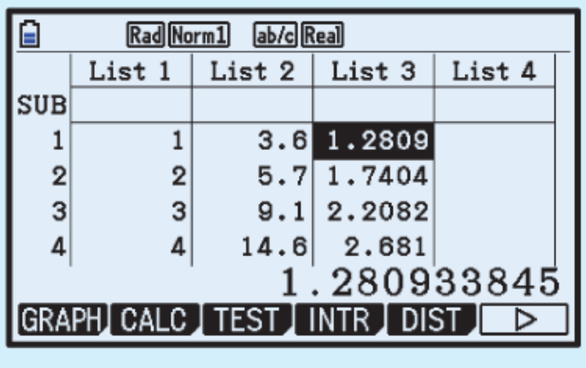


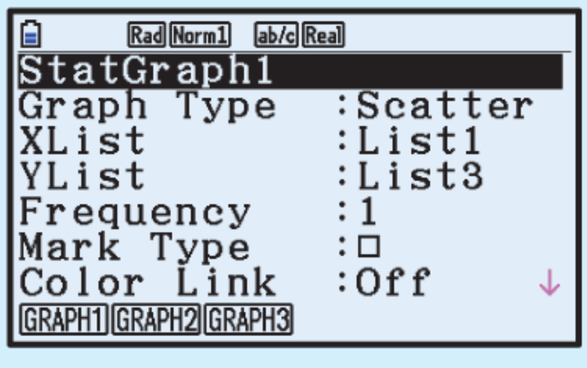


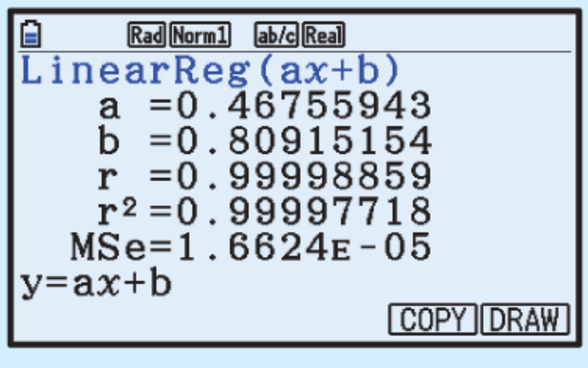


b The graph of $\ln M$ against t appears linear, so an exponential model is appropriate.

c t M $\ln M$







Using technology, the linear model connecting $\ln M$ and t is

$$\ln M \approx 0.468t + 0.809$$
$$\therefore \quad M \approx e^{0.468t+0.809}$$
$$\therefore \quad M \approx e^{0.468t} \times e^{0.809}$$
$$\therefore \quad M \approx e^{0.809} \times (e^{0.468})^t$$
$$\therefore \quad M \approx 2.25 \times (1.60)^t$$

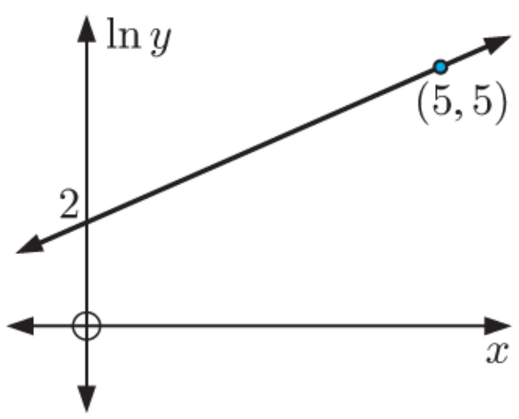
d When $t = 0$, $M \approx 2.25 \times (1.60)^0 \approx 2.25$

So, the original mass of the bacteria was approximately 2.25 g.

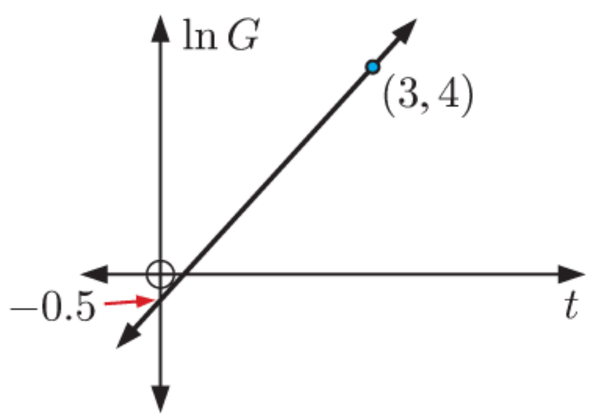
EXERCISE 8B

1 For each of the following graphs, find the exponential model connecting the variables.

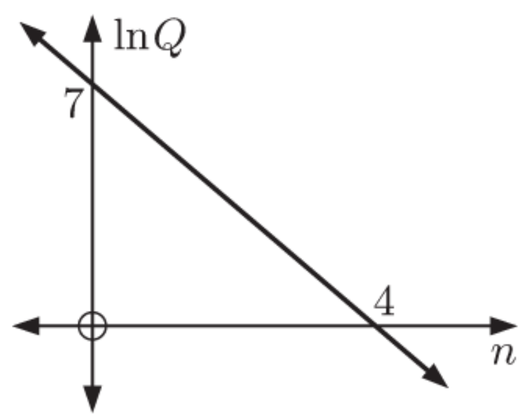
a



b



c



2 The population of a buffalo herd was recorded every 5 years after 1990.

Years since 1990 (x)	5	10	15	20	25
Population (y)	440	597	819	1120	1540

a Draw a scatter diagram of $\ln y$ against x.

b Explain why x and y are related by an exponential model.

c Find the linear model connecting $\ln y$ and x.

d Find the exponential model connecting x and y.

e Estimate the population of the herd in: **i** 1998 **ii** 2025.

3 The table below shows the concentration of venom in the blood of a snake bite victim at various times after an injection was administered to neutralise it.

Time (t minutes)	10	20	30	40	50	60	70
Venom concentration (V units)	104.6	36.5	12.7	4.43	1.55	0.539	0.188

a Draw scatter diagrams of V against t and $\ln V$ against t.

b Explain why the model connecting V and t is exponential.

c Find the linear model connecting $\ln V$ and t.

d Find the exponential model connecting V and t.

e The victim is considered to be "safe" when the venom concentration is less than 0.01 units. How long will it take to reach this "safe" level?



DISCUSSION

1 Will a linear model fitted to x against y necessarily be the same as the linear model fitted to y against x?

2 Notice that for an exponential model:

$$y = kb^x$$
$$\therefore \quad b^x = \frac{y}{k}$$
$$\therefore \quad x \ln b = \ln\left(\frac{y}{k}\right) = \ln y - \ln k$$
$$\therefore \quad x = -\frac{\ln k}{\ln b} + \frac{1}{\ln b} \ln y$$
$$\therefore \quad x = A + B \ln y \quad \text{for some constants } A \text{ and } B.$$

Will a logarithmic model fitted to x against y necessarily be the same as the exponential model fitted to y against x?

C POWER MODELS

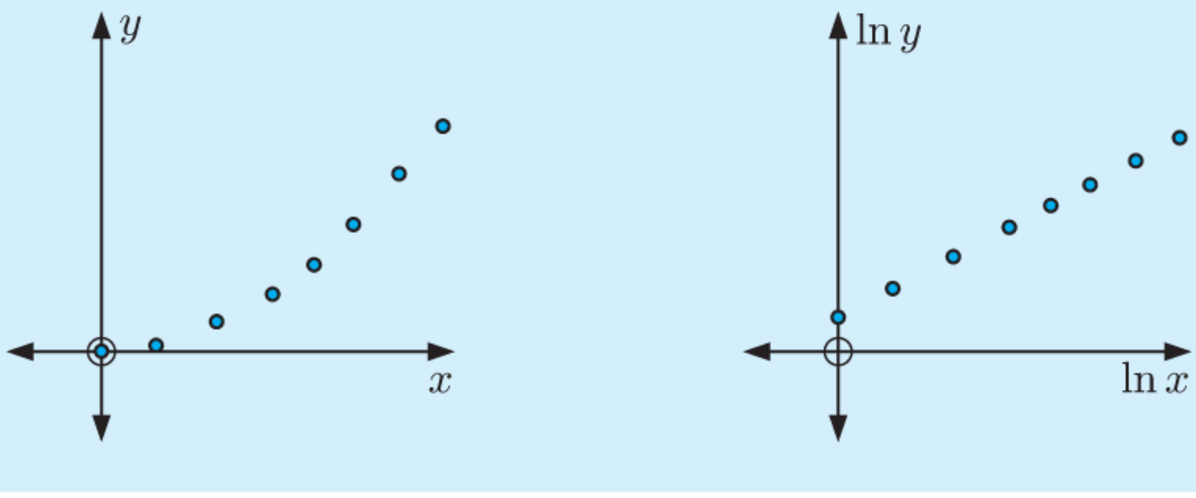
A **power model** has the form $y = ax^n$ where $a > 0$ and $n \neq 0$ are constants.

Using the laws of logarithms, notice that:

If $y = ax^n$, then $\ln y = \ln(ax^n)$ {taking the logarithm of both sides}
$= \ln a + \ln(x^n)$
$= \ln a + n \ln x$

Since $\ln a$ and n are constants, there is a linear relationship between the variables $\ln y$ and $\ln x$.

If the variables x and y are connected by a **power model**, the graph of $\ln y$ against $\ln x$ is linear.

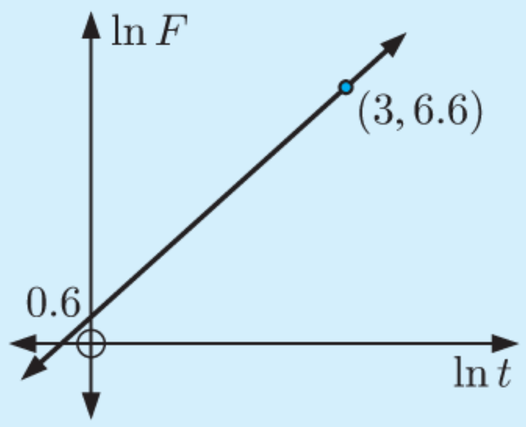


Since $y = ax^n$, all power models pass through the origin. However, since we cannot take the logarithm of 0, we will need to remove data such as $(0, 0)$ before starting our analysis.

Example 5

Self Tutor

Find the power model connecting F and t.



The graph of $\ln F$ against $\ln t$ is linear with

gradient $m = \dfrac{6.6 - 0.6}{3 - 0} = 2$ and y-intercept $c = 0.6$.

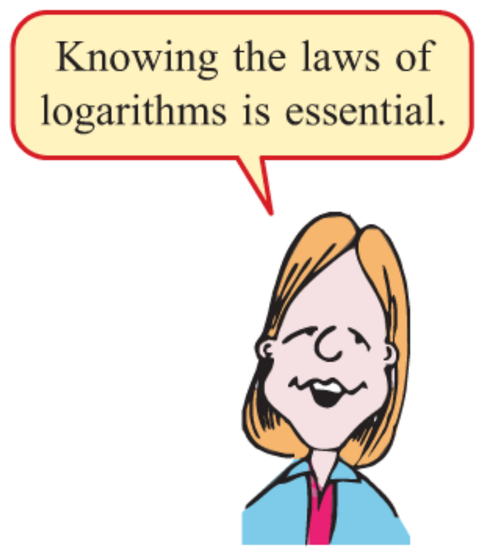
$\therefore \ \ln F = 2 \ln t + 0.6$

$\therefore \ \ln F = \ln(t^2) + \ln(e^{0.6})$

$\therefore \ \ln F = \ln(t^2 \times e^{0.6})$

$\therefore \ F = t^2 \times e^{0.6}$

$\therefore \ F \approx 1.82t^2$



Example 6

Self Tutor

The maximum speed of a canoe with different numbers of rowers is recorded in the table below:

Number of rowers (n)	4	6	10	14	18
Maximum speed (S km h^{-1})	8.7	10.3	12.6	14.2	15.9

a Draw scatter diagrams of S against n and $\ln S$ against $\ln n$.

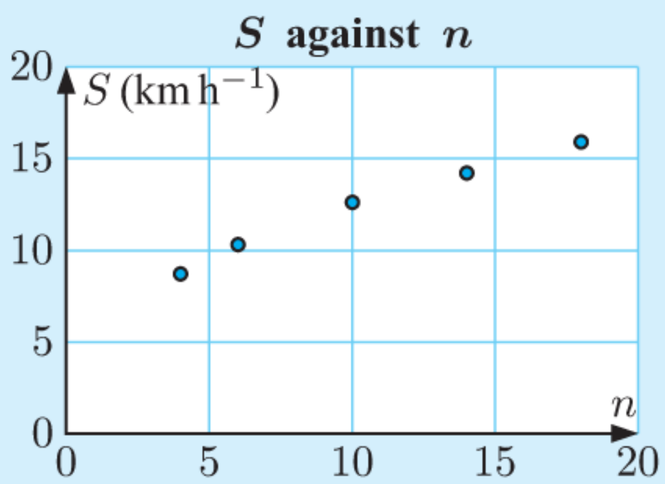
b Explain why S and n are related by a power model.

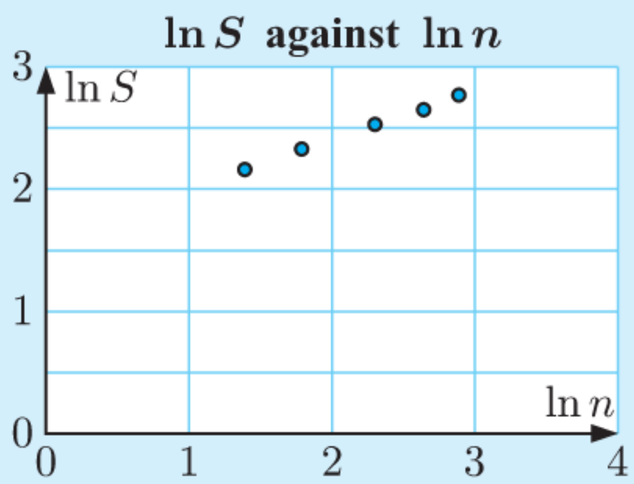
c Find the equation connecting n and S.

d Predict the maximum speed of the canoe if there are 8 rowers.

a

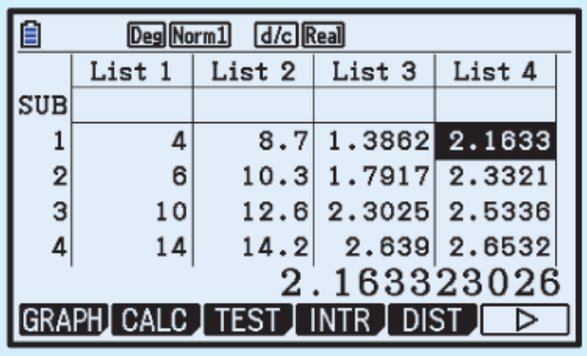
n	4	6	10	14	18
S	8.7	10.3	12.6	14.2	15.9
$\ln n$	1.39	1.79	2.30	2.64	2.89
$\ln S$	2.16	2.33	2.53	2.65	2.77

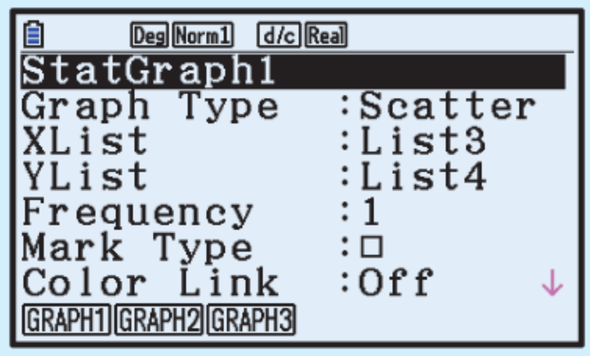


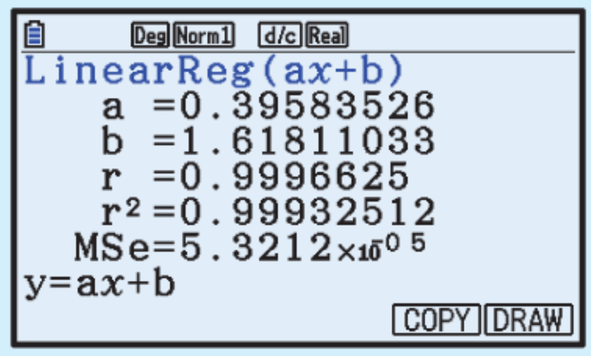


b The graph of $\ln S$ against $\ln n$ appears linear, so a power model is appropriate.

c







Using technology, the linear model connecting $\ln S$ and $\ln n$ is

$$\ln S \approx 0.396 \ln n + 1.618$$
$$\therefore \quad S \approx e^{1.618+0.396 \ln n}$$
$$\therefore \quad S \approx e^{1.618} \times e^{0.396 \ln n}$$
$$\therefore \quad S \approx 5.04 \times n^{0.396}$$

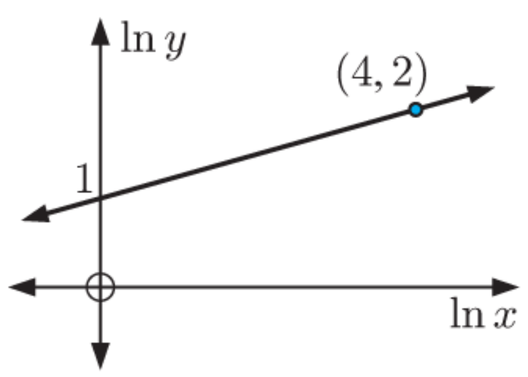
d When $n = 8$, $S \approx 5.04 \times 8^{0.396} \approx 11.5$

So, we would expect the maximum speed of a canoe with 8 rowers to be approximately 11.5 km h^{-1}.

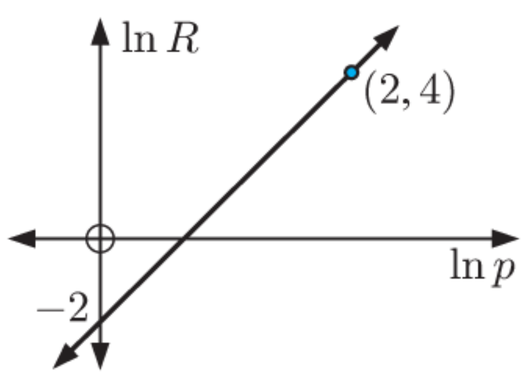
EXERCISE 8C

1 For each of the following graphs, find the power model connecting the variables.

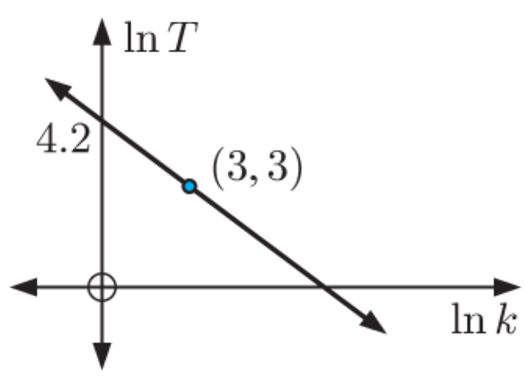
a



b



c



2 A tennis ball is dropped from the top of a cliff. The distance fallen after various times is measured.

Time (t seconds)	1.4	2.1	2.9	3.3	3.6	4.3	4.9
Distance fallen (D m)	10.0	21.3	40.6	52.1	63.4	89.2	117.7

a Draw a scatter diagram of $\ln D$ against $\ln t$.

b Explain why D and t are related by a power model.

c Find the linear model connecting $\ln D$ and $\ln t$.

d Find the power model connecting D and t.

3 When a force is applied to an object, the resulting *pressure* exerted on the object depends on the *surface area* where the force is applied. The pressure that results when 100 newtons of force is applied over various areas, is recorded below.

Area (a m^2)	4	7	15	20	24	30	35	40	45
Pressure (P pascals)	25	14.3	6.7	5	4.2	3.3	2.9	2.5	2.2

a Draw scatter diagrams of P against a and $\ln P$ against $\ln a$.

b Explain why P and a are related by a power model.

c Find the linear model connecting $\ln P$ and $\ln a$.

d Find the power model connecting P and a.

e Estimate the pressure that results when the 100 newton force is applied over an area of 10 m^2. Comment on the reliability of this estimate.

D PROBLEM SOLVING

When we obtain a set of data, we generally do not know which type of model connects the variables. We must use the techniques we have learnt to assess which model is the most appropriate.

To determine whether a **linear**, **logarithmic**, **exponential**, or **power** model is most appropriate to connect variables x and y:

Step 1: Draw scatter diagrams of y against x, y against $\ln x$, $\ln y$ against x, and $\ln y$ against $\ln x$.

Step 2: Find the regression line for each scatter diagram in which the variables appear linearly related. If more than one model appears appropriate, choose the one with the highest r^2 value.

Step 3: Write the equation connecting the variables x and y without logarithms.

EXERCISE 8D

1 When Tristan went fishing, he caught ten salmon. He measured and weighed each one:

Length (x cm)	32	28	15	42	39	22	25	45	35	47
Weight (y g)	300	200	33	680	540	100	150	830	400	950

a Draw scatter diagrams of:

i y against x **ii** y against $\ln x$ **iii** $\ln y$ against x **iv** $\ln y$ against $\ln x$.

b Explain why a power model is most appropriate.

c Find the linear model connecting $\ln y$ and $\ln x$. State the value of r^2.

d Find the power model connecting x and y.

2 The area affected by a forest fire t minutes after a lightning strike is recorded in the table below.

Time (t minutes)	2	3	5	7	9	10	12
Area affected (A hectares)	1.6	2.5	4.9	9.3	18.2	27.3	55

a Draw scatter diagrams of:

i A against t **ii** A against $\ln t$ **iii** $\ln A$ against t **iv** $\ln A$ against $\ln t$.

b Explain why an exponential model is most appropriate.

c Find the linear model connecting $\ln A$ and t. State the value of r^2.

d Find the exponential model connecting t and A.

3 A water balloon is filled from a hose. The diameter of the balloon after t seconds is recorded in the table:

Time (t seconds)	1	2	3	4	5	6	7	8
Diameter (D cm)	5.1	6.4	7.2	8.0	8.6	9.2	9.8	10

a Draw scatter diagrams of:

i D against t **ii** D against $\ln t$ **iii** $\ln D$ against t **iv** $\ln D$ against $\ln t$.

b Which model is most appropriate for the data? Explain your answer.

c Find the model connecting D and t.

d Estimate the diameter of the water balloon after 3.5 seconds. Comment on the reliability of your estimate.

4 A driver was timed to see how long he took to bring his car to rest from varying speeds:

Speed (v km h^{-1})	10	20	30	40	50	60	70	80	90
Stopping time (t seconds)	1.23	1.54	1.88	2.20	2.52	2.83	3.15	3.45	3.83

a Draw scatter diagrams of:

i t against v **ii** t against $\ln v$ **iii** $\ln t$ against v **iv** $\ln t$ against $\ln v$.

b Which model is most appropriate for the data? Explain your answer.

c Find the model connecting t and v.

d Estimate the stopping time from speed 110 km h^{-1}.

5 The table below shows the value of a house t years after it is built.

Time (t years)	1	4	6	10	12	15	20
Value (£$V \times 1000$)	257	281	299	335	354	390	450

a Find an appropriate model connecting V and t.

b Predict the value of the house after 25 years.

6 The German mathematician and astronomer **Johannes Kepler** (1571 - 1630) was an important figure in the 17th century scientific revolution. He used data from observations of planetary orbits to show that these motions are not random, that they obey certain mathematical laws, and that these laws can be written in algebraic form. He took the Earth as his "base unit", so that orbital periods are given as multiples of one Earth year, and orbital radii as multiples of one Earth orbit. Some of his observations are given in this table.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Orbital radius (r)	0.387	0.723	1.000	1.542	5.202	9.539
Orbital period (T)	0.241	0.615	1.000	1.881	11.862	29.457

Kepler showed that this data could be summarised by an equation of the form $T = ar^k$ where a and k are constants.

a Find values for a and k.

b Determine whether this is an appropriate model for the data.

c Kepler stated the relationship as "The square of the planet's year is proportional to the cube of its distance from the sun." Show that your results agree with this conclusion.

ACTIVITY 1 THE PENDULUM

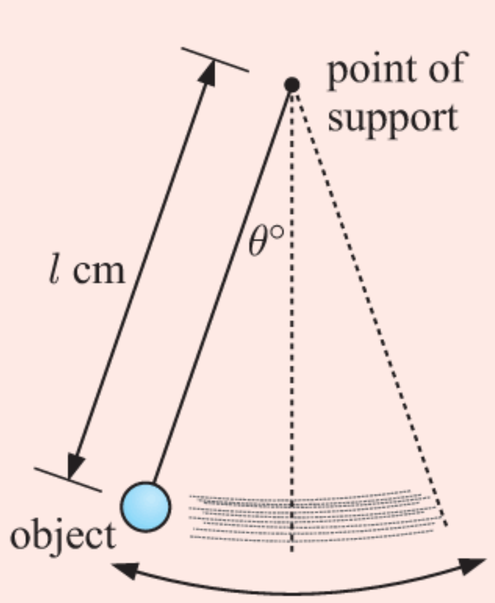
A **pendulum** can be made by tying a solid object to a piece of string and allowing the object to swing back and forth.

The **period** of a pendulum is the time taken for one complete oscillation.

The **length** of the pendulum is the distance from the point of support to the object's centre of mass.

Our aim is to determine a rule which connects the period of the pendulum (T s) with its length (l cm).

The period is independent of the angle θ, so for the purposes of the experiment, use $\theta \approx 15°$.



What to do:

1. Discuss which of the following ideas have merit when finding the period of the pendulum for a particular length:
   - several students should time the period using their stopwatches
   - timing 8 complete swings and finding the average is better than timing one complete swing
   - if several students do the timing, the highest and lowest scores should be removed and the remaining scores averaged.
2. Discuss possible factors which could lead to inaccurate results.
3. After deciding on a method for determining the period, measure the period for pendulum lengths 20 cm, 30 cm, 40 cm, ..., 100 cm. Record your results in a table.
4. Determine the law connecting T and l.

Length (l cm)	*Period* (T s)
20	
30	
40	
⋮	
100	

ACTIVITY 2 SURGE, TERMINAL VELOCITY, AND LOGISTIC MODELS

Click on the icon to investigate other types of models which can all be studied by transforming variables and using linear regression.



E NON-LINEAR REGRESSION

In some cases, we cannot transform the data into a form where we can use linear regression.

For example, we cannot transform the general quadratic model $y = ax^2 + bx + c$ into the linear form $Y = \alpha X + \beta$ where X and Y are functions of x and y, and α and β are constants.

In these cases we can use **non-linear regression** to fit the model.

NON-LINEAR REGRESSION

In the linear regression procedure, our aim was to minimise SS_{res}, the **sum of the squared residuals**. The **residuals** are the *vertical* distances between each point and the line. The values of the coefficients that gave the minimum value of SS_{res}, gave us the line of best fit.

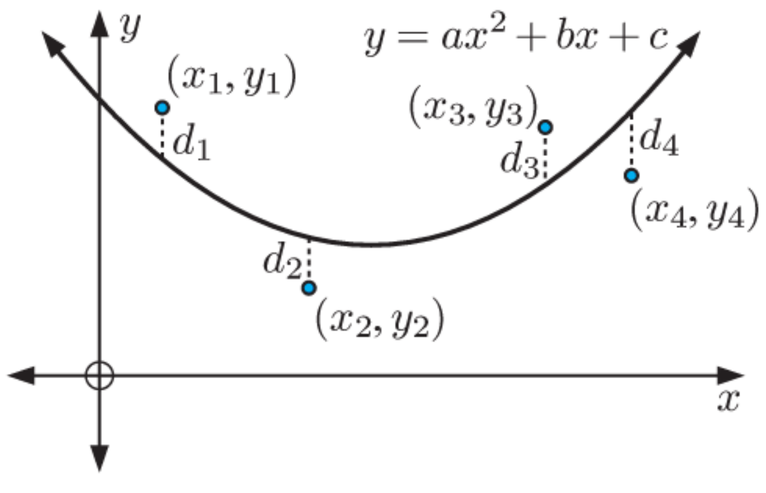
In non-linear regression, we apply these same ideas. However, instead of a line, we consider the graph of a non-linear function.

For example, consider the quadratic model $y = ax^2 + bx + c$.

We want to minimise $SS_{\text{res}} = \sum_{i=1}^{n} {d_i}^2$

$$= \sum_{i=1}^{n} (y_i - \widehat{y_i})^2$$

$$= \sum_{i=1}^{n} \left(y_i - (a{x_i}^2 + bx_i + c)\right)^2$$



In **Chapter 7**, we used quadratic theory to find the coefficients which minimised SS_{res}. This is not possible in this case.

In general, there is no analytic solution to this optimisation problem, so instead we use numerical methods to minimise SS_{res}.

ACTIVITY 3 USING TECHNOLOGY TO PERFORM NON-LINEAR REGRESSION

In this Activity, we will use a spreadsheet to explore non-linear regression.

What to do:

1 The data below was generated using an exponential model $y = e^{\lambda x}$ where $\lambda = 4.2$.

x	-0.3	0.2	0	-0.4	0.1	0.3	-0.5	-0.2	0.4	0.5	-0.1
y	0.20	1.47	1.42	0.13	0.65	3.12	0.07	0.43	4.58	4.52	0.40

Click on this icon to obtain a spreadsheet that performs non-linear regression to fit an exponential model $y = e^{\lambda x}$.

SPREADSHEET



a What do the values in the column on the far left represent?

b Explain how the spreadsheet is estimating λ.

c How can the *accuracy* of the estimate be improved in the spreadsheet?

d Use logarithmic transformations to fit an exponential model to the data. How does your result compare with the value from the spreadsheet?

2 Consider fitting the model $y = \ln(x + c)$ to the data below:

x	0	1	2	3	4	5	6	7	8	9	10
y	0.38	1.05	0.88	1.60	1.74	1.92	2.42	1.73	2.71	2.10	2.06

a Write down an appropriate range that c might lie within.

b Create a spreadsheet similar to that in **1** to fit this model.

c Produce a scatter diagram for the data, and graph the fitted model on the same set of axes.

d Comment on how well the fitted model fits the data. Include the value of SS_{res} in your answer.

3 Click on this icon to obtain a spreadsheet that performs non-linear regression to fit a quadratic model $y = ax^2 + bx + c$ to the data below:

SPREADSHEET



x	-3	-2	0	2	-4	3	-1	5	1	-5	4
y	5.61	0.56	0.99	20.61	18.15	29.8	-2.72	70.57	6.55	30.38	49.38

a Write down the ranges of possible values that are being considered for a, b, and c.

b Find the values of a, b, and c which minimise SS_{res}.

c Produce a scatter diagram for the data and graph the fitted model on the same set of axes.

d Consider the number of rows that were necessary to calculate a, b, and c to just 1 decimal place of accuracy. Discuss the problems with using this spreadsheet approach for non-linear regression.

In examinations, you will only have access to your graphics calculator. This will have functions to fit quadratic, cubic, and quartic models. Click on the icon to obtain instructions.



DISCUSSION

What your calculator does when it fits polynomial models is not actually "non-linear" regression. The regression algorithm that it is performing is still linear, just in more than two variables.

For example, consider the quadratic model $y = ax^2 + bx + c$. If we let $z = x^2$, then we have $y = az + bx + c$, which is linear in terms of the variables x, y, and z.

Does the "linear" in linear regression refer to the linearity of the *variables* or the *coefficients*?

EXERCISE 8E

1 Consider the data in the table below:

x	9.6	7.4	3.6	4.6	5.4	10.6	12.1	4.1	6.8	9.5
y	0.4	0.8	3.6	1.4	0.9	0.4	0.2	2.0	0.6	0.6

The model $y = \dfrac{2.482}{x - 2.904}$ was fitted to the data using non-linear regression.

a Draw a scatter diagram of the data, and sketch the fitted model on the same set of axes.

b Calculate the residual of each data point.

c Hence calculate SS_{res} for the model.

d Comment on how well the model fits the data.

2 The model $y = ax^n$ was fitted to the data below:

x	2.7	3.7	5.7	9.1	2.0	9.0	9.4	6.6	6.3	0.6
y	24.6	44.4	70.2	138.1	20.2	135.1	148.4	85.0	86.7	10.2

Using non-linear regression, it was found that $a \approx 6.894$ and $n \approx 1.359$.

a Draw a scatter diagram of the data, and sketch the fitted model on the same set of axes.

b Calculate the residual of each point.

c Hence calculate SS_{res}.

d Use $\ln x$ and $\ln y$ to fit the power model to the data. How does this model compare to the one above?

3 The table below shows the prices for pizzas at Pinkie's Pizza Parlour:

Diameter (x inches)	6	9	12	15	18	21
Price ($\$y$)	6.75	10.00	13.50	17.50	21.75	26.75

a Draw a scatter diagram for the data.

b Use your graphics calculator to fit a quadratic model of the form $y = ax^2 + bx + c$ to the data.

c Calculate SS_{res} for the model.

d Comment on whether your quadratic model is reasonable.



4 The table below shows the power a motor glider needs in order to overcome air resistance when travelling at various speeds:

Speed (v m s^{-1})	20	25	30	35	40	45	50
Power (P horsepower)	65	124	214	334	503	718	982

a Draw a scatter diagram for the data.

b Use your graphics calculator to fit a cubic model to the data.

c Calculate SS_{res} for the model.

5 Consider fitting a quartic model $y = ax^4 + bx^3 + cx^2 + dx + e$ to the data alongside:

x	1	2	3	4	5
y	9	10	7	5	6

a Use your graphics calculator to fit a quartic model.

b Draw a scatter diagram for the data and plot your fitted model on the same set of axes.

c Calculate SS_{res} for the model. Explain your answer.

DISCUSSION

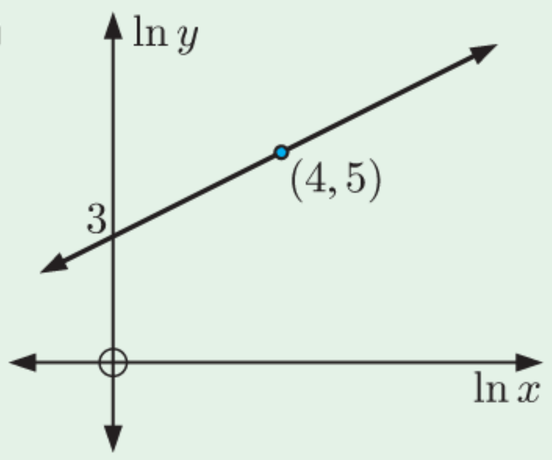
Suppose you have a set of n data points (x_1, y_1), ..., (x_n, y_n) where $n > 4$.

If you obtained a quadratic, a cubic, and a quartic model of best fit for the data, which would you expect to have the lowest SS_{res}? Discuss *why* this will have the lowest SS_{res}.

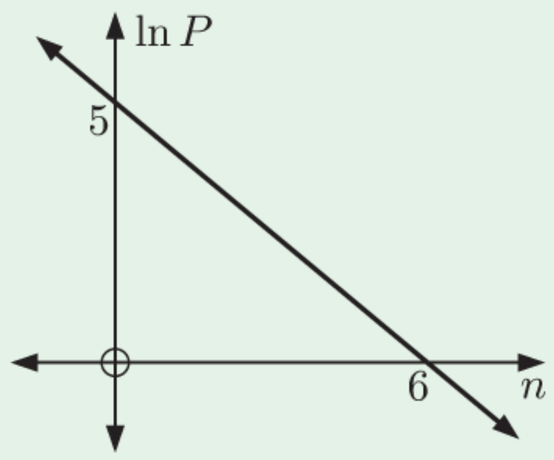
REVIEW SET 8A

1 Find the model connecting the variables:

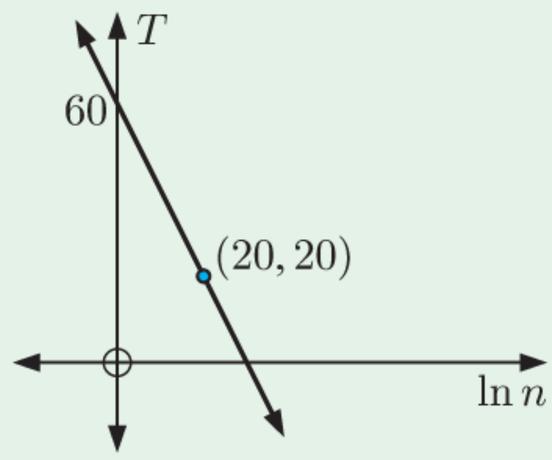
a



b



c



2 A cinema records the average number of audience members for a new action movie, and wants to use the data to estimate future attendance.

Days after release (d)	4	13	18	21	25	30	34	40	47
Audience members (A)	45	34	28	27	24	19	17	13	11

a Draw scatter diagrams of:

i A against d **ii** A against $\ln d$ **iii** $\ln A$ against d **iv** $\ln A$ against $\ln d$.

b Explain why an exponential model is appropriate.

c Find the linear model connecting $\ln A$ and d.

d Hence find the exponential model connecting A and d.

e Estimate the average number of audience members when the movie was first released.

3 The rate of a chemical reaction decreases with time as the reactants are slowly consumed.

Time (t seconds)	75	100	125	150	175	200
Rate of reaction (R)	45.4	42.2	39.3	36.5	34.1	31.8

a Draw scatter diagrams of:
i R against t **ii** R against $\ln t$ **iii** $\ln R$ against t **iv** $\ln R$ against $\ln t$.

b Which model for R and t is most appropriate? Explain your answer.

c Find the model connecting R and t.

d Hence estimate the rate of reaction after 160 seconds. Comment on the reliability of your answer.

4 The table below shows the alarm volume required to wake a sleeping person under the influence of alcohol.

Alcohol reading (a g per 100 mL)	0.02	0.04	0.06	0.08
Volume (V dB)	35.5	40	43	45

a Draw scatter diagrams of:
i V against a **ii** V against $\ln a$ **iii** $\ln V$ against a **iv** $\ln V$ against $\ln a$.

b Which model is most appropriate for the data? Explain your answer.

c Find the model connecting V and a.

d Estimate the alarm volume required to wake a person with an alcohol reading of:
i 0.15 g per 100 mL **ii** 0.05 g per 100 mL.

5 The table below shows the acceleration produced when a 125 newton force acts upon objects with different mass:

Mass (m kg)	10	20	30	40	50	60	70	80
Acceleration (a m s^{-2})	12.4	6.3	4.2	3.1	2.5	2.1	1.8	1.5

a Find the model which best fits the data. Explain your choice.

b According to Newton's second law of motion, the force F, mass m, and acceleration a are related by the equation $F = ma$. How closely does your model fit the theoretical model?

6 Hick's law states that the time T for a person to choose between n choices is given by the equation:

$$T = b\log(n+1)$$

where b is a constant estimated from data.

The table below shows data for a particular person:

n (choices)	2	4	8	16	32	64
T (seconds)	2.51	3.90	5.57	7.49	8.44	10.83

Using non-linear regression, it was found that $b \approx 5.830$.

a Draw a scatter diagram of T against n. Sketch the fitted model on the same set of axes.

b Calculate the residual for each data point.

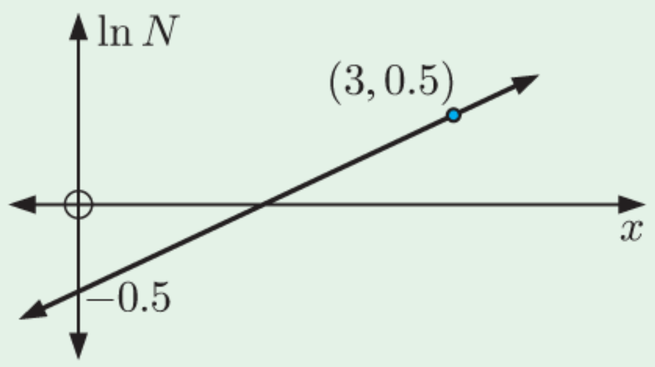
c Hence calculate SS_{res} for the model.

d Comment on how well the model fits the data.

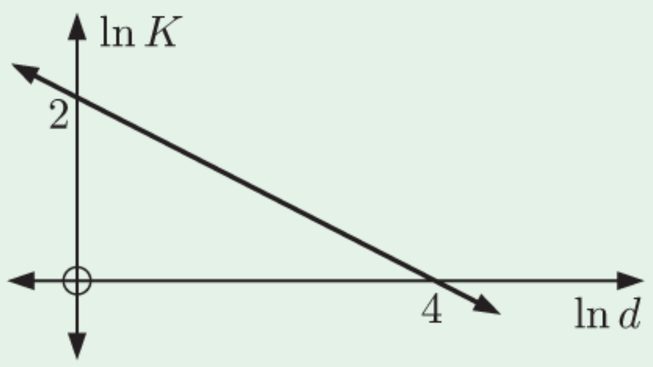
REVIEW SET 8B

1 Find the model connecting the variables:

a



b



2 Warm blooded animals maintain a remarkably constant body temperature by using most of the food they consume to produce heat. The larger the animal, the more heat it needs to produce. The following table gives the heat produced (H joules) for animals of various masses (m kg).

Mass (m kg)	0.02	0.5	4	20	50	100	500
Heat produced (H joules)	25	170	630	1670	2500	4200	16 750

a Draw scatter diagrams of:

i H against m **ii** H against $\ln m$

iii $\ln H$ against m **iv** $\ln H$ against $\ln m$.

b Explain why a power model is most suitable.

c Find the linear model connecting $\ln H$ and $\ln m$, stating the coefficient of determination r^2.

d Find the power model connecting H and m.

e Estimate the heat produced by an elephant of mass 5 tonnes. Comment on the reliability of your estimate.

3 The table below shows the temperature of a cup of coffee as it cools.

Time of cooling (t minutes)	5	10	15	20	25	30	35
Temperature (T °C above room temperature)	31.5	13.3	5.6	2.3	1.0	0.41	0.17

a Draw a scatter diagram of $\ln T$ against t.

b Does an exponential model seem appropriate?

c If the room temperature is kept constant at 25°C, estimate the temperature of the coffee when the experiment started.

d Estimate the temperature of the coffee after it has been cooling for 2 minutes.

4 The pH of a person's blood depends on the partial pressure of carbon dioxide contained within it.

Partial pressure of CO_2 (P torr)	10	15	20	25	30	35	40	45	50	55	60
Blood pH (b)	8.02	7.81	7.70	7.63	7.53	7.44	7.40	7.35	7.31	7.25	7.22

a Explain why a logarithmic model is suitable for this data.

b Find the logarithmic model connecting b and P.

c Hence estimate the partial pressure of CO_2 in a person's blood with pH 7.5 .

5 A bird bath is filled with water. Over time, the water evaporates as shown in the table below:

Time (t hours)	3	6	9	12	15	18	21	24
Water remaining (V litres)	6.7	3.6	2	1.1	0.6	0.32	0.18	0.10

a Draw scatter diagrams of:

i V against t **ii** V against $\ln t$

iii $\ln V$ against t **iv** $\ln V$ against $\ln t$.

b Find an appropriate model connecting V and t.

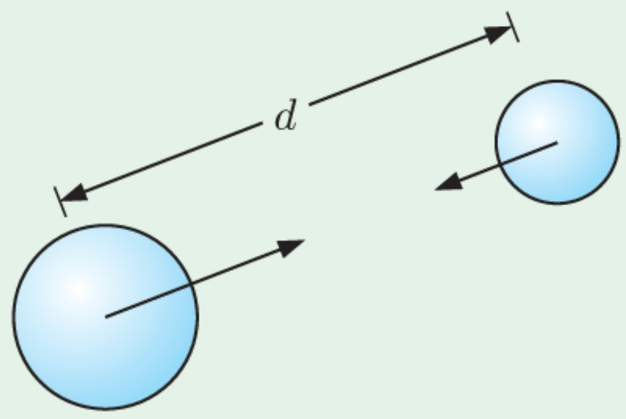
c Estimate the amount of water remaining in the bird bath after 5 hours.

d Estimate the amount of water which has evaporated after 10 hours.



6 Gravity is the force of attraction which exists between any two objects in the universe. The force of attraction F between two spherical masses at various distances d apart is shown below:

d (cm)	24	26	28	30	32	34	36	38	40
F (newtons)	237	202	174	152	133	118	105	95	85



a Find the model which best fits the data. Explain your choice.

b Estimate the force of attraction between the two spheres if they are:

i 29 cm apart **ii** 20 cm apart **iii** 50 m apart.

7 The table below shows the average time T it takes for a computer to sort n items using a **selection sort** algorithm:

n (items)	5	10	50	100	500	1000
T (microseconds)	17.4	28.2	157.6	305.5	2812.9	9701.0

a Draw a scatter diagram of T against n.

b Use your graphics calculator to fit a quadratic model to the data. Graph this model on your scatter diagram.

c Calculate SS_{res} for the model.

d Predict the time it will take to sort 10 000 items using the selection sort algorithm.

e Comment on the reliability of your prediction in **d**.

Chapter 9

Vectors

Contents:

A Vectors and scalars
B Geometric operations with vectors
C Vectors in the plane
D The magnitude of a vector
E Operations with plane vectors
F Vectors in space
G Operations with vectors in space
H The vector between two points
I Parallelism
J The scalar product of two vectors
K The angle between two vectors
L The vector product of two vectors
M Vector components



OPENING PROBLEM

An aeroplane in calm conditions is flying at 800 km h^{-1} due east. A cold wind suddenly blows from the south-west at 35 km h^{-1}, pushing the aeroplane slightly off course.

Things to think about:

a How can we illustrate the plane's movement and the wind using a scale diagram?

b What operation do we need to perform to find the effect of the wind on the aeroplane?

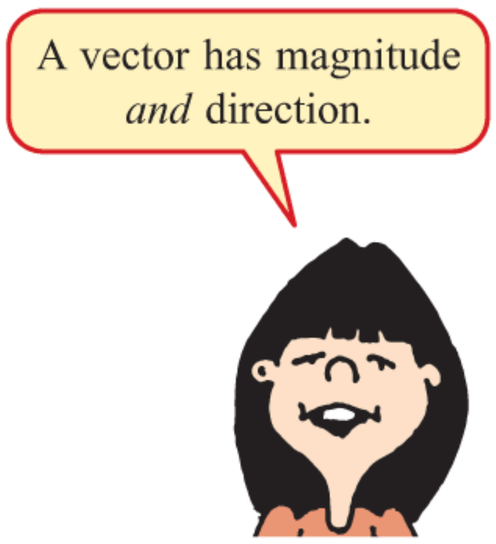
c Can you use a scale diagram to determine the resulting speed and direction of the aeroplane?

A VECTORS AND SCALARS

In the **Opening Problem**, the effect of the wind on the aeroplane is determined by both its speed *and* its direction. The effect would be different if the wind was blowing against the aeroplane rather than from behind it.

Quantities which have only magnitude are called **scalars**.

Quantities which have both magnitude and direction are called **vectors**.



The *speed* of the plane is a scalar. It describes how fast the plane is moving.

The *velocity* of the plane is a vector. It includes both its speed and also its direction.

Other examples of vector quantities are:

- acceleration
- displacement
- force
- momentum

For example, farmer Giles needs to remove a fence post. He starts by pushing on the post sideways to loosen the ground. Giles has a choice of how hard to push the post and in which direction. The force he applies is therefore a vector.



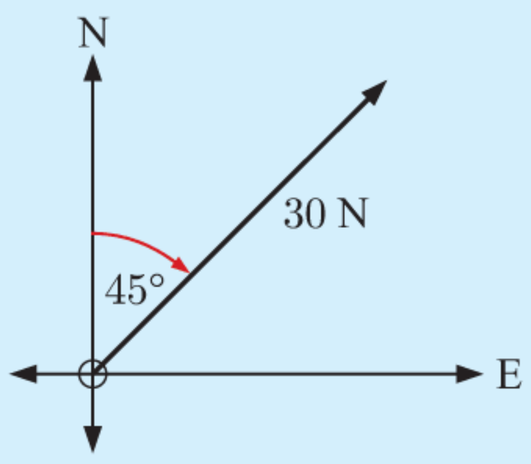
DIRECTED LINE SEGMENT REPRESENTATION

We can represent a vector quantity using a **directed line segment** or **arrow**.

The **length of the arrow** represents the size or magnitude of the quantity, and the **arrowhead** shows its direction.

Example 1 **Self Tutor**

Draw a scale diagram to represent a force of 30 newtons in a north-easterly direction.



Scale:
1 cm ≡ 10 N

EXERCISE 9A.1

1 Using a scale of 1 cm represents 10 units, sketch a vector to represent:

a 30 newtons in a south-easterly direction

b 25 m s^{-1} in a northerly direction

c an excavator digging a tunnel at a rate of 30 cm min^{-1} at an angle of 30° to the ground

d an aeroplane taking off at an angle of 10° to the runway with a speed of 50 m s^{-1}.

2 If $\xrightarrow{\quad 25\,\text{mm} \quad}$ represents a velocity of 50 m s^{-1} due east, draw a directed line segment representing a velocity of:

a 100 m s^{-1} due west **b** 75 m s^{-1} north-east.

3 Draw a scale diagram to represent:

a a force of 30 newtons in a north-westerly direction

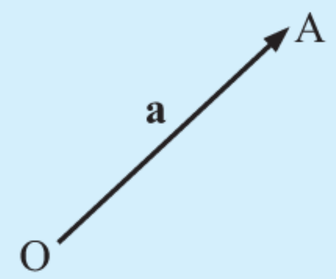
b a velocity of 36 m s^{-1} vertically downwards

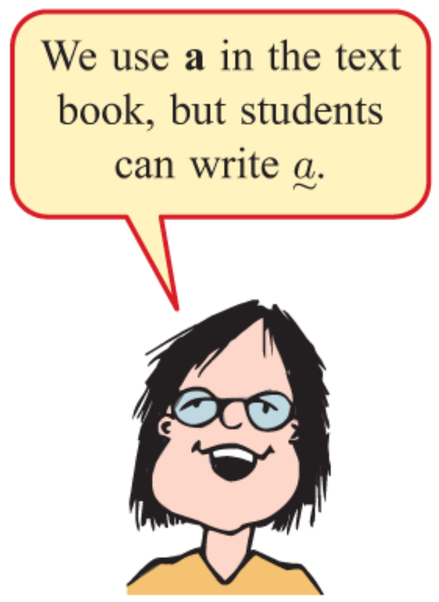
c a displacement of 4 units at an angle of 15° to the positive x-axis

d the velocity of an aeroplane taking off at an angle of 8° to the runway at a speed of 150 km h^{-1}.

VECTOR NOTATION

- The vector from the origin O to point A is called the **position vector** of point A. It is written as $\overrightarrow{\text{OA}}$, **a** or a.
  The **magnitude** or **length** of vector $\overrightarrow{\text{OA}}$ is written as $|\overrightarrow{\text{OA}}|$ or OA or $|\mathbf{a}|$ or $|a|$.





- $\overrightarrow{\text{AB}}$ is the vector which **originates** at A and **terminates** at B, $\overrightarrow{\text{AB}}$ is called the **displacement vector** of B relative to A.

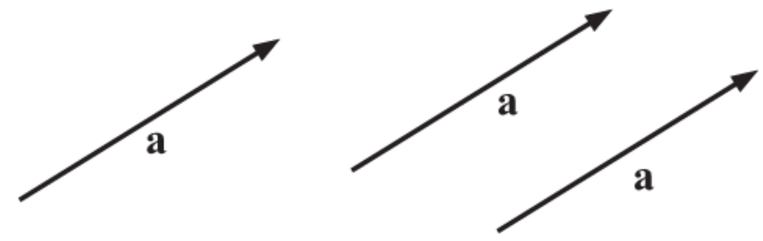


GEOMETRIC VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude *and* direction.

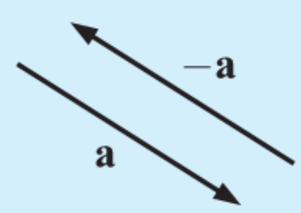
Equal vectors are **parallel** and **in the same direction**, and are **equal in length**. The arrows that represent them are translations of one another.

We can draw a vector with given magnitude and direction from *any* point, so we consider vectors to be **free**.



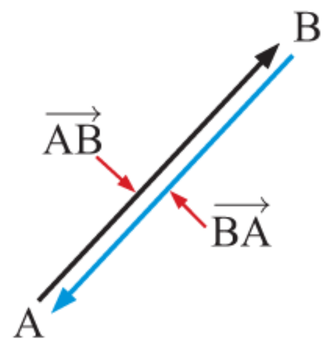
GEOMETRIC NEGATIVE VECTORS

The **negative** of a vector **a** is the vector parallel to **a** and with the same length, but in the opposite direction. It is denoted $-\mathbf{a}$.

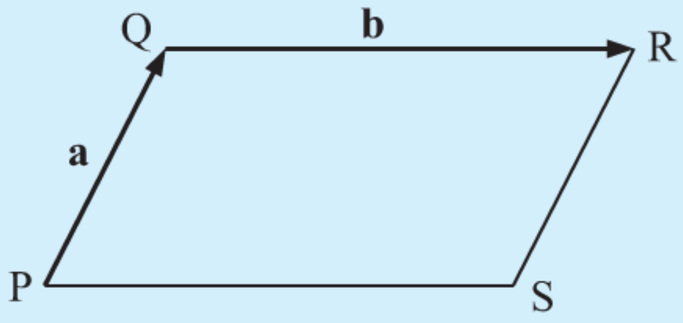


For example, $\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length, but they have opposite directions.

$\overrightarrow{BA}$ is the **negative** of $\overrightarrow{AB}$, and we write $\overrightarrow{BA} = -\overrightarrow{AB}$.



Example 2 **Self Tutor**



PQRS is a parallelogram in which $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$.

Find vector expressions for:

a $\overrightarrow{QP}$ **b** $\overrightarrow{RQ}$ **c** $\overrightarrow{SR}$ **d** $\overrightarrow{SP}$

a $\overrightarrow{QP} = -\mathbf{a}$ {the negative vector of $\overrightarrow{PQ}$}

b $\overrightarrow{RQ} = -\mathbf{b}$ {the negative vector of $\overrightarrow{QR}$}

c $\overrightarrow{SR} = \mathbf{a}$ {parallel to and the same length as $\overrightarrow{PQ}$}

d $\overrightarrow{SP} = -\mathbf{b}$ {parallel to and the same length as $\overrightarrow{QR}$, but opposite in direction}

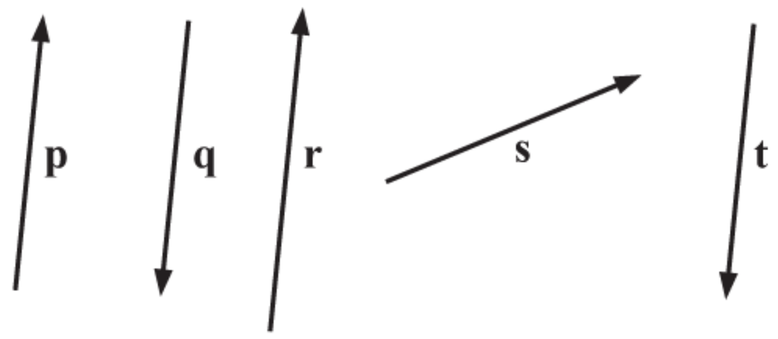
EXERCISE 9A.2

1 State the vectors which are:

a equal in magnitude **b** parallel

c in the same direction **d** equal

e negatives of one another.



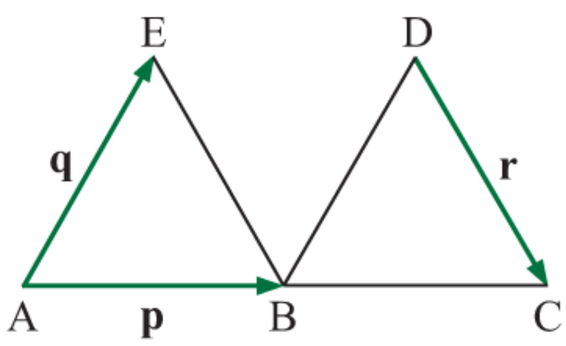
2 The figure shown consists of two equilateral triangles. A, B, and C are collinear.

$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AE} = \mathbf{q}$, and $\overrightarrow{DC} = \mathbf{r}$.

Which of the following statements are true?

a $\overrightarrow{EB} = \mathbf{r}$ **b** $|\mathbf{p}| = |\mathbf{q}|$ **c** $\overrightarrow{BC} = \mathbf{r}$

d $\overrightarrow{DB} = \mathbf{q}$ **e** $\overrightarrow{ED} = \mathbf{p}$ **f** $\mathbf{p} = \mathbf{q}$



3 ABCDEF is a regular hexagon.

a Write down the vector which:

i originates at B and terminates at C

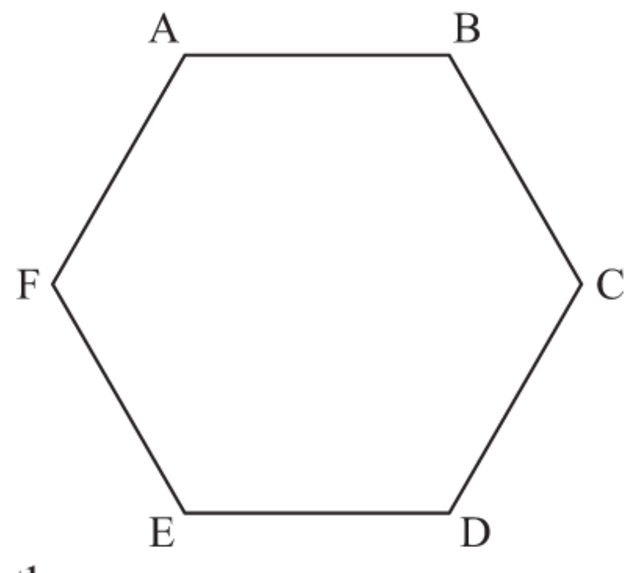
ii is equal to $\overrightarrow{AB}$.

b Write down *all* vectors which:

i are the negative of $\overrightarrow{EF}$

ii have the same length as $\overrightarrow{ED}$.

c Write down a vector which is parallel to $\overrightarrow{AB}$ and twice its length.



DISCUSSION

- Does a vector of length 0 have a direction?
- Is it possible to represent such a vector geometrically?

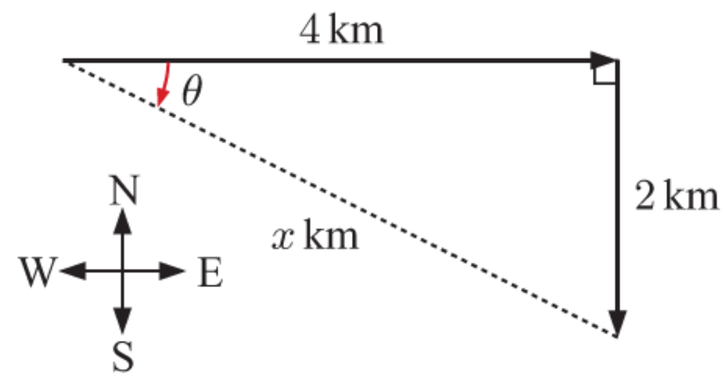
B GEOMETRIC OPERATIONS WITH VECTORS

In previous years we have often used trigonometry and Pythagoras' theorem to solve problems involving distances and directions.

A typical problem could be:

A runner runs east for 4 km and then south for 2 km.

How far is she from her starting point and in what direction?



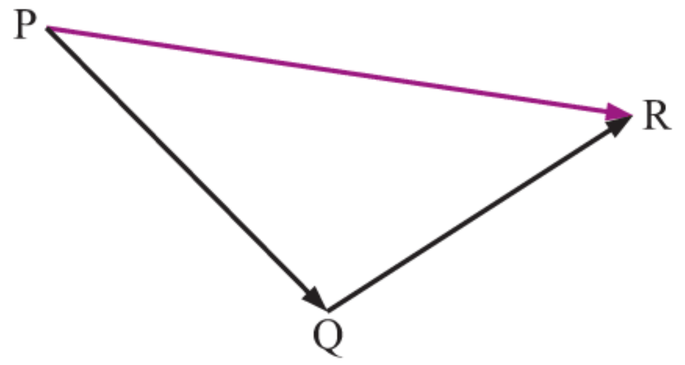
Problems like these can be well represented by vectors. In this case, the vectors are displacements. There is a displacement of 4 km to the east followed by a displacement of 2 km to the south. We want to know the *sum* of these displacements, including distance *and* direction.

GEOMETRIC VECTOR ADDITION

Suppose we have three towns P, Q, and R.

A trip from P to Q followed by a trip from Q to R has the same origin and destination as a trip from P to R.

This can be expressed in vector form as the sum
$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.



From examples like this, we can define vector addition geometrically as follows:

To construct $\mathbf{a} + \mathbf{b}$:

Step 1: Draw $\mathbf{a}$.

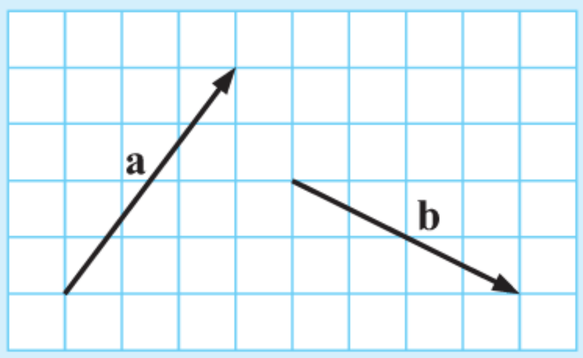
Step 2: At the arrowhead end of $\mathbf{a}$, draw $\mathbf{b}$.

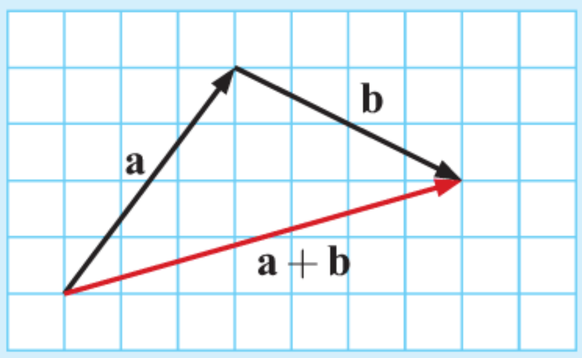
Step 3: Join the beginning of $\mathbf{a}$ to the arrowhead end of $\mathbf{b}$. This is vector $\mathbf{a} + \mathbf{b}$.



Example 3 **Self Tutor**

Given vectors $\mathbf{a}$ and $\mathbf{b}$ shown, construct $\mathbf{a} + \mathbf{b}$.





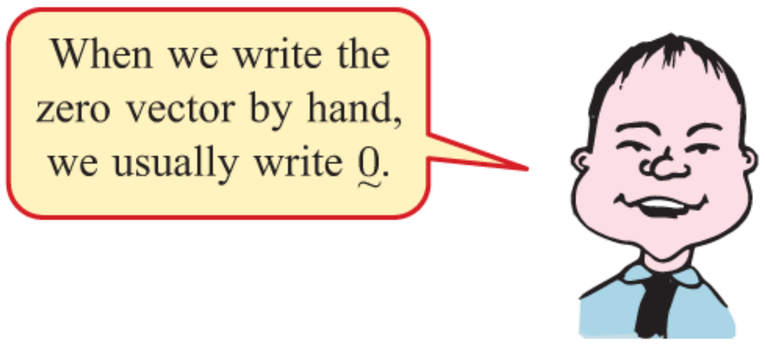
THE ZERO VECTOR

Having defined vector addition, we are now able to state that:

The **zero vector 0** is a vector of length 0.

For any vector **a**: $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$.



Example 4 — Self Tutor

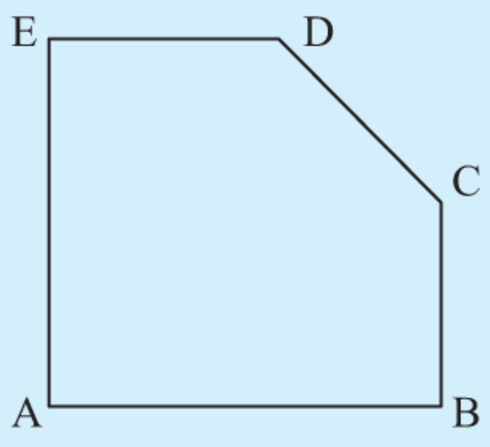
Find a single vector which is equal to:

a $\overrightarrow{BC} + \overrightarrow{CA}$

b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC}$

c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}$



a $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

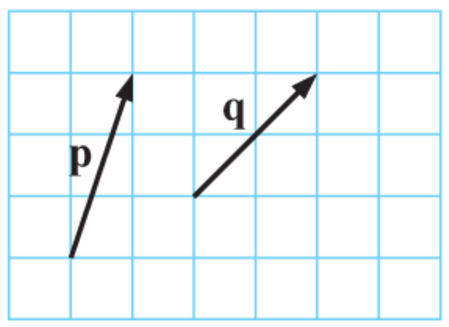
b $\overrightarrow{BA} + \overrightarrow{AE} + \overrightarrow{EC} = \overrightarrow{BC}$

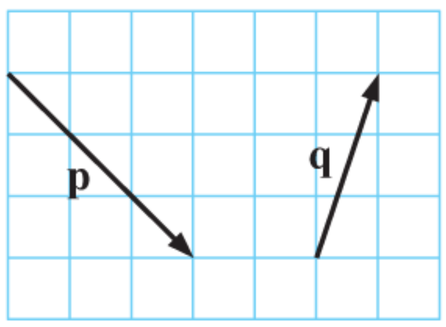
c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AA} = \mathbf{0}$

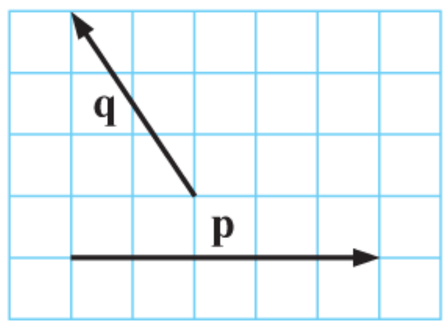
d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{AE}$

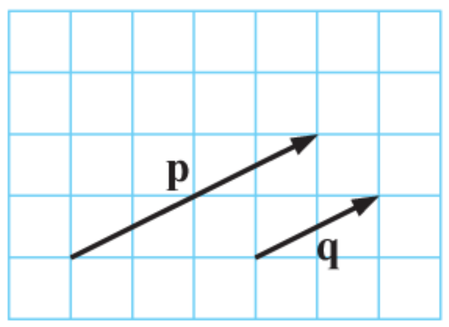
EXERCISE 9B.1

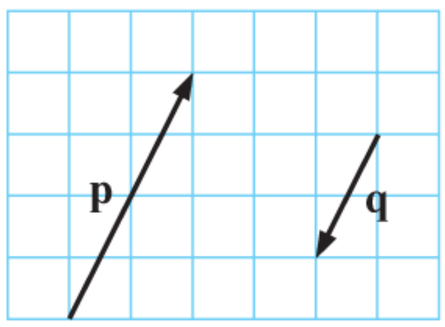
1 Use the given vectors $\mathbf{p}$ and $\mathbf{q}$ to construct $\mathbf{p} + \mathbf{q}$:

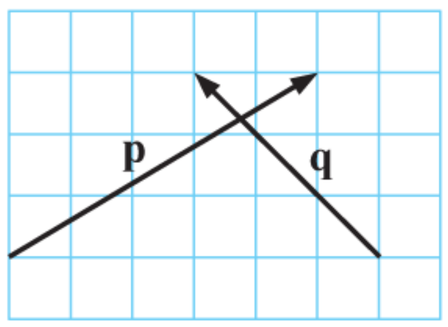
a 

b 

c 

d 

e 

f 

2 Find a single vector which is equal to:

a $\overrightarrow{AB} + \overrightarrow{BC}$

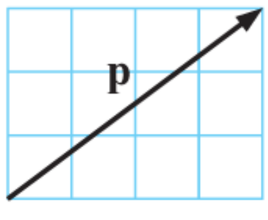
b $\overrightarrow{BC} + \overrightarrow{CD}$

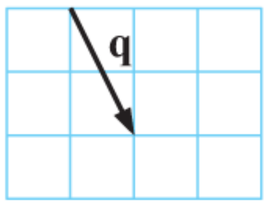
c $\overrightarrow{AB} + \overrightarrow{BA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

e $\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$

f $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}$

3 a Given 

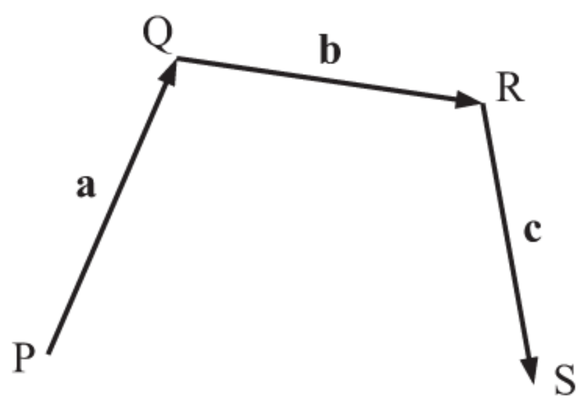
and 

use vector diagrams to find:

i $\mathbf{p} + \mathbf{q}$ ii $\mathbf{q} + \mathbf{p}$.

b For any two vectors $\mathbf{p}$ and $\mathbf{q}$, is $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$? Illustrate your answer.

4 Consider:



One way of finding $\overrightarrow{PS}$ is:

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{PR} + \overrightarrow{RS} \\ &= (\mathbf{a} + \mathbf{b}) + \mathbf{c}.\end{aligned}$$

Use the diagram to show that $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

5 Answer the **Opening Problem** on page **194**.

6 Susan and Cathy are pushing a heavy trolley containing groceries. Susan pushes the trolley with force 40 N in the direction 017°. Cathy pushes the trolley with force 35 N in the direction 335°.

a Use a scale diagram to estimate the resultant force from the two girls pushing.

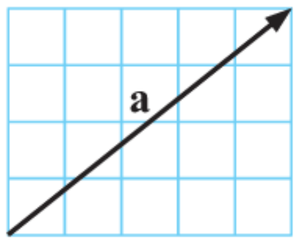
b Use trigonometry to find the resultant force more accurately.

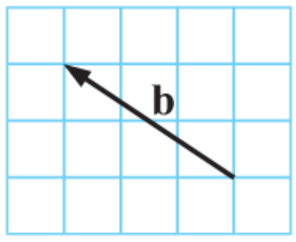


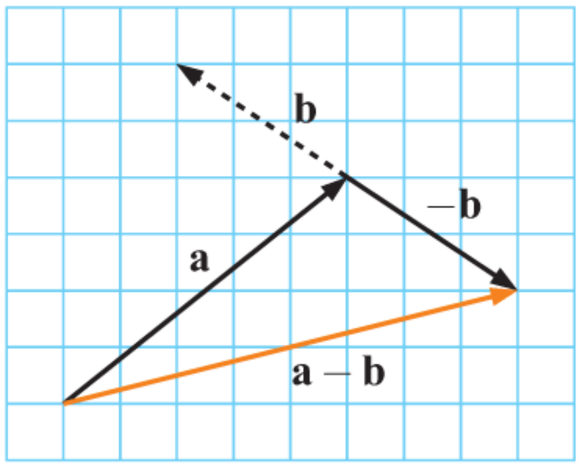
GEOMETRIC VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**. $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$

For example,

given 

and 

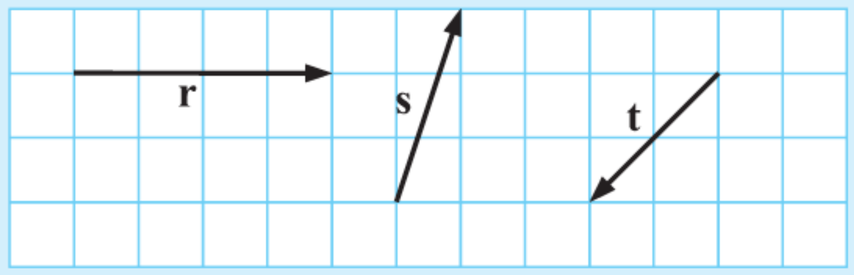
then 

Example 5 **Self Tutor**

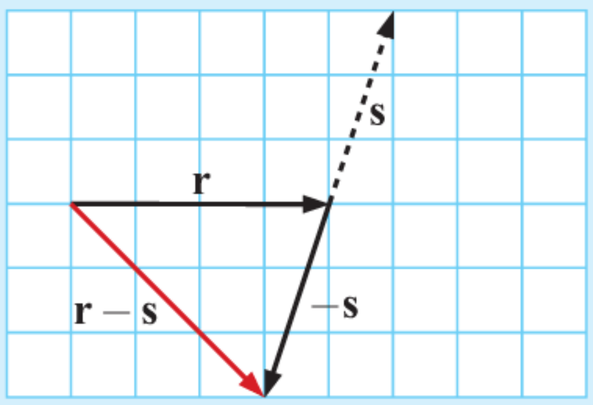
For the vectors illustrated, show how to construct:

a $\mathbf{r} - \mathbf{s}$

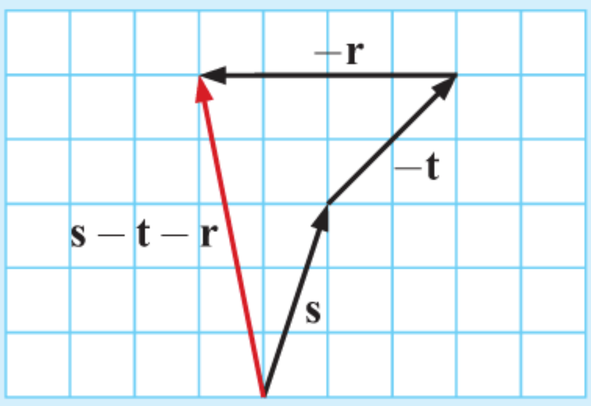
b $\mathbf{s} - \mathbf{t} - \mathbf{r}$



a



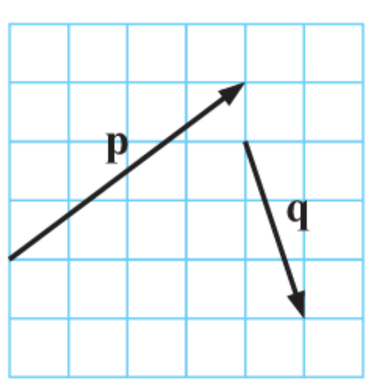
b



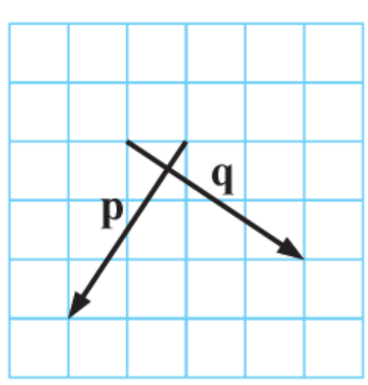
EXERCISE 9B.2

1 For the following vectors $\mathbf{p}$ and $\mathbf{q}$, show how to construct $\mathbf{p} - \mathbf{q}$:

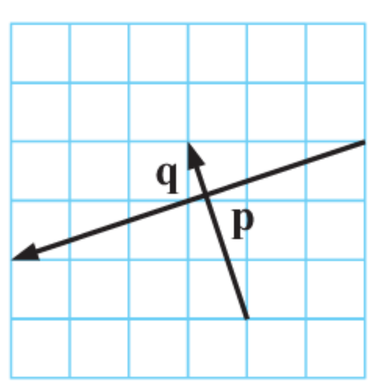
a



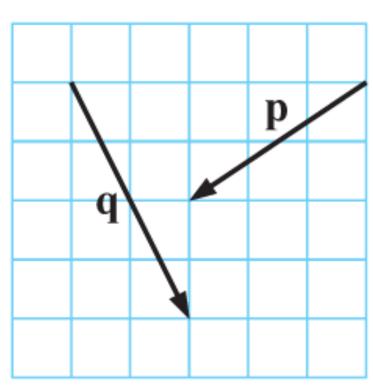
b



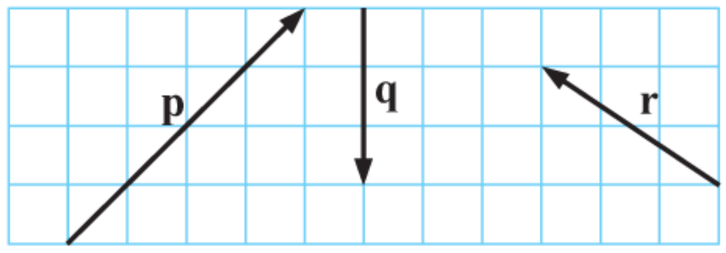
c



d



2 For the vectors illustrated, show how to construct:



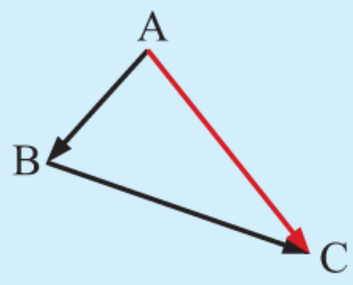
a $\mathbf{p} + \mathbf{q} - \mathbf{r}$ **b** $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **c** $\mathbf{r} - \mathbf{q} - \mathbf{p}$

Example 6 — Self Tutor

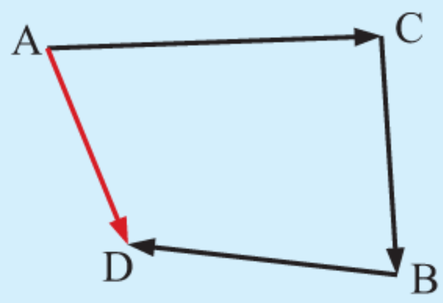
For points A, B, C, and D, simplify the following vector expressions:

a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB}$

a
$$\begin{aligned} &\overrightarrow{AB} - \overrightarrow{CB} \\ &= \overrightarrow{AB} + \overrightarrow{BC} \quad \{\text{as } \overrightarrow{BC} = -\overrightarrow{CB}\} \\ &= \overrightarrow{AC} \end{aligned}$$



b
$$\begin{aligned} &\overrightarrow{AC} - \overrightarrow{BC} - \overrightarrow{DB} \\ &= \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD} \\ &= \overrightarrow{AD} \end{aligned}$$



3 For points A, B, C, and D, simplify the following vector expressions:

a $\overrightarrow{AC} + \overrightarrow{CB}$ **b** $\overrightarrow{AD} - \overrightarrow{BD}$ **c** $\overrightarrow{AC} + \overrightarrow{CA}$

d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **e** $\overrightarrow{BA} - \overrightarrow{CA} + \overrightarrow{CB}$ **f** $\overrightarrow{AB} - \overrightarrow{CB} - \overrightarrow{DC}$

GEOMETRIC SCALAR MULTIPLICATION

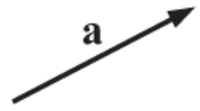
A **scalar** is a non-vector quantity. It has magnitude but no direction.

We can multiply vectors by scalars such as 2 and -3, or in fact any $k \in \mathbb{R}$.

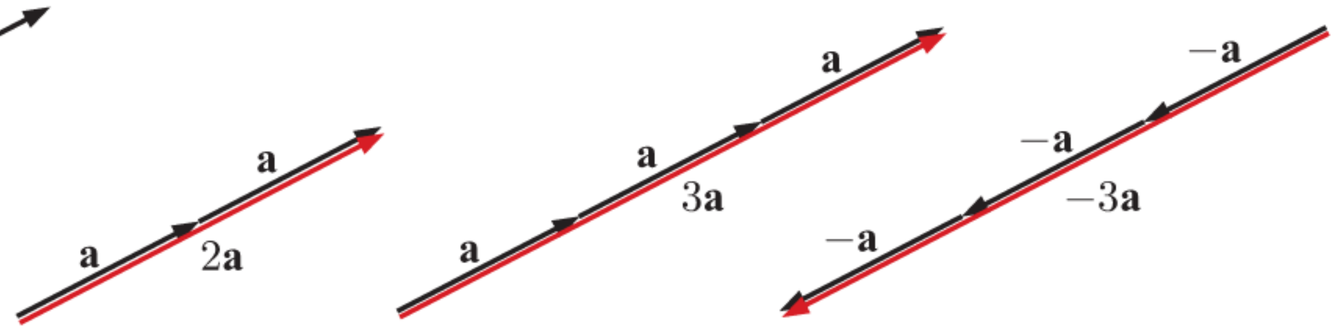
If $\mathbf{a}$ is a vector, we define:

- $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ has the same direction as $\mathbf{a}$ but is twice the length
- $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$ has the same direction as $\mathbf{a}$ but is three times the length
- $-3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a})$ has the opposite direction to $\mathbf{a}$ and is three times the length.



If **a** is 

then we have:



If **a** is a vector and k is a scalar, then $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.

- If $k > 0$, $k\mathbf{a}$ and **a** have the same direction.
- If $k < 0$, $k\mathbf{a}$ and **a** have opposite directions.
- If $k = 0$, $k\mathbf{a} = \mathbf{0}$, the zero vector.

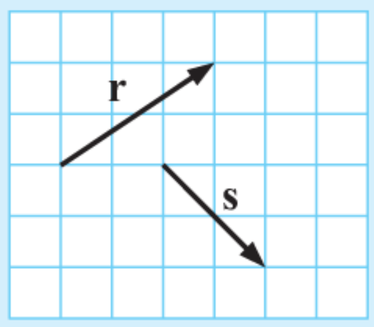
VECTOR SCALAR MULTIPLICATION



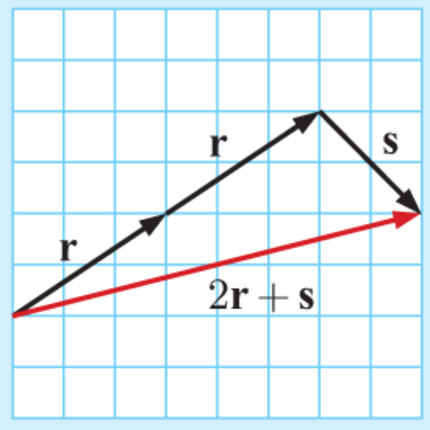
Example 7 **Self Tutor**

Given the vectors below, construct:

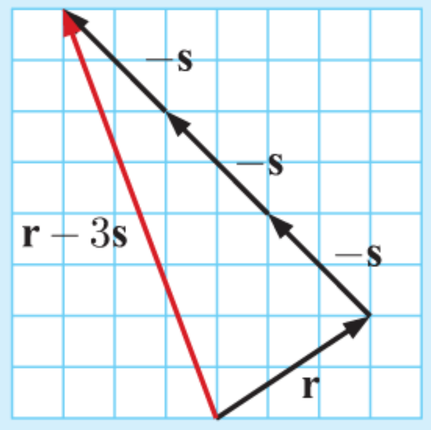
a $2\mathbf{r} + \mathbf{s}$ **b** $\mathbf{r} - 3\mathbf{s}$



a

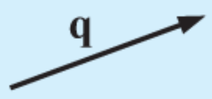


b

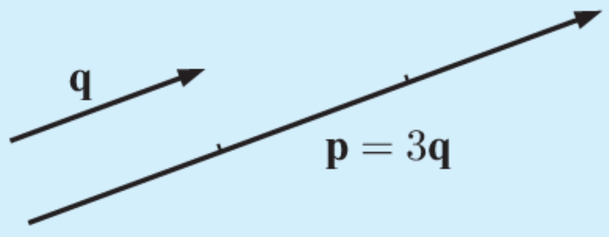


Example 8 **Self Tutor**

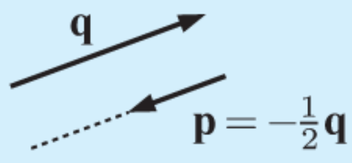
Sketch vectors **p** and **q** if: **a** $\mathbf{p} = 3\mathbf{q}$ **b** $\mathbf{p} = -\frac{1}{2}\mathbf{q}$

Suppose **q** is: 

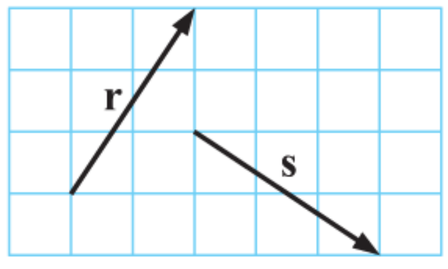
a



b



EXERCISE 9B.3

1 Given vectors 

, construct geometrically:

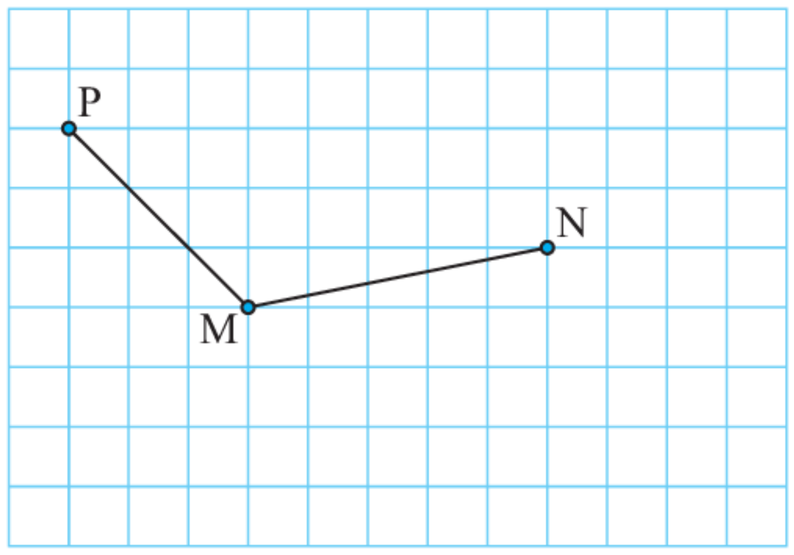
a $-\mathbf{r}$ **b** $2\mathbf{s}$ **c** $\frac{1}{2}\mathbf{r}$ **d** $-\frac{3}{2}\mathbf{s}$

e $2\mathbf{r} - \mathbf{s}$ **f** $2\mathbf{r} + 3\mathbf{s}$ **g** $\frac{1}{2}\mathbf{r} + 2\mathbf{s}$ **h** $\frac{1}{2}(\mathbf{r} + 3\mathbf{s})$

2 Sketch vectors **p** and **q** if:

a $\mathbf{p} = \mathbf{q}$ **b** $\mathbf{p} = -\mathbf{q}$ **c** $\mathbf{p} = 2\mathbf{q}$ **d** $\mathbf{p} = \frac{1}{3}\mathbf{q}$ **e** $\mathbf{p} = -3\mathbf{q}$

3 **a** Copy this diagram and on it mark the points:



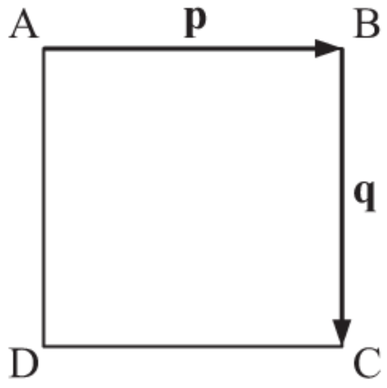
i X such that $\overrightarrow{MX} = \overrightarrow{MN} + \overrightarrow{MP}$

ii Y such that $\overrightarrow{MY} = \overrightarrow{MN} - \overrightarrow{MP}$

iii Z such that $\overrightarrow{PZ} = 2\overrightarrow{PM}$.

b What type of figure is MNYZ?

4

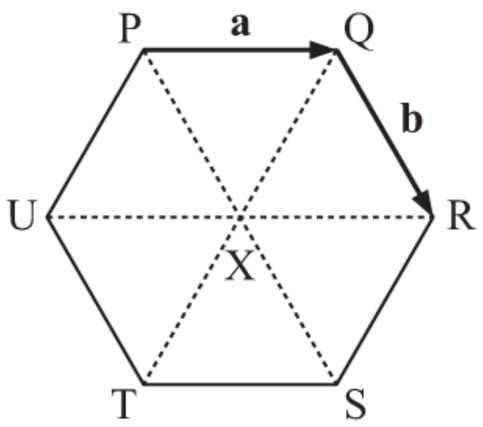


ABCD is a square. Its diagonals [AC] and [BD] intersect at M.

If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{BC} = \mathbf{q}$, find in terms of $\mathbf{p}$ and $\mathbf{q}$:

a $\overrightarrow{CD}$ **b** $\overrightarrow{AC}$ **c** $\overrightarrow{AM}$ **d** $\overrightarrow{BM}$

5

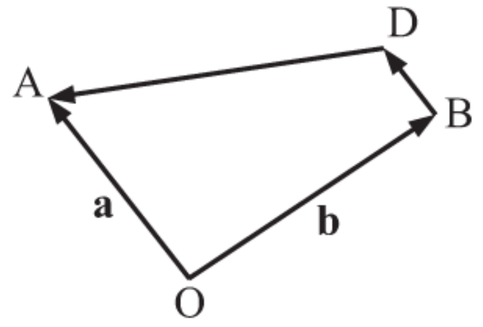


PQRSTU is a regular hexagon.

If $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{QR} = \mathbf{b}$, find in terms of $\mathbf{a}$ and $\mathbf{b}$:

a $\overrightarrow{PX}$ **b** $\overrightarrow{PS}$ **c** $\overrightarrow{QX}$ **d** $\overrightarrow{RS}$

6



In the given figure [BD] is parallel to [OA] and half its length.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, vector expressions for:

a $\overrightarrow{BD}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

d $\overrightarrow{OD}$ **e** $\overrightarrow{AD}$ **f** $\overrightarrow{DA}$

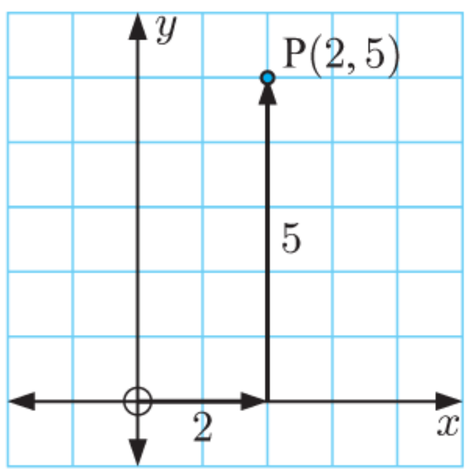
DISCUSSION

- What difficulties are there in using scale diagrams to solve vector problems?
- Can you extend your vector diagrams to higher dimensions?
- Can you draw the zero vector $\mathbf{0}$?

C VECTORS IN THE PLANE

When we plot points in the Cartesian plane, we move first in the x-direction and then in the y-direction.

For example, to plot the point $P(2, 5)$, we start at the origin, move 2 units in the x-direction, and then 5 units in the y-direction.

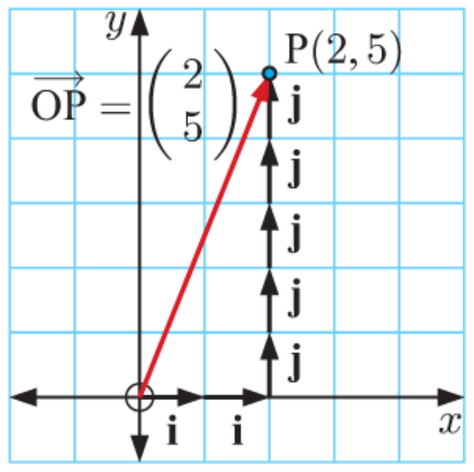


We have previously seen in transformation geometry that translating a point a units in the x-direction and b units in the y-direction can be achieved using the *translation vector* $\begin{pmatrix} a \\ b \end{pmatrix}$.

So, the vector from O to P is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Let $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be a translation 1 unit in the positive x-direction

and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be a translation 1 unit in the positive y-direction.



Moving from O to P is equivalent to two lots of **i** plus 5 lots of **j**.

$$\overrightarrow{OP} = 2\mathbf{i} + 5\mathbf{j}$$

$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

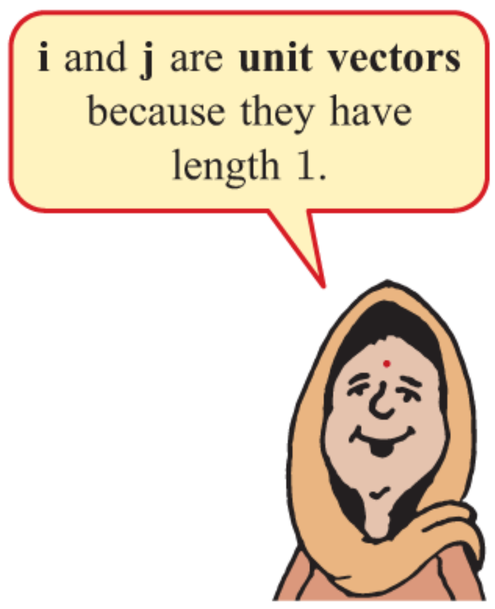
In the 2-dimensional plane, the point $P(x, y)$ has **position vector**

$$\overrightarrow{OP} = \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}} = \underbrace{x\mathbf{i} + y\mathbf{j}}$$

component form unit vector form

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the **base unit vector** in the x-direction

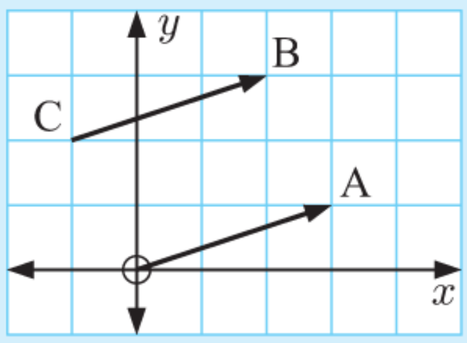
and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the **base unit vector** in the y-direction.



VECTOR EQUALITY

Two vectors are **equal** if their components are equal.

Example 9 **Self Tutor**



a Write vectors $\overrightarrow{OA}$ and $\overrightarrow{CB}$ in component form and in unit vector form.

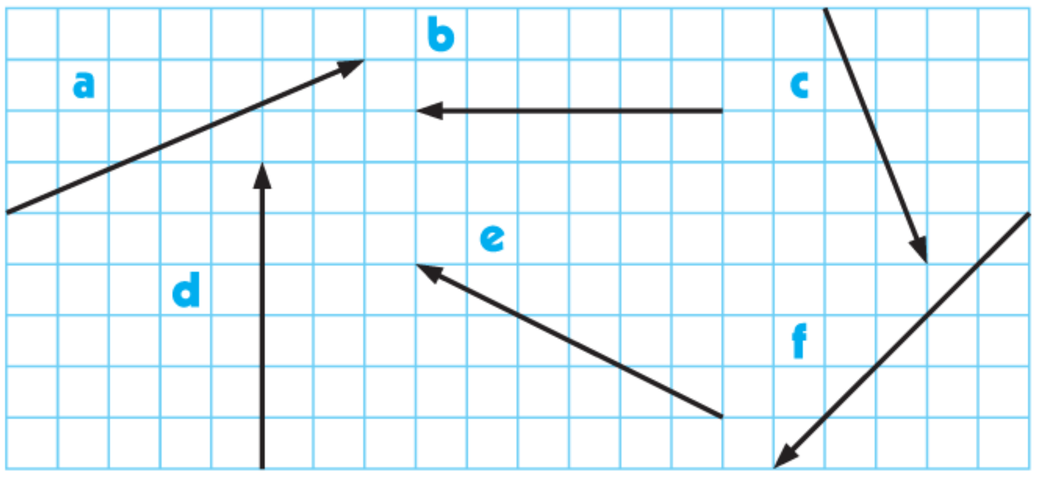
b Comment on your answers in **a**.

a $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$ $\quad$ $\overrightarrow{CB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$

b The vectors $\overrightarrow{OA}$ and $\overrightarrow{CB}$ are equal.

EXERCISE 9C

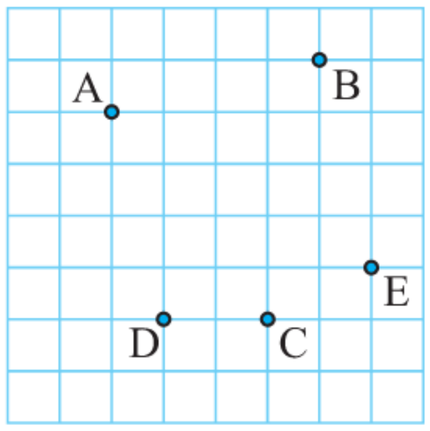
1 Write the illustrated vectors in component form and in unit vector form:



2 Write each vector in unit vector form, and illustrate it using an arrow diagram:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

3



a Write in component form and in unit vector form:

i $\overrightarrow{BA}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{DC}$

iv $\overrightarrow{AC}$ **v** $\overrightarrow{CA}$ **vi** $\overrightarrow{DB}$

vii $\overrightarrow{AE}$ **viii** $\overrightarrow{CE}$ **ix** $\overrightarrow{ED}$

b Use the given points to describe as many pairs of equal vectors as you can.

4 Write in component form, and illustrate using a directed line segment:

a $\mathbf{i} + 2\mathbf{j}$ **b** $-\mathbf{i} + 3\mathbf{j}$ **c** $-5\mathbf{j}$ **d** $4\mathbf{i} - 2\mathbf{j}$

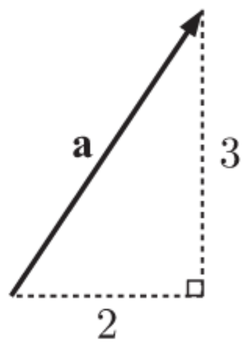
5 Write the zero vector $\mathbf{0}$ in component form.

6 Find the unknowns if:

a $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ **b** $\begin{pmatrix} a+1 \\ 2b-8 \end{pmatrix} = \begin{pmatrix} 9-a \\ a \end{pmatrix}$ **c** $\begin{pmatrix} 2x+3y \\ x-2 \end{pmatrix} = \begin{pmatrix} 11 \\ 2y \end{pmatrix}$

D THE MAGNITUDE OF A VECTOR

Consider vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\mathbf{i} + 3\mathbf{j}$.



The **magnitude** or **length** of $\mathbf{a}$ is represented by $|\mathbf{a}|$.

By Pythagoras, $|\mathbf{a}|^2 = 2^2 + 3^2 = 4 + 9 = 13$

$\therefore \quad |\mathbf{a}| = \sqrt{13}$ units $\quad \{\text{since } |\mathbf{a}| > 0\}$

The **magnitude** or **length** of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{{a_1}^2 + {a_2}^2}$.

Example 10 Self Tutor

If $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ and $\mathbf{q} = 2\mathbf{i} - 5\mathbf{j}$ find: **a** $|\mathbf{p}|$ **b** $|\mathbf{q}|$

a $\mathbf{p} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$

$\therefore \quad |\mathbf{p}| = \sqrt{3^2 + (-5)^2}$

$= \sqrt{34}$ units

b As $2\mathbf{i} - 5\mathbf{j} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$,

$|\mathbf{q}| = \sqrt{2^2 + (-5)^2}$

$= \sqrt{29}$ units

UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

For example, we have seen that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the base unit vectors in the positive x and y-directions respectively.

Example 11 **Self Tutor**

Find k given that $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector.

Since $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector, $\sqrt{(-\frac{1}{3})^2 + k^2} = 1$

$\therefore \sqrt{\frac{1}{9} + k^2} = 1$

$\therefore \frac{1}{9} + k^2 = 1$

$\therefore k^2 = \frac{8}{9}$

$\therefore k = \pm\frac{\sqrt{8}}{3}$

EXERCISE 9D

1 Find the magnitude of:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2 Find the length of:

a $\mathbf{i} + \mathbf{j}$ **b** $5\mathbf{i} - 12\mathbf{j}$ **c** $-\mathbf{i} + 4\mathbf{j}$ **d** $3\mathbf{i}$ **e** $k\mathbf{j}$

3 Which of the following are unit vectors?

a $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ **c** $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ **e** $\begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$

4 Find k such that the given vector is a unit vector:

a $\begin{pmatrix} 0 \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} k \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} k \\ k \end{pmatrix}$ **e** $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

5 Given $\mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{73}$ units, find the possible values of p.

E OPERATIONS WITH PLANE VECTORS

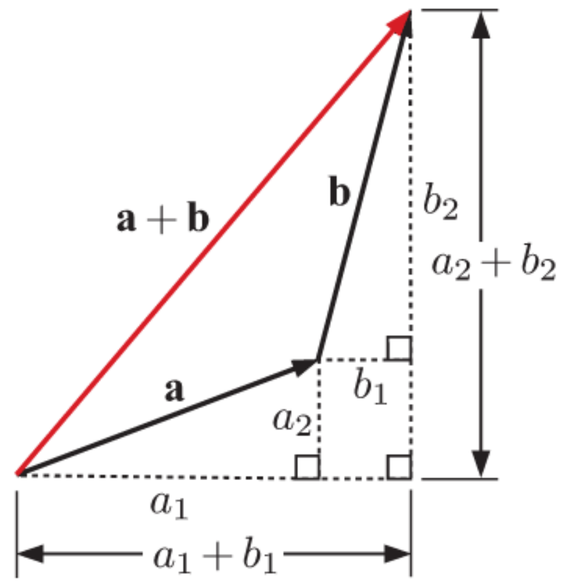
ALGEBRAIC VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Notice that:

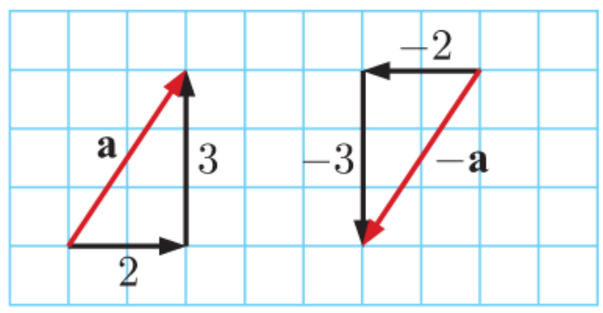
- the horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$
- the vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.



ALGEBRAIC NEGATIVE VECTORS

In the diagram we see the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and its negative $-\mathbf{a} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.



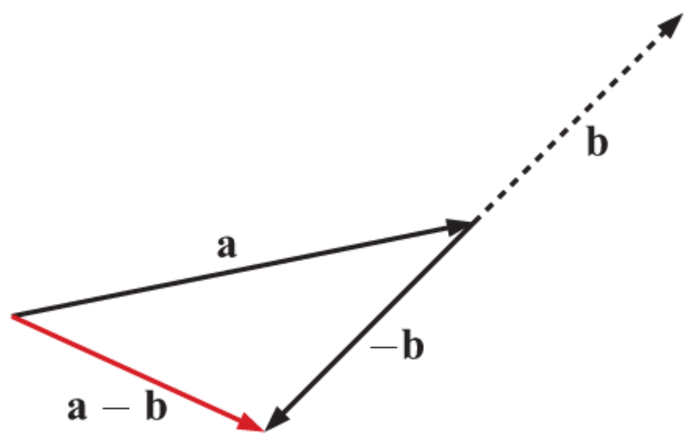
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.

ALGEBRAIC VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.

Suppose $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\therefore \quad \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$
$$= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix}$$
$$= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$$



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

Example 12 — Self Tutor

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$, find:

a $\mathbf{p} + \mathbf{r}$ **b** $\mathbf{q} - \mathbf{p}$ **c** $\mathbf{p} - \mathbf{q} - \mathbf{r}$

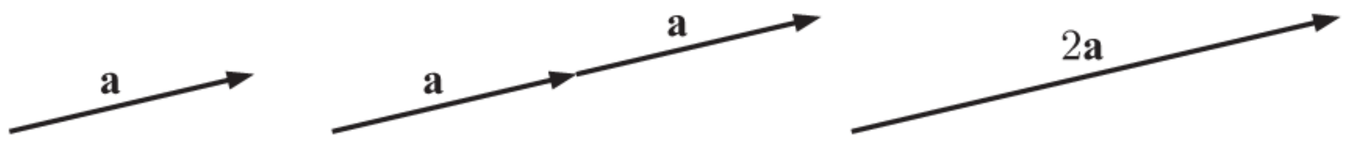
a $\mathbf{p} + \mathbf{r}$
$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix}$
$= \begin{pmatrix} 3 + (-2) \\ -2 + (-5) \end{pmatrix}$
$= \begin{pmatrix} 1 \\ -7 \end{pmatrix}$

b $\mathbf{q} - \mathbf{p}$
$= \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$
$= \begin{pmatrix} 1 - 3 \\ 4 - (-2) \end{pmatrix}$
$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}$

c $\mathbf{p} - \mathbf{q} - \mathbf{r}$
$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix}$
$= \begin{pmatrix} 3 - 1 - (-2) \\ -2 - 4 - (-5) \end{pmatrix}$
$= \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

ALGEBRAIC SCALAR MULTIPLICATION

We have already seen a geometric approach for integer scalar multiplication:



Consider $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

$$2\mathbf{a} = \mathbf{a} + \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \times 1 \\ 2 \times 3 \end{pmatrix}$$

$$3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 3 \times 1 \\ 3 \times 3 \end{pmatrix}$$

If k is any scalar and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$.

Notice that:

- $(-1)\mathbf{v} = \begin{pmatrix} (-1)\,v_1 \\ (-1)\,v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}$
- $(0)\mathbf{v} = \begin{pmatrix} (0)\,v_1 \\ (0)\,v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Example 13 — Self Tutor

For $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ find:

a $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a
$$3\mathbf{q} = 3\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$

b
$$\mathbf{p} + 2\mathbf{q} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{pmatrix} = \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

c
$$\tfrac{1}{2}\mathbf{p} - 3\mathbf{q} = \tfrac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{pmatrix} = \begin{pmatrix} -4 \\ 9\frac{1}{2} \end{pmatrix}$$

EXERCISE 9E

1 If $\mathbf{a} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$ find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{b}$

e $\mathbf{a} + \mathbf{c}$ **f** $\mathbf{c} + \mathbf{a}$ **g** $\mathbf{a} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ find:

a $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$

d $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **e** $\mathbf{q} - \mathbf{r} - \mathbf{p}$ **f** $\mathbf{r} + \mathbf{q} - \mathbf{p}$

3 Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

a Use vector addition to show that $\mathbf{a} + \mathbf{0} = \mathbf{a}$.

b Use vector subtraction to show that $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

4 For $\mathbf{p} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ find:

a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} - 2\mathbf{q}$

e $\mathbf{p} - \frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p} + 3\mathbf{r}$ **g** $2\mathbf{q} - 3\mathbf{r}$ **h** $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$

5 Consider $\mathbf{p} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

a Find geometrically:

i $\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q}$ **ii** $\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q}$ **iii** $\mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q}$

b Comment on your results in **a**.

6 Find k such that:

a $\begin{pmatrix} 2 \\ -5 \end{pmatrix} + k\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 20 \\ -11 \end{pmatrix}$ **b** $\frac{1}{3}\begin{pmatrix} -1 \\ k \end{pmatrix} + \frac{4}{3}\begin{pmatrix} 7 \\ k+1 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$

Example 14 **Self Tutor**

If $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j}$, find $|\mathbf{p} - 2\mathbf{q}|$.

$$\begin{aligned} \mathbf{p} - 2\mathbf{q} &= 3\mathbf{i} - 5\mathbf{j} - 2(-\mathbf{i} - 2\mathbf{j}) \\ &= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{i} + 4\mathbf{j} \\ &= 5\mathbf{i} - \mathbf{j} \end{aligned}$$

$$\therefore\ |\mathbf{p} - 2\mathbf{q}| = \sqrt{5^2 + (-1)^2} = \sqrt{26} \text{ units}$$

7 If $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{s} = -\mathbf{i} + 4\mathbf{j}$, find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|\mathbf{r} - \mathbf{s}|$

e $|\mathbf{s} - 2\mathbf{r}|$ **f** $|2\mathbf{r}|$ **g** $|\mathbf{r} + 2\mathbf{s}|$ **h** $|2\mathbf{r} - \mathbf{s}|$

8 **a** If $\mathbf{p} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, find:

i $|\mathbf{p}|$ **ii** $|2\mathbf{p}|$ **iii** $|-2\mathbf{p}|$ **iv** $|3\mathbf{p}|$ **v** $|-3\mathbf{p}|$

b By letting $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, prove that $|k\mathbf{v}| = |k||\mathbf{v}|$.

9 Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

Show by equating components, that if $k\mathbf{x} = \mathbf{a}$ then $\mathbf{x} = \frac{1}{k}\mathbf{a}$.

ACTIVITY **SKI RACER**

Click on the icon to practise your skills with vectors.



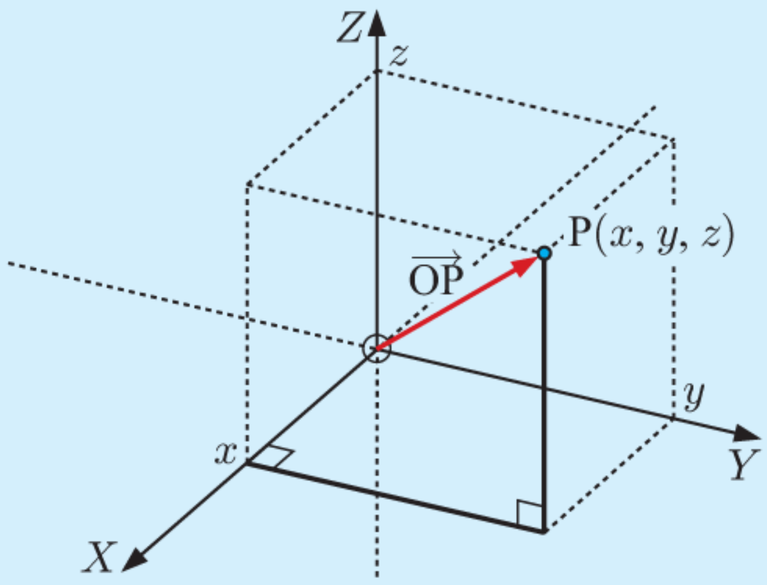
F VECTORS IN SPACE

In **Chapter 10** of the **Core Topics HL** book, we saw that we can specify a point in 3-dimensional space using 3 mutually perpendicular axes called the X, Y, and Z axes.

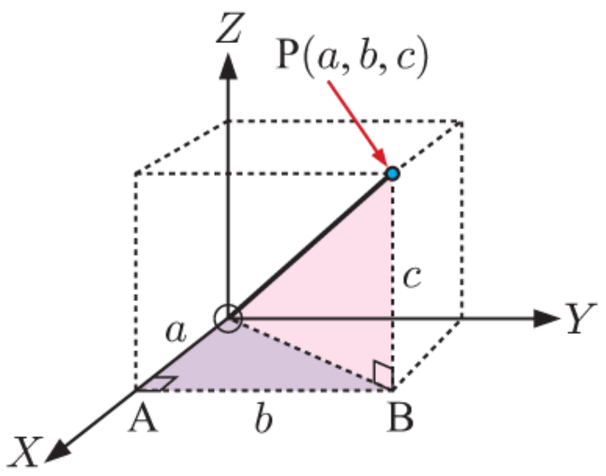
In 3-dimensional space, the **position vector** of $\mathrm{P}(x, y, z)$ is $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

are the **base unit vectors** in the X, Y, and Z directions respectively.



THE MAGNITUDE OF A VECTOR



Triangle OAB is right angled at A

$\therefore \quad \mathrm{OB}^2 = a^2 + b^2 \quad \ldots. \ (1) \quad \{\text{Pythagoras}\}$

Triangle OBP is right angled at B

$\therefore \quad \mathrm{OP}^2 = \mathrm{OB}^2 + c^2 \quad \{\text{Pythagoras}\}$

$\therefore \quad \mathrm{OP}^2 = a^2 + b^2 + c^2 \quad \{\text{using (1)}\}$

$\therefore \quad \mathrm{OP} = \sqrt{a^2 + b^2 + c^2}$

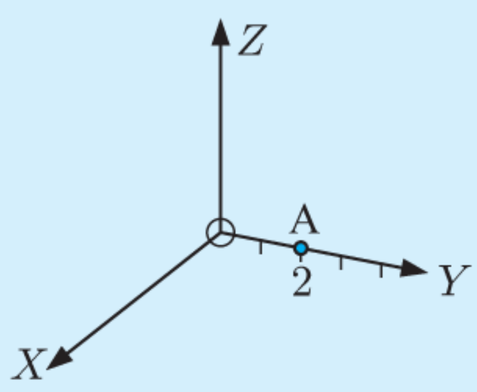
The magnitude or length of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Example 15 Self Tutor

Illustrate each point and find its distance from the origin O:

a A(0, 2, 0) **b** B(3, 0, 2) **c** C(−1, 2, 3)

a

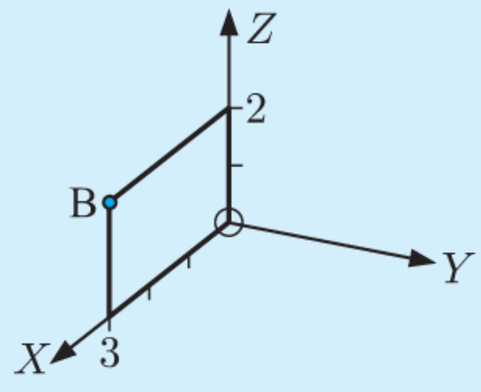


$|\overrightarrow{\mathrm{OA}}|$

$= \sqrt{0^2 + 2^2 + 0^2}$

$= 2$ units

b

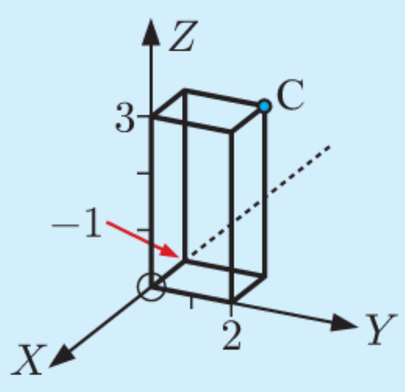


$|\overrightarrow{\mathrm{OB}}|$

$= \sqrt{3^2 + 0^2 + 2^2}$

$= \sqrt{13}$ units

c



$|\overrightarrow{\mathrm{OC}}|$

$= \sqrt{(-1)^2 + 2^2 + 3^2}$

$= \sqrt{14}$ units

EXERCISE 9F

1 Consider the point $T(3, -1, 4)$.

a Draw a diagram to locate the position of T in space. **b** Find $\overrightarrow{OT}$.

c Find the distance from O to T.

2 For each point P, write $\overrightarrow{OP}$ in component form and unit vector form:

a $P(2, 0, 4)$ **b** $P(-3, 1, 2)$ **c** $P(0, -2, 0)$

3 Find the length of:

a $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \\ -1 \end{pmatrix}$

4 Illustrate P and find its distance from the origin O:

a $P(0, 0, -3)$ **b** $P(0, -1, 2)$ **c** $P(3, 1, 4)$ **d** $P(-1, -2, 3)$

5 Write each vector in component form, and find its length:

a $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ **b** $\mathbf{b} = -3\mathbf{i} + 4\mathbf{k}$ **c** $\mathbf{c} = 2\mathbf{j} - 3\mathbf{k}$

6 Find a, b, and c if:

a $\begin{pmatrix} a-4 \\ b-3 \\ c+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ **b** $\begin{pmatrix} a-5 \\ b-2 \\ c+3 \end{pmatrix} = \begin{pmatrix} 3-a \\ 2-b \\ 5-c \end{pmatrix}$ **c** $\begin{pmatrix} 3a+4 \\ b-c \\ 4-5b \end{pmatrix} = \begin{pmatrix} 12-a \\ a-4 \\ 5a-22 \end{pmatrix}$

7 Which of the following are unit vectors?

a $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ **c** $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ **d** $\begin{pmatrix} \frac{1}{3\sqrt{3}} \\ \frac{5}{3\sqrt{3}} \\ -\frac{1}{3\sqrt{3}} \end{pmatrix}$



8 Find k given the unit vector:

a $\begin{pmatrix} -\frac{1}{2} \\ k \\ \frac{1}{4} \end{pmatrix}$ **b** $\begin{pmatrix} k \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ **c** $\begin{pmatrix} k \\ k \\ k \end{pmatrix}$ **d** $\begin{pmatrix} \frac{1}{k} \\ \frac{2}{k} \\ \frac{3}{k} \end{pmatrix}$

9 Find the possible values of m, given that:

a $\begin{pmatrix} -2 \\ m \\ 1 \end{pmatrix}$ has length $\sqrt{14}$ units **b** $\begin{pmatrix} 4 \\ -3 \\ m \end{pmatrix}$ has length 6 units.

10 Suppose $\left|\begin{pmatrix} a \\ b \end{pmatrix}\right| = 5$ units and $\left|\begin{pmatrix} b \\ c \end{pmatrix}\right| = 3$ units. Find the range of possible values of $\left|\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right|$.

G OPERATIONS WITH VECTORS IN SPACE

For 3-dimensional vectors, it is difficult to illustrate operations. Instead, we perform operations using component form.

The rules for algebra with vectors readily extend from 2 dimensions to 3 dimensions:

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then:

- $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$
- $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}$
- $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{pmatrix}$
- $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}$ for any scalar k.

Example 16 **Self Tutor**

If $\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$, find:

a $\mathbf{a} + \mathbf{b}$ **b** $2\mathbf{a} - \mathbf{b}$ **c** $|\mathbf{a}|\mathbf{b}$

a
$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1+3 \\ 3+0 \\ 2+(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

b
$$2\mathbf{a} - \mathbf{b} = 2\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2-3 \\ 6-0 \\ 4-(-1) \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ 5 \end{pmatrix}$$

c
$$|\mathbf{a}|\mathbf{b} = \sqrt{(-1)^2 + 3^2 + 2^2} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \sqrt{14} \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3\sqrt{14} \\ 0 \\ -\sqrt{14} \end{pmatrix}$$

EXERCISE 9G

1 Let $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$. Find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $\mathbf{b} + 2\mathbf{c}$

d $\mathbf{c} - \frac{1}{2}\mathbf{a}$ **e** $\mathbf{a} - \mathbf{b} - \mathbf{c}$ **f** $2\mathbf{b} - \mathbf{c} + \mathbf{a}$

2 Let $\mathbf{a} = \mathbf{i} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find:

a $|\mathbf{a}|$ **b** $|\mathbf{b}|$ **c** $2|\mathbf{a}|$ **d** $|2\mathbf{a}|$

e $-3|\mathbf{b}|$ **f** $|-3\mathbf{b}|$ **g** $|\mathbf{a}+\mathbf{b}|$ **h** $|\mathbf{a}-\mathbf{b}|$

3 Let $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}$. Find:

a $2\mathbf{a} - \mathbf{c}$ **b** $3\mathbf{b} + \frac{1}{2}\mathbf{c}$ **c** $-\mathbf{a} + 3\mathbf{b}$ **d** $\mathbf{a} - 5\mathbf{b} + 4\mathbf{c}$

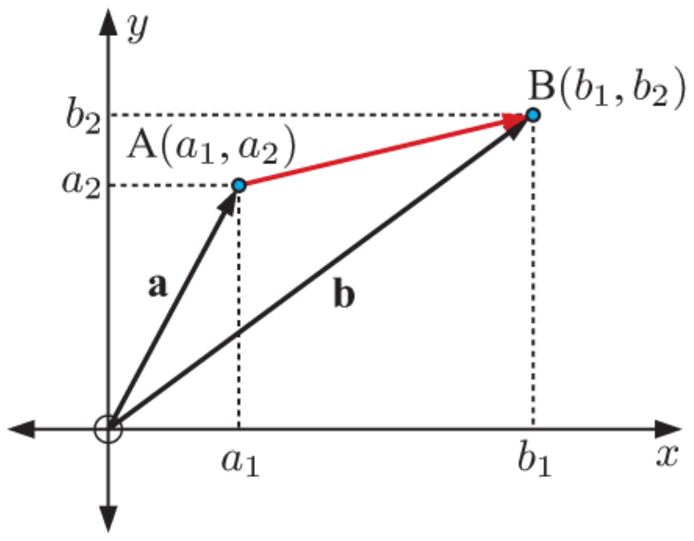
e $|\mathbf{b}+\mathbf{c}|$ **f** $|\mathbf{a}-\mathbf{c}|$ **g** $|\mathbf{a}|\,\mathbf{b}$ **h** $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$

4 Find scalars a, b, and c such that:

a $2\begin{pmatrix} 1 \\ 0 \\ 3a \end{pmatrix} = \begin{pmatrix} b \\ c-1 \\ 2 \end{pmatrix}$ **b** $a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

c $a\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ c \\ 2 \end{pmatrix}$ **d** $2\begin{pmatrix} a \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} b \\ c \\ 4 \end{pmatrix} = \begin{pmatrix} c \\ -c \\ ab \end{pmatrix}$

H THE VECTOR BETWEEN TWO POINTS

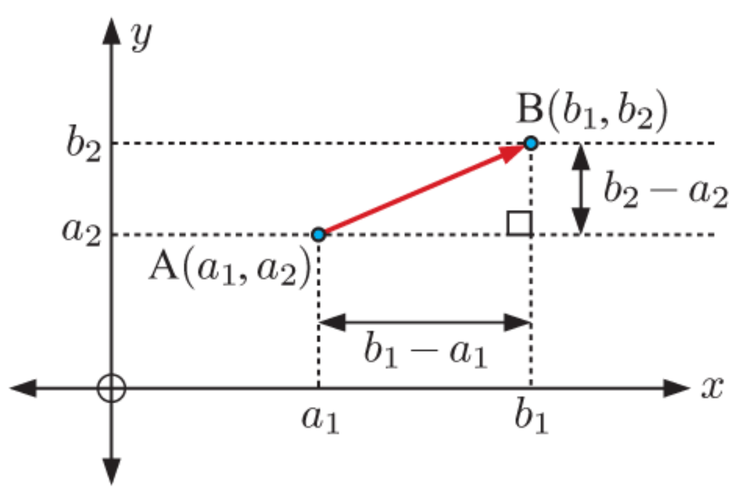


In the diagram, point A has position vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, and point B has position vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\therefore \quad \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\mathbf{a} + \mathbf{b}$$
$$= \mathbf{b} - \mathbf{a}$$
$$= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
$$= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

The **displacement vector of B relative to A** is $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

Notice that $\overrightarrow{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$ is the *translation vector* required to move point A to point B.



Example 17 **Self Tutor**

Given points $A(-1, 2)$, $B(3, 4)$, and $C(4, -5)$, find the displacement vector of:

a B from O **b** B from A **c** A from C.

a The displacement vector of B relative to O is $\overrightarrow{OB} = \begin{pmatrix} 3-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b The displacement vector of B relative to A is $\overrightarrow{AB} = \begin{pmatrix} 3-(-1) \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

c The displacement vector of A relative to C is $\overrightarrow{CA} = \begin{pmatrix} -1-4 \\ 2-(-5) \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

We can define a similar result for vectors in space:

If $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ are two points in space with position vectors **a** and **b** respectively, then the **displacement vector of B relative to A** is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix} \begin{matrix} \leftarrow X\text{-step} \\ \leftarrow Y\text{-step} \\ \leftarrow Z\text{-step} \end{matrix}$$

Example 18 **Self Tutor**

Given $P(-3, 1, 2)$ and $Q(1, -1, 3)$, find: **a** $\overrightarrow{OP}$ **b** $\overrightarrow{PQ}$ **c** $|\overrightarrow{PQ}|$

a $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

b $\overrightarrow{PQ} = \begin{pmatrix} 1-(-3) \\ -1-1 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$

c $|\overrightarrow{PQ}| = \sqrt{4^2 + (-2)^2 + 1^2}$
$= \sqrt{21}$ units

EXERCISE 9H

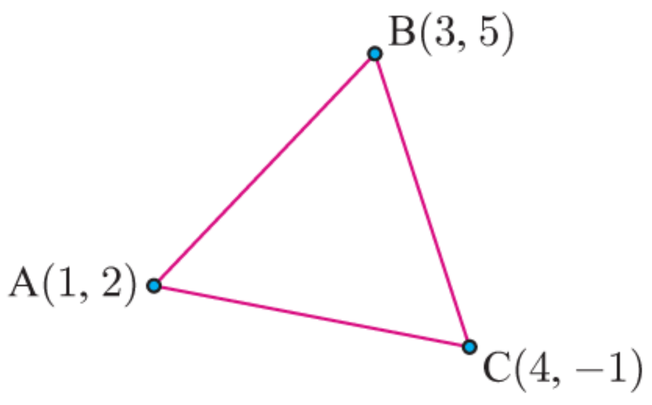
1 Find $\overrightarrow{AB}$ given:

a $A(2, 3)$ and $B(4, 7)$ **b** $A(3, -1)$ and $B(1, 4)$ **c** $A(-2, 7)$ and $B(1, 4)$.

2 Given $A(2, 3)$ and $B(6, 2)$, find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{BA}$ **c** $|\overrightarrow{AB}|$

3



a Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Explain why $\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$.

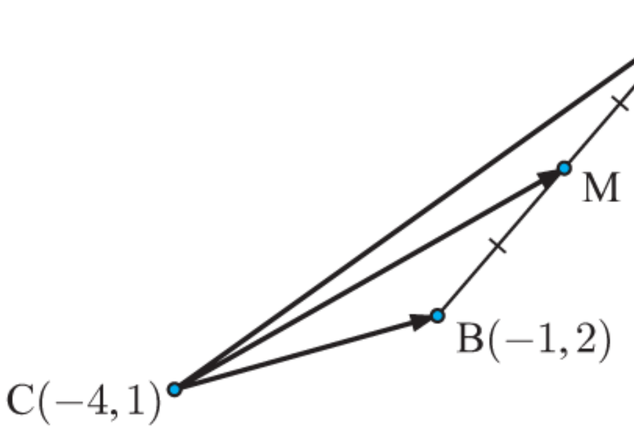
c Hence find $\overrightarrow{BC}$.

d Check your answer to **c** by direct evaluation.

4 **a** Given $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $\overrightarrow{AC}$.

b Given $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{SP}$.

5



a Find the coordinates of M.

b Find vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$, and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

6 Find $\overrightarrow{PQ}$ given:

a P(1, 0, 2) and Q(3, 5, 4)

b P(5, 2, 3) and Q(6, 2, −1)

c P(−2, 3, 0) and Q(1, 4, −3)

d P(4, −1, 5) and Q(−1, −5, 3).

7 If $\overrightarrow{OA} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$, find $\overrightarrow{AB}$ and hence the distance from A to B.

8 Given A(−3, 1, 2) and B(1, 0, −1), find:

a $\overrightarrow{AB}$ and $\overrightarrow{BA}$

b the lengths $|\overrightarrow{AB}|$ and $|\overrightarrow{BA}|$.

9 Given M(4, −2, −1) and N(−1, 2, 0), find:

a the displacement vector of M relative to N

b the displacement vector of N relative to M

c the distance between M and N.

10 For A(−1, 3, −2) and B(3, −2, 1) find:

a $\overrightarrow{AB}$ in terms of **i**, **j**, and **k**

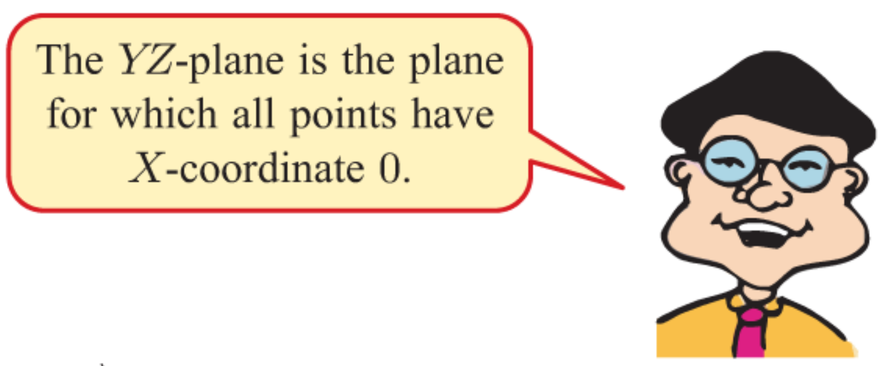
b the magnitude of $\overrightarrow{AB}$.

11 Find the shortest distance from Q(3, 1, −2) to:

a the Y-axis

b the origin

c the YZ-plane.



12 If $\overrightarrow{AB} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\overrightarrow{BC} = -2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, find $\overrightarrow{AC}$ in terms of **i**, **j**, and **k**.

13 If $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, and $\overrightarrow{BD} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, find:

a $\overrightarrow{AD}$

b $\overrightarrow{CB}$

c $\overrightarrow{CD}$

14 Consider the points A(−1, 3) and B(3, k).

a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Given that A and B are 5 units apart, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

Example 19 — Self Tutor

[AB] is the diameter of a circle with centre C$(-1, 2)$. If B is $(3, 1)$, find:

a $\overrightarrow{BC}$ **b** the coordinates of A.

a $\overrightarrow{BC} = \begin{pmatrix} -1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

b If A has coordinates (a, b), then

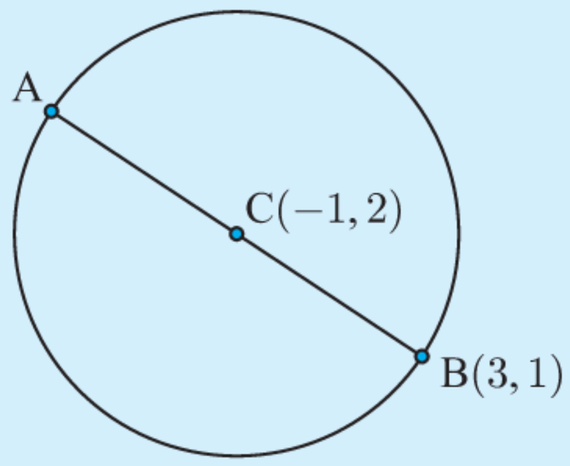
$\overrightarrow{CA} = \begin{pmatrix} a-(-1) \\ b-2 \end{pmatrix} = \begin{pmatrix} a+1 \\ b-2 \end{pmatrix}$

But $\overrightarrow{CA} = \overrightarrow{BC}$, so $\begin{pmatrix} a+1 \\ b-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore\ a+1=-4$ and $b-2=1$

$\therefore\ a=-5$ and $b=3$

$\therefore$ A is $(-5, 3)$.



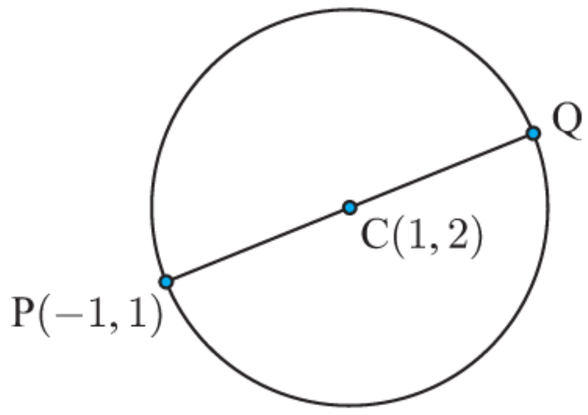
15 Consider the point A$(1, 4)$. Find the coordinates of:

a B given $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ **b** C given $\overrightarrow{CA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

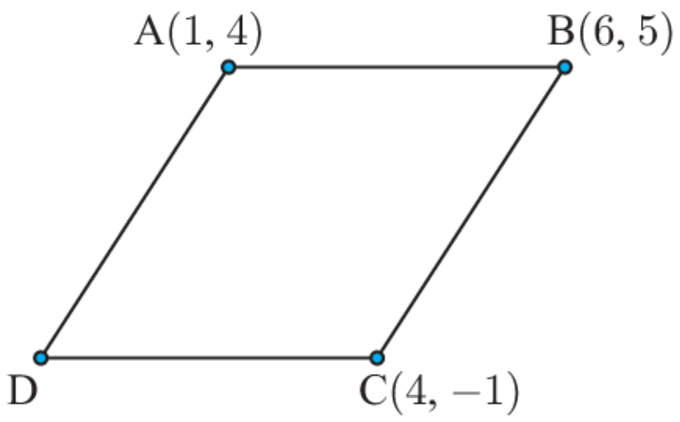
16 [PQ] is the diameter of a circle with centre C.

a Find $\overrightarrow{PC}$.

b Hence find the coordinates of Q.



17



ABCD is a parallelogram.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

18 A sphere has centre C$(-1, 2, 4)$ and diameter [AB] where A is $(-2, 1, 3)$. Find the coordinates of B and the radius of the sphere.

19 Consider A$(-1, 2, 5)$, B$(2, 0, 3)$, and C$(-3, 1, 0)$.

a Find the displacement vector $\overrightarrow{AB}$ and its length AB.

b Find the displacement vector $\overrightarrow{AC}$ and its length AC.

c Find the displacement vector $\overrightarrow{CB}$ and its length CB.

d Hence classify triangle ABC.

20 Show that P$(0, 4, 4)$, Q$(2, 6, 5)$, and R$(1, 4, 3)$ are vertices of an isosceles triangle.

21 The vertices of triangle ABC are $A(5, 6, -2)$, $B(6, 12, 9)$, and $C(2, 4, 2)$.

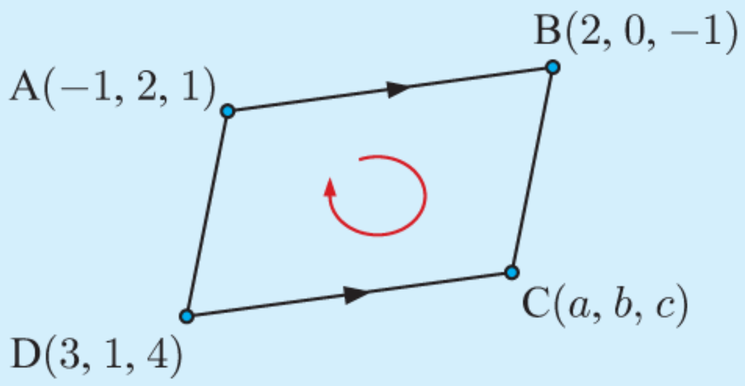
a Show that the triangle is right angled. **b** Hence find the area of the triangle.

Example 20 **Self Tutor**

ABCD is a parallelogram. A is $(-1, 2, 1)$, B is $(2, 0, -1)$, and D is $(3, 1, 4)$. Find the coordinates of C.

Let C be (a, b, c).

[AB] is parallel to [DC], and they have the same length, so $\overrightarrow{DC} = \overrightarrow{AB}$



$$\therefore \begin{pmatrix} a-3 \\ b-1 \\ c-4 \end{pmatrix} = \begin{pmatrix} 2-(-1) \\ 0-2 \\ -1-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$$

$\therefore\ a-3=3, \quad b-1=-2, \quad$ and $\quad c-4=-2$

$\therefore\ a=6, \quad b=-1, \quad$ and $\quad c=2$

So, C is $(6, -1, 2)$.

Check: The midpoint of [DB] is $\left(\frac{3+2}{2}, \frac{1+0}{2}, \frac{4+-1}{2}\right)$ which is $\left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

The midpoint of [AC] is $\left(\frac{-1+6}{2}, \frac{2+-1}{2}, \frac{1+2}{2}\right)$ which is $\left(\frac{5}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

The midpoints are the same, so the diagonals of the parallelogram bisect. ✓

22 $A(-1, 3, 4)$, $B(2, 5, -1)$, $C(-1, 2, -2)$, and $D(r, s, t)$ are four points in space. Find r, s, and t if:

a $\overrightarrow{AC} = \overrightarrow{BD}$ **b** $\overrightarrow{AB} = \overrightarrow{DC}$

23 A quadrilateral has vertices $A(1, 2, 3)$, $B(3, -3, 2)$, $C(7, -4, 5)$, and $D(5, 1, 6)$.

a Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$.

b What can be deduced about the quadrilateral ABCD?

24 PQRS is a parallelogram. P is $(-1, 2, 3)$, Q is $(1, -2, 5)$, and R is $(0, 4, -1)$.

a Use vectors to find the coordinates of S.

b Use the midpoints of the diagonals to check your answer.

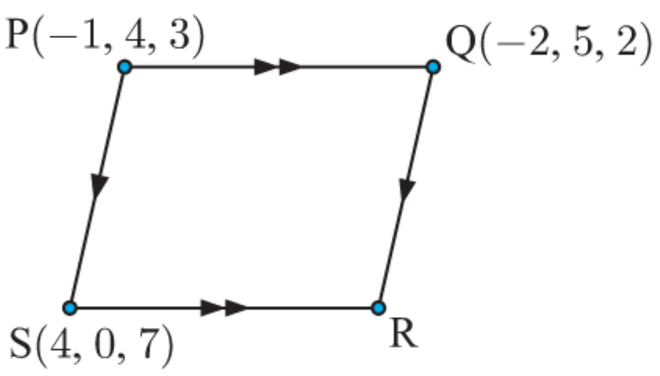
25 Use vectors to determine whether ABCD is a parallelogram:

a $A(5, 0, 3)$, $B(-1, 2, 4)$, $C(4, -3, 6)$, and $D(10, -5, 5)$

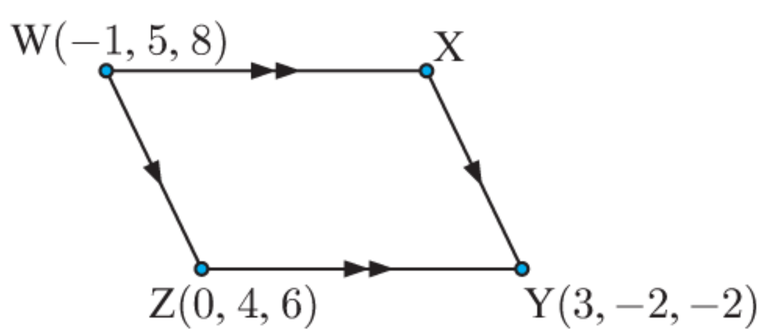
b $A(2, -3, 2)$, $B(1, 4, -1)$, $C(-2, 6, -2)$, and $D(-1, -1, 2)$.

26 Use vector methods to find the remaining vertex of:

a



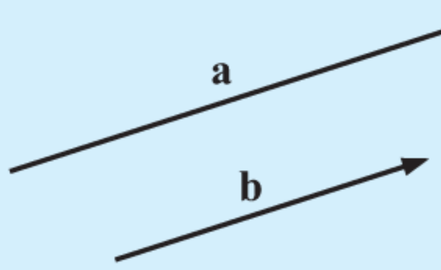
b



I PARALLELISM

Two non-zero vectors are **parallel** if and only if one is a scalar multiple of the other.

Given any non-zero vector $\mathbf{v}$ and non-zero scalar k, the vector $k\mathbf{v}$ is parallel to $\mathbf{v}$.



- If $\mathbf{a}$ is parallel to $\mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - $\mathbf{a}$ is parallel to $\mathbf{b}$, and
  - $|\mathbf{a}| = |k||\mathbf{b}|$.

$|k|$ is the absolute value of k, whereas $|\mathbf{a}|$ is the length of vector $\mathbf{a}$.



Example 21 **Self Tutor**

Find r and s given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a} = k\mathbf{b}$ for some scalar k.

$$\therefore \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix} = k \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$$

$\therefore\ 2 = ks,\ -1 = 2k,$ and $r = -3k$

Consequently, $k = -\frac{1}{2}$

$\therefore\ 2 = -\frac{1}{2}s$ and $r = -3(-\frac{1}{2})$

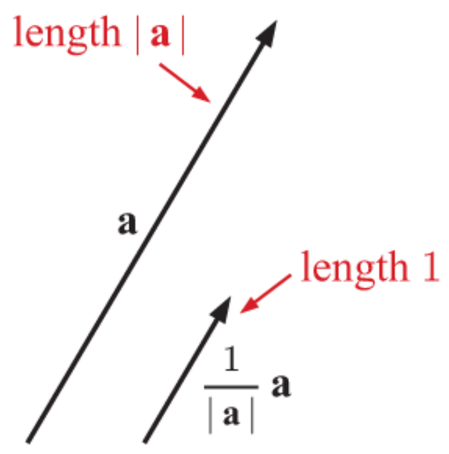
$\therefore\ s = -4$ and $r = \frac{3}{2}$

UNIT VECTORS

Given a non-zero vector $\mathbf{a}$, its magnitude $|\mathbf{a}|$ is a scalar quantity.

If we multiply $\mathbf{a}$ by the scalar $\frac{1}{|\mathbf{a}|}$, we obtain the parallel vector $\frac{1}{|\mathbf{a}|}\mathbf{a}$.

The length of this vector is $\left|\frac{1}{|\mathbf{a}|}\right| |\mathbf{a}| = \frac{|\mathbf{a}|}{|\mathbf{a}|} = 1$, so $\frac{1}{|\mathbf{a}|}\mathbf{a}$ is a unit vector in the direction of $\mathbf{a}$. We write this unit vector as $\widehat{\mathbf{a}}$.



- A unit vector in the direction of $\mathbf{a}$ is $\widehat{\mathbf{a}} = \frac{1}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k in the same direction as $\mathbf{a}$ is $\mathbf{b} = k\widehat{\mathbf{a}} = \frac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector $\mathbf{b}$ of length k which is *parallel to* $\mathbf{a}$ could be $\mathbf{b} = \pm k\widehat{\mathbf{a}} = \pm\frac{k}{|\mathbf{a}|}\mathbf{a}$.

Example 22 — Self Tutor

Find a vector **b** of length 7 units in the opposite direction to $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

The unit vector in the direction of **a** is

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \frac{1}{\sqrt{4+1+1}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

$$\therefore \quad \mathbf{b} = -\frac{7}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

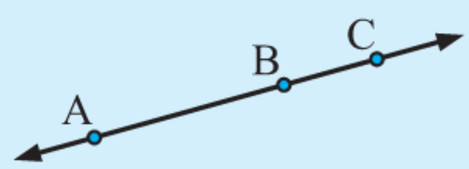
We multiply the unit vector by -7. The negative reverses the direction. The 7 gives the required length.



COLLINEAR POINTS

Three or more points are said to be **collinear** if they lie on the same straight line.

A, B, and C are **collinear** if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some scalar k.



Example 23 — Self Tutor

Prove that A$(-1, 2, 3)$, B$(4, 0, -1)$, and C$(14, -4, -9)$ are collinear.

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 10 \\ -4 \\ -8 \end{pmatrix} = 2\begin{pmatrix} 5 \\ -2 \\ -4 \end{pmatrix}$$

$\therefore \quad \overrightarrow{BC} = 2\overrightarrow{AB}$

$\therefore$ [BC] is parallel to [AB].

Since B is common to both, A, B, and C are collinear.

EXERCISE 9I

1 Determine whether each pair of vectors is parallel:

a $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 15 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -10 \end{pmatrix}$ **c** $\begin{pmatrix} 6 \\ -12 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 8 \end{pmatrix}$

d $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -8 \\ 6 \end{pmatrix}$ **e** $\begin{pmatrix} 8 \\ 0 \\ -20 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 0 \\ 15 \end{pmatrix}$ **f** $\begin{pmatrix} 6 \\ 10 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 24 \\ 40 \\ 8 \end{pmatrix}$

2 Find a such that:

a $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ a \end{pmatrix}$ are parallel **b** $\begin{pmatrix} a \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ are parallel.

3 Find r and s given that $\mathbf{a}$ and $\mathbf{b}$ are parallel:

a $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ r \\ s \end{pmatrix}$ **b** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} r \\ 2 \\ s \end{pmatrix}$

4 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$

5 The position vectors of P, Q, R, and S are $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$ respectively.

Show that [PR] and [QS] are parallel, and state the ratio PR : QS.

6 If $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, write down the vector:

a in the same direction as $\mathbf{a}$ and with twice its length

b in the opposite direction to $\mathbf{a}$ and with half its length.

7 Find the 2-dimensional unit vector in the direction:

a $\mathbf{i} + 2\mathbf{j}$ **b** $2\mathbf{i} - 3\mathbf{j}$ **c** $-2\mathbf{i} + 2\mathbf{j}$

8 Find the 3-dimensional unit vector in the direction:

a $2\mathbf{i} + 3\mathbf{k}$ **b** $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ **c** $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

9 Find a vector $\mathbf{v}$ which has:

a the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and with length 3 units

b the opposite direction to $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and with length 2 units

c the same direction as $\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and with length 6 units

d the opposite direction to $-\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and with length 5 units.

10 **a** Find the vectors of length 1 unit which are parallel to $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$.

b Find vectors of length 2 units which are parallel to $\mathbf{b} = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$.

11 A is $(3, 2)$ and point B is 4 units from A in the direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a Find $\overrightarrow{AB}$. **b** Find $\overrightarrow{OB}$ using $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

c Hence state the coordinates of B.

12 Find the coordinates of point X which is:

a 5 units from $(1, -5)$ in the direction $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$

b 6 units from $(1, 3, -2)$ in the direction $\begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$

c 4 units from $(2, -1, 4)$ in the direction $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

13 Show that the following sets of points are collinear:

a $A(-1, 1)$, $B(4, 6)$, $C(1, 3)$

b $P(3, 3)$, $Q(6, -3)$, $R(1, 7)$

c $A(-2, 1, 4)$, $B(4, 3, 0)$, $C(19, 8, -10)$

d $P(2, 1, 1)$, $Q(5, -5, -2)$, $R(-1, 7, 4)$

14 Find a and b if:

a $A(2, -3, 4)$, $B(11, -9, 7)$, and $C(-13, a, b)$ are collinear

b $K(1, -1, 0)$, $L(4, -3, 7)$, and $M(a, 2, b)$ are collinear.

15 Let $P(x, y, z)$ be a point on a sphere with radius r units and centre $C(X, Y, Z)$.

a Write down the vector $\overrightarrow{CP}$.

b Use the distance formula for CP to write a Cartesian equation for the sphere.

INVESTIGATION 1 LINEAR COMBINATIONS

Any vector $\begin{pmatrix} r \\ s \end{pmatrix}$ in the plane can be written in the form $r\mathbf{i} + s\mathbf{j}$. We say that $\begin{pmatrix} r \\ s \end{pmatrix}$ is a **linear combination** of **i** and **j**.

Click on the icon to explore linear combinations further.



J THE SCALAR PRODUCT OF TWO VECTORS

For ordinary numbers a and b we can write the product of a and b as ab or $a \times b$. There is only one interpretation for this product, so we can use power notation $a^2 = a \times a$, $a^3 = a \times a \times a$, and so on as shorthand.

However, there are *two* different types of product involving two vectors. These are:

- The **scalar product** of 2 vectors, which results in a **scalar** answer and has the notation $\mathbf{a} \bullet \mathbf{b}$ (read "**a** dot **b**").
- The **vector product** of 2 vectors, which results in a **vector** answer and has the notation $\mathbf{a} \times \mathbf{b}$ (read "**a** cross **b**").

Consequently, for vector **a**, we do not use the terms $\mathbf{a}^2$ or $(\mathbf{a})^2$, as it is not clear which of the vector products it refers to.

For 2-dimensional vectors in a plane, we only need to talk about scalar product. The vector cross product will be important for 3-dimensional vectors in space.

SCALAR PRODUCT

The **scalar product** of two vectors is also known as the **dot product** or **inner product**.

▶ $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2$

▶ $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \bullet \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$

Be careful not to confuse the **scalar product**, which is the product of two vectors to give a scalar answer, with **scalar multiplication**, which is the product of a scalar and a vector to give a parallel vector. They are quite different.

Example 24 — Self Tutor

Find: **a** $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

a $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 4 \end{pmatrix}$
$= 2 \times 1 + 3 \times 4$
$= 14$

b $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$
$= -1 \times 3 + 2 \times 1 + 4 \times 2$
$= 7$

ALGEBRAIC PROPERTIES OF THE SCALAR PRODUCT

1 $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$

2 $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$

3 $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

4 $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$

PROOFS OF ALGEBRAIC PROPERTIES



EXERCISE 9J

1 Find:

a $\begin{pmatrix} 2 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 5 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ **d** $\begin{pmatrix} -4 \\ -5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 9 \\ -6 \\ -1 \end{pmatrix}$

2 For $\mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, find:

a $\mathbf{q} \bullet \mathbf{p}$ **b** $\mathbf{q} \bullet \mathbf{r}$ **c** $\mathbf{q} \bullet (\mathbf{p} + \mathbf{r})$ **d** $3\mathbf{r} \bullet \mathbf{q}$

e $2\mathbf{p} \bullet 2\mathbf{p}$ **f** $\mathbf{i} \bullet \mathbf{p}$ **g** $\mathbf{q} \bullet \mathbf{j}$ **h** $\mathbf{i} \bullet \mathbf{i}$

3 For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** $\mathbf{b} \bullet \mathbf{a}$ **c** $|\mathbf{a}|^2$

d $\mathbf{a} \bullet \mathbf{a}$ **e** $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$ **f** $\mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

4 Find:

a $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \bullet (2\mathbf{j} + \mathbf{k})$ **b** $\mathbf{i} \bullet \mathbf{i}$ **c** $(3\mathbf{i} - 2\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j})$

d $(2\mathbf{i} + \mathbf{k}) \bullet (-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ **e** $\mathbf{i} \bullet \mathbf{j}$ **f** $(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \bullet (3\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$

5 Let $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$, and $\mathbf{d} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$.

a Find:

i $\mathbf{a} \bullet \mathbf{c}$ **ii** $\mathbf{a} \bullet \mathbf{d}$ **iii** $\mathbf{b} \bullet \mathbf{c}$ **iv** $\mathbf{b} \bullet \mathbf{d}$

b Hence show that $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{c} + \mathbf{d}) = \mathbf{a} \bullet \mathbf{c} + \mathbf{a} \bullet \mathbf{d} + \mathbf{b} \bullet \mathbf{c} + \mathbf{b} \bullet \mathbf{d}$.

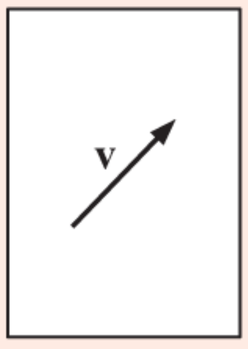
6 Explain why $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is meaningless.

7 If $|\mathbf{a}| = 5$ and $|\mathbf{b}| = 3$, find $(\mathbf{a} + \mathbf{b}) \bullet (\mathbf{a} - \mathbf{b})$.

K THE ANGLE BETWEEN TWO VECTORS

DISCUSSION

1

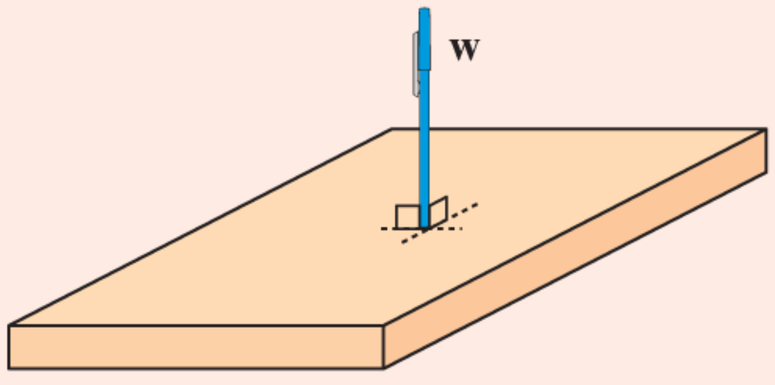


Elaine has drawn a vector $\mathbf{v}$ on a *plane* which is a sheet of paper. It is therefore a 2-dimensional vector.

a How many vectors can she draw which are *perpendicular* to $\mathbf{v}$?

b Are all of these vectors parallel?

2 Edward is thinking about vectors in *space*. These are 3-dimensional vectors. He is holding his pen vertically on his desk to represent a vector $\mathbf{w}$.



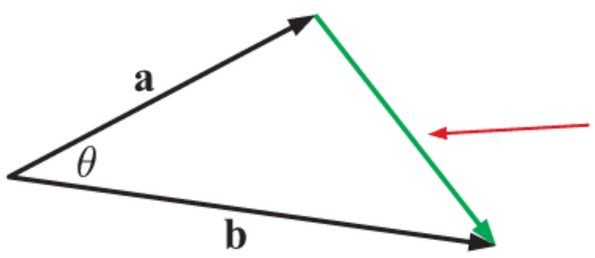
a How many vectors are there which are *perpendicular* to $\mathbf{w}$?

b If Edward was to draw a vector (in pencil) on his desk, would it be perpendicular to $\mathbf{w}$?

c Are all of the vectors which are perpendicular to $\mathbf{w}$, parallel to one another?

THE ANGLE BETWEEN TWO VECTORS

Suppose θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.



This vector is $-\mathbf{a} + \mathbf{b}$ or $\mathbf{b} - \mathbf{a}$.
Its length is $|\mathbf{b} - \mathbf{a}|$.

Using the cosine rule, $|\mathbf{b}-\mathbf{a}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos\theta$

Letting $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$$\mathbf{b}-\mathbf{a} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1-a_1 \\ b_2-a_2 \\ b_3-a_3 \end{pmatrix}$$

$\therefore\ (b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2 = (a_1^2+a_2^2+a_3^2) + (b_1^2+b_2^2+b_3^2) - 2|\mathbf{a}||\mathbf{b}|\cos\theta$

$\therefore\ 2a_1b_1 + 2a_2b_2 + 2a_3b_3 = 2|\mathbf{a}||\mathbf{b}|\cos\theta$ {simplifying}

$\therefore\ a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$

$\therefore\ \mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ can be found using $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$.

Example 25 **Self Tutor**

If $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, find the angle between $\mathbf{p}$ and $\mathbf{q}$.

$$\mathbf{p}\bullet\mathbf{q} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$
$$= 2(-1) + 3(0) + (-1)2$$
$$= -2 + 0 - 2$$
$$= -4$$

Now $\cos\theta = \dfrac{\mathbf{p}\bullet\mathbf{q}}{|\mathbf{p}||\mathbf{q}|}$

$$= \frac{-4}{\sqrt{4+9+1}\sqrt{1+0+4}}$$
$$= \frac{-4}{\sqrt{70}}$$

$\therefore\ \theta = \cos^{-1}\left(\frac{-4}{\sqrt{70}}\right) \approx 119°$

GEOMETRIC PROPERTIES OF THE SCALAR PRODUCT

Using the formula $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, we notice that:

- If $\mathbf{a}$ is perpendicular to $\mathbf{b}$ then $\theta = 90°$.

  $\therefore\ \mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
  $= |\mathbf{a}||\mathbf{b}|\cos 90°$
  $= 0$

- If $\mathbf{a}$ is parallel to $\mathbf{b}$ then $\theta = 0°$ or $180°$.

  $\therefore\ \mathbf{a}\bullet\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$
  $= |\mathbf{a}||\mathbf{b}|\cos 0°$ or $|\mathbf{a}||\mathbf{b}|\cos 180°$
  $= \pm|\mathbf{a}||\mathbf{b}|$

  $\therefore\ |\mathbf{a}\bullet\mathbf{b}| = |\mathbf{a}||\mathbf{b}|$

We are hence able to state the following **geometric properties of the scalar product**:

For non-zero vectors **a** and **b**:

1 $\mathbf{a} \bullet \mathbf{b} = 0 \quad \Leftrightarrow \quad$ **a** and **b** are **perpendicular**.

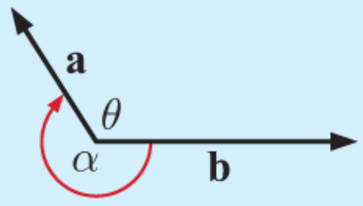
2 $|\mathbf{a} \bullet \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \quad \Leftrightarrow \quad$ **a** and **b** are **parallel vectors**.

3 If θ is the angle between vectors **a** and **b** then $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$.

If θ is acute, $\cos\theta > 0$ and so $\mathbf{a} \bullet \mathbf{b} > 0$

If θ is obtuse, $\cos\theta < 0$ and so $\mathbf{a} \bullet \mathbf{b} < 0$.

The angle between two vectors is always taken as the angle θ such that $0° \leqslant \theta \leqslant 180°$, rather than reflex angle α.



EXERCISE 9K

1 If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** the angle between **a** and **b**.

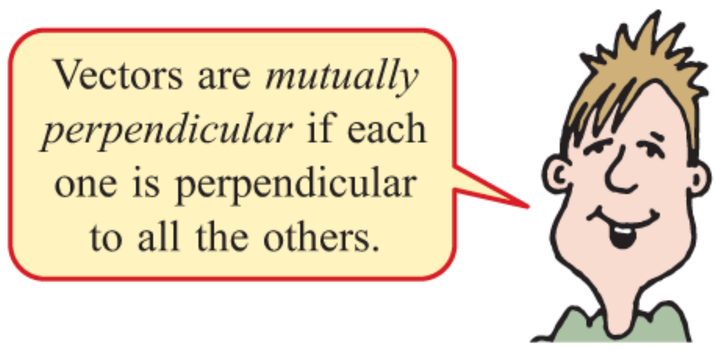
2 Find the angle between **r** and **s**, given:

a $\mathbf{r} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

c $\mathbf{r} = \mathbf{i} - \mathbf{j}$ and $\mathbf{s} = 2\mathbf{i} - \mathbf{j}$ **d** $\mathbf{r} = \mathbf{i}$ and $\mathbf{s} = \mathbf{i} + \mathbf{j}$

3 Show that $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are perpendicular.

4 Show that $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$ are mutually perpendicular.



5 Find the angle between:

a $\begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$ **c** $2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + 2\mathbf{k}$

6 Find t given that the following pairs of vectors are perpendicular:

a $\mathbf{p} = \begin{pmatrix} 3 \\ t \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

c $\mathbf{a} = \begin{pmatrix} t \\ t+2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2-3t \\ t \end{pmatrix}$ **d** $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ t \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2t \\ -3 \\ -4 \end{pmatrix}$

e $\mathbf{p} = \begin{pmatrix} 3 \\ t \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1-t \\ -3 \\ 4 \end{pmatrix}$ **f** $\mathbf{a} = 2\mathbf{i} + t\mathbf{j} + (t-2)\mathbf{k}$, $\mathbf{b} = t\mathbf{i} + 3\mathbf{j} + t\mathbf{k}$

7 Find r, s, and t such that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ 2 \\ r \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} s \\ t \\ 1 \end{pmatrix}$ are mutually perpendicular.

Example 26 **Self Tutor**

Find a vector of length 3 units which is perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

$\begin{pmatrix} -1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 1 \end{pmatrix} = -4 + 4 = 0$

$\therefore$ vectors of the form $k\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ are perpendicular to $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

Now $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ has length $\sqrt{16+1} = \sqrt{17}$ units

$\therefore$ $k = \frac{1}{\sqrt{17}}$ for a unit vector.

$\therefore$ a vector of length 3 units is $\frac{3}{\sqrt{17}}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, which is $\begin{pmatrix} \frac{12}{\sqrt{17}} \\ \frac{3}{\sqrt{17}} \end{pmatrix}$.

8 Find a vector of length:

a 10 units which is perpendicular to $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

b $3\sqrt{2}$ units which is perpendicular to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c $\sqrt{20}$ units which is perpendicular to $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$.

9 Find *two* vectors of length 5 units which are perpendicular to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Example 27 **Self Tutor**

$\mathbf{a}$ and $\mathbf{b}$ have lengths 5 units and $\sqrt{7}$ units respectively, and the angle between them is $110°$. Find $\mathbf{a} \bullet \mathbf{b}$.

$$\begin{aligned} \mathbf{a} \bullet \mathbf{b} &= |\mathbf{a}||\mathbf{b}|\cos\theta \\ &= 5 \times \sqrt{7} \times \cos 110° \\ &\approx -4.525 \end{aligned}$$

10 $\mathbf{a}$ and $\mathbf{b}$ have lengths $\sqrt{10}$ units and 3 units respectively and the angle between them is $71°$. Find $\mathbf{a} \bullet \mathbf{b}$.

11 Find $\mathbf{p} \bullet \mathbf{q}$ for:

a $|\mathbf{p}| = 2$, $|\mathbf{q}| = 5$, and $\theta = 60°$

b $|\mathbf{p}| = 6$, $|\mathbf{q}| = 3$, and $\theta = \frac{2\pi}{3}$

12 Suppose $\mathbf{a} \bullet \mathbf{b} = -12$ and $\mathbf{b}$ is a unit vector.

a Explain why $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular.

b Given that $\mathbf{a}$ and $\mathbf{b}$ are parallel, find $|\mathbf{a}|$.

13 Suppose $|\mathbf{c}| = |\mathbf{d}| = \sqrt{5}$. What can be deduced about $\mathbf{c}$ and $\mathbf{d}$ if:

a $\mathbf{c} \bullet \mathbf{d} = 5$ **b** $\mathbf{c} \bullet \mathbf{d} = -5$?

14 Find three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in space such that $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet \mathbf{c}$ but $\mathbf{b} \neq \mathbf{c}$.

15 Find any two vectors which are not parallel, but which are *both* perpendicular to $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

16 Consider triangle ABC in which A is $(5, 1, 2)$, B is $(6, -1, 0)$, and C is $(3, 2, 0)$. Using scalar product only, show that the triangle is right angled.

17 A$(2, 4, 2)$, B$(-1, 2, 3)$, C$(-3, 3, 6)$, and D$(0, 5, 5)$ are vertices of a quadrilateral.

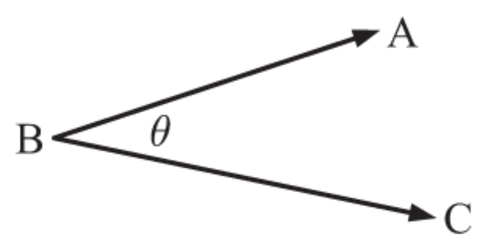
a Prove that ABCD is a parallelogram.

b Find $|\overrightarrow{AB}|$ and $|\overrightarrow{BC}|$. What can be said about ABCD?

c Find $\overrightarrow{AC} \bullet \overrightarrow{BD}$ and discuss your answer.

18 Find the angle ABC of triangle ABC for A$(3, 0, 1)$, B$(-3, 1, 2)$, and C$(-2, 1, -1)$.

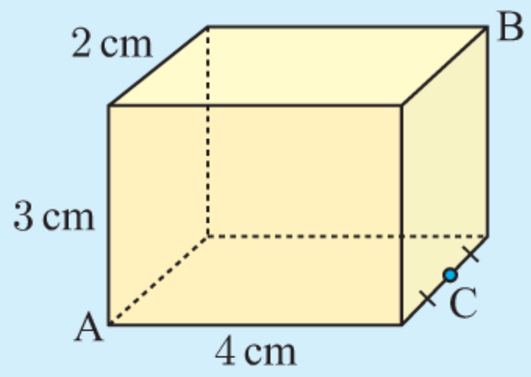
Hint: To find the angle at B, use $\overrightarrow{BA}$ and $\overrightarrow{BC}$.
What angle is found if $\overrightarrow{BA}$ and $\overrightarrow{CB}$ are used?



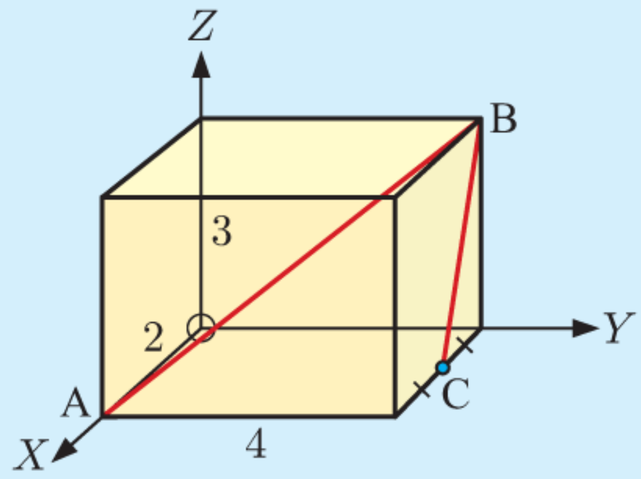
Example 28

Self Tutor

Use vector methods to determine the measure of $A\widehat{B}C$.



Placing the coordinate axes as illustrated, A is $(2, 0, 0)$, B is $(0, 4, 3)$, and C is $(1, 4, 0)$.



$$\therefore \quad \overrightarrow{BA} = \begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

$$\text{Now} \quad \cos(A\widehat{B}C) = \frac{\overrightarrow{BA} \bullet \overrightarrow{BC}}{|\overrightarrow{BA}| \ |\overrightarrow{BC}|}$$

$$= \frac{2(1) + (-4)(0) + (-3)(-3)}{\sqrt{4+16+9}\sqrt{1+0+9}}$$

$$= \frac{11}{\sqrt{290}}$$

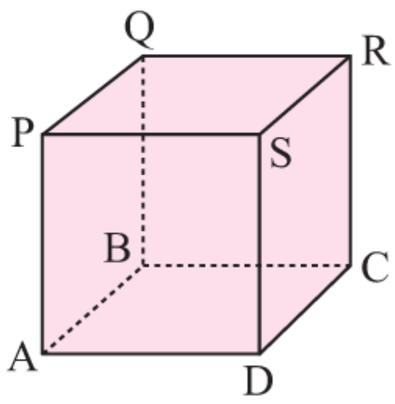
$$\therefore \quad A\widehat{B}C = \cos^{-1}\left(\frac{11}{\sqrt{290}}\right) \approx 49.8°$$

The vectors used must both be away from B (or towards B). If this is not done you will be finding the exterior angle at B.



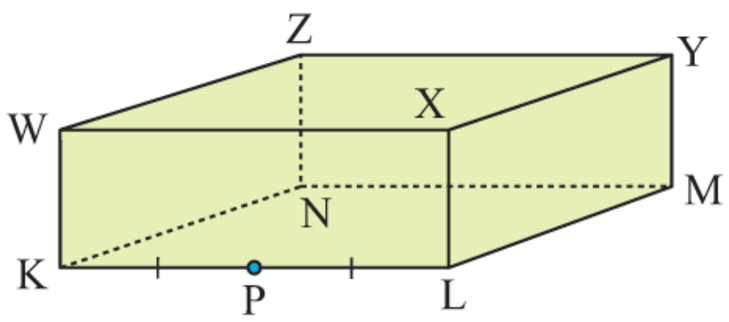
19 The cube alongside has edges of length 2 cm. Find, using vector methods, the measure of:

a $A\hat{B}S$ **b** $R\hat{B}P$ **c** $P\hat{B}S$



20 [KL], [LM], and [LX] are 8, 5, and 3 units long respectively. P is the midpoint of [KL]. Find, using vector methods, the measure of:

a $Y\hat{N}X$ **b** $Y\hat{N}P$



L THE VECTOR PRODUCT OF TWO VECTORS

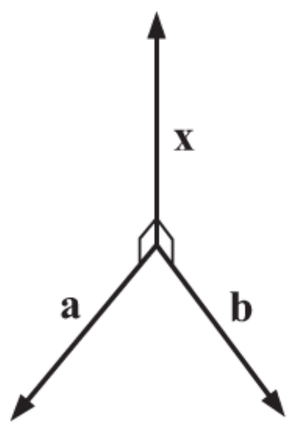
We have seen how the scalar product of two vectors results in a **scalar**.

The second form of product between two vectors is the **vector product** or **vector cross product**, and this results in a **vector**.

The vector cross product is only defined for 3-dimensional vectors.

The vector product arises when we attempt to find a vector which is *perpendicular to two other known vectors*.

Suppose $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is perpendicular to both $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.



$$\therefore \quad \begin{cases} a_1x + a_2y + a_3z = 0 \\ b_1x + b_2y + b_3z = 0 \end{cases} \quad \{\text{as dot products are zero}\}$$

$$\therefore \quad \begin{cases} a_1x + a_2y = -a_3z & \ldots. \ (1) \\ b_1x + b_2y = -b_3z & \ldots. \ (2) \end{cases}$$

We will now try to solve these two equations simultaneously to obtain expressions for x and y in terms of z.

To eliminate x, we multiply (1) by $-b_1$ and (2) by a_1.

$$\begin{array}{rll} -a_1b_1x - a_2b_1y = a_3b_1z & & \{-b_1 \times (1)\} \\ a_1b_1x + a_1b_2y = -a_1b_3z & & \{a_1 \times (2)\} \\ \hline (a_1b_2 - a_2b_1)y = (a_3b_1 - a_1b_3)z & & \end{array}$$

Adding these gives

$\therefore \quad y = (a_3b_1 - a_1b_3)t$ and $z = (a_1b_2 - a_2b_1)t$ for any non-zero t.

Substituting into (1), $a_1x + a_2(a_3b_1 - a_1b_3)t = -a_3(a_1b_2 - a_2b_1)t$

$$\therefore \quad a_1x = (-a_1a_3b_2 + \cancel{a_2a_3b_1} - \cancel{a_2a_3b_1} + a_1a_2b_3)t$$

$$\therefore \quad a_1x = a_1(a_2b_3 - a_3b_2)t$$

$$\therefore \quad x = (a_2b_3 - a_3b_2)t$$

Letting $t = 1$, we obtain $\mathbf{x} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$, and we call this vector the **vector cross product** of **a** and **b**.

The **vector cross product** of vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

is $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$.

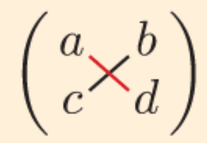
A formula like this will be given in your exam formula booklet. However, the formula may be confusing to use if, for example, you were finding $\mathbf{b} \times \mathbf{a}$. For this reason we recommend an alternative method for calculating vector cross product. We will discover it in the following **Investigation**.

INVESTIGATION 2 THE VECTOR CROSS PRODUCT FORMULA

In matrix algebra, we can define a **determinant** for any square matrix. In particular:

- The determinant of the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

  It is the product of the terms on the main (red) diagonal minus the product of the other terms.



- The determinant of the 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Notice that:

- The sign of the middle term is negative.
- The *size* of the ith term is a_{1i} times the 2×2 determinant seen by covering the top row and the ith column.



What to do:

Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be vectors.

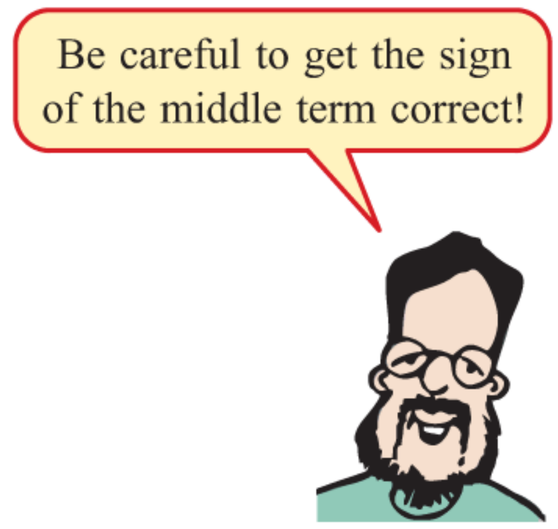
Show that $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$

is equal to the vector cross product $\mathbf{a} \times \mathbf{b}$.

From the **Investigation**, you should have found that:

The vector cross product of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$



SCALAR TRIPLE PRODUCT

For non-zero 3-dimensional vectors **a**, **b**, and **c**, $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$ is called a **scalar triple product.**

Example 29 — Self Tutor

For $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, find:

a $\mathbf{b} \times \mathbf{c}$ **b** $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$

a $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 0 & 4 \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \mathbf{k}$

$= \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$

or $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$

$= \begin{pmatrix} (2)(4) - (3)(0) \\ (3)(2) - (1)(4) \\ (1)(0) - (2)(2) \end{pmatrix}$

$= \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$

b $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 8 \\ 2 \\ -4 \end{pmatrix}$

$= 16 + 2 + 4$

$= 22$



ALGEBRAIC PROPERTIES OF THE VECTOR CROSS PRODUCT

The vector cross product has the following algebraic properties for non-zero 3-dimensional vectors **a**, **b**, **c**, and **d**:

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all **a**.
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for all **a** and **b**.
   This means that $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$ have the same length but opposite direction.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
4. $(\mathbf{a} + \mathbf{b}) \times (\mathbf{c} + \mathbf{d}) = (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} \times \mathbf{d}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{d})$.



EXERCISE 9L.1

1 Calculate:

a $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ -5 \end{pmatrix}$

d $(\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (\mathbf{i} - \mathbf{k})$ **e** $(2\mathbf{i} - \mathbf{k}) \times (\mathbf{j} + 3\mathbf{k})$ **f** $(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \times (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

2 Let $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$. Show that:

a $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ **b** $\mathbf{y} \times \mathbf{y} = \mathbf{0}$ **c** $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$

3 Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$.

a Find $\mathbf{a} \times \mathbf{b}$. **b** Hence determine $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b})$ and $\mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$.

c Explain your results.

4 $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the base unit vectors parallel to the coordinate axes.

a Find $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$, and $\mathbf{k} \times \mathbf{k}$. Comment on your results.

b Find $\mathbf{i} \times \mathbf{j}$ and $\mathbf{j} \times \mathbf{i}$. Comment on your results.

c Predict what the following vector cross products will be, then check your prediction:

i $\mathbf{j} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{j}$ **ii** $\mathbf{i} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$.

5 Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$. Find:

a $\mathbf{b} \times \mathbf{c}$ **b** $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$

6 Suppose $\mathbf{a} = \mathbf{i} + 2\mathbf{k}$, $\mathbf{b} = -\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} - \mathbf{k}$. Find:

a $\mathbf{a} \times \mathbf{b}$ **b** $\mathbf{a} \times \mathbf{c}$

c $(\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$ **d** $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$

Example 30

Self Tutor

Find *all* vectors perpendicular to both $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & 0 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \mathbf{k}$$

$$= -6\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

$$= -2(3\mathbf{i} - \mathbf{j} + \mathbf{k})$$

or

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$= \begin{pmatrix} (2)(-3) - (-1)(0) \\ (-1)(1) - (1)(-3) \\ (1)(0) - (2)(1) \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

The vectors have the form $k\begin{pmatrix}3\\-1\\1\end{pmatrix}$ where k is any non-zero real number.

7 Find *all* vectors perpendicular to both:

a $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\end{pmatrix}$ **b** $\begin{pmatrix}-1\\3\\4\end{pmatrix}$ and $\begin{pmatrix}5\\0\\2\end{pmatrix}$

c $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}-\mathbf{j}-\mathbf{k}$ **d** $\mathbf{i}-\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

8 Find all vectors perpendicular to both $\mathbf{a}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$.

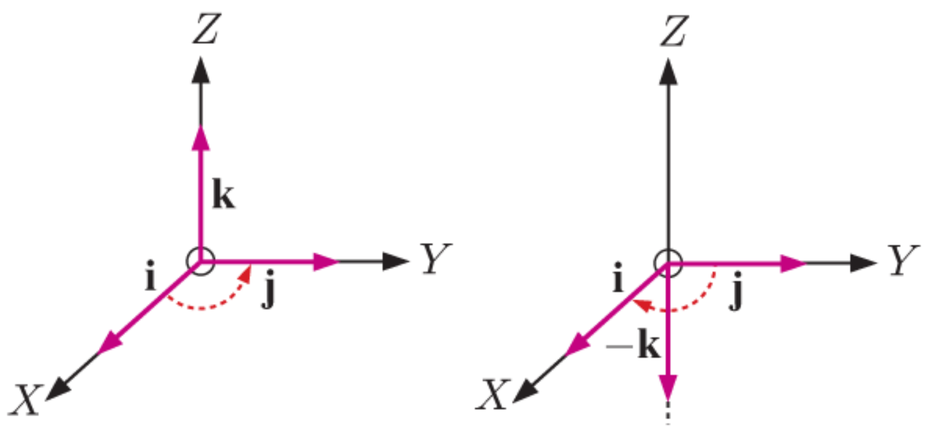
Hence find *two* vectors of length 5 units which are perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

THE DIRECTION OF $\mathbf{a}\times\mathbf{b}$

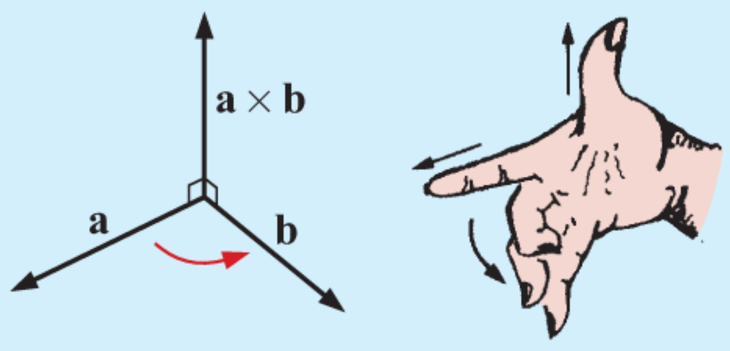
We have already observed that $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$, so $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ are in opposite directions.

However, what is the direction of each?

In the last Exercise, we saw that $\mathbf{i}\times\mathbf{j}=\mathbf{k}$ and $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$.



In general, the **direction** of $\mathbf{a}\times\mathbf{b}$ is determined by the **right hand rule**:



If the fingers on your right hand turn from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{a}\times\mathbf{b}$.

THE LENGTH OF $\mathbf{a}\times\mathbf{b}$

Using $\mathbf{a}\times\mathbf{b}=\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$,

$$|\mathbf{a}\times\mathbf{b}|=\sqrt{(a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2}.$$

However, another very useful form of the length of $\mathbf{a}\times\mathbf{b}$ exists:

$$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta \quad \text{where } \theta \text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}.$$

Proof:

$$
\begin{aligned}
|\mathbf{a}|^2\,|\mathbf{b}|^2\sin^2\theta &= |\mathbf{a}|^2\,|\mathbf{b}|^2(1-\cos^2\theta)\\
&= |\mathbf{a}|^2\,|\mathbf{b}|^2 - |\mathbf{a}|^2\,|\mathbf{b}|^2\cos^2\theta\\
&= |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a}\bullet\mathbf{b})^2\\
&= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2\\
&\qquad \text{which on expanding and then factorising gives}\\
&= (a_2b_3-a_3b_2)^2+(a_3b_1-a_1b_3)^2+(a_1b_2-a_2b_1)^2\\
&= |\mathbf{a}\times\mathbf{b}|^2
\end{aligned}
$$

$$\therefore\quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta \qquad \{\text{as}\quad \sin\theta > 0\}$$

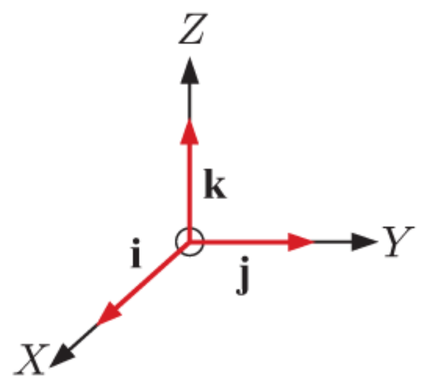
The immediate consequences of this result are:

- If $\mathbf{u}$ is a **unit vector** in the direction of $\mathbf{a}\times\mathbf{b}$ then $\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\mathbf{u}$.
  In some texts this is the **geometric definition** of $\mathbf{a}\times\mathbf{b}$.
- If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\mathbf{a}\times\mathbf{b} = \mathbf{0} \Leftrightarrow$ $\mathbf{a}$ is parallel to $\mathbf{b}$.

EXERCISE 9L.2

1 **a** Find $\mathbf{i}\times\mathbf{k}$ and $\mathbf{k}\times\mathbf{i}$ using the original definition of $\mathbf{a}\times\mathbf{b}$. Check that the **right hand rule** correctly gives the direction of $\mathbf{i}\times\mathbf{k}$ and $\mathbf{k}\times\mathbf{i}$.

b Verify that $\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\mathbf{u}$, where $\mathbf{u}$ is the unit vector in the direction of $\mathbf{a}\times\mathbf{b}$, is true for $\mathbf{i}\times\mathbf{k}$ and $\mathbf{k}\times\mathbf{i}$.



2 Let $\mathbf{a} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{b} = \mathbf{i}-\mathbf{k}$.

a Find $\mathbf{a}\bullet\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$.

b Find $\cos\theta$ using $\mathbf{a}\bullet\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$.

c Find $\sin\theta$ using $\sin^2\theta+\cos^2\theta = 1$.

d Find $\sin\theta$ using $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta$.

3 Prove the property:
"If $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors then $\mathbf{a}\times\mathbf{b} = \mathbf{0} \Leftrightarrow$ $\mathbf{a}$ is parallel to $\mathbf{b}$."

4 Suppose $\mathbf{p} = \begin{pmatrix}2\\1\\-5\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}3\\-1\\1\end{pmatrix}$.

a Show that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

b Find $\mathbf{p}\times\mathbf{q}$.

c Verify that $|\mathbf{p}\times\mathbf{q}| = |\mathbf{p}|\,|\mathbf{q}|$.

5 A, B, and C are 3 distinct points with non-zero position vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ respectively.

a If $\mathbf{a}\times\mathbf{c} = \mathbf{b}\times\mathbf{c}$, what can be deduced about $\overrightarrow{OC}$ and $\overrightarrow{AB}$?

b If $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, what relationship exists between $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{c}$?

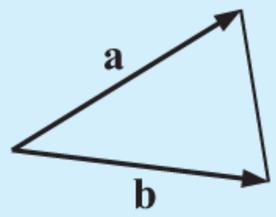
c If $\mathbf{c}\neq\mathbf{0}$ and $\mathbf{b}\times\mathbf{c} = \mathbf{c}\times\mathbf{a}$, prove that $\mathbf{a}+\mathbf{b} = k\mathbf{c}$ for some scalar k.

GEOMETRIC INTERPRETATION OF $|\mathbf{a} \times \mathbf{b}|$

We can use the vector cross product to find the area of a triangle or parallelogram defined by two vectors.

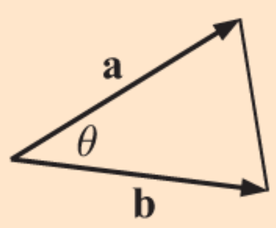
TRIANGLES

If a triangle has defining vectors $\mathbf{a}$ and $\mathbf{b}$ then its area is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units^2.



Proof:

$$\begin{aligned}\text{Area} &= \tfrac{1}{2} \times \text{product of two sides} \times \text{sine of included angle} \\ &= \tfrac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta \\ &= \tfrac{1}{2}|\mathbf{a} \times \mathbf{b}|\end{aligned}$$



Example 31 **Self Tutor**

Find the area of triangle ABC given A$(-1, 2, 3)$, B$(2, 1, 4)$, and C$(0, 5, -1)$.

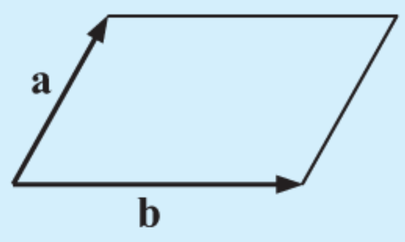
The defining vectors from A are $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

$$\text{Now } \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 1 \\ 1 & 3 & -4 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 3 & -4 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & -4 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{i} + 13\mathbf{j} + 10\mathbf{k}$$

$$\therefore \text{ area} = \tfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \tfrac{1}{2}\sqrt{1 + 169 + 100}$$
$$= \tfrac{1}{2}\sqrt{270} \text{ units}^2$$

PARALLELOGRAMS

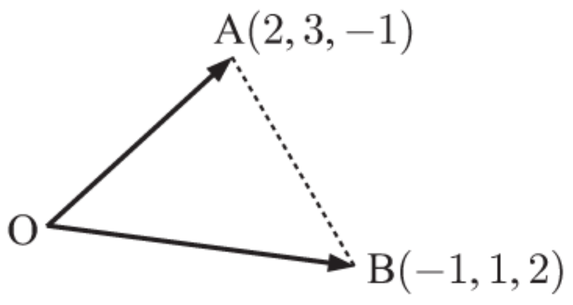
If a parallelogram has defining vectors $\mathbf{a}$ and $\mathbf{b}$ then its area is $|\mathbf{a} \times \mathbf{b}|$ units^2.



The proof follows directly from that of a triangle, as the parallelogram consists of two congruent triangles with defining vectors $\mathbf{a}$ and $\mathbf{b}$.

EXERCISE 9L.3

1 Consider the triangle OAB shown. Find:



a $\overrightarrow{OA}$ and $\overrightarrow{OB}$

b $\overrightarrow{OA} \times \overrightarrow{OB}$

c the area of $\triangle OAB$.

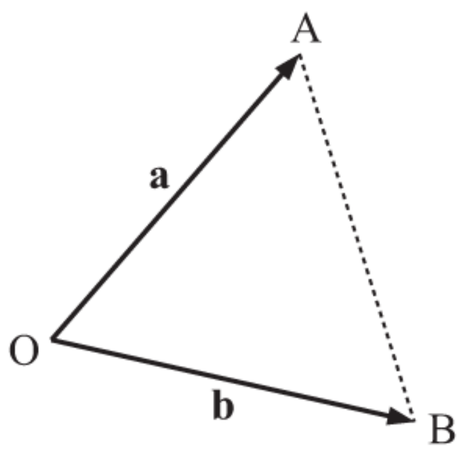
2 Calculate the area of triangle ABC given:

a A$(2, 1, 1)$, B$(4, 3, 0)$, C$(1, 3, -2)$

b A$(0, 0, 0)$, B$(-1, 2, 3)$, C$(1, 2, 6)$

c A$(1, 3, 2)$, B$(2, -1, 0)$, C$(1, 10, 6)$

3



In the triangle OAB alongside, the defining vectors from O are **a** and **b**.

a Write down the area of $\triangle$OAB in terms of **a** and **b**.

b The defining vectors from B are $\overrightarrow{BO}$ and $\overrightarrow{BA}$.

i Write these vectors in terms of **a** and **b**.

ii Hence write down the area of $\triangle$OAB using the defining vectors from B.

c Use the properties of the vector product to show that the areas in **a** and **b ii** are equal.

4 $A(-1, 1, 2)$, $B(2, 0, 1)$, and $C(x, 2, -1)$ are three points in space. The area of triangle ABC is $\sqrt{88}$ units^2. Find x.

5 $A(2, 4, 1)$, $B(1, 9, 1)$, and $C(-1, 5, 0)$ are three points in space.

a Show that OABC is a parallelogram. **b** Hence find the area of OABC.

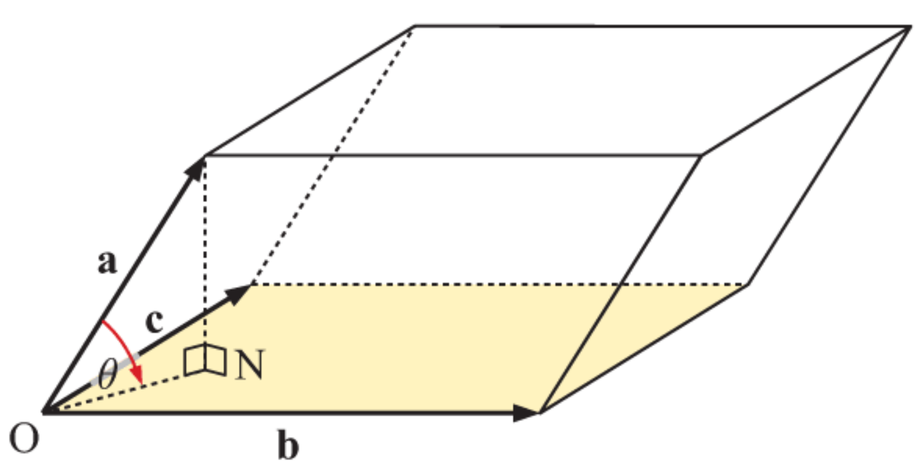
6 Calculate the area of parallelogram ABCD given $A(-1, 2, 2)$, $B(2, -1, 4)$, and $C(0, 1, 0)$.

7 ABCD is a parallelogram with vertices $A(-1, 3, 2)$, $B(2, 0, 4)$, and $C(-1, -2, 5)$. Find:

a the coordinates of D **b** the area of the parallelogram.

8 The 3-dimensional shape alongside is a **parallelepiped** with defining vectors **a**, **b**, and **c**.

Let θ be the angle between **a** and the base plane.



a Find:

i the area of the base plane

ii the perpendicular height of the parallelepiped.

b Hence show that the volume of the parallelepiped is $|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|$ units^3, the scalar triple product.

c $A(3, 0, 0)$, $B(0, 1, 0)$, and $C(1, 2, 3)$ are the vertices of a parallelepiped which are adjacent to another vertex at $O(0, 0, 0)$. Find the volume of the parallelepiped.

DISCUSSION

Discuss the following statement:

"Scalar product is a measure of the *parallelism* of two vectors, whereas vector product is a measure of the *perpendicularity* of two vectors."

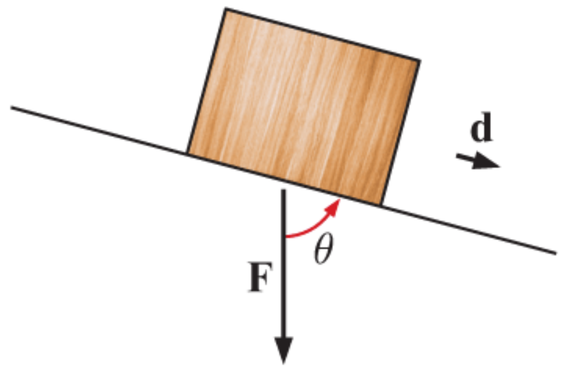
M VECTOR COMPONENTS

We have seen that vectors are commonly written in *component form* using the base unit vectors **i**, **j**, and **k**.

For the vector $\mathbf{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, the components x, y, and z are scalars which tell us the components of **a** in the **i**, **j**, and **k** directions respectively. Notice that x, y, and z may each be positive or negative.

In many vector problems, particularly in Physics, we are interested in the component of a vector which acts in a particular direction.

For example, when a block of wood rests on a slope, gravity exerts a force **F** downwards on the block. In order for the block to not slide, friction must balance the component of **F** in the downhill direction **d**.

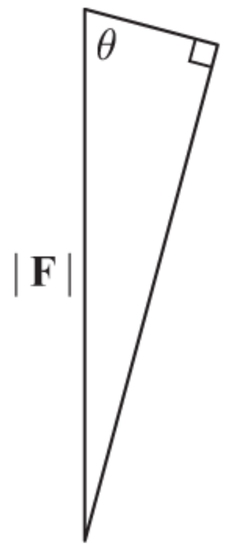


Suppose the angle between **F** and **d** is θ.

The component of **F** in the direction **d** is $|\mathbf{F}|\cos\theta$

$$= \frac{|\mathbf{F}||\mathbf{d}|\cos\theta}{|\mathbf{d}|}$$

$$= \frac{\mathbf{F} \bullet \mathbf{d}}{|\mathbf{d}|}.$$



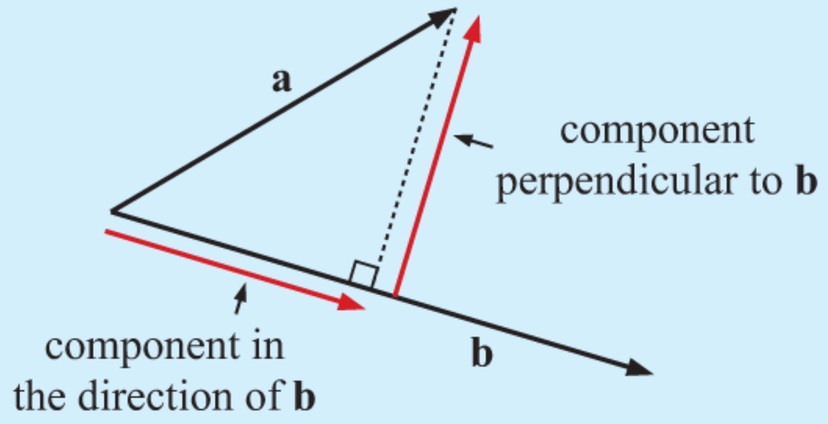
The component of the force *normal* or *perpendicular* to the surface is $|\mathbf{F}|\sin\theta = \dfrac{|\mathbf{F}||\mathbf{d}|\sin\theta}{|\mathbf{d}|}$

$$= \frac{|\mathbf{F} \times \mathbf{d}|}{|\mathbf{d}|}.$$

We can generalise these results as follows:

For 3-dimensional vectors **a** and **b**:

- the component of **a** in the direction of **b** is $\dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}$
- the component of **a** perpendicular to **b** is $\dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$.



These definitions include cases for which the angle between the vectors is obtuse. The component of **a** in the direction of **b** is therefore *signed*, being positive for acute θ and negative for obtuse θ.

DISCUSSION

Why is it sensible that the component of **a** *perpendicular* to **b** is *not* signed?

Example 32

Self Tutor

Suppose $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$. Find the component of **a**:

a in the direction of **b** **b** perpendicular to **b**.

a component of **a** in the direction of $\mathbf{b} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} = \dfrac{1(3) + 2(5) + 0(-1)}{\sqrt{3^2 + 5^2 + (-1)^2}}$

$$= \frac{13}{\sqrt{35}}$$

b $\mathbf{a} \times \mathbf{b}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 3 & 5 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 5 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \mathbf{k}$$

$$= -2\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$\therefore$ component of $\mathbf{a}$ perpendicular to $\mathbf{b}$

$$= \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$$

$$= \frac{\sqrt{(-2)^2 + 1^2 + (-1)^2}}{\sqrt{3^2 + 5^2 + (-1)^2}}$$

$$= \frac{\sqrt{6}}{\sqrt{35}}$$

EXERCISE 9M

1 For the vectors $\mathbf{a}$ and $\mathbf{b}$, find the component of $\mathbf{a}$:

i in the direction of $\mathbf{b}$ **ii** perpendicular to $\mathbf{b}$.

a $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

b $\mathbf{a} = \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$

c $\mathbf{a} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{k}$

d $\mathbf{a} = 2\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

2 For the vectors $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$, find the component of $\mathbf{a}$:

a in the direction of $\mathbf{b}$ **b** perpendicular to $\mathbf{b}$.

Explain your answers.

3 **a** For the vectors $\mathbf{p} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, find:

i the component of $\mathbf{p}$ in the direction of $\mathbf{q}$

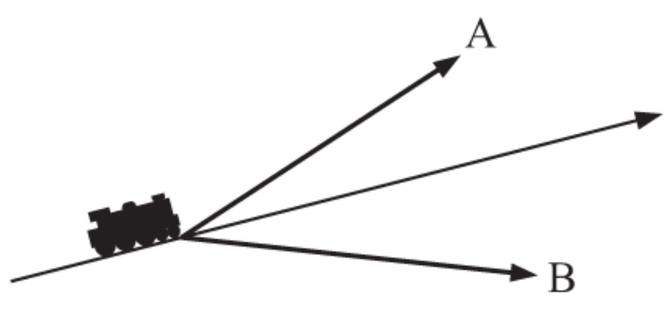
ii the component of $\mathbf{q}$ in the direction of $\mathbf{p}$.

b Draw a diagram to explain why your answers in **a** are different.

4 Two strongmen are pulling a train along a track in the direction $7\mathbf{i} + 6\mathbf{j} + \mathbf{k}$.

Strongman A is exerting the force $220\mathbf{i} + 200\mathbf{j} + 40\mathbf{k}$ N.

Strongman B is exerting the force $180\mathbf{i} + 240\mathbf{j} + 60\mathbf{k}$ N.



a Find the magnitude of the force exerted by each strongman.

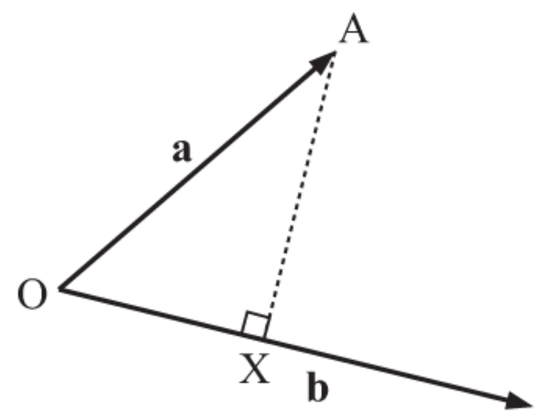
b For each strongman, find the component of the force in the direction of the train's motion.

c Is the strongman who is exerting the most force contributing the most to the movement of the train? Explain your answer.

5 **a** Explain why the component of vector $\mathbf{a}$ acting perpendicular to vector $\mathbf{b}$ has direction $\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$.

Hint: All vectors in the plane containing $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to $\mathbf{a} \times \mathbf{b}$.

b Consider the vectors $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.



i Find the vector $\overrightarrow{OX}$, where OX is the component of **a** in the direction of **b**.

ii Use **a** to find the vector $\overrightarrow{XA}$, where XA is the component of **a** perpendicular to **b**.

iii Verify that:

(1) $\overrightarrow{OX}$ is perpendicular to $\overrightarrow{XA}$ **(2)** $\overrightarrow{OX} + \overrightarrow{XA} = \mathbf{a}$.

REVIEW SET 9A

1 Using a scale of 1 cm $\equiv$ 10 units, sketch a vector to represent:

a an aeroplane taking off at an angle of $8°$ to a runway with a speed of $60\ \text{m s}^{-1}$

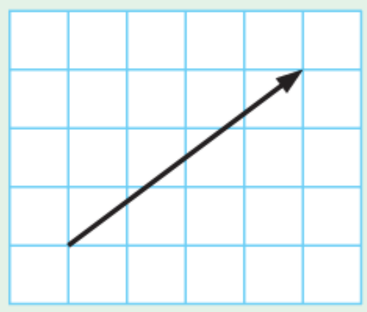
b a displacement of 45 m in a north-easterly direction.

2 Simplify:

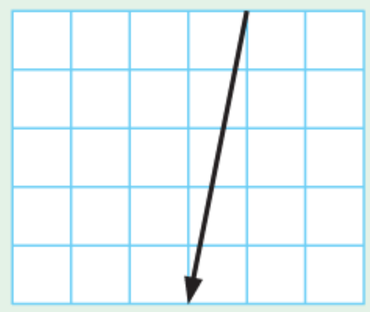
a $\overrightarrow{AB} - \overrightarrow{CB}$ **b** $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{DC}$

3 Write the illustrated vectors in component form and in unit vector form:

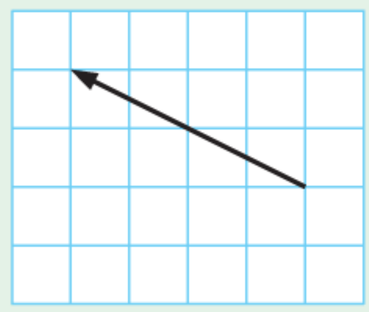
a



b



c



4 Illustrate P and find its distance from the origin O:

a P(1, 3, 2) **b** P(2, -1, 3)

5 If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$, find $\overrightarrow{SP}$.

6 Given $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -1 \\ -4 \end{pmatrix}$, find:

a $\mathbf{a} - \mathbf{b}$ **b** $|\mathbf{a}|$ **c** $2\mathbf{a} + 3\mathbf{b}$

d $-\frac{1}{3}\mathbf{a}$ **e** $|\mathbf{b}|\,\mathbf{a}$ **f** $|\mathbf{b} - 2\mathbf{a}|$

7 Find scalars a, b, and c such that $3\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} + a\begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} b \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ c \\ 7 \end{pmatrix}$.

8 The points A, B, and C have position vectors $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $5\mathbf{i} - \mathbf{k}$, and $-\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ respectively.

a M is the midpoint of BC. Find the magnitude of $\overrightarrow{AM}$.

b Point D is such that $\overrightarrow{AC} = \overrightarrow{BD}$. Find the position vector of D.

9 Find m and n if $\begin{pmatrix} 3 \\ m \\ n \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \\ 2 \end{pmatrix}$ are parallel vectors.

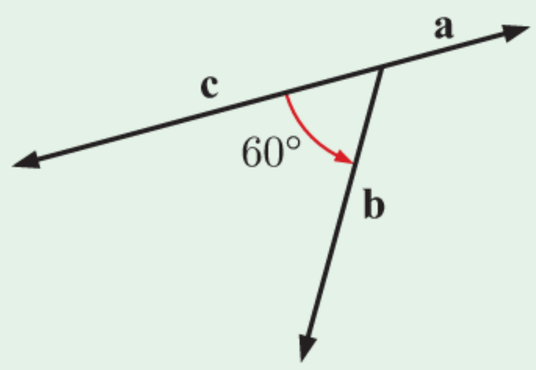
10 Prove that $P(-6, 8, 2)$, $Q(4, 6, 8)$, and $R(19, 3, 17)$ are collinear.

11 Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$.

12 If $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, find: **a** $\mathbf{p} \bullet \mathbf{q}$ **b** $\mathbf{q} \bullet (\mathbf{p} - \mathbf{r})$

13 Suppose $|\mathbf{a}| = 2$, $|\mathbf{b}| = 4$, and $|\mathbf{c}| = 5$.
Find:

a $\mathbf{a} \bullet \mathbf{b}$ **b** $\mathbf{b} \bullet \mathbf{c}$ **c** $\mathbf{a} \bullet \mathbf{c}$



14 If $\mathbf{u} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$, find:

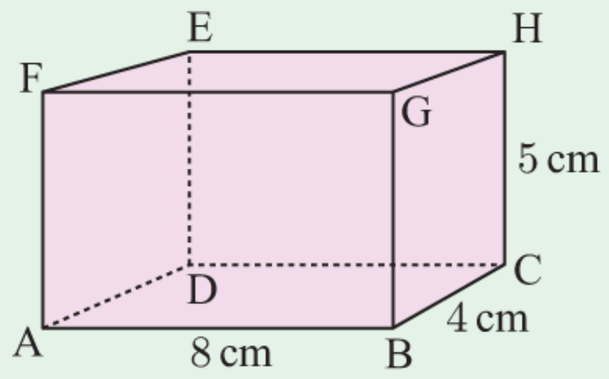
a $\mathbf{u} \bullet \mathbf{v}$ **b** the angle between $\mathbf{u}$ and $\mathbf{v}$.

15 Find the angle between the vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$.

16 Find t given that $\begin{pmatrix} 2-t \\ 3 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 4 \\ t+1 \end{pmatrix}$ are perpendicular.

17 Find a unit vector which is parallel to $\mathbf{i} + r\mathbf{j} + 2\mathbf{k}$ and perpendicular to $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

18 Use vector methods to find the measure of $G\widehat{A}C$:

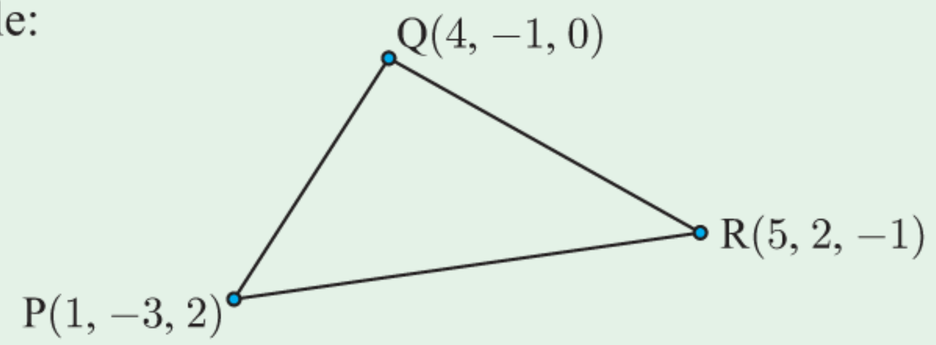


19 Given $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$, find:

a $\mathbf{a} \times \mathbf{b}$ **b** $(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{c}$ **c** $\mathbf{b} \times \mathbf{c}$

20 Find *two* vectors of length 3 units which are perpendicular to both $-\mathbf{i} + 3\mathbf{k}$ and $2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

21 Find the area of this triangle:

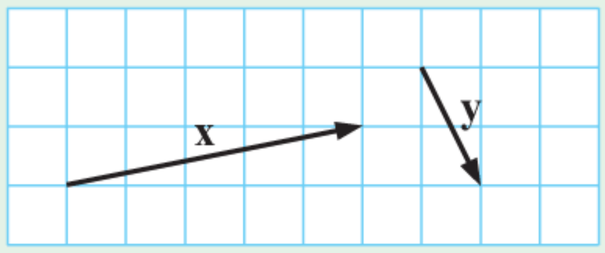


22 Given $\mathbf{a} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, find the component of $\mathbf{a}$:

a in the direction of $\mathbf{b}$ **b** perpendicular to $\mathbf{b}$.

REVIEW SET 9B

1 Copy the given vectors and find geometrically:

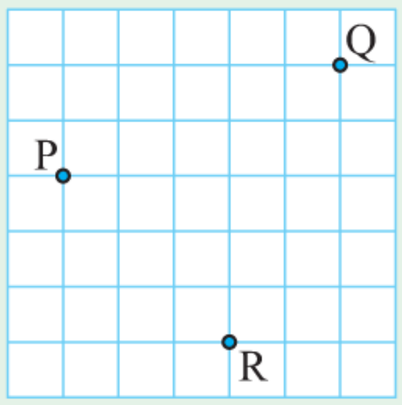


a $\mathbf{x} + \mathbf{y}$ **b** $\mathbf{y} - 2\mathbf{x}$

2 Find a single vector which is equal to:

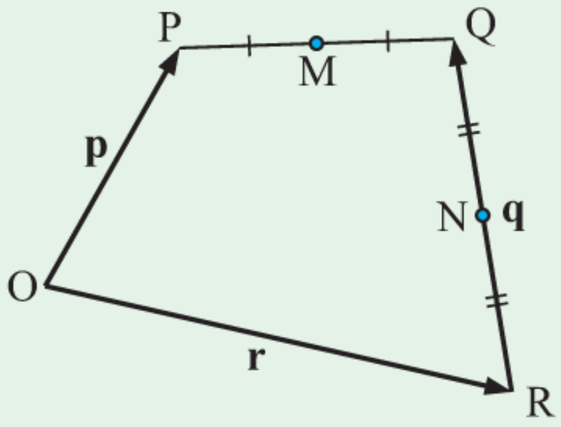
a $\overrightarrow{PR} + \overrightarrow{RQ}$ **b** $\overrightarrow{PS} + \overrightarrow{SQ} + \overrightarrow{QR}$

3 Find in component form and in unit vector form:



a $\overrightarrow{PQ}$ **b** $\overrightarrow{RP}$

4



In the figure alongside, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$, and $\overrightarrow{RQ} = \mathbf{q}$. M and N are the midpoints of [PQ] and [QR] respectively. Find, in terms of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$:

a $\overrightarrow{OQ}$ **b** $\overrightarrow{PQ}$ **c** $\overrightarrow{ON}$ **d** $\overrightarrow{MN}$

5 For $\mathbf{m} = \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$, $\mathbf{n} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$, and $\mathbf{p} = \begin{pmatrix} -1 \\ 3 \\ 6 \end{pmatrix}$, find:

a $\mathbf{m} - \mathbf{n} + \mathbf{p}$ **b** $2\mathbf{n} - 3\mathbf{p}$ **c** $|\mathbf{m} + \mathbf{p}|$

6 If $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -7 \\ 4 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -6 \\ 1 \\ -3 \end{pmatrix}$, find $\overrightarrow{CB}$.

7 Given P$(2, -5, 6)$ and Q$(-1, 7, 9)$, find:

a the displacement vector of Q relative to P **b** the distance from P to Q

c the distance from P to the X-axis.

8 Find k given that $k\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} - k\mathbf{k}$ is a unit vector.

9 Find a and b if J$(-4, 1, 3)$, K$(2, -2, 0)$, and L$(a, b, 2)$ are collinear.

10 Find *two* vectors of length 4 units which are parallel to $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

11 Find the coordinates of point X which is 12 units from $(-2, 1, -5)$, in the direction $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

12 Suppose $|\mathbf{v}| = 3$ and $|\mathbf{w}| = 2$. If $\mathbf{v}$ is parallel to $\mathbf{w}$, what values might $\mathbf{v} \bullet \mathbf{w}$ take?

13 Using $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, verify that $\mathbf{p} \bullet (\mathbf{q} - \mathbf{r}) = \mathbf{p} \bullet \mathbf{q} - \mathbf{p} \bullet \mathbf{r}$.

14 P$(-1, 2, 3)$ and Q$(4, 0, -1)$ are two points in space. Find:

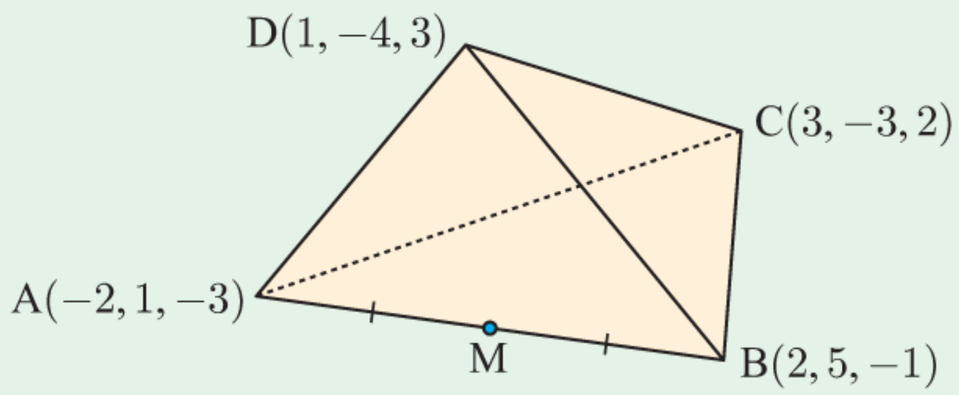
a $\overrightarrow{\text{PQ}}$ **b** the angle that $\overrightarrow{\text{PQ}}$ makes with the X-axis.

15 Given $\mathbf{p} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{q} = -\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, find the angle between $\mathbf{p}$ and $\mathbf{q}$.

16 Find t given $\begin{pmatrix} -4 \\ t+2 \\ t \end{pmatrix}$ and $\begin{pmatrix} t \\ 1+t \\ -3 \end{pmatrix}$ are perpendicular vectors.

17 Find the angle between $2\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.

18 Find the measure of $\text{D}\widehat{\text{M}}\text{C}$.



19 Find all vectors perpendicular to both $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

20 Given $|\mathbf{u}| = 3$, $|\mathbf{v}| = 5$, and $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$, find the possible values of $\mathbf{u} \bullet \mathbf{v}$.

21 Point A has position vector $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and point B has position vector $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$.

a Find $\mathbf{a} \times \mathbf{b}$.

b Point C is 6 units from the origin O, in the direction opposite to that of $\mathbf{a} \times \mathbf{b}$. Find the coordinates of C.

c Find the area of the triangle OAB.

d Hence find the volume of the tetrahedron OABC.

Chapter 10

Vector applications

Contents:

A Lines in 2 and 3 dimensions
B The angle between two lines
C Constant velocity problems
D The shortest distance from a point to a line
E The shortest distance between two objects
F Intersecting lines



OPENING PROBLEM

Haley and Liam are flying their drones in a park. Haley's drone takes off from the point $(5,\ 1,\ 0)$ and moves at $2.5\ \text{m s}^{-1}$ in the direction $\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix}$. Liam's drone takes off from the point $(-12,\ -3,\ 0)$ and moves at $1.5\ \text{m s}^{-1}$ in the direction $\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}$.



Things to think about:

a Can you write an equation to describe the position of each drone after t seconds?

b How far apart are the drones after 2 seconds?

c Do the paths of the drones intersect?

d At what time are the drones closest to one another? How far apart are the drones at this time?

There are many applications of vectors in geometry. We do not use some of these applications in 2-dimensional planar geometry because we have other tools.

In 3-dimensional space, however, vector methods are very efficient, in particular when we consider the relationships between lines.

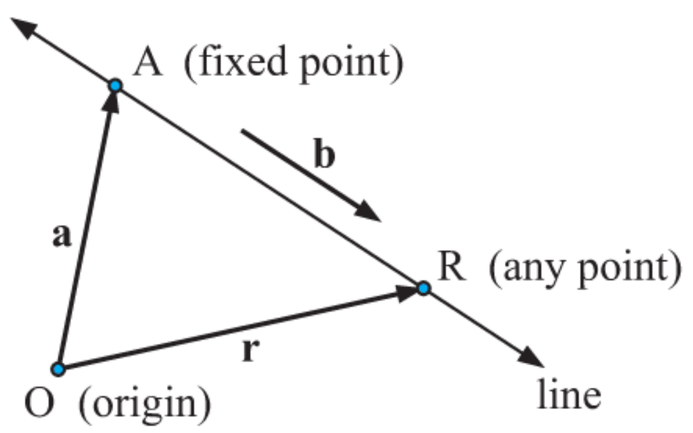
A LINES IN 2 AND 3 DIMENSIONS

In both 2-dimensional and 3-dimensional geometry we can determine the **equation of a line** using its **direction** and any **fixed point** on the line.

In 2 dimensions we are dealing with a **line in a plane**. In 3 dimensions we are dealing with a **line in space**.

Suppose a line passes through a fixed point A with position vector $\mathbf{a}$, and the line is parallel to the vector $\mathbf{b}$.

Consider a point R on the line with position vector $\mathbf{r}$.



DEMO



By vector addition, $\mathbf{r} = \mathbf{a} + \overrightarrow{\text{AR}}$.

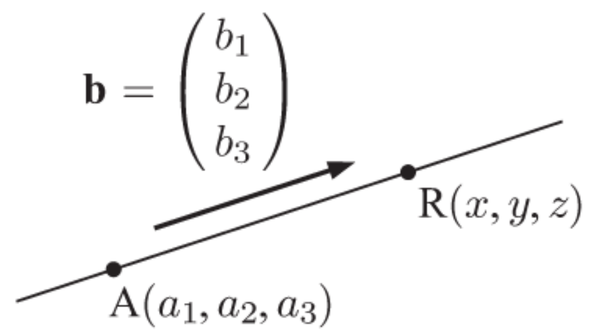
Since $\overrightarrow{\text{AR}}$ is parallel to $\mathbf{b}$,

$$\overrightarrow{\text{AR}} = \lambda\mathbf{b} \quad \text{for some scalar } \lambda \in \mathbb{R}$$
$$\therefore\ \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

So, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}$ is the **vector equation** of the line.

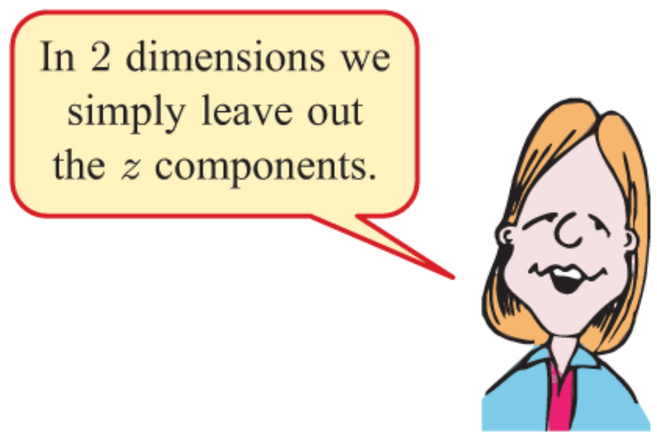
Suppose $R(x, y, z)$ is any point on the line,
$A(a_1, a_2, a_3)$ is the known or fixed point on the line,

and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **direction vector** of the line.



- $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is the **vector equation** of the line.
- The **parametric equations** of the line are:

$$\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases} \quad \text{where } \lambda \in \mathbb{R} \text{ is the } \textbf{parameter}.$$



DISCUSSION

In 2 dimensions we talk about the gradient of a line.

- If a line in 2 dimensions has direction vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, what is its gradient?
- Does it make sense to talk about the gradient of a line in space?

Example 1 **Self Tutor**

A line in space passes through $(1, -2, 3)$ in the direction $4\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}$.
Describe the line using:

a a vector equation **b** parametric equations.

a The vector equation is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

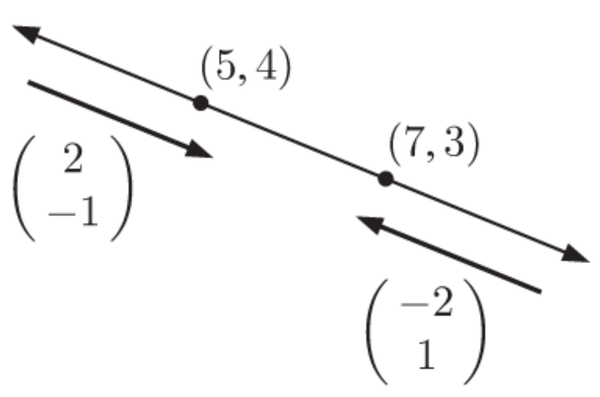
$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

b The parametric equations are: $x = 1 + 4\lambda, \quad y = -2 + 5\lambda, \quad z = 3 - 6\lambda, \quad \lambda \in \mathbb{R}$.

NON-UNIQUENESS OF THE VECTOR EQUATION OF A LINE

Consider the line passing through $(5, 4)$ and $(7, 3)$. When writing the equation of the line, we could use either point to give the position vector $\mathbf{a}$.

Similarly, we could use the direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, but we could also use $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ or any non-zero scalar multiple of these vectors.



We could thus write the equation of the line as

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R} \quad \text{or} \quad \mathbf{r} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \mu \in \mathbb{R} \quad \text{and so on.}$$

Notice how we use different parameters λ and μ when we write these equations. This is because the parameters are clearly not the same: when $\lambda = 0$, we have the point $(5, 4)$
when $\mu = 0$, we have the point $(7, 3)$.

In fact, the parameters are related by $\mu = 1 - \lambda$.

Example 2 **Self Tutor**

Find parametric equations for the line through $A(2, -1, 4)$ and $B(-1, 0, 2)$.

We require a direction vector for the line, either $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$$\overrightarrow{AB} = \begin{pmatrix} -1-2 \\ 0-(-1) \\ 2-4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$$

Using point A, the equations are:

$$x = 2 - 3\lambda, \quad y = -1 + \lambda, \quad z = 4 - 2\lambda, \quad \lambda \in \mathbb{R}$$

or using point B, the equations are:

$$x = -1 - 3\mu, \quad y = \mu, \quad z = 2 - 2\mu, \quad \mu \in \mathbb{R}.$$

EXERCISE 10A

1 Describe each of the following lines using:

i a vector equation **ii** parametric equations.

a a line with direction $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ which passes through $(1, 4)$

b a line passing through $(5, 2)$ which is perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

c a line parallel to $3\mathbf{i} + 7\mathbf{j}$ which cuts the x-axis at -6

d a line perpendicular to $3\mathbf{i} - \mathbf{j}$ which cuts the y-axis at 2

e a line passing through $(3, 0)$ and $(7, 2)$

f a line passing through $(-2, 5)$ and $(4, -6)$.

2 A line passes through $(4, -3)$ with direction vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

a Write parametric equations for the line using the parameter λ.

b Find the points on the line for which $\lambda = 0, 1, 2, -1$, and -3.

3 Line L has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$.

a Locate the point on the line corresponding to $\lambda = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

4 Describe each of the following lines using:

i a vector equation **ii** parametric equations.

a a line parallel to $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ which passes through $(1, 3, -7)$

b a line which passes through $(0, 1, 2)$ with direction vector $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

c a line parallel to the X-axis which passes through $(-2, 2, 1)$

d a line parallel to $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ which passes through $(0, 2, -1)$

e a line perpendicular to the XY-plane which passes through $(3, 2, -1)$.

5 Find the vector equation of the line which passes through:

a A$(1, 2, 1)$ and B$(-1, 3, 2)$ **b** C$(0, 1, 3)$ and D$(3, 1, -1)$

c E$(1, 2, 5)$ and F$(1, -1, 5)$ **d** G$(0, 1, -1)$ and H$(5, -1, 3)$.

6 Find the direction vector of the line:

a $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ **b** $x = 5 - t, \quad y = 1 + t, \quad z = 2 - 3t, \quad t \in \mathbb{R}$

7 Find the coordinates of the point where the line with parametric equations $x = 1 - \lambda$, $y = 3 + \lambda$, and $z = 3 - 2\lambda$, $\lambda \in \mathbb{R}$ meets:

a the YZ-plane $x = 0$ **b** the XZ-plane $y = 0$ **c** the XY-plane $z = 0$.

8 The parametric equations of a line are $x = x_0 + \lambda l$, $y = y_0 + \lambda m$, $z = z_0 + \lambda n$ for $\lambda \in \mathbb{R}$.

a Find the coordinates of the point on the line corresponding to $\lambda = 0$.

b Find the direction vector of the line.

9 A line has vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad t \in \mathbb{R}$.

a Show that A$(3, 3, 4)$ lies on the line.

b Find the coordinates of two points on the line which are 9 units from A.

10 Find points on the line with parametric equations $x = 2 - \lambda$, $y = 3 + 2\lambda$, and $z = 1 + \lambda$, $\lambda \in \mathbb{R}$ which are $5\sqrt{3}$ units from the point $(1, 0, -2)$.

11 A line has parametric equations $x = 1 + \lambda$, $y = 2 - \lambda$, $z = 3 + \lambda$.

a Find the distance from P$(1, 1, 2)$ to a general point X on the line.

b Use quadratic theory to find the value of λ which minimises the distance PX.

c Show that when the distance PX is minimised, $\overrightarrow{\text{PX}}$ is perpendicular to the line.

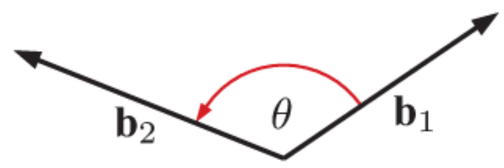
12 Line L_1 has equation $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$. Line L_2 passes through $(2, 0, 3)$ and $(-2, 1, 3)$.

a Find the equation of L_2.

b The line L_3 is perpendicular to both L_1 and L_2, and cuts the Z-axis at -2. Find the vector equation of L_3.

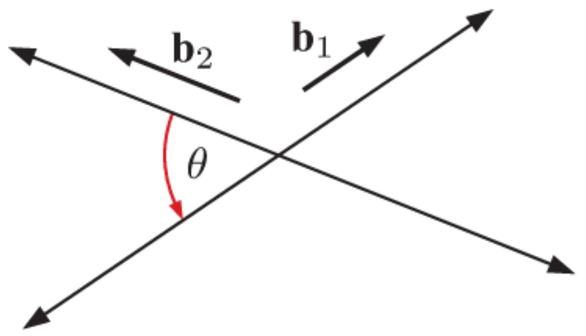
B THE ANGLE BETWEEN TWO LINES

In **Chapter 9** we saw that the angle between two vectors is measured in the range $0° \leqslant \theta \leqslant 180°$, and found using the formula $\cos\theta = \dfrac{\mathbf{b}_1 \bullet \mathbf{b}_2}{|\mathbf{b}_1||\mathbf{b}_2|}$.



Since lines continue infinitely in both directions, we talk about the *acute* angle between them. For an acute angle, $\cos\theta > 0$, so we use the formula

$$\cos\theta = \frac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$$



where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the direction vectors of the given lines.

Example 3 — Self Tutor

Find the angle between the lines L_1: $x = 2 - 3t$, $y = -1 + t$ and L_2: $x = 1 + 2s$, $y = -4 + 3s$, where $s, t \in \mathbb{R}$

L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Now $\cos\theta = \dfrac{|\mathbf{b}_1 \bullet \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|} = \dfrac{|-6+3|}{\sqrt{10}\sqrt{13}} \approx 0.2631$

$\therefore \ \theta \approx 74.7°$

$\therefore$ the angle between L_1 and L_2 is about $74.7°$.

PARALLEL AND PERPENDICULAR LINES

Consider two lines $\mathbf{r}_1 = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mu\mathbf{b}_2$.

- $\mathbf{r}_1$ and $\mathbf{r}_2$ are **parallel** if $\mathbf{b}_1 = k\mathbf{b}_2$ for some scalar k. We write $\mathbf{r}_1 \parallel \mathbf{r}_2$.
- $\mathbf{r}_1$ and $\mathbf{r}_2$ are **perpendicular** if $\mathbf{b}_1 \bullet \mathbf{b}_2 = 0$. We write $\mathbf{r}_1 \perp \mathbf{r}_2$.

Example 4 — Self Tutor

Find the angle between the lines:

$$L_1: \ \mathbf{r}_1 = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}, \ \lambda \in \mathbb{R} \quad \text{and} \quad L_2: \ \mathbf{r}_2 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}, \ \mu \in \mathbb{R}.$$

L_1 has direction vector $\mathbf{b}_1 = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$. L_2 has direction vector $\mathbf{b}_2 = \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}$.

Since $\mathbf{b}_1 = -2\mathbf{b}_2$, the lines are parallel. The angle between them is $0°$.

EXERCISE 10B

1 Find the angle between the lines:

a L_1: $\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$ and L_2: $\mathbf{r}_2 = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + s\begin{pmatrix} -4 \\ -2 \end{pmatrix}$, $s \in \mathbb{R}$

b L_1: $\mathbf{r}_1 = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ -10 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and L_2: $\mathbf{r}_2 = \begin{pmatrix} -4 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} -2 \\ 5 \end{pmatrix}$, $\mu \in \mathbb{R}$

c L_1: $\mathbf{r}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and L_2: $\mathbf{r}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$, $\mu \in \mathbb{R}$

d L_1: $\mathbf{r}_1 = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and L_2: $\mathbf{r}_2 = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$, $\mu \in \mathbb{R}$.

2 Find the angle between the lines L_1: $x = -4 + 12t$, $y = 3 + 5t$, $t \in \mathbb{R}$
and L_2: $x = 3s$, $y = -6 - 4s$, $s \in \mathbb{R}$.

3 Show that the lines L_1: $x = 2 + 5p$, $y = 19 - 3p$, $z = 9 + 2p$, $p \in \mathbb{R}$
and L_2: $x = 3 + 4r$, $y = 7 + 10r$, $z = -3 + 5r$, $r \in \mathbb{R}$ are perpendicular.

4 The line L_1 passes through $(-6, 3)$ and is parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.
The line L_2 has direction $5\mathbf{i} + 4\mathbf{j}$ and cuts the y-axis at 8.
Find the angle between the lines.

5 **a** Find the angle between the lines:

L_1: $\mathbf{r}_1 = \begin{pmatrix} 8 \\ 9 \\ 10 \end{pmatrix} + s\begin{pmatrix} 3 \\ -16 \\ 7 \end{pmatrix}$, $s \in \mathbb{R}$ and L_2: $\mathbf{r}_2 = \begin{pmatrix} 15 \\ 29 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$, $t \in \mathbb{R}$.

b A third line L_3 is perpendicular to L_1 and has direction vector $-3\mathbf{j} + a\mathbf{k}$.

i Find the value of a. **ii** Find the angle between L_2 and L_3.

6 The line L_1 passes through $(-1, 5, 2)$ and $(0, 2, -2)$, and the line L_2 is parallel to $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.
Find the angle between the lines.

7 The line L_1 has equation $\mathbf{r}_1 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$. The line L_2 meets L_1 on the X-axis and passes through $(5, -3, 1)$. Find the angle between the lines.

C CONSTANT VELOCITY PROBLEMS

An object moving with a constant velocity will travel in a straight line. To model the position using vectors:

- the **velocity vector** of the motion gives the direction vector of the line
- **time** is the parameter
- the **initial position** of the object gives a fixed point on the line.

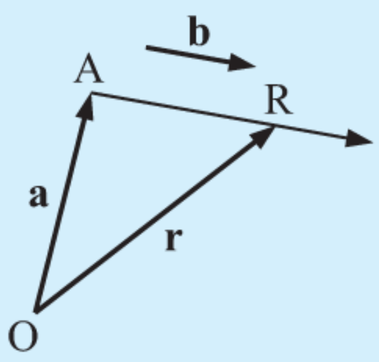
Make sure the units of distance, time, and velocity correspond.



If an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\mathbf{b}|$.



Example 5 **Self Tutor**

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the vector equation of the path of an object.

The time t is in seconds, $t \geqslant 0$. The distance units are metres.

a Find the object's initial position.

b Plot the path of the object for $t = 0, 1, 2, 3$.

c Find the velocity vector of the object.

d Find the object's speed.

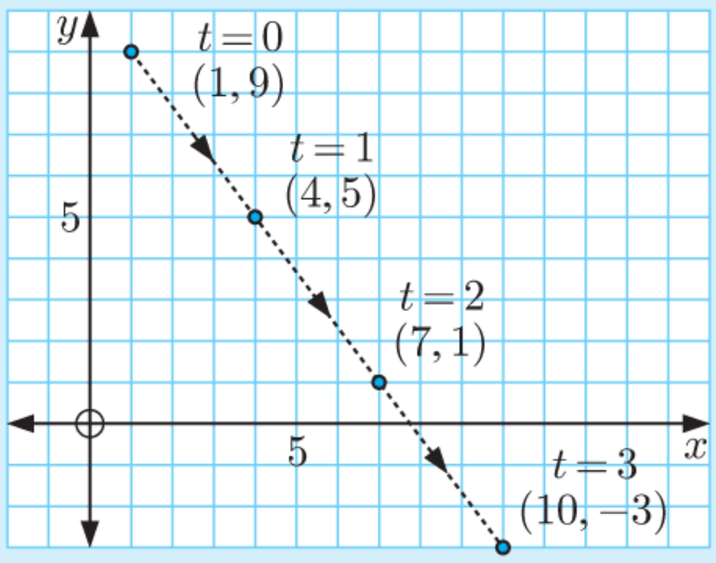
e If the object continues in the same direction but increases its speed to 30 m s^{-1}, state its new velocity vector.



a At $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

$\therefore$ the object is at $(1, 9)$.

b



c The velocity vector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

d The speed is $\left|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right| = \sqrt{3^2 + (-4)^2}$

$= 5$ m s^{-1}.

c Previously, the speed was 5 m s^{-1} and the velocity vector was $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$\therefore$ the new velocity vector is $\frac{30}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$.

Velocity is a vector. Speed is a scalar.



Example 6 **Self Tutor**

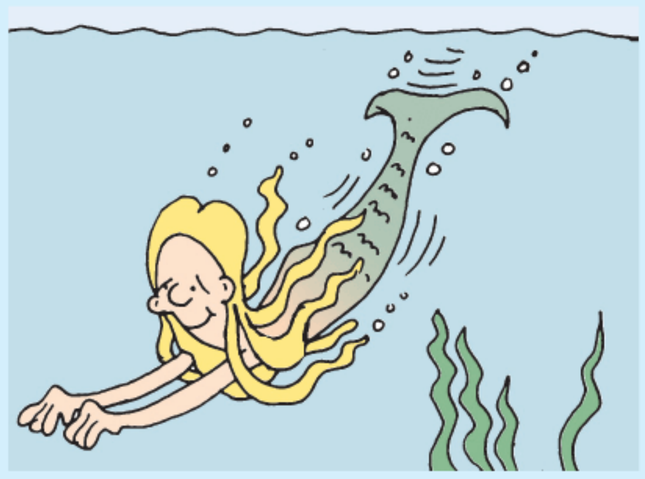
A mermaid is initially at $(15, 8, -1)$. She swims with velocity vector $3\mathbf{i} - 5\mathbf{j} - \frac{9}{2}\mathbf{k}$ metres per second.

Find:

a the position of the mermaid after t seconds

b the speed of the mermaid

c the time when the mermaid reaches depth 28 m.



a $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 15 \\ 8 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ -5 \\ -\frac{9}{2} \end{pmatrix} = \begin{pmatrix} 15+3t \\ 8-5t \\ -1-\frac{9}{2}t \end{pmatrix}, \quad t \in \mathbb{R}$$

After t seconds the mermaid is at $(15+3t,\ 8-5t,\ -1-\frac{9}{2}t)$.

b The speed of the mermaid

$$= \sqrt{3^2 + (-5)^2 + (-\tfrac{9}{2})^2}$$
$$= \sqrt{54\tfrac{1}{4}}$$
$$\approx 7.37 \text{ m s}^{-1}$$

c The mermaid reaches depth 28 m when

$$z(t) = -1 - \tfrac{9}{2}t = -28$$
$$\therefore \ \tfrac{9}{2}t = 27$$
$$\therefore \ t = 6 \text{ seconds}$$

EXERCISE 10C

1 A particle at $P(x(t), y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

a Find the initial position of P.

b Plot the path of P for $t = 0, 1, 2, 3$.

c Find the velocity vector of P.

d Find the speed of P.

e A second particle is travelling in the opposite direction, but with speed 8 cm s^{-1}. Find its velocity vector.

2 Each of the following vector equations represents the path of a moving object. t is measured in seconds, $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

a $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \\ 6 \end{pmatrix}$

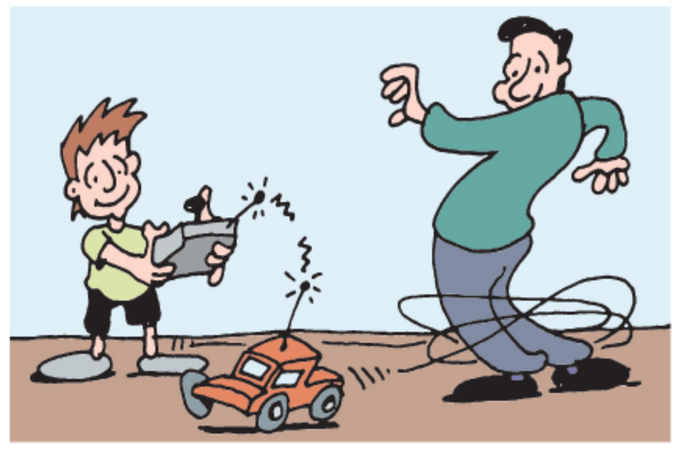
b $x = 3 + 2t,\ y = -t,\ z = 4 - 2t$

3 **a** Find the vector equation of a boat initially at $(2, 3)$, which travels with velocity vector $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$. The grid units are kilometres, and the time is in hours.

b Locate the boat's position after 90 minutes.

c How long will it take for the boat to reach the point $(5, -0.75)$?

4 A remote controlled toy car is initially at the point $(-3, -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.



a Write an expression for the position vector of the car at any time $t \geqslant 0$.

b Hence find the position vector of the car at time $t = 2.5$.

c Find when the car is **i** due north **ii** due west of the observation point $(0, 0)$.

d Plot the car's position at times $t = 0, 0.5, 1, 1.5, 2, 2.5$.

5 Find the velocity vector of:

a a speed boat moving parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with speed 150 km h^{-1}

b a jogger moving parallel to $\begin{pmatrix} -5 \\ 12 \end{pmatrix}$ with speed 7.8 km h^{-1}

c a ferry moving parallel to $6\mathbf{i} + 7\mathbf{j}$ with speed 25 km h^{-1}

d a hot air balloon moving parallel to $4\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ with speed 33 km h^{-1}

e a swooping eagle moving in the direction $-2\mathbf{i} + 5\mathbf{j} - 14\mathbf{k}$ with speed 90 km h^{-1}.

6 Yacht A moves according to $x_A(t) = 4 + t$, $y_A(t) = 5 - 2t$, $t \geqslant 0$. Yacht B moves according to $x_B(t) = 1 + 2t$, $y_B(t) = -8 + t$, $t \geqslant 0$. The distance units are kilometres, and the time units are hours.

a Find the initial position of each yacht. **b** Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant, and state these speeds.

d Show, using vector algebra, that the paths of the yachts are at right angles to each other.

7 A cable car leaves from $(10, 3, 0)$. It moves in the direction $2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ at 4.5 m s^{-1}.



a Find the velocity vector of the cable car.

b Find the position of the cable car after 30 seconds.

c The destination of the cable car has X-coordinate 550. Find the length of the cable car ride.

d At what angle to the horizontal does the cable car travel?

8 A helicopter at $A(6, 9, 3)$ moves with constant velocity. 10 minutes later it is at $B(3, 10, 2.5)$. Distances are in kilometres.

a Find $\overrightarrow{AB}$. **b** Find the helicopter's speed.

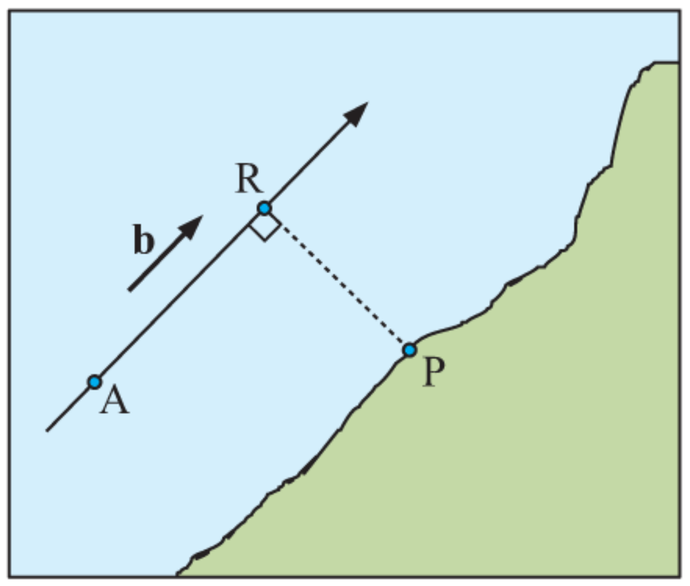
c Determine the vector equation which represents the path of the helicopter.

d The helicopter is travelling directly towards its helipad, which has Z-coordinate 0. Find the total time taken for the helicopter to land.

e At what angle to the horizontal is the helicopter flying?

D THE SHORTEST DISTANCE FROM A POINT TO A LINE

A ship R sails through point A in the direction **b** and continues past a port P.



The ship is *closest* to the port when [PR] is perpendicular to [AR]. In this situation, point R is called the **foot of the perpendicular from P to the line**.

We find the foot of the perpendicular using $\overrightarrow{PR} \bullet \mathbf{b} = 0$, and hence find the distance PR.

For problems in 3 dimensions, the shortest distance could also be calculated as the component of **b** perpendicular to $\overrightarrow{AP}$, which is $\dfrac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|}$. This is very efficient if all you want to know is the distance, but it will not tell you either the foot of the perpendicular or the value of the parameter corresponding to this point.

Example 7 **Self Tutor**

A line has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$. Find the shortest distance from the point $P(5, -1)$ to the line.

Let N be the point on the line closest to P.

N has coordinates $(1 + 3t, 2 - t)$ for some t, and $\overrightarrow{PN}$ is $\begin{pmatrix} 1 + 3t - 5 \\ 2 - t - (-1) \end{pmatrix} = \begin{pmatrix} 3t - 4 \\ 3 - t \end{pmatrix}$.

The distance between P and the line is minimised when $\overrightarrow{PN} \bullet \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$.

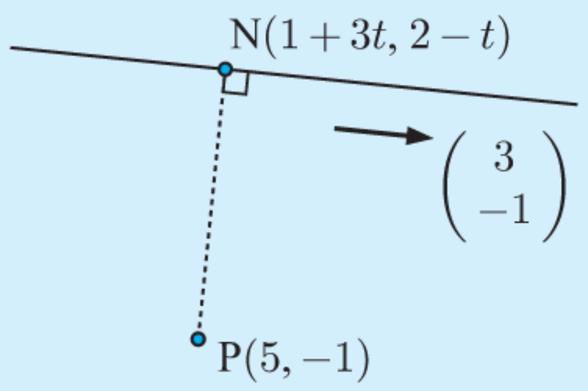
$$\therefore \begin{pmatrix} 3t - 4 \\ 3 - t \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0$$

$$\therefore 3(3t - 4) - (3 - t) = 0$$

$$\therefore 9t - 12 - 3 + t = 0$$

$$\therefore 10t = 15$$

$$\therefore t = \tfrac{15}{10} = \tfrac{3}{2}$$



Thus $\overrightarrow{PN} = \begin{pmatrix} \frac{9}{2} - 4 \\ 3 - \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\therefore |\overrightarrow{PN}| = \frac{1}{2}\sqrt{1^2 + 3^2} = \frac{\sqrt{10}}{2}$ units

$\therefore$ the shortest distance from the point P to the line is $\frac{\sqrt{10}}{2}$ units.

EXERCISE 10D

1 Find the shortest distance from:

a $P(3, 2)$ to the line with parametric equations $x = 2 + t,\ y = 3 + 2t,\ t \in \mathbb{R}$

b $Q(-1, 1)$ to the line with parametric equations $x = t,\ y = 1 - t,\ t \in \mathbb{R}$

c $R(-3, -1)$ to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ s \in \mathbb{R}$

d $S(5, -2)$ to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -7 \end{pmatrix},\ t \in \mathbb{R}$.

2 An ocean liner is at $(6, -6)$, cruising at $10\ \text{km h}^{-1}$ in the direction $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.
A fishing boat is anchored at $(0, 0)$. Distances are in kilometres.

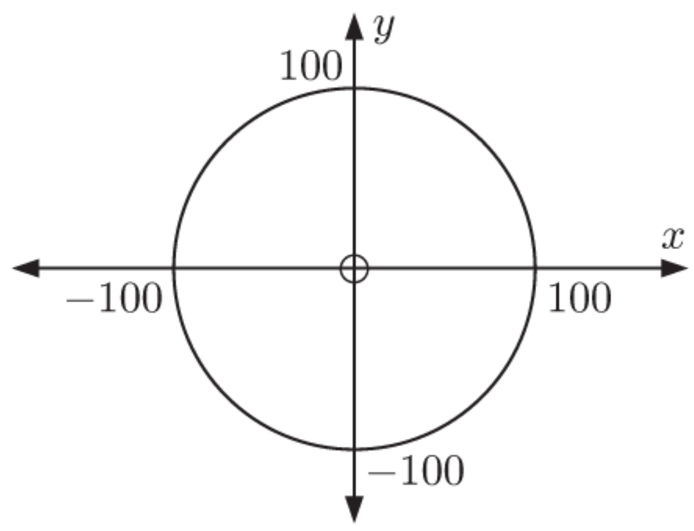
a Find, in terms of **i** and **j**, the initial position vector of the liner from the fishing boat.

b Write an expression for the position vector of the liner at any time t hours after it has sailed from $(6, -6)$.

c When will the liner be due east of the fishing boat?

d Find the time and position of the liner when it is nearest to the fishing boat.

3 Let **i** represent a displacement 1 km due east
and **j** represent a displacement 1 km due north.
The control tower of an airport is at $(0, 0)$. Aircraft within 100 km of $(0, 0)$ are visible on the radar screen at the control tower.
At 12:00 noon an aircraft is 200 km east and 100 km north of the control tower. It is flying parallel to the vector $\mathbf{b} = -3\mathbf{i} - \mathbf{j}$ with speed $40\sqrt{10}\ \text{km h}^{-1}$.



a Write down the velocity vector of the aircraft.

b Write a vector equation for the path of the aircraft using t to represent the time in hours that has elapsed since 12:00 noon.

c Find the position of the aircraft at 1:00 pm.

d Show that the aircraft first becomes visible on the radar screen at 1:00 pm.

e Find the time when the aircraft is closest to the control tower, and find the distance between the aircraft and the control tower at this time.

f At what time will the aircraft disappear from the radar screen?

4 Consider again the line and point in **Example 7**. Assume they all lie in the plane $z = 0$, so we can now write the problem in 3 dimensions as the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix},\ t \in \mathbb{R}$ and point $P(5, -1, 0)$.

a Find a point A on the line by letting $t = 0$.

b State the direction vector **b** of the line.

c Calculate the shortest distance between the point and the line using $\dfrac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|}$.

Example 8 **Self Tutor**

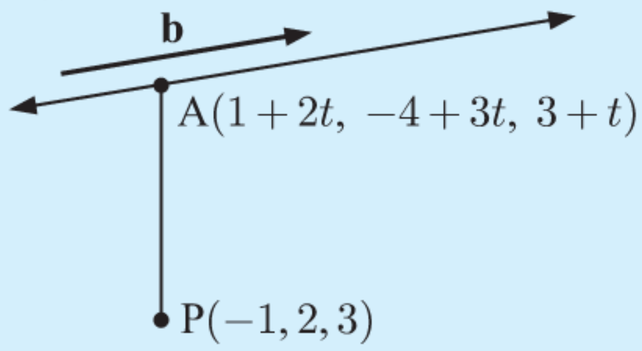
Consider the point $P(-1, 2, 3)$ and the line with parametric equations $x = 1+2t$, $y = -4+3t$, $z = 3+t$, $t \in \mathbb{R}$.

a Find the coordinates of the foot of the perpendicular from P to the line.

b Find the shortest distance from the point to the line.

a The line has direction vector $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Let $A(1+2t,\ -4+3t,\ 3+t)$ be any point on the given line.



$$\therefore \quad \overrightarrow{PA} = \begin{pmatrix} 1+2t-(-1) \\ -4+3t-2 \\ 3+t-3 \end{pmatrix} = \begin{pmatrix} 2+2t \\ -6+3t \\ t \end{pmatrix}$$

If A is the closest point on the line to P, then $\overrightarrow{PA} \perp \mathbf{b}$.

$$\therefore \quad \overrightarrow{PA} \bullet \mathbf{b} = 0$$

$$\therefore \quad \begin{pmatrix} 2+2t \\ -6+3t \\ t \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$$

$$\therefore \quad 2(2+2t) + 3(-6+3t) + 1(t) = 0$$

$$\therefore \quad 4 + 4t - 18 + 9t + t = 0$$

$$\therefore \quad 14t = 14$$

$$\therefore \quad t = 1$$

Substituting $t = 1$ into the parametric equations, the foot of the perpendicular is $(3, -1, 4)$.

b Using $t = 1$, $\quad \overrightarrow{PA} = \begin{pmatrix} 2+2 \\ -6+3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$

$$\therefore \quad |\overrightarrow{PA}| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{26} \text{ units}$$

So, the shortest distance from P to the line is $\sqrt{26}$ units.

5 **a** Find the coordinates of the foot of the perpendicular from $(3, 0, -1)$ to the line with parametric equations $x = 2+3t$, $y = -1+2t$, $z = 4+t$, $t \in \mathbb{R}$.

b Hence find the shortest distance from the line to the point.

6 **a** Find the coordinates of the foot of the perpendicular from $(1, 1, 3)$ to the line with vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

b Hence find the shortest distance from the point to the line.

c Check your calculation of the shortest distance by considering $A(1, -1, 2)$, $P(1, 1, 3)$, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, and finding $\dfrac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|}$.

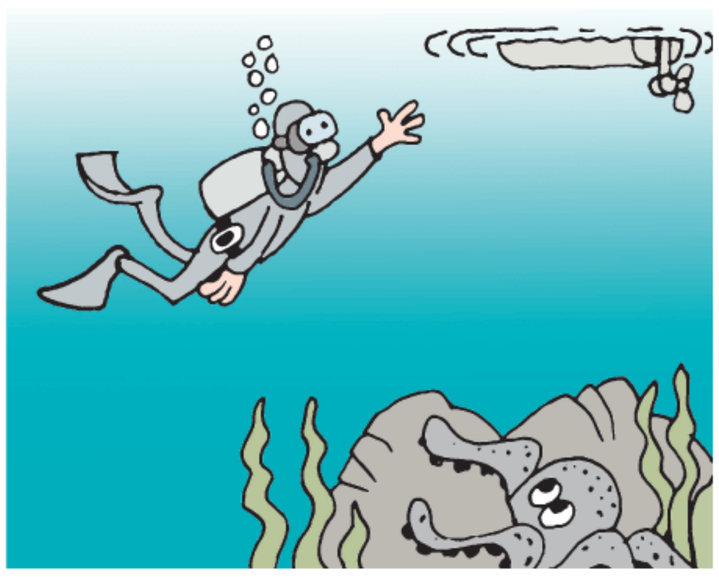
7 A diver swims from $(12, 25, -20)$ back to his boat at $(7, -15, 0)$, at a speed of 0.9 m s^{-1}.

a Find the velocity vector of the diver.

b An octopus watches the diver from its home at $(12, -8, -5)$.

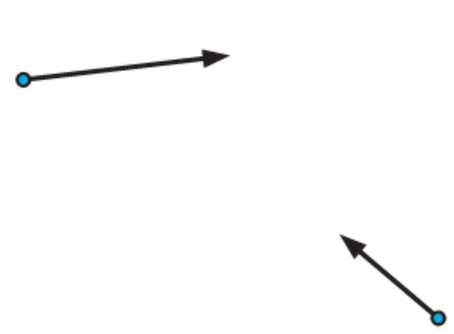
i At what time is the diver closest to the octopus?

ii Find the shortest distance from the diver to the octopus.



E THE SHORTEST DISTANCE BETWEEN TWO OBJECTS

Consider two objects each moving in a straight line. Unless the objects are moving in the same direction and with the same speed, the distance between the objects will vary over time.



If each object moves with constant velocity, we can use quadratic theory to find the shortest distance between the objects, and the time when it occurs.

Example 9 Self Tutor

Cyclist A moves with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 12 \\ -5 \end{pmatrix} + t\begin{pmatrix} -5 \\ 6 \end{pmatrix}$, and cyclist B moves with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \end{pmatrix} + t\begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

The distance units are metres, and $t \geqslant 0$ is the time in seconds.

Find the shortest distance between the cyclists, and the time when it occurs.

At time t, A is at $(12 - 5t, -5 + 6t)$, and B is at $(7 - 3t, 10 + 5t)$.

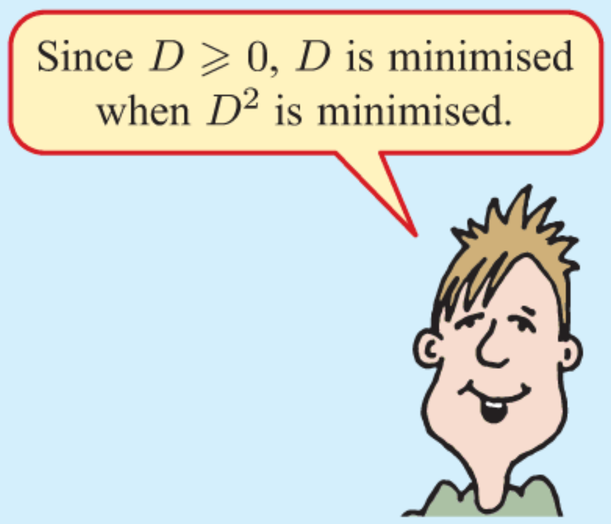
The distance between A and B is
$$\begin{aligned} D &= \sqrt{[(7-3t)-(12-5t)]^2 + [(10+5t)-(-5+6t)]^2} \\ &= \sqrt{(-5+2t)^2 + (15-t)^2} \\ &= \sqrt{25 - 20t + 4t^2 + 225 - 30t + t^2} \\ &= \sqrt{5t^2 - 50t + 250}\text{ m} \end{aligned}$$

Now D is minimised when $D^2 = 5t^2 - 50t + 250$ is minimised.

This occurs when $t = -\dfrac{-50}{2 \times 5} = 5$

When $t = 5$, $D = \sqrt{5(5)^2 - 50(5) + 250} = \sqrt{125} \approx 11.2$

The shortest distance between the cyclists is ≈ 11.2 m, and this occurs after 5 seconds.



EXERCISE 10E

1 A runner moves with vector equation $\mathbf{r} = \begin{pmatrix} -10 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $t \geqslant 0$, and a walker moves with vector equation $\mathbf{r} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, $t \geqslant 0$. The time units are seconds, and the distance units are metres.

a Find the distance between the people after:

i 3 seconds **ii** 10 seconds.

b Write an expression for the distance D between the people after t seconds.

c Find the shortest distance between the people, and the time when it occurs.

2 At a wildlife park, a tour bus enters the giraffe enclosure at $(-50,\ 30)$, and moves with velocity vector $4\mathbf{i} - 3\mathbf{j}$. At the same time, a giraffe in the enclosure moves with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -30 \\ -15 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t \geqslant 0$. The time is in seconds, and the distance units are metres.

a Find the distance between the bus and the giraffe when the bus enters the enclosure.

b A tourist wants to photograph the giraffe when it is closest to the bus.

i When should the tourist take the photograph?

ii How far is the giraffe from the bus at this time?

iii Find the position of the giraffe at this time.

3 At noon, truck A leaves from $(68,\ -21)$ with speed 90 km h^{-1}. At the same time, truck B leaves from $(110,\ -20)$ with speed 85 km h^{-1}. Both trucks travel in straight lines which pass through X$(-40,\ 60)$.

a Find the velocity vector of each truck.

b At what time will each truck pass through X?

c **i** At what time will the trucks be closest to one another? Round your answer to the nearest minute.

ii Find the distance between the trucks at this time.

4 Object P moves according to $\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$, and object Q moves according to $\mathbf{r} = \begin{pmatrix} -4 \\ -3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. The distance units are metres, and $t \geqslant 0$ is the time in seconds.

a Find the initial distance between the objects.

b Find the shortest distance between the objects, and the time when this occurs.

5 Alex and Yvonne are having a water balloon fight. From $(4,\ 2,\ 0)$, Alex runs with velocity vector $\begin{pmatrix} a \\ -3 \\ 0 \end{pmatrix}$. At the same time, Yvonne throws a water balloon from $(8,\ -20,\ 2)$ with velocity vector $4\mathbf{i} + b\mathbf{j} - \mathbf{k}$. The water balloon travels at 9 m s^{-1} in a direction perpendicular to Alex's path.

a Find a and b, giving reasons for your answers.

b Find the time when the water balloon is closest to Alex.

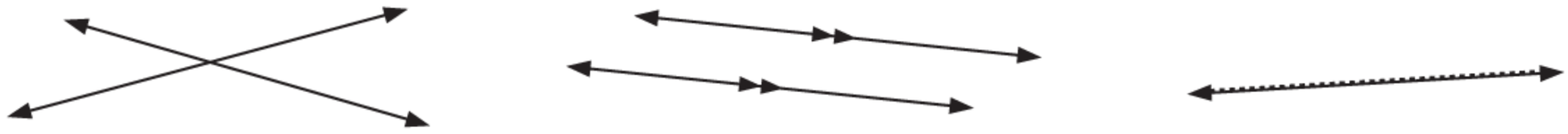
c Find the shortest distance between the water balloon and Alex. Interpret your answer.

F INTERSECTING LINES

INTERSECTING LINES IN 2 DIMENSIONS

There are three possible relationships between two lines in a plane:

intersecting **parallel** **coincident**



Vector equations of lines in 2 dimensions can be **solved simultaneously** to find the point where the lines meet.

Example 10 **Self Tutor**

Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad s \in \mathbb{R}.$

Line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$

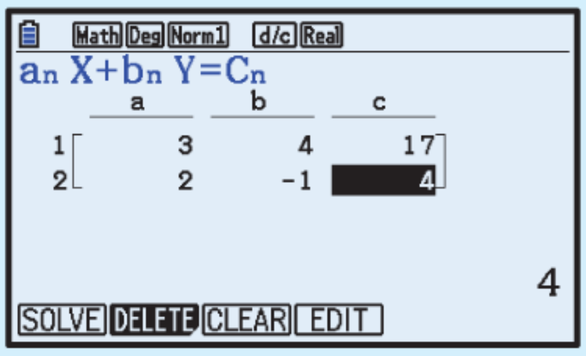
Find the point where the two lines meet.

The lines meet where $\begin{pmatrix} -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + t\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$\therefore \quad -2 + 3s = 15 - 4t$ and $1 + 2s = 5 + t$

$\therefore \quad 3s + 4t = 17 \quad \ldots. (1)$ and $2s - t = 4 \quad \ldots. (2)$

Solving (1) and (2) simultaneously using technology gives $s = 3, \ t = 2$.



Using line 1, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

Checking in line 2, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 5 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$ ✓

$\therefore$ the lines meet at $(7, 7)$.

EXERCISE 10F.1

1 Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \end{pmatrix} + r\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad r \in \mathbb{R}.$

Line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}.$

Find the point where the two lines meet.

2 The isosceles triangle ABC is defined by three lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} + s\begin{pmatrix} -1 \\ -2 \end{pmatrix}$, and

(AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where $r, s, t \in \mathbb{R}$.

a Use vector methods to find the coordinates of A, B, and C.

b Find all side lengths of the triangle. Hence determine which two sides are equal in length.

3 Quadrilateral ABCD is formed by these lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + r\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} + s\begin{pmatrix} -8 \\ 32 \end{pmatrix}$,

(CD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 25 \end{pmatrix} + t\begin{pmatrix} -8 \\ -2 \end{pmatrix}$, and (AD) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + u\begin{pmatrix} -3 \\ 12 \end{pmatrix}$,

where $r, s, t, u \in \mathbb{R}$.

a Use vector methods to find the coordinates of A, B, C, and D.

b Find:

i $\overrightarrow{AC}$ and $\overrightarrow{DB}$ **ii** $|\overrightarrow{AC}|$ and $|\overrightarrow{DB}|$ **iii** $\overrightarrow{AC} \bullet \overrightarrow{DB}$

c What do the answers to **b** tell you about quadrilateral ABCD?

4 Try to find the point of intersection of the lines

L_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$ and L_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} + s\begin{pmatrix} 3 \\ -6 \end{pmatrix}$, $s \in \mathbb{R}$.

Explain your answer.

Example 11

Self Tutor

Boat A moves according to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $t \geqslant 0$ seconds. Boat B moves according to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ -16 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $t \geqslant 0$ seconds.

a Find the point at which the paths of the boats intersect.

b Will the boats actually collide at this point? Explain your answer.

a At time t, boat A is at $(-1 + 2t, -5 + 3t)$ and boat B is at $(15 - t, -16 + 2t)$.

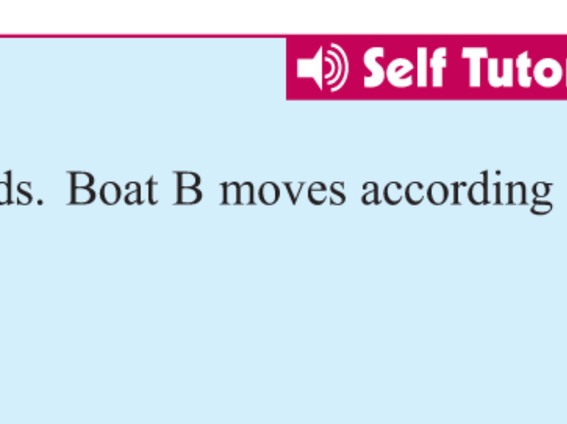
Their paths are not parallel, so they must intersect.

Suppose boat A reaches the intersection point at time t_A
and boat B reaches the intersection point at time t_B.

$\therefore\ -1 + 2t_A = 15 - t_B$ and $-5 + 3t_A = -16 + 2t_B$

$\therefore\ 2t_A + t_B = 16$ .... (1) and $3t_A - 2t_B = -11$ .... (2)

Solving (1) and (2) simultaneously using technology gives $t_A = 3,\ t_B = 10$.



Using boat A, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -5 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$\therefore$ the paths intersect at $(5, 4)$.

Checking with boat B, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ -16 \end{pmatrix} + 10\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ ✓

b The boats will not actually collide at this point, as boat A passes through the intersection point after $t_A = 3$ seconds, and boat B passes through this point after $t_B = 10$ seconds.

5 Particle A's position is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -13 \\ -1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $t \geqslant 0$ seconds.

Particle B's position is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ -13 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $t \geqslant 0$ seconds.

a Find the point at which the paths of the particles intersect.

b Will the particles collide at this point? Explain your answer.

6 Jove is riding a scooter from $(-4, -1)$ with speed 2.5 m s^{-1} in the direction $3\mathbf{i} - 4\mathbf{j}$. Fiona is riding a skateboard with position $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $t \geqslant 0$. The time units are seconds, and the distance units are metres.

a Who is moving faster?

b Find the vector equation from Jove's position after t seconds.

c Find the point at which Jove and Fiona's paths intersect.

d Will Jove and Fiona collide at this point? Explain your answer.

7 At 1:34 pm, submarine P fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ from the point $(-5, 4)$.

k minutes later submarine Q fires a torpedo with velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ from the point $(15, 7)$.

Distances are measured in kilometres, and time is in minutes.

a Show that the position of P's torpedo can be written as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

b What is the speed of P's torpedo?

c Show that the position of Q's torpedo can be written as $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t - k)\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

d Find the angle between the paths of the torpedoes.

e Q's torpedo is successful in knocking out P's torpedo.

i At what time did Q fire its torpedo?

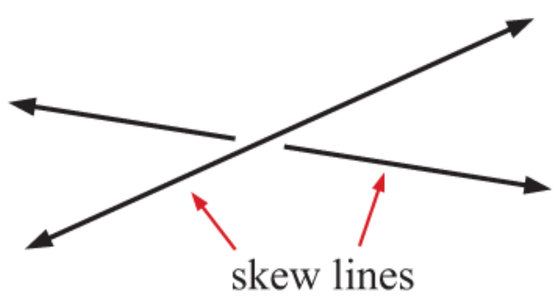
ii At what time did the collision occur?

iii Where did the collision occur?

INTERSECTING LINES IN 3 DIMENSIONS

When we attempt to find the intersection point of lines in 3 dimensions, we obtain a system of three equations in two unknowns, which must be solved simultaneously.

If the system has no solutions, then the lines do not intersect. This may occur because the lines are **parallel**, or because the lines are **skew**, which means they do not lie in the same plane.



Example 12 **Self Tutor**

Line 1 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \quad s \in \mathbb{R}.$

Line 2 has equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}, \quad t \in \mathbb{R}.$

Line 3 has parametric equations $x = -1 + 2u, \ y = 1 - 2u, \ z = 1 + 4u, \ u \in \mathbb{R}.$

Find, if possible, the intersection point of:

a line 1 and line 2 **b** line 2 and line 3.

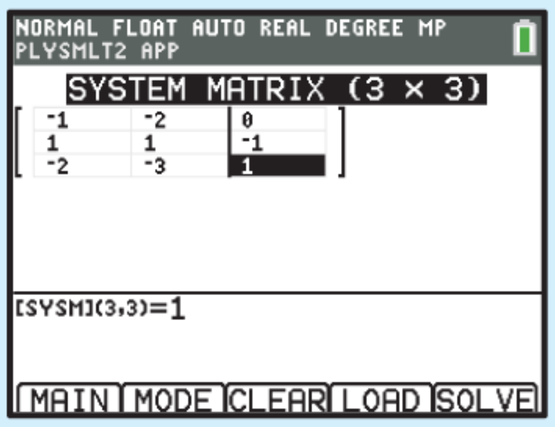
a Equating x, y, and z values in lines 1 and 2 gives

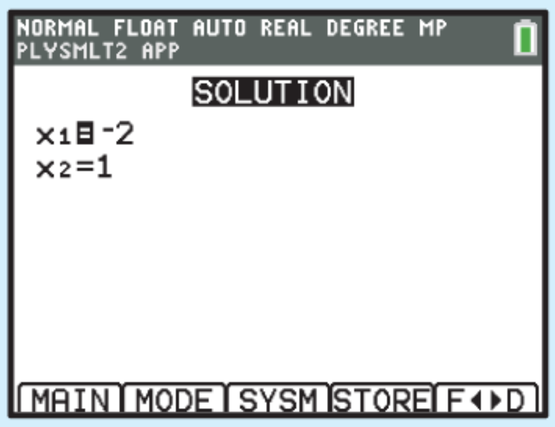
$1 - s = 1 + 2t$, $s = -1 - t$, and $3 - 2s = 4 + 3t$

$\therefore \ -s - 2t = 0 \ \ldots. (1)$, $s + t = -1 \ \ldots. (2)$, and $-2s - 3t = 1 \ \ldots. (3)$

Solving (1), (2), and (3) simultaneously gives $s = -2, \ t = 1$.

Substituting $t = 1$ into line 2, we find that lines 1 and 2 intersect at $(1 + 2(1), -1 - (1), 4 + 3(1))$ which is $(3, -2, 7)$.





b Equating x, y, and z values in lines 2 and 3 gives

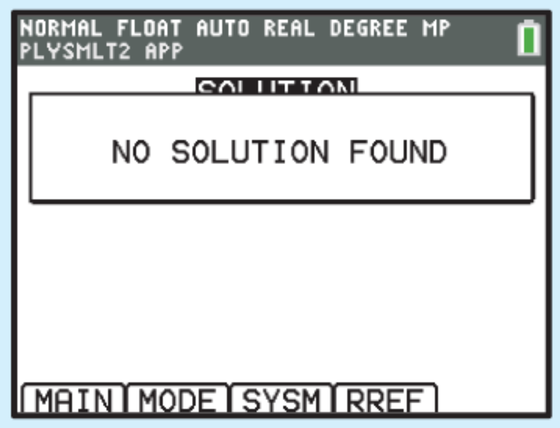
$1 + 2t = -1 + 2u$, $-1 - t = 1 - 2u$, and $4 + 3t = 1 + 4u$

$\therefore \ 2t - 2u = -2 \ \ldots. (1)$, $-t + 2u = 2 \ \ldots. (2)$, and $3t - 4u = -3 \ \ldots. (3)$

Solving (1), (2), and (3) simultaneously gives no solution.

So, the lines do not intersect.

Since $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$, we conclude that the lines are skew.



EXERCISE 10F.2

1 Find, if possible, the intersection point of:

a L_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}$ and L_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \ s \in \mathbb{R}$

b L_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -12 \\ 12 \end{pmatrix}, \ \lambda \in \mathbb{R}$ and L_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}, \ \mu \in \mathbb{R}$

c L_1: $x = 6t, \ y = 3 + 8t, \ z = -1 + 2t, \ t \in \mathbb{R}$
and L_2: $x = 2 + 3s, \ y = 4s, \ z = 1 + s, \ s \in \mathbb{R}$

d L_1: $x = 1 + \lambda, \ y = 2 - \lambda, \ z = 3 + 2\lambda, \ \lambda \in \mathbb{R}$
and L_2: $x = 2 + 3\mu, \ y = 3 - 2\mu, \ z = -5 + \mu, \ \mu \in \mathbb{R}$.

2 A bird flies according to $\mathbf{r}_1 = \begin{pmatrix} -2 \\ 6 \\ 9 \end{pmatrix} + t \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$ and a bat flies according to $\mathbf{r}_2 = \begin{pmatrix} 31 \\ -20 \\ 5 \end{pmatrix} + t \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$.



The time $t \geqslant 0$ is in seconds, and the distance units are metres.

a Find the speed of: **i** the bird **ii** the bat.

b Show that the paths of the bird and the bat intersect.

c Will the bird and the bat collide at the intersection point? Explain your answer.

3 A seal swims through the ocean according to $\mathbf{r}_1 = \begin{pmatrix} -7 \\ 4 \\ -22 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix}$, $t \geqslant 0$, while a shark swims from $(-35, -6, -10)$ at 14 m s^{-1} in the direction $6\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$. The time is in seconds, and the distance units are metres.

a Find the velocity vector of the shark.

b Find the point at which the paths of the seal and the shark intersect.

c Find the distance between the seal and the shark when the seal passes through the intersection point.

4 A clay target shooter is positioned at $(20, 6, 1)$. The target is released from $(8, 16, 1)$ with velocity vector $14\mathbf{i} + 9\mathbf{j} + 8\mathbf{k}$. The shooter fires the bullet with speed 360 m s^{-1} in the direction $4\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$.

a Find the velocity vector of the bullet. **b** Find the speed of the target.

c Show that the paths of the target and the bullet intersect.

d Given that the bullet hit the target:

i Find how long after the target was released that the bullet was fired.

ii Find how far the bullet travelled before it hit the target.

THEORY OF KNOWLEDGE

In his 1827 book entitled *The Barycentric Calculus*, German mathematician **August Möbius** demonstrated the use of directed line segments in projective geometry. He discussed the addition and scalar multiplication of directed line segments, but he did not name them vectors.

Gauss, **Grassmann**, **Laplace**, and **Lagrange** all contributed to build the field we know as vector analysis today.

An important contribution was made in 1843 by **Sir William Hamilton**, who defined a *quaternion* as the quotient of two directed line segments.

The American mathematician **Josiah Willard Gibbs** (1839 - 1903) began teaching a course on vector analysis at Yale University in 1879. He published *Elements of Vector Analysis* in two halves, and in 1901 his work was summarised by his student **Edwin Wilson** in the book *Vector Analysis*.



Sir William Hamilton

Oliver Heaviside (1850 - 1925) developed vector analysis separately from Gibbs, receiving a copy of Gibbs' work later, in 1888. Heaviside wrote of Gibbs:

> "... the invention of quaternions must be regarded as a most remarkable feat of human ingenuity. Vector analysis, without quaternions, could have been found by any mathematician by carefully examining the mechanics of the Cartesian mathematics; but to find out quaternions required a genius."

1 Are geometry and algebra two separate domains of knowledge?

2 Given that Gibbs and Heaviside developed vector analysis independently, who can claim to be the founder of this field?

3 Does mathematics *evolve*?

REVIEW SET 10A

1 For the line that passes through $(-6, 3)$ with direction $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, write down the corresponding:

a vector equation **b** parametric equations.

2 $(-3, m)$ lies on the line with vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 18 \\ -2 \end{pmatrix} + t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$, $t \in \mathbb{R}$.
Find m.

3 Consider $A(2, -1, 3)$ and $B(0, 1, -1)$.

a Find the vector equation of the line through A and B.

b Hence find the coordinates of C on (AB) which is 2 units from A.

4 The line L_1 passes through $(5, 2)$ and is perpendicular to $2\mathbf{i} - 7\mathbf{j}$.
The line L_2 passes through $(-1, 4)$ and $(3, -1)$.
Find the angle between the lines.

5 A moving particle has coordinates $P(x(t), y(t), z(t))$ where $x(t) = -4 + 8t$, $y(t) = 3 + 6t$, and $z(t) = 1 - 2t$. The distance units are metres, and $t \geqslant 0$ is the time in seconds. Find the:

a initial position of the particle **b** position of the particle after 4 seconds

c particle's velocity vector **d** speed of the particle.

6 Line 1 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Line 2 has vector equation $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, $\mu \in \mathbb{R}$.

Find the point where the two lines meet.

7 Find the shortest distance from:

a $P(1, 4)$ to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$

b $Q(-2, -5)$ to the line $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, $t \in \mathbb{R}$.

8 A yacht is sailing with constant speed $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i} - 3\mathbf{j}$. Initially it is at the point $(-6, 10)$. A beacon is at $(0, 0)$ at the centre of a tiny atoll. Distances are in kilometres.



a Find, in terms of **i** and **j**, the:

i initial position vector of the yacht

ii velocity vector of the yacht

iii position vector of the yacht after t hours, $t \geqslant 0$.

b Find the time when the yacht is closest to the beacon.

c If there is a reef of radius 8 km around the atoll, will the yacht hit the reef?

9 Submarine X23 is at $(2, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ at exactly 2:17 pm.

Submarine Y18 is at $(11, 3)$. It fires a torpedo with velocity vector $\begin{pmatrix} -1 \\ a \end{pmatrix}$ at 2:19 pm to intercept the torpedo from X23. Distance units are kilometres, and time units are minutes.

a Find $x_1(t)$ and $y_1(t)$ for the torpedo fired from submarine X23.

b Find $x_2(t)$ and $y_2(t)$ for the torpedo fired from submarine Y18.

c At what time does the interception occur?

d What was the direction and speed of the intercepting torpedo?

e Find the acute angle between the paths of the torpedoes.

10 Given $A(-1, 2, 3)$, $B(1, 0, -1)$, and $C(1, 3, 0)$, find:

a the vector equation of (AB)

b the coordinates of the foot of the perpendicular from C to (AB).

11 At time t seconds, object A is at $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 14 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ and object B is at $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -16 \\ 8 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$, $t \geqslant 0$.

a Find the position of each object after 2 seconds.

b Find the point at which the paths of the objects intersect.

c Will the objects collide at this point? Explain your answer.

12 On an ice hockey rink, Chelsea hits a puck from $(1, 5)$ with velocity vector $3\mathbf{i} + 6\mathbf{j}$. At the same time, Vivian hits a puck from $(21, 10)$ at 10 m s^{-1} in the direction $-3\mathbf{i} + 4\mathbf{j}$. The time is in seconds, and the distance units are metres.

a Find the velocity vector of Vivian's puck.

b Find the distance between the pucks after 1 second.

c Find the shortest distance between the pucks, and the position of Chelsea's puck when this occurs.

REVIEW SET 10B

1 Line L has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $t \in \mathbb{R}$.

a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

2 Write down **i** a vector equation **ii** parametric equations for the line passing through:

a $(2, -3, 1)$ with direction $\begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$ **b** $(-1, 6, 3)$ and $(5, -2, 0)$.

3 Find the angle between line L_1 passing through $(0, 3)$ and $(5, -2)$, and line L_2 passing through $(-2, 4)$ and $(-6, 7)$.

4 Find the angle between the lines:
L_1: $x = 1 - 4t$, $y = 3t$ and L_2: $x = 2 + 5s$, $y = 5 - 12s$, where $t, s \in \mathbb{R}$.

5 Consider A$(3, -1, 1)$ and B$(0, 2, -2)$.

a Find $|\overrightarrow{AB}|$.

b Show that the line passing through A and B can be described by $\mathbf{r} = 2\mathbf{j} - 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - \mathbf{k})$ where $\lambda \in \mathbb{R}$.

c Find the angle between (AB) and the line with equation $\mathbf{r} = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$, $t \in \mathbb{R}$.

6 Find the velocity vector of an object moving in the direction $3\mathbf{i} - \mathbf{j}$ with speed 20 km h^{-1}.

7 The vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ 12 \end{pmatrix} + t\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$

represents the path of a person zip-lining through a forest. t is measured in seconds, $t \geqslant 0$. Distances are measured in metres. Find the:



a initial position **b** velocity vector

c speed of the zip-liner

d time taken to reach the end of the line at $z = 0$

e location of the endpoint of the line

f position of the person when he is closest to his friend watching from $(52, 144, 3)$.

8 Triangle ABC is formed by three lines:

(AB) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, (BC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and

(AC) is $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + u\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where s, t, and u are scalars.

a Use vector methods to find the coordinates of A, B, and C.

b Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$, and $|\overrightarrow{AC}|$.

c Classify triangle ABC.

9 Jet ski A leaves from $(150, -30)$ and travels at 10 m s^{-1} in the direction $-24\mathbf{i} + 7\mathbf{j}$. At the same time, jet ski B leaves from $(99, 133)$ and travels with velocity vector $-3\mathbf{i} - 9\mathbf{j}$. The time units are seconds and the distance units are metres.

a Are the jet skis travelling at right angles to one another? Explain your answer.

b Find the velocity vector of jet ski A.

c Find the point at which the paths of the jet skis intersect.

d Find the distance between A and B when jet ski B passes through the intersection point.

e At what time is jet ski B closest to jet ski A's starting point?

10 Object P moves according to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} -3 \\ 2 \end{pmatrix}$, $t \geqslant 0$.

Object Q moves according to $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 22 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \end{pmatrix}$, $t \geqslant 0$.

The time units are seconds, and the distance units are centimetres.
Find the shortest distance between the objects, and the time when it occurs.

11 If possible, find the intersection point of:

a L_1: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $t \in \mathbb{R}$ and L_2: $\mathbf{r} = \begin{pmatrix} -8 \\ 0 \\ 7 \end{pmatrix} + s\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$, $s \in \mathbb{R}$

b L_1: $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $t \in \mathbb{R}$ and L_2: $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + s\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$, $s \in \mathbb{R}$.

12 Answer the **Opening Problem** on page **242**.

Chapter 11

Complex numbers

Contents:

A Real quadratics with $\Delta < 0$
B Complex numbers
C Operations with complex numbers
D Equality of complex numbers
E The complex plane
F Modulus and argument
G Geometry in the complex plane
H Polar form
I Exponential form
J Frequency and phase

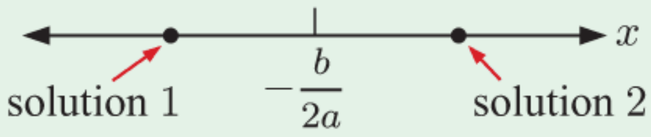


OPENING PROBLEM

The quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ is said to be *real* if $a, b, c \in \mathbb{R}$.

We have seen previously that solutions to this equation can be found using the **quadratic formula** $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$ is called the **discriminant**.

The average of these solutions is $x = -\dfrac{b}{2a}$ which corresponds to the axis of symmetry of the corresponding quadratic function.



Suppose $a > 0$ so the quadratic opens upwards, and that $\Delta > 0,$ so the quadratic equation has two real distinct solutions.

Suppose we start to increase the value of c, so Δ starts to decrease.

Things to think about:

a While Δ is still positive but getting smaller, what happens to the solutions?

b When $\Delta = 0,$ the solution to the equation is $x = -\dfrac{b}{2a}$. Is it still useful to think of this as two *identical* solutions?

c Once $\Delta < 0$ we know that the equation has no *real* solutions. However, suppose the square root of a negative number has some other meaning. If so, then the quadratic formula still gives us two solutions.

i Does the average of these solutions still correspond to the axis of symmetry of the corresponding quadratic function?

ii Do the sum and product of the solutions give us $-\dfrac{b}{a}$ and $\dfrac{c}{a}$, respectively?

d Is it *useful* to take the square root of a negative number?

A REAL QUADRATICS WITH $\Delta < 0$

We have previously seen that:

If $ax^2 + bx + c = 0,\ a \neq 0$ and $a, b, c \in \mathbb{R},$ then the solutions or roots are found using the formula $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$ is known as the **discriminant**.

We also observed that if $\Delta < 0,$ the equation has no real solutions. There are no real numbers whose squares are negative, so we cannot place the square root of any negative number on the real number line.

However, mathematicians define an *imaginary* number to allow us to consider such square roots.

The **imaginary number** is defined as $\boldsymbol{i = \sqrt{-1}}$ with the property that $i \times i = \sqrt{-1} \times \sqrt{-1} = -1$.

It is called "imaginary" because we cannot place it on the real number line.

HISTORICAL NOTE

Heron or **Hero of Alexandria** (10 - 70) was a mathematician and engineer in Roman Egypt. He is most commonly known for **Heron's formula** for finding the area of a triangle, but he also developed an iterative process called **Heron's method** or the **Babylonian method** for computing square roots.

Heron is the first known person to consider whether it might make sense to take the square root of a negative number. In 50 AD, while finding the height of a pyramid using a known formula, he obtained the following result which was published in the book *Stereometrica*:

$$\begin{aligned} h &= \sqrt{(15)^2 - 2\left(\frac{28-4}{2}\right)^2} \\ &= \sqrt{225 - 2(12)^2} \\ &= \sqrt{225 - 144 - 144} \\ &= \sqrt{81 - 144} \end{aligned}$$

Since this result had no use in the context of a real pyramid, Heron did not consider the concept further.

Nearly 1500 years later, the square root of a negative number was considered again by **Gerolamo Cardano** (1501 - 1576). In his book *Ars Magna* published in 1545, Cardano gives the solutions to an equation as $5 \pm \sqrt{-15}$. Rather than dismissing the result, he went on to find the product of the two solutions:

$$(5 + \sqrt{-15}) \times (5 - \sqrt{-15}) = 25 - (-15) = 40$$

It was **Rafael Bombelli** who defined the imaginary number $i = \sqrt{-1}$ in 1572.



Rafael Bombelli

With $i = \sqrt{-1}$ defined, we can expand our rules for square roots to include negative numbers.

Example 1 **Self Tutor**

Write in terms of i:

a $\sqrt{-81}$ **b** $-\sqrt{-3}$

a
$$\begin{aligned} &\sqrt{-81} \\ &= \sqrt{81} \times \sqrt{-1} \\ &= 9i \end{aligned}$$

b
$$\begin{aligned} &-\sqrt{-3} \\ &= -\sqrt{3}\sqrt{-1} \\ &= -i\sqrt{3} \end{aligned}$$

This allows us to obtain solutions for quadratic equations with $\Delta < 0$.

Example 2 — Self Tutor

Solve the quadratic equations:

a $x^2 = -4$ **b** $x^2 + x + 2 = 0$

a
$x^2 = -4$
$\therefore\ x = \pm\sqrt{-4}$
$\therefore\ x = \pm\sqrt{4}\sqrt{-1}$
$\therefore\ x = \pm 2i$

b Using the quadratic formula,
$x = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)}$
$\therefore\ x = \dfrac{-1 \pm \sqrt{-7}}{2}$
$\therefore\ x = \dfrac{-1 \pm i\sqrt{7}}{2} = -\frac{1}{2} \pm \frac{\sqrt{7}}{2}i$

EXERCISE 11A

1 Write in terms of i:

a $\sqrt{-9}$ **b** $\sqrt{-64}$ **c** $\sqrt{-121}$ **d** $\sqrt{-\frac{1}{4}}$ **e** $\sqrt{-5}$

f $\sqrt{-17}$ **g** $\sqrt{-8}$ **h** $-\sqrt{-36}$ **i** $-\sqrt{-11}$ **j** $-\sqrt{-\frac{1}{2}}$

2 Solve for x:

a $x^2 = 25$ **b** $x^2 = -25$ **c** $x^2 = 5$

d $x^2 = -5$ **e** $4x^2 = 9$ **f** $4x^2 = -9$

3 Solve for x using the quadratic formula:

a $x^2 - 10x + 29 = 0$ **b** $x^2 + 6x + 25 = 0$ **c** $x^2 + 14x + 50 = 0$

d $x^2 - 3x + 5 = 0$ **e** $x^2 + 4 = x$ **f** $3x^2 + 6x + 5 = 0$

g $2x^2 + 5 = 6x$ **h** $x^2 - 2\sqrt{3}x + 4 = 0$ **i** $2x + \dfrac{1}{x} = 1$

B COMPLEX NUMBERS

The solutions to quadratic equations with $\Delta < 0$ have the form $x = a + bi$, where a and b are real.

Any number of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, is called a **complex number**.

Notice that:

- All real numbers can be written as complex numbers by simply choosing $b = 0$.
- In the special case $a = 0$, we have a number of the form bi where $b \in \mathbb{R}$. We say this number is **purely imaginary**.

REAL AND IMAGINARY PARTS

If $z = a + bi$ where $a, b \in \mathbb{R}$, then:

- a is the **real part** of z, written $\mathcal{Re}(z)$
- b is the **imaginary part** of z, written $\mathcal{Im}(z)$.

For example:

- If $z = 2 + 3i$, then $\mathcal{Re}(z) = 2$ and $\mathcal{Im}(z) = 3$.
- If $z = -\sqrt{2}i$, then $\mathcal{Re}(z) = 0$ and $\mathcal{Im}(z) = -\sqrt{2}$.

COMPLEX CONJUGATES

Suppose $z = a + bi$ where $a, b \in \mathbb{R}$.

The **complex conjugate** of z is $z^* = a - bi$.

$\mathcal{Re}(z^*) = \mathcal{Re}(z)$ and $\mathcal{Im}(z^*) = -\mathcal{Im}(z)$

For example, if $z = 2 + 3i$ then $z^* = 2 - 3i$.

EXERCISE 11B

1 Copy and complete:

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$3 + 2i$			
$5 - i$			
3			
$4i$			
0			

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$-3 + 4i$			
$-7 - 2i$			
$-11i$			
$i\sqrt{3}$			
$1 - i\sqrt{2}$			

2 Which of the numbers in **1** are:

a real **b** purely imaginary?

3 Given the complex numbers $z = -1 + 4i$ and $w = 6 - 5i$, find:

a $\mathcal{Re}(z)$ **b** $\mathcal{Im}(w)$ **c** z^*

d w^* **e** $\mathcal{Im}(z^*)$ **f** $\mathcal{Re}(w^*)$

4 Show that if z is purely imaginary, then z^* is also purely imaginary.

5 Show that $(z^*)^* = z$.

C OPERATIONS WITH COMPLEX NUMBERS

Operations with complex numbers are identical to those with radicals, but with $i^2 = -1$ rather than $(\sqrt{2})^2 = 2$ or $(\sqrt{3})^2 = 3$.

For example:

- addition:
$$(2 + \sqrt{3}) + (4 + 2\sqrt{3}) = (2 + 4) + (1 + 2)\sqrt{3} = 6 + 3\sqrt{3}$$
$$(2 + i) \quad + \quad (4 + 2i) = (2 + 4) + (1 + 2)i \quad = 6 + 3i$$
- multiplication:
$$(2 + \sqrt{3})(4 + 2\sqrt{3}) = 8 + 4\sqrt{3} + 4\sqrt{3} + 2(\sqrt{3})^2 = 8 + 8\sqrt{3} + 6 = 14 + 8\sqrt{3}$$
$$(2 + i)(4 + 2i) = 8 + \ 4i \ + \ 4i \ + \ 2i^2 \quad = 8 + \ 8i \ - 2 = 6 + 8i$$

$$(a+bi)+(c+di)=(a+c)+(b+d)i \qquad \textbf{addition}$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i \qquad \textbf{subtraction}$$
$$(a+bi)(c+di)=ac+adi+bci+bdi^2 \qquad \textbf{multiplication}$$

When we perform division with radicals, we use the radical conjugate of the denominator to achieve a rational denominator.

For example, $\dfrac{1+2\sqrt{3}}{4-\sqrt{3}}=\left(\dfrac{1+2\sqrt{3}}{4-\sqrt{3}}\right)\left(\dfrac{4+\sqrt{3}}{4+\sqrt{3}}\right)=\dfrac{4+\sqrt{3}+8\sqrt{3}+6}{16-3}=\dfrac{10+9\sqrt{3}}{13}$

In a similar way, when we perform division with complex numbers, we use the *complex conjugate* of the denominator to achieve a real denominator.

For example, $\dfrac{1+2i}{4-i}=\left(\dfrac{1+2i}{4-i}\right)\left(\dfrac{4+i}{4+i}\right)=\dfrac{4+i+8i+2i^2}{16-i^2}=\dfrac{2+9i}{17}$

$$\frac{a+bi}{c+di}=\left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)=\frac{ac-adi+bci-bdi^2}{c^2+d^2} \qquad \textbf{division}$$

Example 3 — Self Tutor

If $z=3+2i$ and $w=4-i$ find:

a $z+w$ **b** $z-w$ **c** zw **d** $\dfrac{z}{w}$

a
$$\begin{aligned} z+w &= (3+2i)+(4-i)\\ &= 7+i \end{aligned}$$

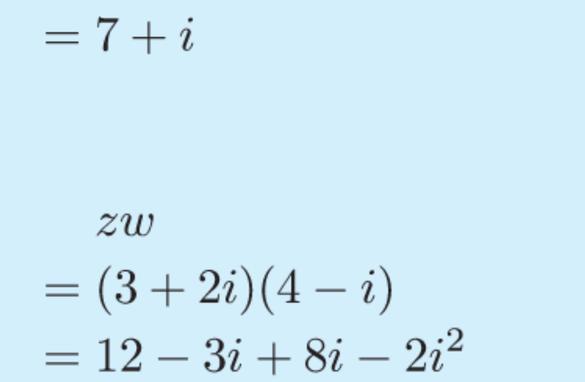
b
$$\begin{aligned} z-w &= (3+2i)-(4-i)\\ &= 3+2i-4+i\\ &= -1+3i \end{aligned}$$

GRAPHICS CALCULATOR INSTRUCTIONS

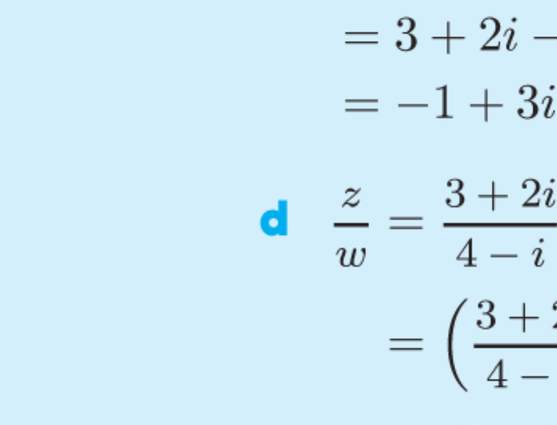
c
$$\begin{aligned} zw &= (3+2i)(4-i)\\ &= 12-3i+8i-2i^2\\ &= 12+5i+2\\ &= 14+5i \end{aligned}$$

d
$$\begin{aligned} \frac{z}{w} &= \frac{3+2i}{4-i}\\ &= \left(\frac{3+2i}{4-i}\right)\left(\frac{4+i}{4+i}\right)\\ &= \frac{12+3i+8i+2i^2}{16-i^2}\\ &= \frac{10+11i}{17}\\ &= \frac{10}{17}+\frac{11}{17}i \end{aligned}$$

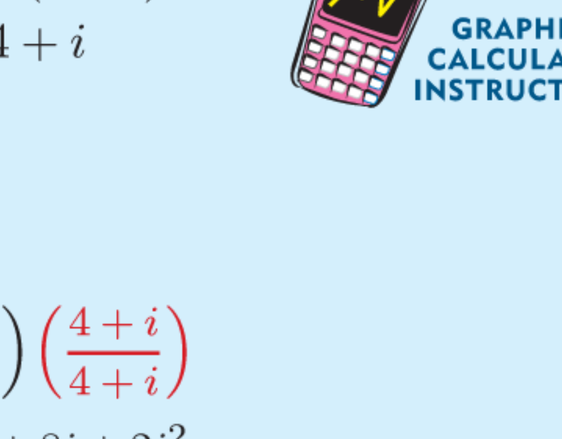
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EXERCISE 11C



1 If $z = 5 - 2i$ and $w = 2 + i$, find in simplest form:

a $z + w$ **b** $2z$ **c** iw **d** $z - w$

e $2z - 3w$ **f** zw **g** w^2 **h** z^2

2 For $z = 1 + i$ and $w = -2 + 3i$, find in simplest form:

a $z + 2w$ **b** z^2 **c** z^3 **d** iz

e w^2 **f** zw **g** z^2w **h** izw

3 Prove that the sum of a complex number and its conjugate is always real.

4 **a** Simplify i^n for $n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ and also for $n = -1, -2, -3, -4, -5$.

b Hence simplify i^{4n+3} where n is any integer.

5 **a** Use complex number notation to find $\sqrt{-4} \times \sqrt{-9}$.

b We have previously seen that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for any $a, b \geqslant 0$.
Is this rule also valid for negative a, b? Justify your answer.

6 Suppose $z = 2 - i$ and $w = 1 + 3i$. Write in the form $a + bi$ where $a, b \in \mathbb{R}$ are given exactly:

a $\dfrac{z}{w}$ **b** $\dfrac{i}{z}$ **c** $\dfrac{w}{iz}$ **d** $\dfrac{z^2}{w - i}$

7 Simplify:

a $\dfrac{i}{1 - 2i}$ **b** $\dfrac{i(2 - i)}{3 - 2i}$ **c** $\dfrac{1}{2 - i} - \dfrac{2}{2 + i}$ **d** $\dfrac{i}{x + i} - \dfrac{i}{x - i}$

8 If $z = 2 + i$ and $w = -1 + 2i$, find:

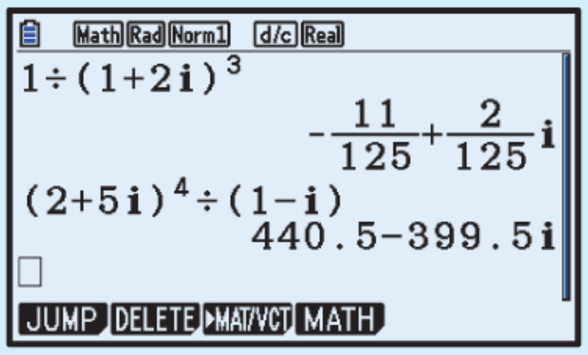
a $\mathcal{Im}(4z - 3w)$ **b** $\mathcal{Re}(zw)$ **c** $\mathcal{Im}(iz^2)$ **d** $\mathcal{Re}\left(\dfrac{z}{w}\right)$

Example 4 — Self Tutor

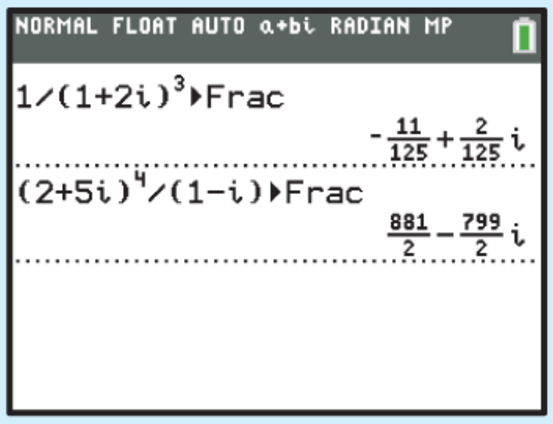
Use technology to calculate:

a $\dfrac{1}{(1 + 2i)^3}$ **b** $\dfrac{(2 + 5i)^4}{1 - i}$

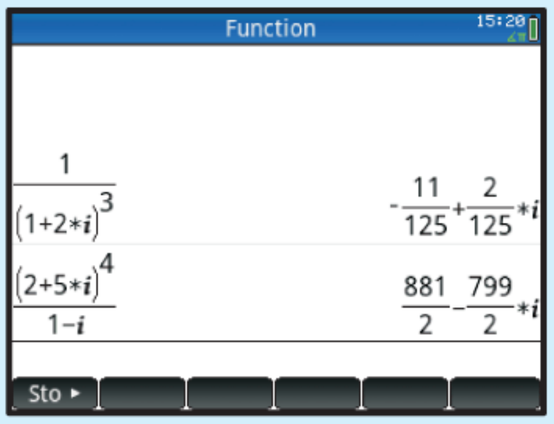
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a $\dfrac{1}{(1 + 2i)^3} = -\dfrac{11}{125} + \dfrac{2}{125}i$

b $\dfrac{(2 + 5i)^4}{1 - i} = \dfrac{881}{2} - \dfrac{799}{2}i$

9 Use technology to find:

a $(5+3i)^3$ **b** $\dfrac{1}{(3-4i)^2}$ **c** $(4-i)^5$

d $\dfrac{i}{(2+i)^3}$ **e** $(6+2i)^{-2}$ **f** $\dfrac{(4+5i)^4}{1-3i}$

10 Express $z = \dfrac{3i}{\sqrt{2}-i} + 1$ in the form $a+bi$ where $a, b \in \mathbb{R}$ are given exactly.

11 Suppose $w = \dfrac{z-1}{z^* + 1}$ where $z = a+bi$ for $a, b \in \mathbb{R}$, and z^* is the complex conjugate of z.

a Write w in the form $x+yi$.

b Hence determine the conditions under which w is purely imaginary.

D EQUALITY OF COMPLEX NUMBERS

Two complex numbers are **equal** when their **real parts** are equal and their **imaginary parts** are equal.

$$\text{If } a+bi = c+di, \text{ then } a=c \text{ and } b=d.$$

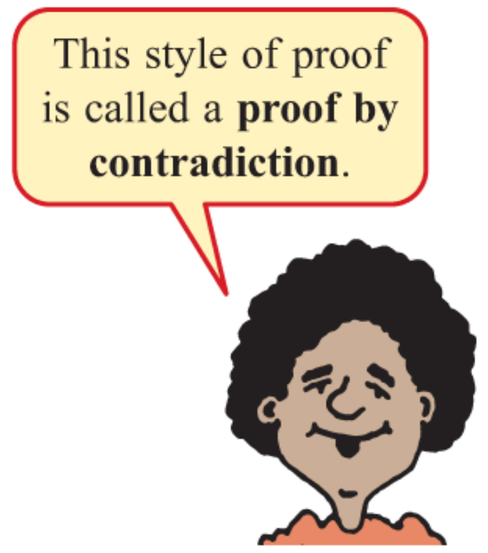
Proof:

Suppose $b \neq d$. Now if $a+bi = c+di$ where a, b, c, and d are real,

$$\text{then } bi - di = c - a$$
$$\therefore i(b-d) = c-a$$
$$\therefore i = \frac{c-a}{b-d} \quad \{\text{as } b \neq d\}$$

This is false as the RHS is real but the LHS is imaginary.

Thus, the supposition is false. Hence $b=d$ and furthermore $a=c$.



Example 5 Self Tutor

Find real numbers x and y such that:

a $(x+yi)(2-i) = -i$ **b** $(x+2i)(1-i) = 5+yi$

a
$$(x+yi)(2-i) = -i$$
$$\therefore x+yi = \frac{-i}{2-i}$$
$$= \left(\frac{-i}{2-i}\right)\left(\frac{2+i}{2+i}\right)$$
$$= \frac{-2i-i^2}{4+1}$$
$$= \frac{1-2i}{5}$$
$$= \tfrac{1}{5} - \tfrac{2}{5}i$$

Equating real and imaginary parts, $x = \frac{1}{5}$ and $y = -\frac{2}{5}$.

b
$$(x+2i)(1-i) = 5+yi$$
$$\therefore x - xi + 2i + 2 = 5 + yi$$
$$\therefore (x+2) + (2-x)i = 5 + yi$$

Equating real and imaginary parts,

$$\therefore x+2 = 5 \text{ and } 2-x = y$$
$$\therefore x = 3 \text{ and } y = -1$$

EXERCISE 11D

1 Find real numbers a and b such that:

a $a + bi = 3 - i$ **b** $a + bi = i - \sqrt{2}$ **c** $a + bi = 0$

2 Find exact real numbers x and y such that:

a $2x + 3yi = -x - 6i$ **b** $x^2 + xi = 4 - 2i$

c $(x + yi)(2 - i) = 8 + i$ **d** $(3 + 2i)(x + yi) = -i$

e $2(x + yi) = x - yi$ **f** $(x + 2i)(y - i) = -4 - 7i$

g $(x + i)(3 - yi) = 1 + 13i$ **h** $(x + yi)(2 + i) = 2x - (y + 1)i$

3 The complex number z satisfies the equation $3z + 17i = iz + 11$. Write z in the form $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

4 Find the real values of m and n for which $3(m + ni) = n - 2mi - (1 - 2i)$.

5 Suppose $(a + bi)^2 = -16 - 30i$ where $a, b \in \mathbb{R}$. Find the possible values of a and b.

6 Given $z^2 = 1 + i + \dfrac{58}{9(3 - 7i)}$, find z in the form $a + bi$.

HISTORICAL NOTE

18th century mathematicians enjoyed playing with these new "imaginary" numbers, but they were regarded as little more than interesting curiosities until the work of **Carl Friedrich Gauss** (1777 - 1855), the German mathematician, astronomer, and physicist.

For centuries mathematicians had attempted to find a method of trisecting an angle using a compass and straight edge. Gauss put an end to this when he used complex numbers to prove the impossibility of such a construction. By his systematic use of complex numbers, he was able to convince mathematicians of their usefulness.

Early last century, the American engineer **Charles Proteus Steinmetz** used complex numbers to solve electrical problems, illustrating that complex numbers did have a practical application.

Complex numbers are now used extensively in electronics, engineering, and physics.

E THE COMPLEX PLANE

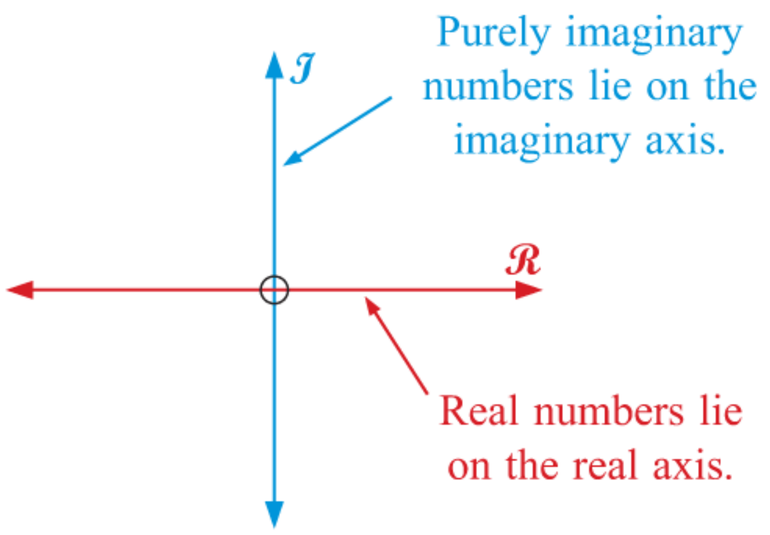
We have seen that a complex number can be written in Cartesian form as $z = a + bi$ where $a = \mathcal{Re}(z)$ and $b = \mathcal{Im}(z)$ are both real numbers.

There is a one-to-one relationship between any complex number $a + bi$ and the point (a, b) in the Cartesian plane.

We can therefore plot any complex number on a plane as a unique ordered number pair. We refer to the plane as the **complex plane** or the **Argand plane**.

On the complex plane, the x-axis is called the **real axis** and the y-axis is called the **imaginary axis**.

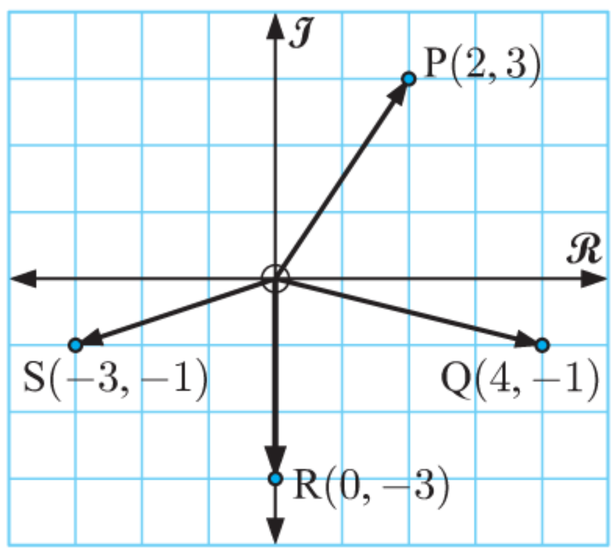
- All **real** numbers with $b = 0$ lie on the real axis.
- All **purely imaginary** numbers with $a = 0$ lie on the imaginary axis.
- The origin $(0, 0)$ lies on both axes. It corresponds to $z = 0$, a real number.
- Complex numbers that are neither real nor purely imaginary (a and b are both not 0) lie in one of the four quadrants.



We saw in **Chapter 9** that any point P on the Cartesian plane can be represented by a vector. The position vector for the point P is $\overrightarrow{OP}$. In the same way, we can illustrate complex numbers using vectors on the Argand plane. We call this an **Argand diagram**.

On an Argand diagram, $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$ represents $x + yi$.

For example:



$\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ represents $2 + 3i$

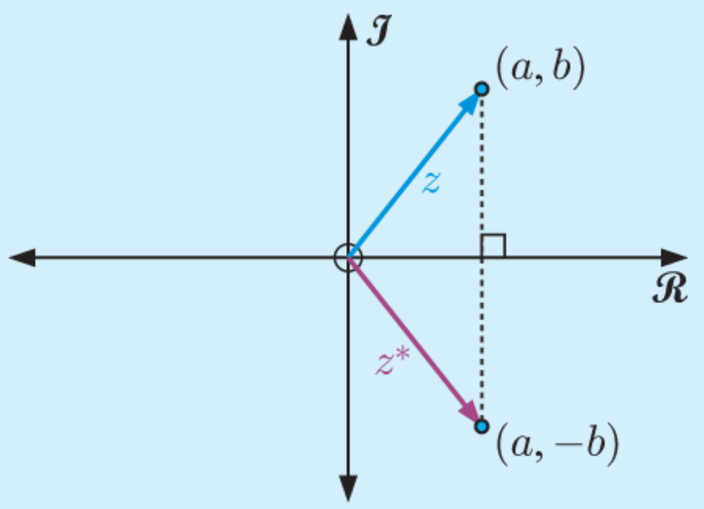
$\overrightarrow{OQ} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$ represents $4 - i$

$\overrightarrow{OR} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$ represents $-3i$

$\overrightarrow{OS} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$ represents $-3 - i$

The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

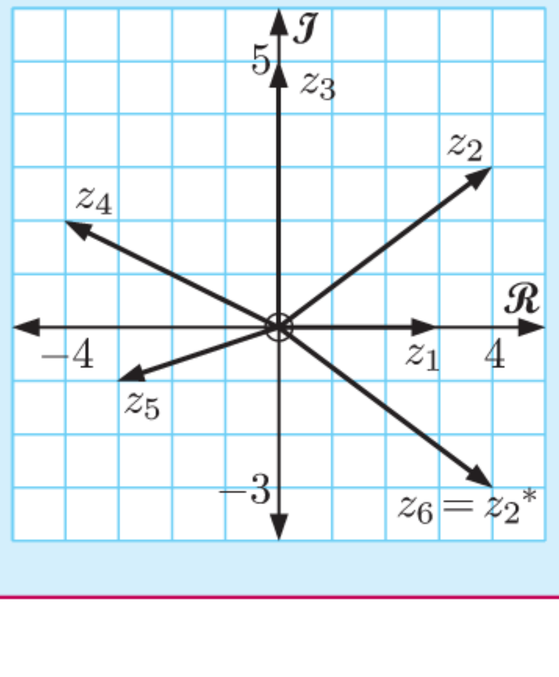
In the complex plane, z^* is the reflection of z in the real axis.



Example 6 **Self Tutor**

Illustrate the positions of the following complex numbers on an Argand diagram:

$z_1 = 3$, $z_2 = 4 + 3i$,
$z_3 = 5i$, $z_4 = -4 + 2i$,
$z_5 = -3 - i$, $z_6 = {z_2}^*$



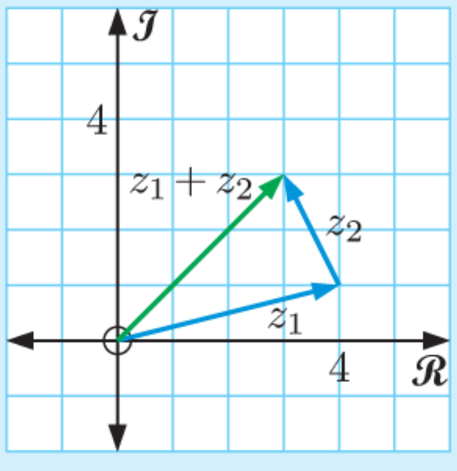
We can apply the vector operations of addition, subtraction, and scalar multiplication to perform these operations with complex numbers.

Example 7 **Self Tutor**

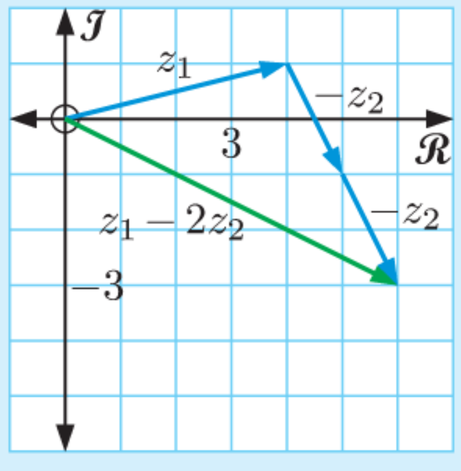
Suppose $z_1 = 4 + i$ and $z_2 = -1 + 2i$. Find using both algebra and vectors:

a $z_1 + z_2$ **b** $z_1 - 2z_2$ **c** $2z_1 + z_2^*$

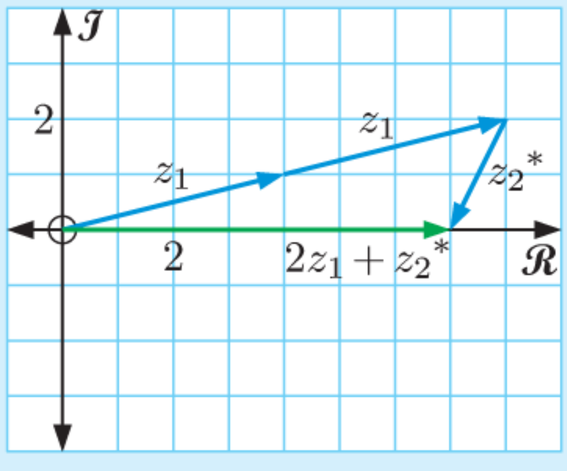
a
$$\begin{aligned} & z_1 + z_2 \\ &= (4 + i) + (-1 + 2i) \\ &= 4 + i - 1 + 2i \\ &= 3 + 3i \end{aligned}$$



b
$$\begin{aligned} & z_1 - 2z_2 \\ &= (4 + i) - 2(-1 + 2i) \\ &= 4 + i + 2 - 4i \\ &= 6 - 3i \end{aligned}$$



c
$$\begin{aligned} & 2z_1 + z_2^* \\ &= 2(4 + i) + (-1 - 2i) \\ &= 8 + 2i - 1 - 2i \\ &= 7 \end{aligned}$$



EXERCISE 11E

1 On an Argand diagram, illustrate the complex numbers:

$$z_1 = 5, \quad z_2 = -1 + 2i, \quad z_3 = -6 - 2i, \quad z_4 = -6i, \quad z_5 = 2 - i, \quad \text{and} \quad z_6 = 4i$$

2 Illustrate on an Argand diagram:

a $z = 3 + 2i$ and $z^* = 3 - 2i$ **b** $z = -2 + 5i$ and $z^* = -2 - 5i$

3 Suppose z is a complex number. Explain, with illustration, how to find geometrically:

a $3z$ **b** $-2z$ **c** $z + 3$ **d** $3i - z$

e $2 - z$ **f** $z^* + i$ **g** $\dfrac{z+2}{3}$ **h** $\dfrac{z-4}{2}$

4 Suppose $z = 1 + 2i$ and $w = 3 - i$. Find using both algebra and vectors:

a $z + w$ **b** $z - w$ **c** $2z - w$ **d** $w - 3z$

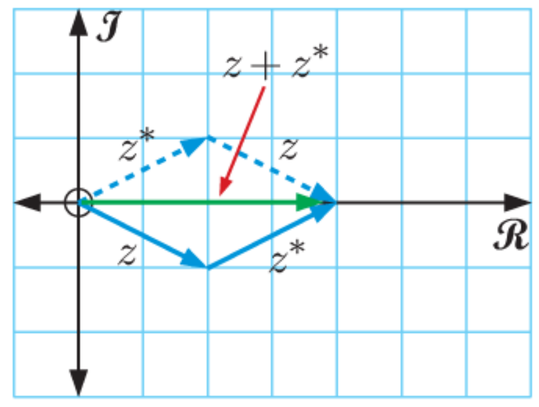
5 Suppose $z_1 = 4 - i$ and $z_2 = 2 + 3i$. Find using both algebra and vectors:

a $z_1 + 1$ **b** $z_1 + 2i$ **c** $z_2 + \frac{1}{2}z_1$ **d** $\dfrac{z_1 + 4}{2}$

6 Suppose $z_1 = -2 + 3i$ and $z_2 = 1 - 2i$. Find using both algebra and vectors:

a $z_1 + z_2^*$ **b** $2z_1^* - z_2$ **c** $z_1^* - 2z_2$ **d** $3z_1 + 2z_2$

7 Suppose $z = 2 - i$. From the diagram we can see that $z + z^* = 4$, which is real.



a Explain, with illustration, why $z + z^*$ is always real for any complex number z.

b Explain, with illustration, why $z - z^*$ is always either purely imaginary or zero. What distinguishes these two cases?

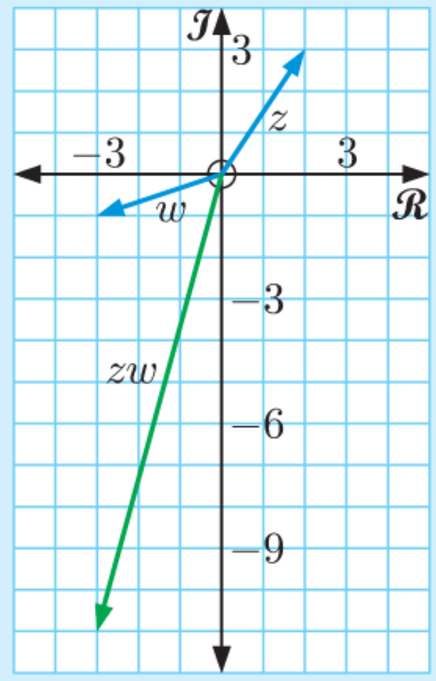
8 Using an Argand diagram, explain why for all complex numbers z and w:

a $(z+w)^* = z^* + w^*$ **b** $(z-w)^* = z^* - w^*$

Example 8 **Self Tutor**

Let $z = 2+3i$ and $w = -3-i$. Find zw, and illustrate z, w, and zw on the same Argand diagram.

$$\begin{aligned} zw &= (2+3i)(-3-i) \\ &= -6-2i-9i+3 \\ &= -3-11i \end{aligned}$$



9 Find zw, and illustrate z, w, and zw on the same Argand diagram:

a $z = 2-i,\ \ w = 1+2i$ **b** $z = -1+i,\ \ w = 3-i$

c $z = 3-2i,\ \ w = 3+2i$ **d** $z = 2i,\ \ w = 2+i$

F MODULUS AND ARGUMENT

Since complex numbers can be represented by vectors in the Argand plane, we can attribute to them both a *magnitude* and a *direction*.

The magnitude of a complex number is called its **modulus**, and its direction is called its **argument**.

MODULUS

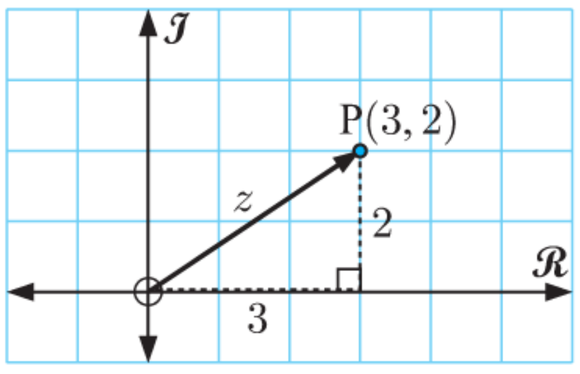
The **modulus** of the complex number $z = a+bi$ is the length of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$, which is the real number $|z| = \sqrt{a^2+b^2}$.

Notice that if $z = a+bi$ then $|z|$ gives the distance of the point $(a,\ b)$ from the origin. This is consistent with the previous definition of modulus that $|x|$ is the distance of real number x from the origin O.

For example, consider the complex number $z = 3+2i$.

The distance from O to P is its modulus, $|z|$.

$\therefore\ \ |z| = \sqrt{3^2+2^2}$ {Pythagoras}



PROPERTIES OF MODULUS

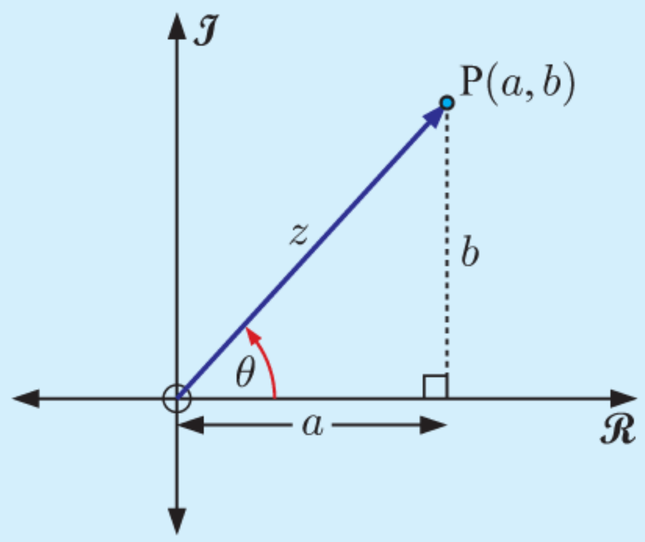
- $|z^*| = |z|$
- $|z|^2 = zz^*$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$
- $|z_1 z_2 z_3 .... z_n| = |z_1||z_2||z_3|....|z_n|$ and $|z^n| = |z|^n$ for $n \in \mathbb{Z}^+$.



ARGUMENT

Suppose the complex number $z = a + bi$ is represented by the vector $\overrightarrow{OP}$.

The **argument** of z, or simply $\arg z$, is the angle θ in the interval $-\pi < \theta \leqslant \pi$ which is measured anticlockwise between the positive real axis and $\overrightarrow{OP}$.



We specify the domain restriction $-\pi < \theta \leqslant \pi$ to avoid confusion with the infinitely many possibilities for θ which are 2π apart. This guarantees that $z \mapsto \arg z = \theta$ is a function.

Notice that:

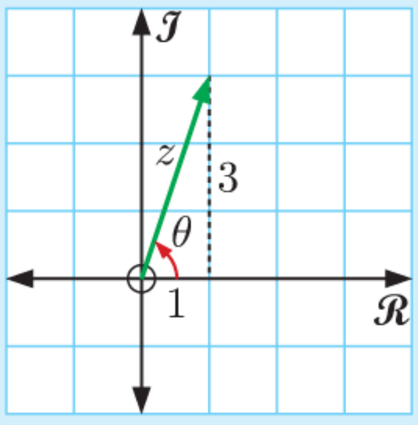
Real numbers (other than zero) have argument 0 or π.
Purely imaginary numbers have argument $\frac{\pi}{2}$ or $-\frac{\pi}{2}$.

Example 9 — Self Tutor

Find $|z|$ and $\arg z$ for z equal to:

a $1 + 3i$ **b** $-1 + 3i$ **c** $-3 - i$

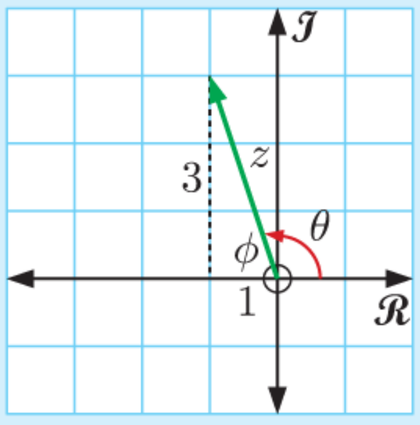
a $|z| = \sqrt{1^2 + 3^2} = \sqrt{10}$



$\tan\theta = \frac{3}{1} = 3$

$\therefore \arg z \approx 1.25$

b $|z| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$



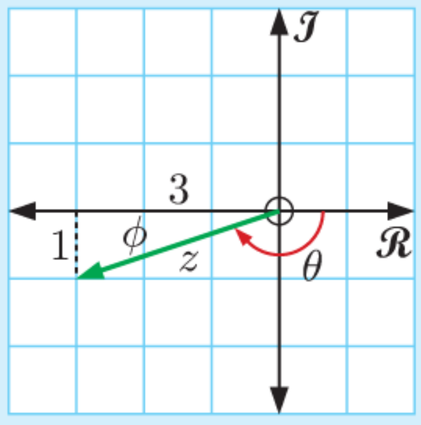
$\tan\phi = \frac{3}{1} = 3$

$\therefore \phi \approx 1.25$

But $\theta = \pi - \phi$

$\therefore \arg z \approx 1.89$

c $|z| = \sqrt{(-3)^2 + (-1)^2} = \sqrt{10}$



$\tan\phi = \frac{1}{3}$

$\therefore \phi \approx 0.322$

But $\theta = -\pi + \phi$

$\therefore \arg z \approx -2.82$

EXERCISE 11F

1 Find $|z|$ for z equal to:

a $3-4i$ **b** $5+12i$ **c** $-8+2i$ **d** $3i$ **e** -4

2 Find $|z|$ and $\arg z$ for z equal to:

a $2+i$ **b** $2-i$ **c** $-2+i$ **d** $-2-i$

3 Find $|w|$ and $\arg w$ for w equal to:

a $3+2i$ **b** $-3+2i$ **c** $2-3i$ **d** $-2+3i$

4 Suppose $z=a-i$ where a is real, and $\arg z=-\frac{5\pi}{6}$. Find the exact value of a.

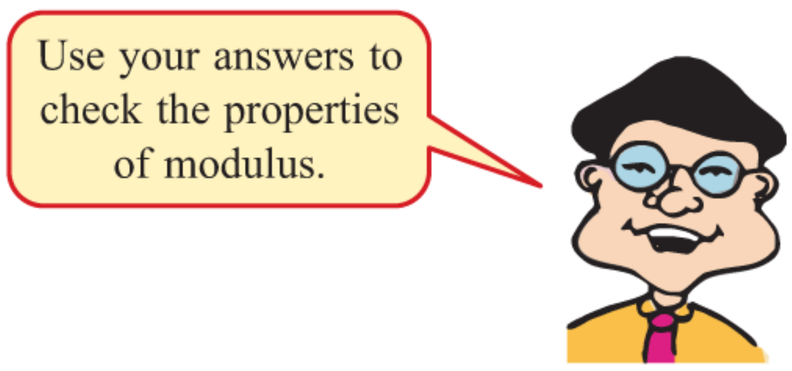
5 If $z=2+i$ and $w=-1+3i$, find:

a $|z|$ **b** $|z^*|$ **c** $|z^*|^2$

d zz^* **e** $|zw|$ **f** $|z||w|$

g $\left|\frac{z}{w}\right|$ **h** $\frac{|z|}{|w|}$ **i** $|z^2|$

j $|z|^2$ **k** $|z^3|$ **l** $|z|^3$



6 Find $|z|$ given:

a $z=\cos\theta+i\sin\theta$ **b** $z=r(\cos\theta+i\sin\theta),\ r\in\mathbb{R}$

7 Given $|z|=2$, use the rules $|zw|=|z||w|$ and $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$ to find:

a $|3z|$ **b** $|-2z|$ **c** $|(2-i)z|$

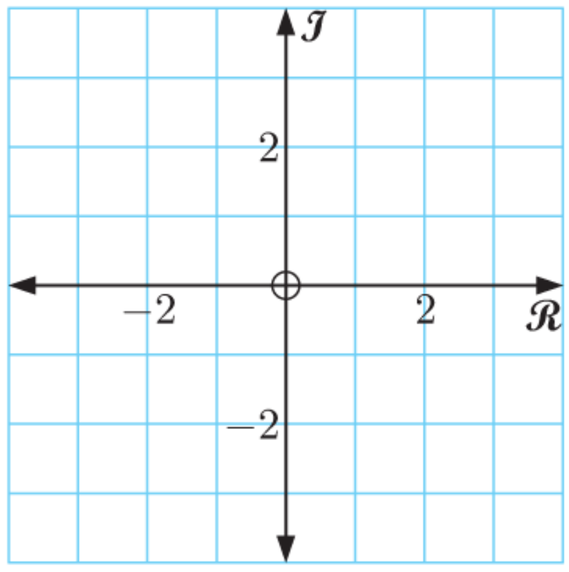
d $|iz|$ **e** $\left|\frac{1}{z}\right|$ **f** $\left|\frac{z^2}{3i}\right|$

8 On separate Argand planes, sketch the set of complex numbers z such that:

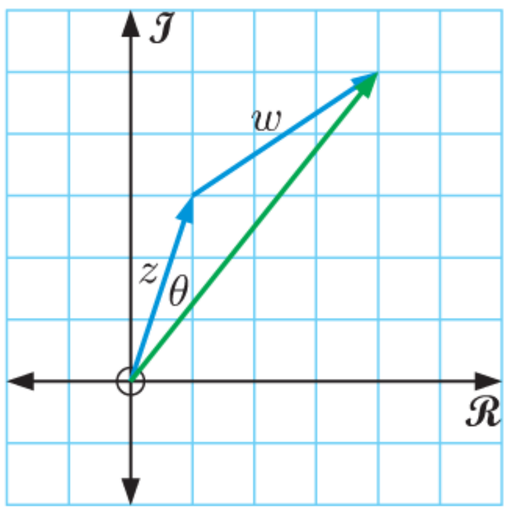
a $|z|=2$ **b** $|z|\leqslant 1$

c $3<|z|<4$ **d** $0\leqslant\arg z\leqslant\pi$

e $\frac{\pi}{4}<\arg z<\frac{3\pi}{4}$ **f** $\arg z=-\frac{\pi}{2}$



9



Let $z=1+3i$ and $w=3+2i$.

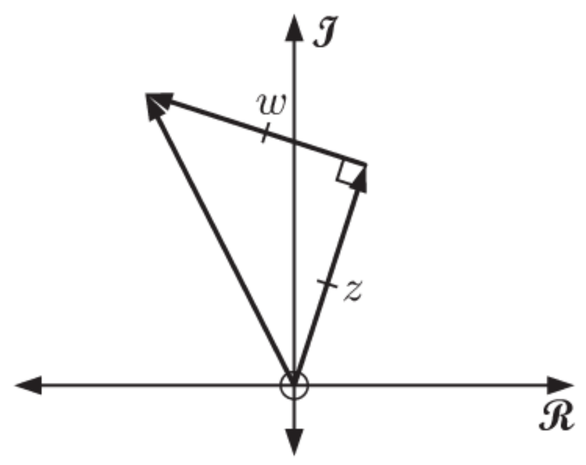
a Find $z+w$.

b Hence find $|z+w|$ and $\arg(z+w)$.

c Use the cosine rule to find θ.

d Hence verify that $\arg(z+w)+\theta=\arg z$.

10 On the Argand diagram alongside, $|z| = 3$ and $\arg z = \frac{2\pi}{5}$. Find:



a $|w|$ **b** $\arg w$

c $|z+w|$ **d** $\arg(z+w)$

11 Suppose $w = \dfrac{z+i}{z-i}$ where $z = a+bi$, $a, b \in \mathbb{R}$.

a Write w in the form $X+Yi$ where X and Y involve a and b.

b If $|z| = 1$, find $\mathcal{Re}(w)$.

12 Find the complex number z such that $|z| = 2\sqrt{10}$ and $\dfrac{50}{z^*} - \dfrac{10}{z} = 2+9i$.

G GEOMETRY IN THE COMPLEX PLANE

When we work in the Cartesian plane, we often need to consider x and y components separately. In the complex plane we have the advantage that both components are recorded in the one number.

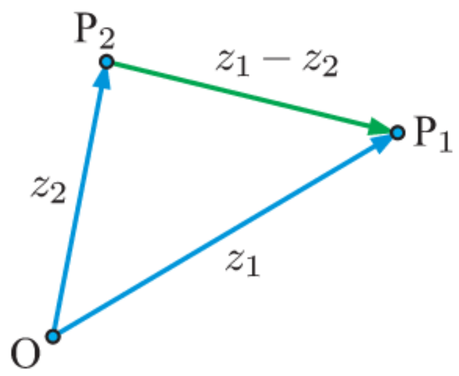
DISTANCES

Suppose P_1 and P_2 are two points in the complex plane which correspond to the complex numbers z_1 and z_2.

$$\begin{aligned}\overrightarrow{P_2P_1} &= \overrightarrow{P_2O} + \overrightarrow{OP_1}\\ &= -z_2 + z_1\\ &= z_1 - z_2\end{aligned}$$

$$\therefore \quad |z_1 - z_2| = |\overrightarrow{P_2P_1}|$$

$$= \text{distance between } P_1 \text{ and } P_2.$$



If $z_1 \equiv \overrightarrow{OP_1}$ and $z_2 \equiv \overrightarrow{OP_2}$ then $|z_1 - z_2|$ is the distance between the points P_1 and P_2.

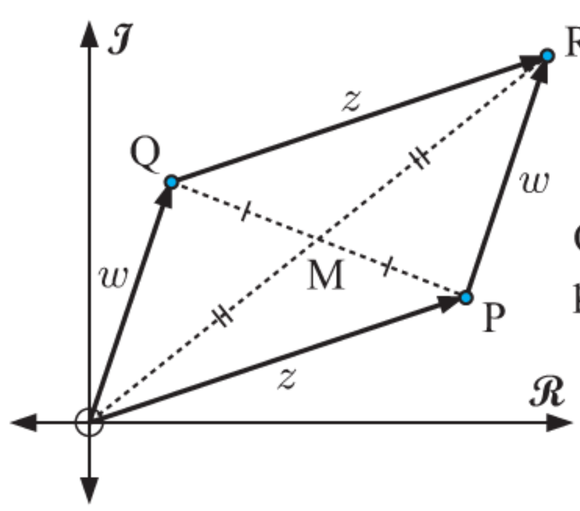
MIDPOINTS

The parallelogram alongside is formed using complex numbers w and z.

The diagonals of the parallelogram are $\overrightarrow{OR}$ and $\overrightarrow{PQ}$, and these diagonals bisect each other.

Now $\overrightarrow{OR} \equiv w + z$

$$\therefore \quad \overrightarrow{OM} \equiv \frac{w+z}{2}.$$



OPRQ is a parallelogram

Example 10 — Self Tutor

P(2, 3) and Q(6, 1) are two points on the Cartesian plane. Use complex numbers to find:

a distance PQ **b** the midpoint of [PQ].

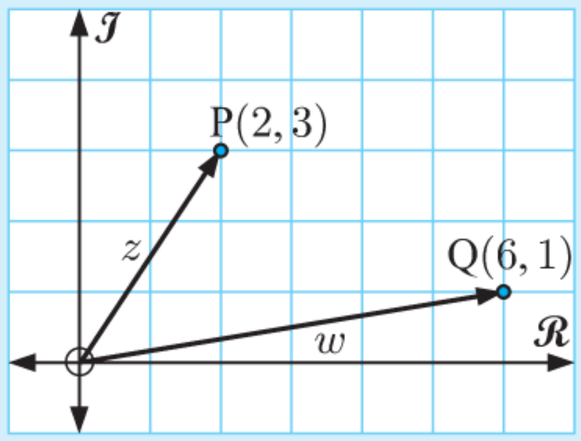
a If $z = 2 + 3i$ and $w = 6 + i$

then $w - z = 6 + i - 2 - 3i$

$= 4 - 2i$

$\therefore |w - z| = \sqrt{4^2 + (-2)^2} = \sqrt{20}$

$\therefore$ PQ $= \sqrt{20}$ units



b $\frac{z+w}{2} = \frac{2+3i+6+i}{2} = 4 + 2i$

$\therefore$ the midpoint of [PQ] is $(4, 2)$.

TRANSFORMATIONS

Example 11 — Self Tutor

Find the transformation which moves z to iz.

If $z = x + iy$, then $iz = i(x + iy)$

$= xi + i^2 y$

$= -y + xi$

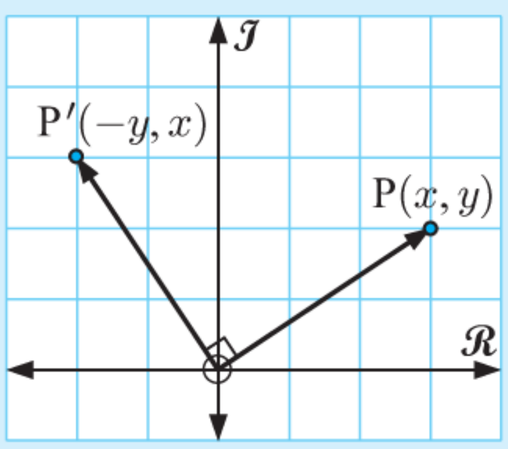
If $z \equiv \text{P}(x, y)$ then $iz \equiv \text{P}'(-y, x)$

We notice that $|z| = \sqrt{x^2 + y^2}$

and $|iz| = \sqrt{(-y)^2 + x^2}$

$= \sqrt{x^2 + y^2}$

$\therefore$ OP$'$ = OP



So, z moves to iz under an anticlockwise rotation of $\frac{\pi}{2}$ about O.

EXERCISE 11G

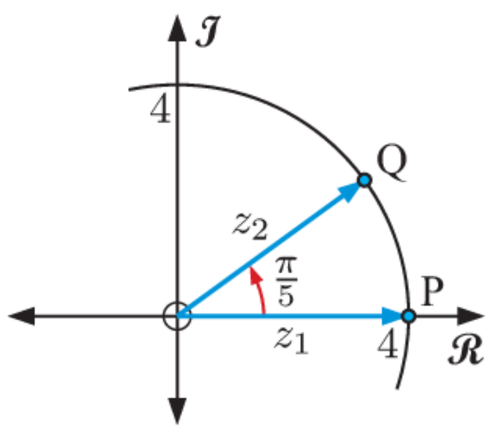
1 Use complex numbers to find **i** distance AB **ii** the midpoint of [AB] for:

a A(3, 6) and B(−1, 2) **b** A(−4, 7) and B(1, −3).

2 z_1 and z_2 correspond to the points P and Q respectively.

a Find the exact value of $|z_1 - z_2|$.

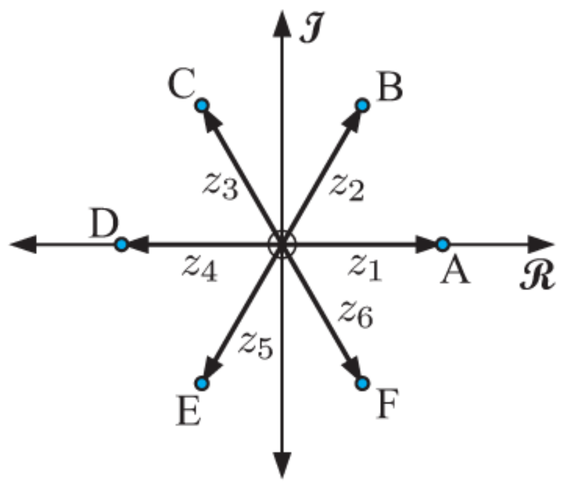
b Find the exact perimeter and area of triangle OPQ.



3 Find the transformation which moves:

a z to z^* **b** z to $-z$ **c** z to $-z^*$ **d** z to $-iz$.

4



ABCDEF is a regular hexagon centred at O, and $|z_1| = 2$.

a Find:

i $|z_1 - z_2|$ **ii** $|z_3 - z_6|$

iii $|z_2 - z_4|$ **iv** $\arg(z_2 - z_1)$

v $\arg(z_4 - z_6)$

b Find $|z_4 - i|$ and interpret your answer geometrically.

ACTIVITY 1 THE TRIANGLE INEQUALITY

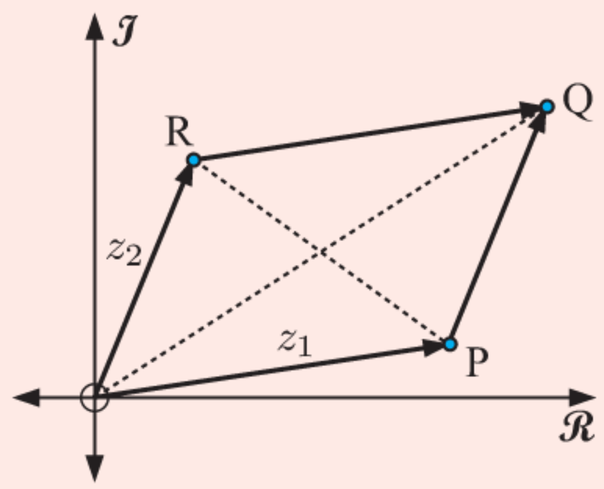
The **triangle inequality** states that for any complex numbers z_1 and z_2, $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

What to do:

1 Verify that $|z_1 + z_2| \leqslant |z_1| + |z_2|$ for:

a $z_1 = 3 + 4i$, $z_2 = 6i$ **b** $z_1 = 2 - 3i$, $z_2 = 4 + 7i$

2 OPQR is a parallelogram. $\overrightarrow{OP}$ represents z_1 and $\overrightarrow{OR}$ represents z_2, where z_1 and z_2 are complex numbers.



a Find, in terms of z_1 and z_2:

i $\overrightarrow{OQ}$ **ii** $\overrightarrow{PR}$

b Use triangle OPQ to explain why $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

Discuss the case of equality.

c Use triangle OPR to explain why $|z_1 - z_2| \geqslant |z_2| - |z_1|$.

Discuss the case of equality.

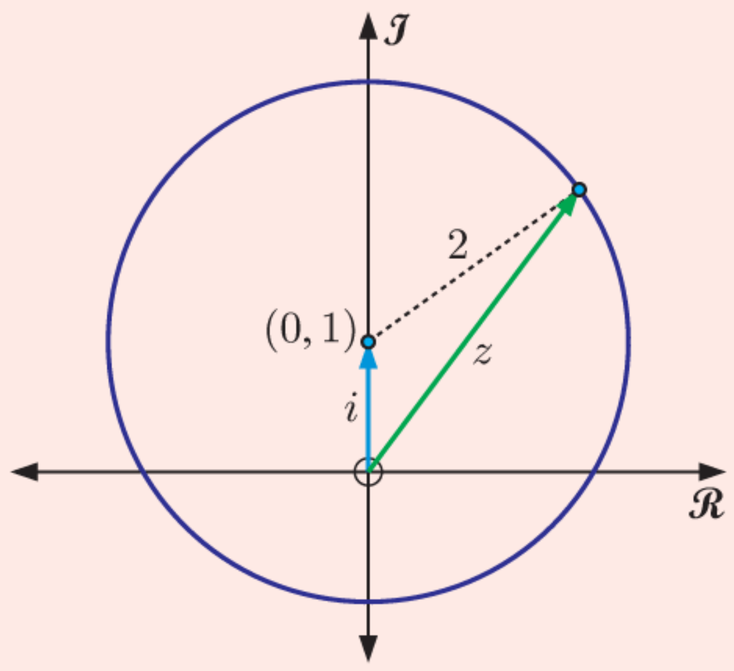
ACTIVITY 2 LOCUS

A **locus** is a curve or other figure formed by the set of points which satisfy a particular equation or inequality.

For example:

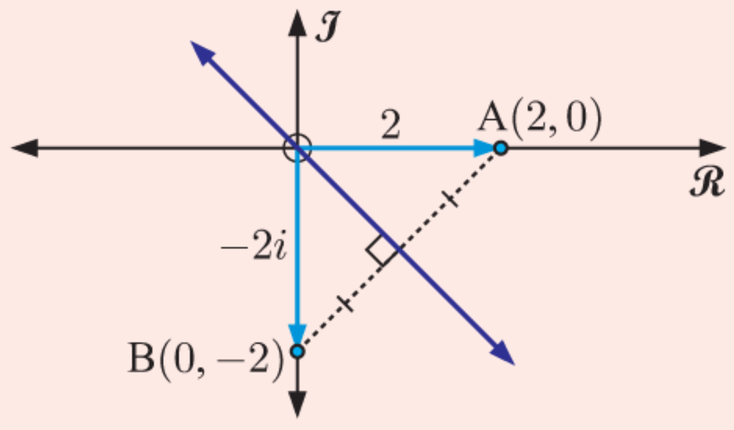
- Consider the equation $|z - i| = 2$.

  The point representing z is 2 units from the point representing i, so z lies on a circle with centre $(0, 1)$ and radius 2 units.



- Consider the equation $|z - 2| = |z + 2i|$.

  The point representing z is equidistant from $A(2, 0)$ and $B(0, -2)$, so z lies on the **perpendicular bisector** of [AB].



What to do:

1 **a** Graph all complex numbers which satisfy $|z| = 1$.

b Write $|z| = 1$ in Cartesian form by letting $z = x + yi$. What does this equation represent?

2 Illustrate the locus of points corresponding to:

a $|z| = 3$ **b** $|z| \geqslant 3$ **c** $|z| \leqslant 3$

d $\arg z = \frac{2\pi}{3}$ **e** $\frac{\pi}{3} \leqslant \arg z \leqslant \frac{2\pi}{3}$ **f** $\arg z = \frac{k\pi}{3}, \ k \in \mathbb{Z}$

3 Illustrate the locus of points corresponding to:

a $|z - 4| = 1$ **b** $|z - 3i| = 4$ **c** $|z + i| = 2$

d $|z - (2 + i)| = 2$ **e** $|z - (3 + 2i)| = 5$ **f** $|z - (-2 + i)| = 3$

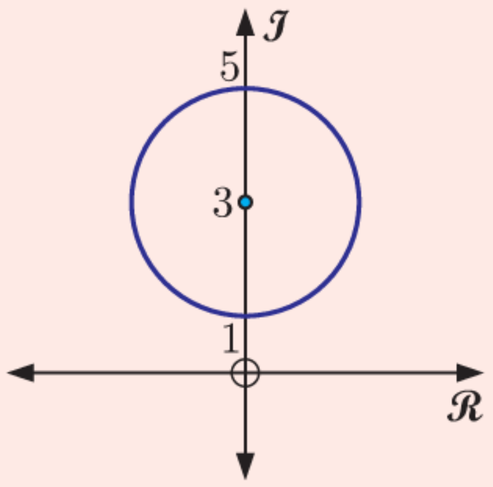
g $|z + 1| \leqslant 2$ **h** $|z - (1 - i)| > 2$ **i** $1 < |z - 3i| < 2$

4 Illustrate:

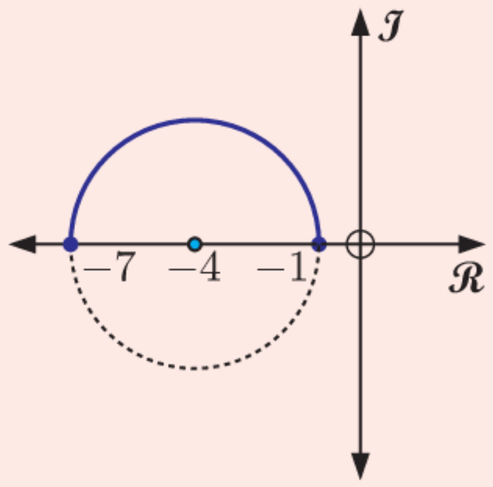
a $\{z \mid |z + i| = 2 \text{ and } \mathcal{R}e(z) \geqslant 0\}$ **b** $\{z \mid |z + 2| \leqslant 3 \text{ and } \mathcal{I}m(z) < 0\}$

5 Write the solution set of each graph:

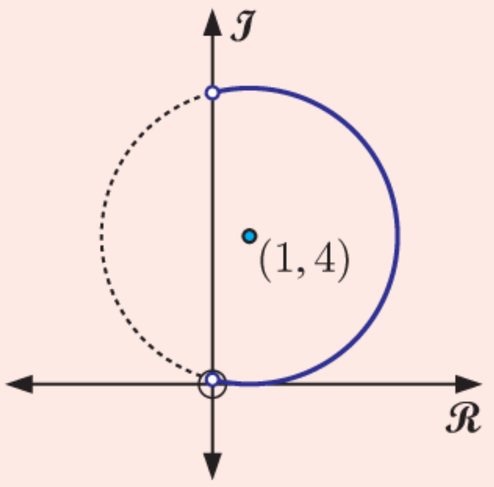
a



b



c



6 Illustrate the locus of points corresponding to:

a $|z + i| = |z - i|$ **b** $|z - 1| = |z - 3|$ **c** $|z + 2i| = |z|$

d $|z - 3| = |z + i|$ **e** $|z - 3| < |z + 5|$ **f** $|z - 4i| \geqslant |z + 4|$

H POLAR FORM

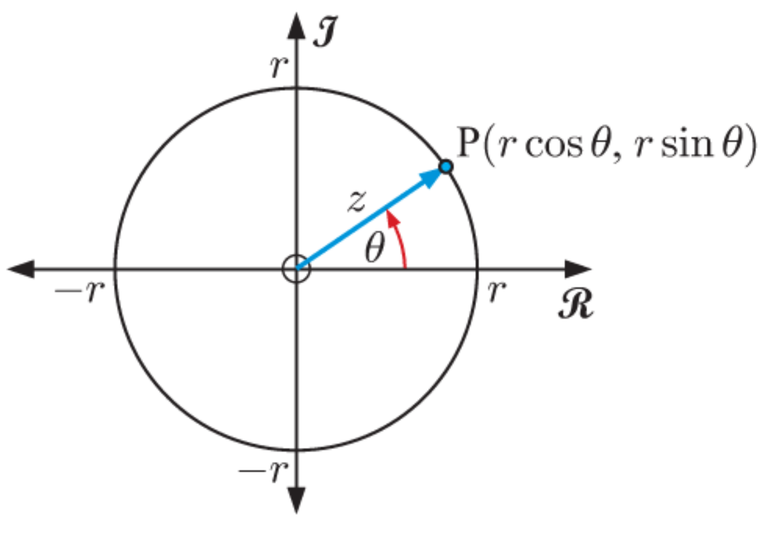
We have seen that the **Cartesian form** of a complex number is $z = a + bi$. However, we can also use a **polar form** which is based on the modulus and argument of z.

Any point P which lies on a circle with centre $O(0, 0)$ and radius r, has Cartesian coordinates $(r\cos\theta, r\sin\theta)$.

$\therefore$ on the Argand plane, the complex number represented by $\overrightarrow{OP}$ is $z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

where $r = |z|$ and $\theta = \arg z$.



We define $\mathbf{cis\,\theta = \cos\theta + i\sin\theta}$ so that $z = |z|\,\text{cis}\,\theta$.

A complex number z has **polar form** $z = |z| \operatorname{cis}\theta$ where $|z|$ is the **modulus** of z, θ is the **argument** of z, and $\operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Polar form is also called **modulus-argument form**.

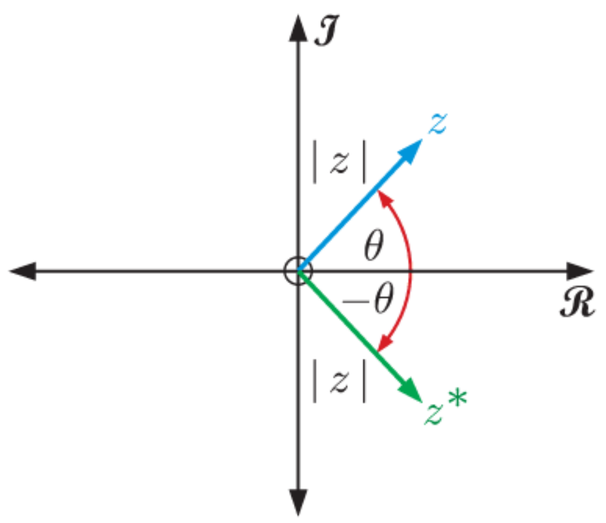
DISCUSSION

How do the definitions of modulus and argument help to ensure that the polar form for any complex number is unique? Why might this be useful?

We will soon see that polar form is extremely powerful for performing multiplication and division with complex numbers, and for quickly finding powers.

The conjugate of z has the same length as z, and its argument is $(-\theta)$.

If $z = |z| \operatorname{cis}\theta$, then $z^* = |z| \operatorname{cis}(-\theta)$.

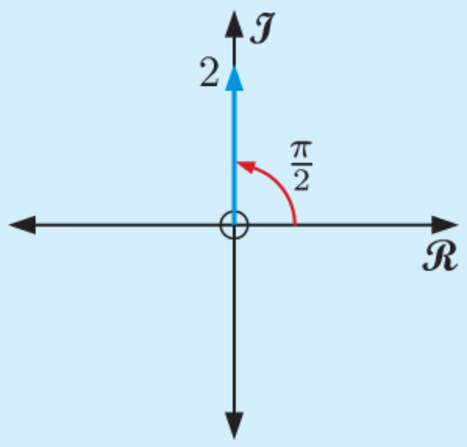


Example 12 — Self Tutor

Write in polar form:

a $2i$ **b** -3 **c** $1-i$

a

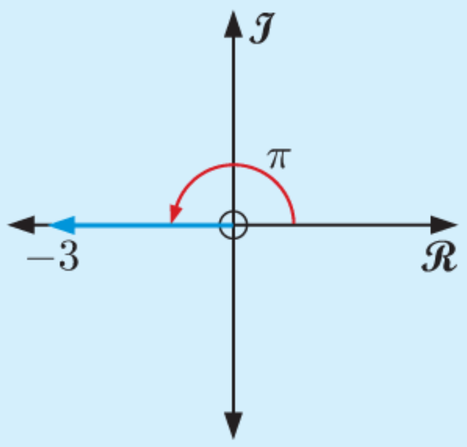


$|2i| = 2$

$\arg(2i) = \frac{\pi}{2}$

$\therefore \quad 2i = 2\operatorname{cis}\frac{\pi}{2}$

b

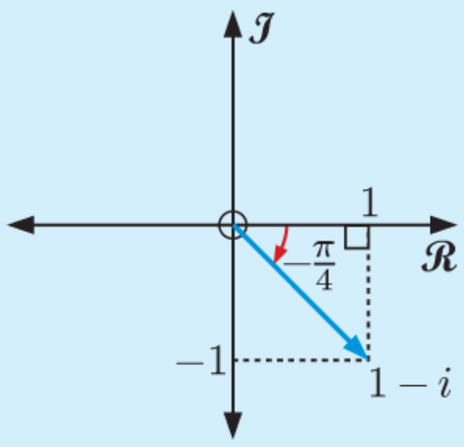


$|-3| = 3$

$\arg(-3) = \pi$

$\therefore \quad -3 = 3\operatorname{cis}\pi$

c



$|1-i| = \sqrt{1+1} = \sqrt{2}$

$\arg(1-i) = -\frac{\pi}{4}$

$\therefore \quad 1-i = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$

EXERCISE 11H.1

1 Write in polar form:

a 4 **b** $4i$ **c** -6 **d** $-3i$

e $1+i$ **f** $2-2i$ **g** $-\sqrt{3}+i$ **h** $2\sqrt{3}+2i$

2 What complex number cannot be written in polar form? Explain your answer.

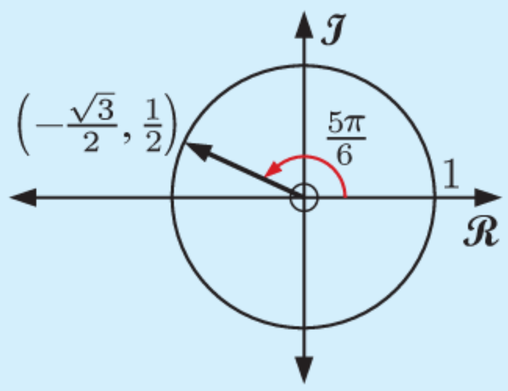
3 Convert $k+ki$ to polar form.

Hint: You must consider $k > 0$, $k = 0$, and $k < 0$ separately.

Example 13 **Self Tutor**

Convert $\sqrt{3}\,\text{cis}\,\frac{5\pi}{6}$ to Cartesian form.

We expand $\text{cis}\,\frac{5\pi}{6}$ using a unit circle diagram.



$$\begin{aligned}
&\sqrt{3}\,\text{cis}\,\tfrac{5\pi}{6}\\
&= \sqrt{3}\left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right)\\
&= \sqrt{3}\left(-\tfrac{\sqrt{3}}{2} + i \times \tfrac{1}{2}\right)\\
&= -\tfrac{3}{2} + \tfrac{\sqrt{3}}{2}i
\end{aligned}$$

4 Convert to Cartesian form without using a calculator:

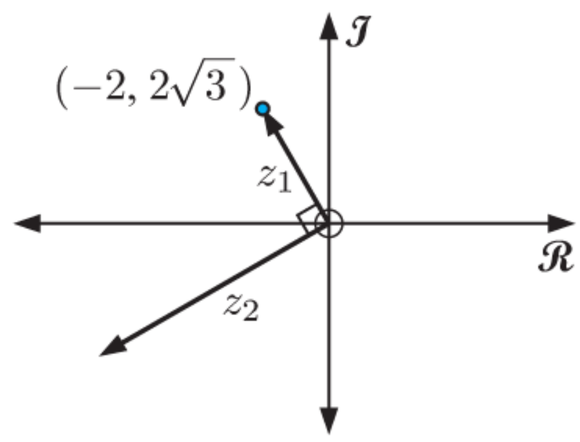
a $2\,\text{cis}\,\frac{\pi}{2}$ **b** $8\,\text{cis}\,\frac{\pi}{4}$ **c** $4\,\text{cis}\,\frac{\pi}{6}$ **d** $3\,\text{cis}\,0$

e $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ **f** $\sqrt{3}\,\text{cis}\,\frac{2\pi}{3}$ **g** $5\,\text{cis}\,\pi$ **h** $10\,\text{cis}\left(-\frac{2\pi}{3}\right)$

5 **a** Write z_1 in polar form.

b z_2 has twice the length of z_1. Write z_2 in:

i polar form **ii** Cartesian form.



6 State the value of:

a $\text{cis}\,\frac{3\pi}{4}$ **b** $\text{cis}\,0$ **c** $|\,\text{cis}\,\theta\,|$ for any θ.

Example 14 **Self Tutor**

Use your calculator to convert:

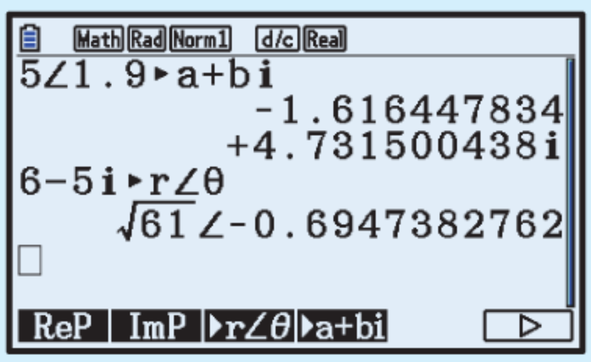
a $5\,\text{cis}(1.9)$ to Cartesian form **b** $6 - 5i$ to modulus-argument form.

a $5\,\text{cis}(1.9) \approx -1.62 + 4.73i$

b $6 - 5i \approx \sqrt{61}\,\text{cis}(-0.695)$



GRAPHICS CALCULATOR INSTRUCTIONS



7 Use your calculator to convert to Cartesian form:

a $4\,\text{cis}(0.7)$ **b** $\sqrt{3}\,\text{cis}(2.5187)$ **c** $\sqrt{11}\,\text{cis}\left(-\frac{3\pi}{8}\right)$

d $2.3\,\text{cis}\left(-\frac{4\pi}{5}\right)$ **e** $3.51\,\text{cis}(3.02)$ **f** $2.836\,49\,\text{cis}(-2.684\,32)$

8 Use your calculator to convert to modulus-argument form:

a $3-4i$ **b** $-5-12i$ **c** $\sqrt{7}+\sqrt{2}i$

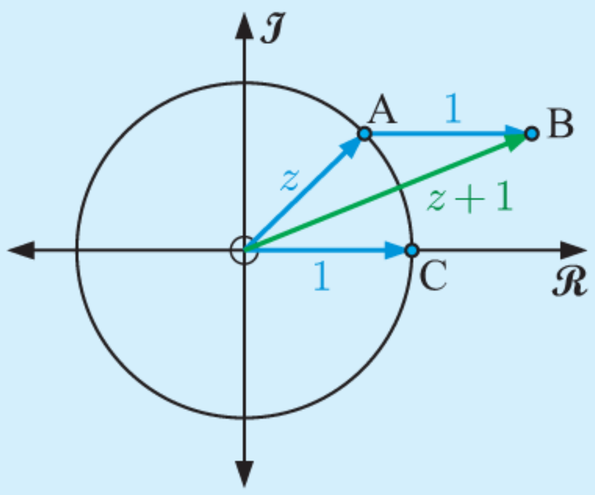
d $-1.4+2.3i$ **e** $2.72-0.57i$ **f** $-11.6814+13.2697i$

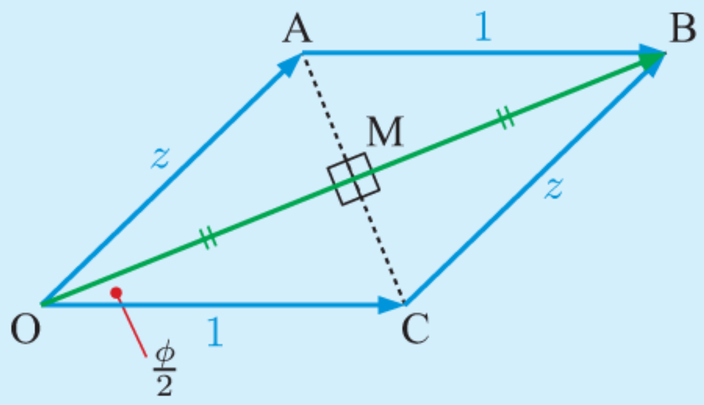
9 Add the following using Cartesian form and convert your answer to polar form:

a $3\,\text{cis}\,\frac{\pi}{4}+\text{cis}\left(-\frac{3\pi}{4}\right)$ **b** $2\,\text{cis}\,\frac{2\pi}{3}+5\,\text{cis}\left(-\frac{2\pi}{3}\right)$

Example 15 **Self Tutor**

Suppose $z=\text{cis}\,\phi$ where ϕ is acute. Find the modulus and argument of $z+1$.





$|z|=1$, so z ends on the unit circle.

$z+1$ is the vector $\overrightarrow{\text{OB}}$ shown.

Now OABC is a rhombus.

$\therefore \arg(z+1)=\frac{\phi}{2}$ {diagonals bisect the angles}

Let M be the midpoint of OB.

$\therefore \cos\frac{\phi}{2}=\frac{\text{OM}}{1}$

$\therefore \text{OM}=\cos\frac{\phi}{2}$

$\therefore \text{OB}=2\cos\frac{\phi}{2}$

$\therefore |z+1|=2\cos\frac{\phi}{2}$

10 **a** If $z=\text{cis}\,\phi$ where ϕ is acute, find the modulus and argument of $z-1$.

b Hence write $z-1$ and $(z-1)^*$ in polar form.

11 **a** Find, in terms of z, the equation which describes the points on the circle illustrated.

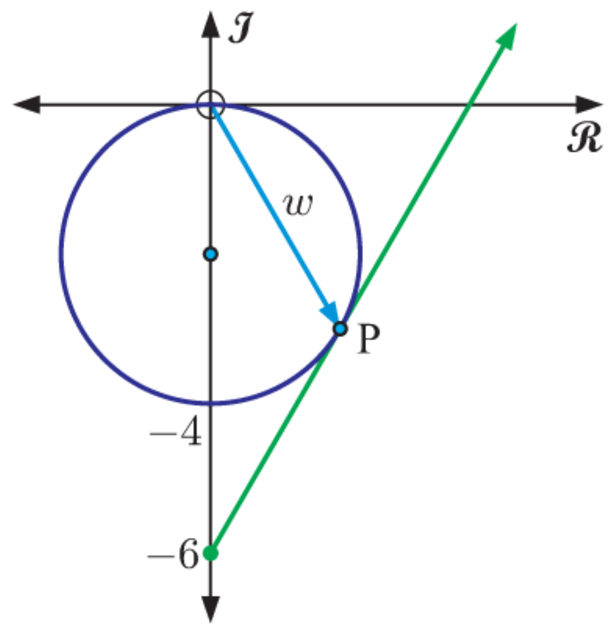
b The tangent to the circle from $(0,\,-6)$ meets the circle at the point P.

Let $w\equiv\overrightarrow{\text{OP}}$.

i Show that $|w|=2\sqrt{3}$.

ii Find $\arg w$.

iii Hence write w in the form $a+bi$.



PROPERTIES OF $\text{cis}\,\theta$

Three useful properties of $\text{cis}\,\theta$ are:

- $\text{cis}\,\theta\times\text{cis}\,\phi=\text{cis}(\theta+\phi)$
- $\dfrac{\text{cis}\,\theta}{\text{cis}\,\phi}=\text{cis}(\theta-\phi)$
- $\text{cis}(\theta+k2\pi)=\text{cis}\,\theta$ for all $k\in\mathbb{Z}$.

We will prove these properties in **Section I**.

Example 16 — Self Tutor

Use the properties of cis to simplify:

a $\text{cis}\,\frac{\pi}{5}\,\text{cis}\,\frac{3\pi}{10}$ **b** $\dfrac{\text{cis}\,\frac{\pi}{5}}{\text{cis}\,\frac{7\pi}{10}}$

a
$$\begin{aligned}\text{cis}\,\tfrac{\pi}{5}\,\text{cis}\,\tfrac{3\pi}{10} &= \text{cis}\left(\tfrac{\pi}{5}+\tfrac{3\pi}{10}\right)\\ &= \text{cis}\,\tfrac{5\pi}{10}\\ &= \text{cis}\,\tfrac{\pi}{2}\\ &= \cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}\\ &= 0+i(1)\\ &= i\end{aligned}$$

b
$$\begin{aligned}\frac{\text{cis}\,\frac{\pi}{5}}{\text{cis}\,\frac{7\pi}{10}} &= \text{cis}\left(\tfrac{\pi}{5}-\tfrac{7\pi}{10}\right)\\ &= \text{cis}\left(-\tfrac{5\pi}{10}\right)\\ &= \text{cis}\left(-\tfrac{\pi}{2}\right)\\ &= \cos\left(-\tfrac{\pi}{2}\right)+i\sin\left(-\tfrac{\pi}{2}\right)\\ &= 0+i(-1)\\ &= -i\end{aligned}$$

EXERCISE 11H.2

1 Use the properties of cis to simplify the following. Convert your answer to Cartesian form where possible.

a $\text{cis}\,\theta\,\text{cis}\,2\theta$ **b** $\dfrac{\text{cis}\,3\theta}{\text{cis}\,\theta}$ **c** $(\text{cis}\,\theta)^3$

d $\text{cis}\,\frac{\pi}{18}\,\text{cis}\,\frac{\pi}{9}$ **e** $2\,\text{cis}\,\frac{\pi}{12}\,\text{cis}\,\frac{\pi}{6}$ **f** $2\,\text{cis}\,\frac{2\pi}{5}\times 4\,\text{cis}\,\frac{8\pi}{5}$

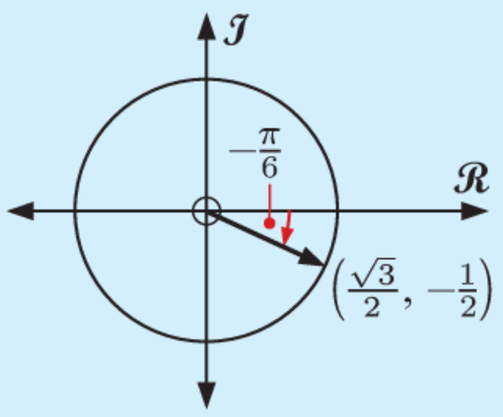
g $\sqrt{3}\,\text{cis}\left(-\frac{\pi}{15}\right)\times 2\,\text{cis}\,\frac{11\pi}{15}$ **h** $\dfrac{\text{cis}\,\frac{\pi}{2}}{\text{cis}\,\frac{\pi}{3}}$ **i** $\dfrac{4\,\text{cis}\,\frac{\pi}{12}}{2\,\text{cis}\,\frac{7\pi}{12}}$

j $\dfrac{\sqrt{2}\,\text{cis}\,\frac{3\pi}{20}}{\text{cis}\,\frac{9\pi}{10}}$ **k** $\dfrac{\sqrt{32}\,\text{cis}\,\frac{\pi}{8}}{\sqrt{2}\,\text{cis}\left(-\frac{7\pi}{8}\right)}$ **l** $\left(\sqrt{2}\,\text{cis}\,\frac{\pi}{8}\right)^4$

Example 17 — Self Tutor

Simplify $\text{cis}\,\frac{107\pi}{6}$.

$$\begin{aligned}\tfrac{107\pi}{6} &= 17\tfrac{5}{6}\pi = 18\pi-\tfrac{\pi}{6}\\ \therefore\ \text{cis}\,\tfrac{107\pi}{6} &= \text{cis}\left(18\pi-\tfrac{\pi}{6}\right)\\ &= \text{cis}\left(-\tfrac{\pi}{6}\right) \qquad \{\text{cis}(\theta+k2\pi)=\text{cis}\,\theta,\ \ k\in\mathbb{Z}\}\\ &= \cos\left(-\tfrac{\pi}{6}\right)+i\sin\left(-\tfrac{\pi}{6}\right)\\ &= \tfrac{\sqrt{3}}{2}-\tfrac{1}{2}i\end{aligned}$$



2 Use the property $\text{cis}(\theta+k2\pi)=\text{cis}\,\theta,\ \ k\in\mathbb{Z}$ to evaluate:

a $\text{cis}\,17\pi$ **b** $\text{cis}(-37\pi)$ **c** $\text{cis}\,\frac{91\pi}{3}$

3 Suppose $z = 2\,\text{cis}\,\theta$.

a State $|z|$ and $\arg z$.

b Write in polar form:

i z^* **ii** $-z$

iii $-z^*$ **iv** $(-z)^*$

$-2\,\text{cis}\,\theta$ is *not* in polar form since the modulus of a complex number cannot be negative.



Example 18 **Self Tutor**

Use the properties of $\text{cis}\,\theta$ to find the transformation which moves z to iz.

Let $z = r\,\text{cis}\,\theta$ and use $i = 1\,\text{cis}\,\frac{\pi}{2}$.

$\therefore\ iz = \text{cis}\,\frac{\pi}{2} \times r\,\text{cis}\,\theta$

$= r\,\text{cis}\left(\theta + \frac{\pi}{2}\right)$

$\therefore$ z has been rotated anticlockwise by $\frac{\pi}{2}$ about O.

4 **a** Write $-i$ in polar form.

b Suppose $z = r\,\text{cis}\,\theta$ is any complex number. Write $-iz$ in polar form.

c What transformation maps z to $-iz$?

5 **a** Write -1 in polar form.

b Suppose $z = r\,\text{cis}\,\theta$ is any complex number. Write $-z$ in polar form.

c What transformation maps z to $-z$?

6 Suppose $z = r\,\text{cis}\,\theta$ is a complex number. Use a geometric interpretation of z^* to help write z^* in polar form.

MULTIPLICATION OF COMPLEX NUMBERS

Suppose $z = |z|\,\text{cis}\,\theta$ and $w = |w|\,\text{cis}\,\phi$

Then $zw = |z|\,\text{cis}\,\theta \times |w|\,\text{cis}\,\phi$

$= \underbrace{|z|\,|w|}_{\text{non-negative}}\,\text{cis}(\theta + \phi)$ {property of cis}

$\therefore\ |zw| = |z|\,|w|$ and $\arg zw = \theta + \phi = \arg z + \arg w$.

If a complex number z is multiplied by $r\,\text{cis}\,\theta$ then its modulus is *multiplied* by r and its argument is *increased* by θ.

Example 19 **Self Tutor**

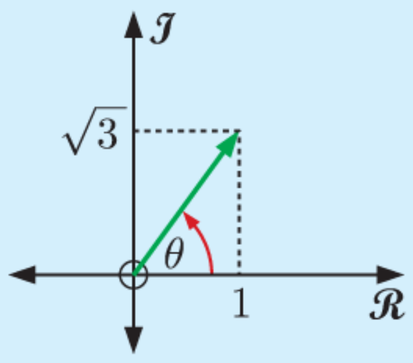
a Write $z = 1 + \sqrt{3}i$ in polar form.

b Hence multiply z by $2\,\text{cis}\,\frac{\pi}{6}$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $2\,\text{cis}\,\frac{\pi}{6}$?

a



If $z = 1 + \sqrt{3}i$, then $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

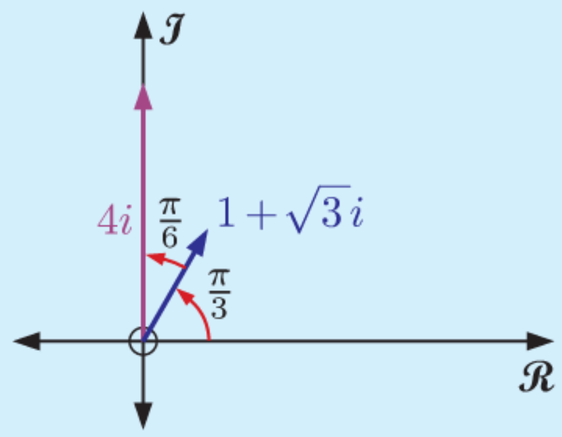
Now $\tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\therefore\ \arg z = \frac{\pi}{3}$ $\{\mathcal{Re}(z),\ \mathcal{Im}(z) > 0\}$

$\therefore\ z = 2\,\text{cis}\,\frac{\pi}{3}$

b $(1+\sqrt{3}i) \times 2\operatorname{cis}\frac{\pi}{6} = 2\operatorname{cis}\frac{\pi}{3} \times 2\operatorname{cis}\frac{\pi}{6}$
$= 4\operatorname{cis}\left(\frac{\pi}{3}+\frac{\pi}{6}\right)$
$= 4\operatorname{cis}\frac{\pi}{2}$
$= 4(0+1i)$
$= 4i$

c



d When z was multiplied by $2\operatorname{cis}\frac{\pi}{6}$, it was dilated with scale factor 2, then rotated anticlockwise through $\frac{\pi}{6}$.

EXERCISE 11H.3

1 **a** Write $z = 2-2i$ in polar form. **b** Hence multiply z by $\frac{1}{3}\operatorname{cis}\frac{\pi}{3}$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $\frac{1}{3}\operatorname{cis}\frac{\pi}{3}$?

2 **a** Write $z = -\sqrt{3}+i$ in polar form. **b** Hence multiply z by $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $\frac{1}{2}\operatorname{cis}\left(-\frac{\pi}{3}\right)$?

3 Suppose $z = -1+i$ and $w = 1-\sqrt{3}i$.

a Write z and w in polar form. **b** Hence find zw in polar form.

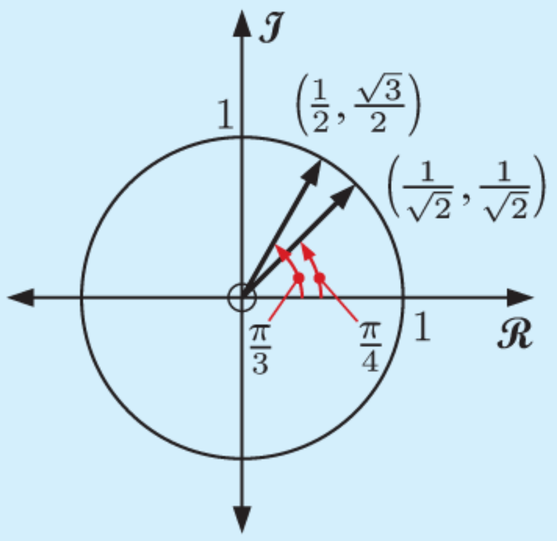
c Describe the transformation to z when it is multiplied by w.

Example 20

Self Tutor

Use complex numbers to write $\cos\frac{7\pi}{12}$ and $\sin\frac{7\pi}{12}$ in simplest surd form.

$\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}$
$= \operatorname{cis}\frac{7\pi}{12}$
$= \operatorname{cis}\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right)$
$= \operatorname{cis}\frac{\pi}{4} \times \operatorname{cis}\frac{\pi}{3}$ $\quad\{\operatorname{cis}(\theta+\phi) = \operatorname{cis}\theta \times \operatorname{cis}\phi\}$
$= \left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) \times \left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$
$= \left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$
$= \left(\frac{1}{2\sqrt{2}}-\frac{\sqrt{3}}{2\sqrt{2}}\right)+i\left(\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\right)$



Equating real parts: $\cos\frac{7\pi}{12} = \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right) \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$

Equating imaginary parts: $\sin\frac{7\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$

4 Use complex number methods to find, in simplest surd form:

a $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$ **b** $\cos\frac{11\pi}{12}$ and $\sin\frac{11\pi}{12}$

5 Use polar form to explain why:

a $\frac{z}{w}$ has modulus $\frac{|z|}{|w|}$ and argument $\arg z - \arg w$ provided $w \neq 0$

b z^n has modulus $|z|^n$ and argument $n \arg z$ for any $n \in \mathbb{Z}^+$.

6 Suppose $z = 2\,\text{cis}\,\frac{\pi}{5}$ and $w = 5\,\text{cis}\left(-\frac{2\pi}{7}\right)$. Find the modulus and argument of:

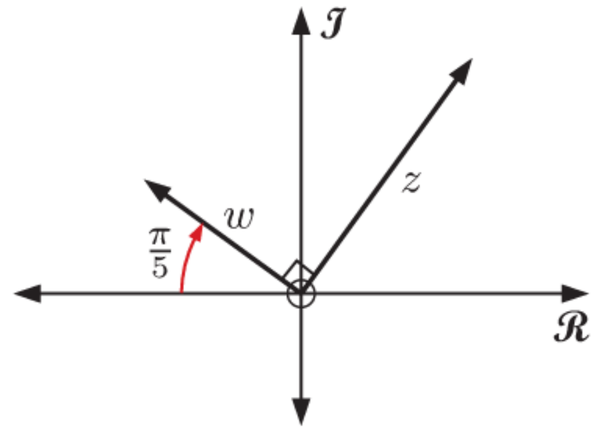
a z **b** w **c** zw **d** $\frac{z}{w}$ **e** z^4 **f** w^3

7 In the Argand diagram alongside, $|z| = 3$ and $|w| = 2$.

a Write z and w in polar form.

b Describe the transformation to z when it is multiplied by w.

c Draw z, w, and zw on an Argand diagram.



Example 21 — Self Tutor

Suppose $z = \sqrt{2}\,\text{cis}\,\theta$ where θ is obtuse. Find the modulus and argument of:

a $2z$ **b** $\frac{z}{i}$ **c** $(1-i)z$

a $2z = 2\sqrt{2}\,\text{cis}\,\theta$

$\therefore\ |2z| = 2\sqrt{2}$ and $\arg 2z = \theta$



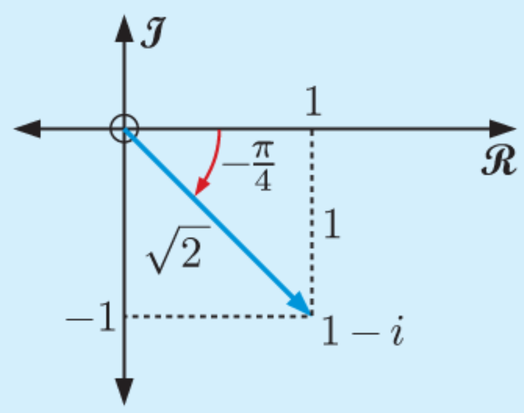
b $i = \text{cis}\,\frac{\pi}{2}$

$\therefore\ \frac{z}{i} = \frac{\sqrt{2}\,\text{cis}\,\theta}{\text{cis}\,\frac{\pi}{2}}$

$= \sqrt{2}\,\text{cis}\left(\theta - \frac{\pi}{2}\right)$

$\therefore\ \left|\frac{z}{i}\right| = \sqrt{2}$ and $\arg\left(\frac{z}{i}\right) = \theta - \frac{\pi}{2}$ {since θ is obtuse, $-\pi < \theta - \frac{\pi}{2} \leqslant \pi$}

c



$1 - i = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$\therefore\ (1-i)z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right) \times \sqrt{2}\,\text{cis}\,\theta$

$= 2\,\text{cis}\left(-\frac{\pi}{4} + \theta\right)$

$\therefore\ |(1-i)z| = 2$ and $\arg((1-i)z) = \theta - \frac{\pi}{4}$

8 Suppose $z = 3\,\text{cis}\,\theta$ where θ is acute. Find the modulus and argument of:

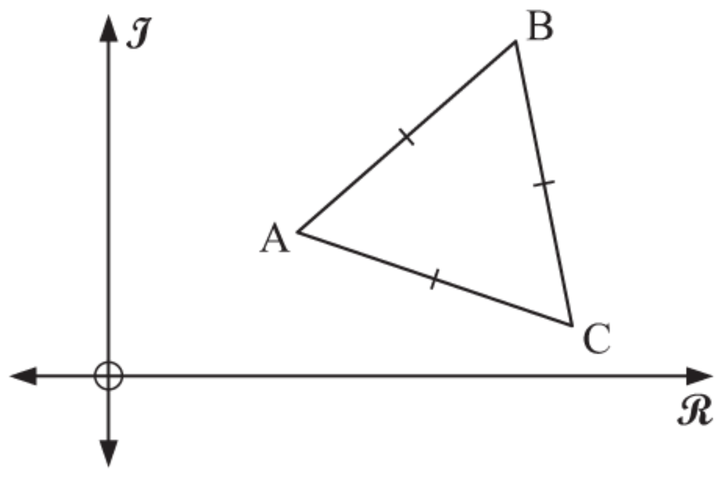
a $-z$ **b** z^2 **c** iz **d** $(1+i)z$ **e** $\frac{z}{i}$ **f** $\frac{z}{1-i}$

9 ABC is an equilateral triangle. Suppose z_1 represents $\overrightarrow{OA}$, z_2 represents $\overrightarrow{OB}$, and z_3 represents $\overrightarrow{OC}$.

a Explain what vectors represent $z_2 - z_1$ and $z_3 - z_2$.

b Find $\left|\dfrac{z_2 - z_1}{z_3 - z_2}\right|$ and $\arg\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right)$.

c Hence find the value of $\left(\dfrac{z_2 - z_1}{z_3 - z_2}\right)^3$.



I EXPONENTIAL FORM

HISTORICAL NOTE — EULER'S BEAUTIFUL EQUATION

One of the most remarkable results in mathematics is known as **Euler's beautiful equation** $e^{i\pi} = -1$ named after **Leonhard Euler**.

It is called beautiful because it links together three great constants of mathematics: Euler's constant e, the imaginary number i, and the ratio of a circle's circumference to its diameter, which is π.

Harvard lecturer **Benjamin Pierce** said of $e^{i\pi} = -1$,

> "Gentlemen, that is surely true, it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth."

Euler's beautiful equation is a special case of a more general result that Euler proved:

For any $\theta \in \mathbb{R}$, $\boldsymbol{e^{i\theta} = \cos\theta + i\sin\theta}$.

This identity allows us to write any complex number $z = |z|\operatorname{cis}\theta$ in the **exponential form** or **Euler form** $z = |z|e^{i\theta}$.

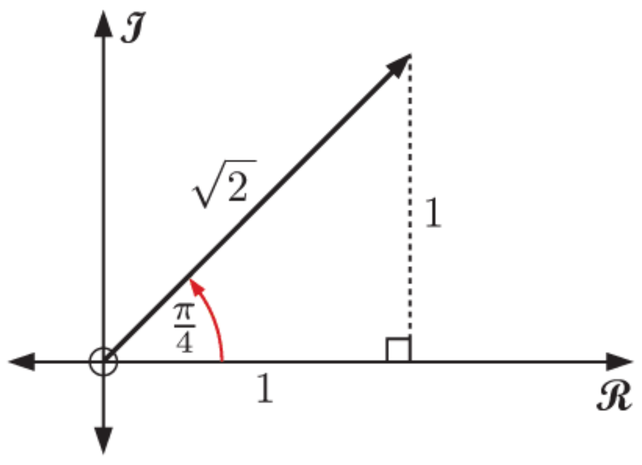
For example, consider $z = 1 + i$.

$|z| = \sqrt{2}$ and $\theta = \frac{\pi}{4}$,

$\therefore \quad 1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4} = \sqrt{2}e^{i\frac{\pi}{4}}$

So, $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$ is the **polar form** of $1 + i$

and $\sqrt{2}e^{i\frac{\pi}{4}}$ is the **exponential form** of $1 + i$.



INVESTIGATION — EXPONENTIAL FORM

We cannot prove the Euler identity for ourselves until we study calculus. However, in this Investigation we put together our knowledge of exponentials and complex numbers to justify why it is reasonable.

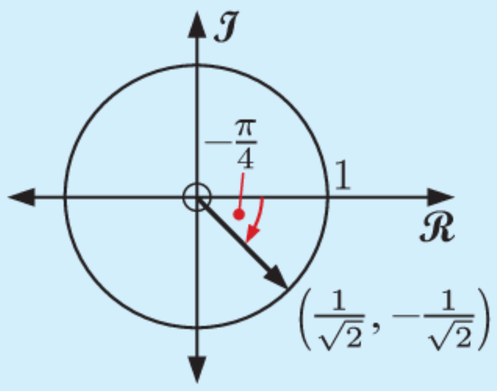


Example 22 **Self Tutor**

Evaluate: **a** $e^{-i\frac{\pi}{4}}$ **b** i^{-i}

a $e^{-i\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$

$= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$



b Now $i = 0 + 1i$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= e^{i\frac{\pi}{2}}$

$\therefore\ i^{-i} = \left(e^{i\frac{\pi}{2}}\right)^{-i}$

$= e^{-i^2\frac{\pi}{2}}$

$= e^{\frac{\pi}{2}}$

EXERCISE 11I

1 Use $e^{i\theta} = \cos\theta + i\sin\theta$ to write each of the following complex numbers in Cartesian form:

a $e^{i\pi}$ **b** $e^{i\frac{\pi}{3}}$ **c** $e^{-i\frac{\pi}{2}}$

2 Write:

a $-1 + i$ in polar form and exponential form

b $3\,\text{cis}\left(-\frac{\pi}{6}\right)$ in Cartesian form and exponential form

c $2e^{i\frac{2\pi}{3}}$ in Cartesian form and polar form.

3 Use your calculator to convert to Cartesian form:

a $e^{1.2i}$ **b** $4e^{-0.7i}$ **c** $8.4e^{2.63i}$ **d** $0.62e^{-1.02i}$

4 Use your calculator to convert to exponential form:

a $5 + 2i$ **b** $-9 + 4i$ **c** $-2.6 - 1.8i$ **d** $\sqrt{5} - i\sqrt{19}$



GRAPHICS CALCULATOR INSTRUCTIONS

5 Use $\text{cis}\,\theta = e^{i\theta}$ to prove that:

a $\text{cis}\,\theta\,\text{cis}\,\phi = \text{cis}(\theta + \phi)$ **b** $\frac{\text{cis}\,\theta}{\text{cis}\,\phi} = \text{cis}(\theta - \phi)$

c $(\text{cis}\,\theta)^n = \text{cis}\,n\theta$ **d** $\text{cis}(\theta + 2k\pi) = \text{cis}\,\theta,\ \ k \in \mathbb{Z}$

6 Suppose $z = \text{cis}\,\theta$, find the argument of:

a z^3 **b** $\sqrt{z}$ **c** iz^4 **d** $-iz^{\frac{2}{5}}$

7 Given $z = 2e^{i\theta}$ and $w = 3e^{i\phi}$, find the modulus and argument of:

a $\sqrt{w}$ **b** z^4w **c** $\frac{z}{w^2}$ **d** $w^3 \times \sqrt{z}$

8 Evaluate:

a e^i **b** e^{-i} **c** 3^i **d** i^i

9 Prove that for any angle θ:

a $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ **b** $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$

10 Let $w = 2e^{\frac{5\pi}{6}i}$.

a Draw w, w^2, and w^3 on an Argand diagram.

b Find the smallest integer $k > 3$ such that w^k is purely imaginary.

11 Given $Ae^{i\frac{\pi}{7}} + \sqrt{2}Be^{-i\frac{\pi}{4}} = e^{i\frac{\pi}{5}}$, find the exact values of A and B.

J FREQUENCY AND PHASE

The exponential form is particularly useful in applied mathematics because:

- it connects the exponential and trigonometric functions
- we can apply the usual exponent laws
- we can consider just real parts or just imaginary parts whenever we wish.

In this Section we consider the addition of trigonometric functions with the same **frequency** but different **phase**.

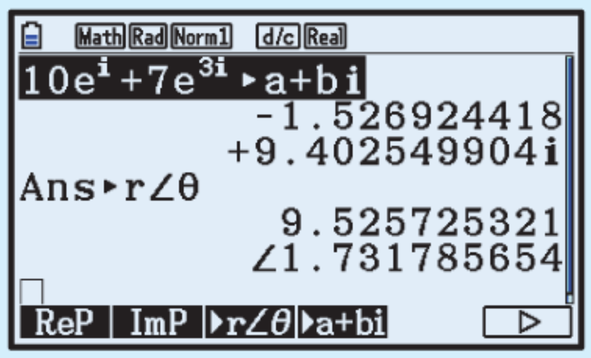
For example, $\sin\left(3t - \frac{\pi}{3}\right)$ and $\sin\left(3t + \frac{\pi}{6}\right)$ have the same period and therefore oscillate with the same *frequency*. However, they are not in the same *phase* because their maxima do not occur at the same time.

Example 23 **Self Tutor**

Write $10\cos(2t+1) + 7\cos(2t+3)$ in the form $A\cos(2t+B)$ where $A > 0$, $-\pi < B \leqslant \pi$.

$$10\cos(2t+1) + 7\cos(2t+3) = \mathcal{Re}\left(10\,\text{cis}(2t+1) + 7\,\text{cis}(2t+3)\right)$$
$$= \mathcal{Re}\left(10e^{(2t+1)i} + 7e^{(2t+3)i}\right)$$

Now $10e^{(2t+1)i} + 7e^{(2t+3)i}$
$= e^{2ti}\left(10e^{i} + 7e^{3i}\right)$
$\approx e^{2ti}(-1.53 + 9.40i)$
$\approx e^{2ti}\left(9.53e^{1.73i}\right)$
$\approx 9.53e^{(2t+1.73)i}$



So, $10\cos(2t+1) + 7\cos(2t+3) \approx \mathcal{Re}\left(9.53e^{(2t+1.73)i}\right)$
$\approx 9.53\cos(2t+1.73)$

EXERCISE 11J

1 Write:

a $4\cos(3t+1) + 3\cos(3t+4)$ in the form $A\cos(3t+B)$ where $A > 0$, $-\pi < B \leqslant \pi$

b $7\cos 2t + 5\cos(2t+3)$ in the form $A\cos(2t+B)$ where $A > 0$, $-\pi < B \leqslant \pi$.

2 **a** Show that $5\sin(t+1) + 6\sin(t+5) = \mathcal{Im}\left(5e^{(t+1)i} + 6e^{(t+5)i}\right)$.

b Hence write $5\sin(t+1)+6\sin(t+5)$ in the form $A\sin(t+B)$ where $A > 0$, $-\pi < B \leqslant \pi$.

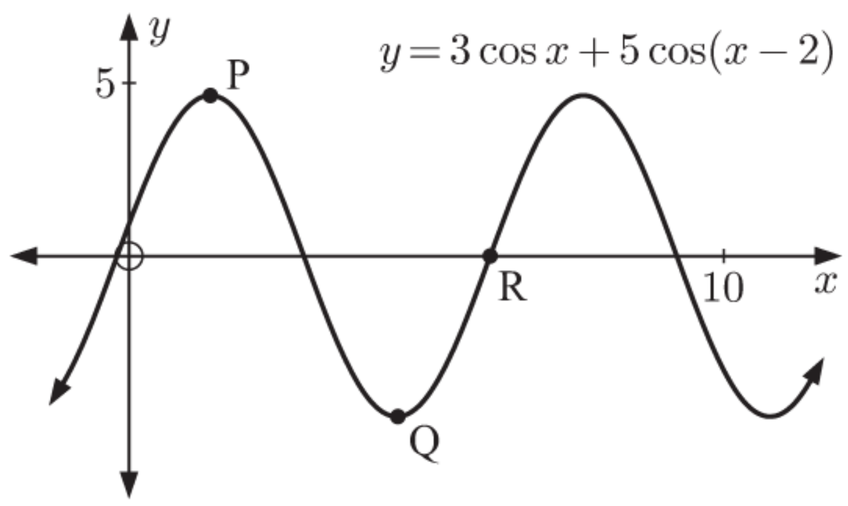
3 Write:

a $\sin 4t + 8\sin(4t+3)$ in the form $A\sin(4t+B)$

b $10\sin(2t+3) + 6\sin(2t-1)$ in the form $A\sin(2t+B)$.

4 **a** Write $3\cos x + 5\cos(x - 2)$ in the form $A\cos(x + B)$.

b Hence find the coordinates of the labelled points.



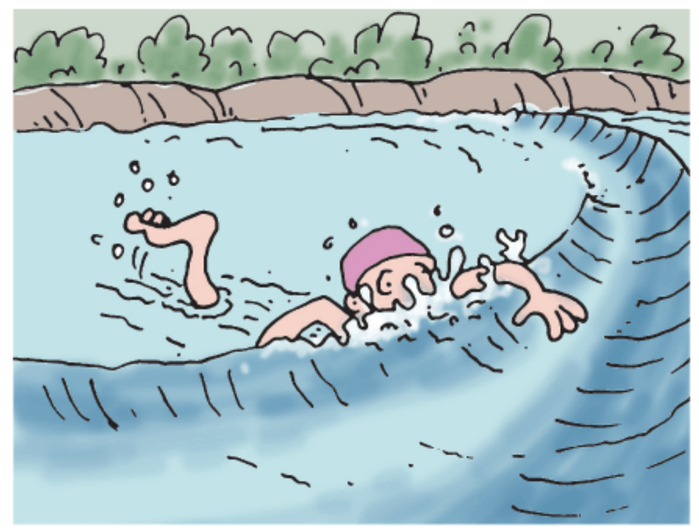
5 Write $6\cos x + 8\sin x$ in the form $A\cos(x + B)$.

6 Two voltage sources are connected to a circuit. At time t milliseconds, the voltage from source A is $V_A(t) = 10\cos(30t)$ volts, and the voltage from source B is $15\cos(30t + 5)$ volts.

a Write, in the form $V(t) = A\cos(30t + B)$, an expression for the total voltage in the circuit at time t milliseconds.

b Hence find the highest voltage in the circuit.

7 Gracie is swimming in a large lake. At time t seconds, she is affected by waves with height $H_1(t) = 0.2\sin 2t$ m from the wake of a speed boat, and waves of height $H_2(t) = 0.05\sin(2t + 1)$ m from the wake of a kayak.



a Write, in the form $H(t) = A\sin(2t + B)$, an expression for the total height of the wave affecting Gracie at time t seconds.

b At what times in the first 10 seconds is the wave not affecting Gracie at all?

c **i** Find the first time at which the wave affecting Gracie has maximum height.

ii At this time, what percentage of the wave is caused by the speed boat?

REVIEW SET 11A

1 Write in terms of i:

a $\sqrt{-49}$ **b** $\sqrt{-23}$ **c** $-\sqrt{-\frac{1}{9}}$

2 Solve for x using the quadratic formula:

a $x^2 - 6x + 11 = 0$ **b** $x^2 + 3 = 2\sqrt{2}x$ **c** $4x^2 + 3x + 1 = 0$

3 Given $z = 6 - 2i$ and $w = 3 + 7i$, find:

a z^* **b** $\mathcal{I}m(w^*)$ **c** $\mathcal{R}e(z - 3w)$

4 If $z = 3 + i$ and $w = -2 - i$, find in simplest form:

a $2z - 3w$ **b** $\frac{z^*}{w}$ **c** $\frac{w}{w^*}$

5 Find exactly the real and imaginary parts of $\frac{3}{i + \sqrt{3}} + \sqrt{3}$.

6 Find a complex number z such that $2z - 1 = iz - i$. Write your answer in the form $z = a + bi$ where $a, b \in \mathbb{R}$.

7 Find rationals a and b such that $\frac{2 - 3i}{2a + bi} = 3 + 2i$.

8 On an Argand diagram, illustrate the complex numbers:

a $3-2i$ **b** $-1+5i$

9 Suppose $z=3+2i$ and $w=-2+i$. Find, using both algebra and vectors:

a $z+w$ **b** $2z-w$ **c** z^* **d** $3w^*-z^*$

10 Find $|z|$ and $\arg z$ for z equal to:

a $5+2i$ **b** $-5+2i$ **c** $5-2i$ **d** $-2+5i$

11 Given $|z|=5$, find:

a $|5z|$ **b** $|-4z^*|$ **c** $|(3+i)z|$

d $\left|\frac{i}{z}\right|$ **e** $\left|\frac{3}{z^2}\right|$ **f** $\left|\frac{3+4i}{z}\right|$

12 Write in polar form:

a $3+3i$ **b** $-7i$ **c** $-6\sqrt{3}+6i$

13 Use your calculator to convert:

a $-4-7i$ to polar form **b** $2.5\,\text{cis}\,\frac{\pi}{9}$ to Cartesian form.

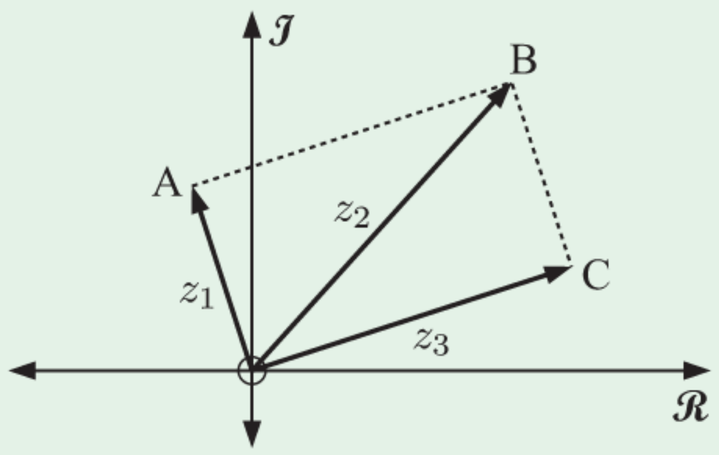
14 OABC is a rectangle.

$z_1=\text{cis}\,\frac{3\pi}{5}$ and $z_3=\sqrt{3}\,\text{cis}\,\phi$.

a Find ϕ.

b Find $\arg(z_3-z_2)$.

c Write z_2 in polar form.



15 If $z=\text{cis}\,\phi$ and $w=\dfrac{1+z}{1+z^*}$, show that $w=\text{cis}\,\phi$ also.

16 **a** Write $z=-1-i$ in polar form.

b Hence multiply z by $3\,\text{cis}\,\frac{\pi}{4}$.

c Illustrate what has happened on an Argand diagram.

d What transformations have taken place when multiplying z by $3\,\text{cis}\,\frac{\pi}{4}$?

17 Let $z_1=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ and $z_2=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$.

Express $\left(\dfrac{z_1}{z_2}\right)^3$ in the form $z=a+bi$.

18 If $z=4+i$ and $w=2-3i$, find:

a $2w^*-iz$ **b** $|w-z^*|$ **c** $|z^{10}|$ **d** $\arg(w-z)$

19 Write $2-2\sqrt{3}i$ in polar form. Hence find all values of n for which $\left(2-2\sqrt{3}i\right)^n$ is purely imaginary.

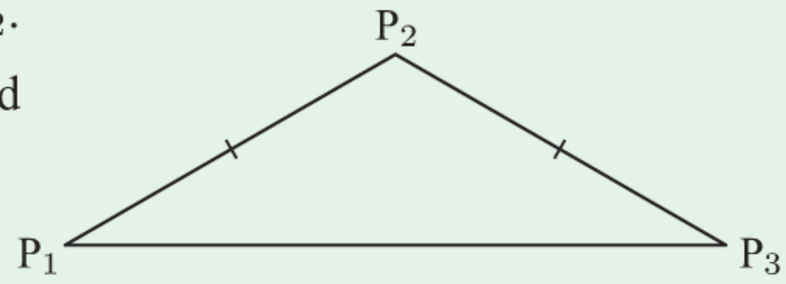
20 If $z=4\,\text{cis}\,\theta$, find the modulus and argument of:

a z^3 **b** $\frac{1}{z}$ **c** iz^*

21 Illustrate the region defined by:

a $\{z\mid 2\leqslant|z|\leqslant 5\}$ **b** $\{z\mid -\frac{\pi}{4}<\arg z\leqslant\frac{\pi}{2}\}$

22 $P_1P_2P_3$ is an isosceles triangle where $P_1P_3 = \sqrt{3}\,P_1P_2$.
O is an origin such that $\overrightarrow{OP_1} \equiv z_1$, $\overrightarrow{OP_2} \equiv z_2$, and $\overrightarrow{OP_3} \equiv z_3$.
Suppose $\arg(z_2 - z_1) = \alpha$.



a Show that $\arg(z_3 - z_2) = \alpha - \frac{\pi}{3}$.

b Find the modulus and argument of $\dfrac{z_2 - z_1}{z_3 - z_2}$.

23 Write in Cartesian form: **a** $e^{i\frac{2\pi}{3}}$ **b** $5e^{-i\pi}$

24 Given $z = 4e^{i\frac{\pi}{4}}$ and $w = 3e^{-i\frac{\pi}{3}}$, write zw in polar form.

25 Write $\sin 3t + 4\sin\left(3t - \frac{3\pi}{4}\right)$ in the form $A\sin(3t + B)$, where $A > 0$ and $-\pi < B \leqslant \pi$.

REVIEW SET 11B

1 Solve for x:

a $2x^2 = -18$ **b** $x^2 - 5x + 7 = 0$ **c** $-2x^2 + 6x - 9 = 0$

2 Find the complex conjugate of:

a $-3 - 8i$ **b** $\frac{1}{2} + \frac{5}{2}i$ **c** $(2 + i)(4 - 5i)$

3 For $z = 2 + 3i$ and $w = -6 + i$, find:

a $z + 3w$ **b** zw **c** $\dfrac{w}{z}$ **d** $\dfrac{z^*}{i}$

4 Use technology to find:

a $(5 - 8i)^2$ **b** $\dfrac{4 + 2i}{(3 - i)^2}$ **c** $(2 - 3i)^4$ **d** $(1 + i)^{-4}$

5 z and w are two complex numbers such that $2z + w = i$ and $z - 3w = 7 - 10i$.
Find $z + w$ in the form $a + bi$.

6 Find real x and y such that:

a $x + iy = 0$ **b** $(3 - 2i)(x + i) = 17 + yi$

c $(x + iy)^2 = x - iy$ **d** $(3x + 2yi)(1 - i) = (3y + 1)i - x$

7 Suppose $z^2 = 5 - 12i$. Write z in the form $a + bi$ where $a, b \in \mathbb{R}$.

8 Let $z = a + bi$ be a complex number and k be a real constant. Prove that $(kz)^* = kz^*$.

9 z and w are non-real complex numbers with the property that both $z + w$ and zw are real.
Prove that $z^* = w$.

10 Suppose $z = 4 - i$. Show, on a single Argand diagram:

a z **b** z^* **c** $z + 3i$ **d** $\dfrac{z + 1}{2}$

11 Given $|z| = 4$, find:

a $|3z|$ **b** $|2iz|$ **c** $\left|\dfrac{i + 1}{z}\right|$ **d** $|(3 - i)z|$

12 Let $z = 1 + i$ and $w = -2 - i$.
Find zw, and illustrate z, w, and zw on the same Argand diagram.

13 If $z = -1 + 3i$ and $w = 3 + i$, find:

a $|z|$ **b** $|w|$ **c** $|z^*|^2$ **d** zz^*

e $|zw|$ **f** $\left|\frac{z}{w}\right|$ **g** $|z^2|$ **h** $|z^3|$

14 On an Argand diagram, illustrate the set of points corresponding to:

a $|z| \geqslant 1$ **b** $\frac{\pi}{2} \leqslant \arg z \leqslant \pi$ **c** $\arg z = \pm\frac{\pi}{4}$

15 Write in polar form:

a $-5i$ **b** $2 - 2i\sqrt{3}$ **c** $k - ki$ where $k < 0$

16 Convert to Cartesian form:

a $4\,\text{cis}\,\frac{\pi}{2}$ **b** $\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$ **c** $8\,\text{cis}\left(-\frac{\pi}{3}\right)$

17 Simplify:

a $\text{cis}\,\frac{\pi}{5}\,\text{cis}\,\frac{2\pi}{3}$ **b** $\dfrac{3\,\text{cis}\left(-\frac{\pi}{10}\right)}{\text{cis}\,\frac{\pi}{4}}$ **c** $\text{cis}\,\frac{73\pi}{3}$

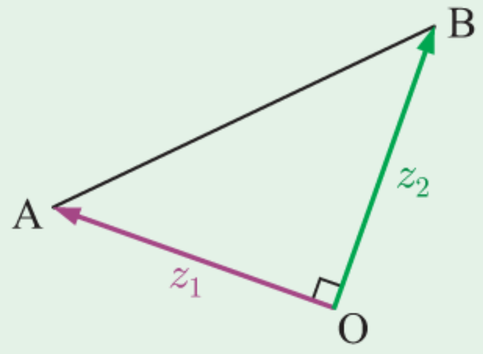
18 Suppose $z = r\,\text{cis}\,\frac{4\pi}{5}$ where $r > 0$. Write in polar form:

a $3z$ **b** $-z$ **c** z^* **d** z^2 **e** iz

19 Suppose $z = (1 + bi)^2$ where b is real and positive. Find the exact value of b if $\arg z = \frac{\pi}{3}$.

20 If $z = 2\sqrt{2}\,\text{cis}\,\alpha$, write $(1 - i)z$ in polar form. Hence find $\arg[(1 - i)z]$.

21 $z_1 \equiv \overrightarrow{OA}$ and $z_2 \equiv \overrightarrow{OB}$ represent two sides of a right angled isosceles triangle OAB.



a Determine the modulus and argument of $\dfrac{{z_1}^2}{{z_2}^2}$.

b Hence deduce that ${z_1}^2 + {z_2}^2 = 0$.

22 Points A and B represent the numbers $z = 2 - 2i$ and $w = -1 - \sqrt{3}i$ respectively in the complex plane.

a Given the origin O, find $\widehat{\text{AOB}}$ in terms of π.

b Calculate the argument of zw in terms of π.

c Find $k > 0$ given that the real powers of $\dfrac{z}{kw}$ all lie on a circle centred at O.

23 Write:

a $-3 - 3i$ in polar form and exponential form

b $5\,\text{cis}\left(-\frac{5\pi}{6}\right)$ in Cartesian form and exponential form

c $7e^{i\frac{\pi}{2}}$ in Cartesian form and polar form.

24 In an unbalanced three-phase electrical circuit, the current at time t ms is given by $I(t) = 5\sin 10t + 7\sin\left(10t - \frac{2\pi}{3}\right) + 8\sin\left(10t - \frac{4\pi}{3}\right)$ milliamperes.

a Write $I(t)$ in the form $A\cos(10t + B)$.

b Find the highest current flowing through the circuit, and the time at which it first occurs.

Chapter 12

Matrices

Contents:

A Matrix structure
B Matrix equality
C Addition and subtraction
D Scalar multiplication
E Matrix algebra
F Matrix multiplication
G The inverse of a matrix
H Simultaneous linear equations



OPENING PROBLEM

Aakriti owns a stationery shop. She sells two brands of pen, in three colours. Her sales for one week are shown in the table below.

	Brand	
Colour	*Pentex*	*Rollerball*
Blue	32	24
Black	25	16
Red	13	9



Pentex pens sell for \$1.19 each, and Rollerball pens sell for \$1.55 each.

Things to think about:

a How can we convert the table into a 3×2 quantities matrix **Q**?

b How can we display the prices in a price matrix **P**?

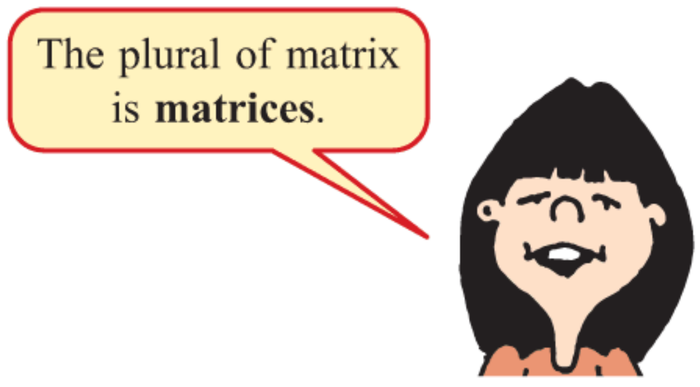
c How can we multiply the matrices **Q** and **P**? What does the matrix **QP** represent?

d Can you find the total revenue for Aakriti in pen sales for the week?

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

The vectors we have just studied are a special class of matrix, having just a single column. They are often called **column vectors** or **column matrices**.

In type, we generally use a bold capital letter such as **A** to represent a matrix. However, we can also use a bold lower case letter if we particularly want to indicate a column vector.



A MATRIX STRUCTURE

Matrices are used extensively to summarise the numerical information from a table. They are fundamental to all digital storage, and are hence one of the most used mathematical tools in existence.

For example, we can write these two tables of information as matrices by extracting the numbers and placing them in brackets.

Shopping list	
Bread	2 loaves
Juice	1 carton
Eggs	6
Cheese	1

Furniture inventory			
	chairs	tables	beds
Flat	6	1	2
Unit	9	2	3
House	10	3	4

Shopping list matrix $\mathbf{S} = \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$.

Furniture inventory matrix $\mathbf{F} = \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$.

Notice how the organisation of the data is maintained in matrix form.

MATRIX ORDER

The **order** of a matrix with m rows and n columns is $m \times n$.

If the matrix **A** has order $m \times n$, we can write

$\mathbf{A} = (a_{ij})$ where $i = 1, 2, 3, ..., m$

$j = 1, 2, 3, ..., n$

and a_{ij} is the element in the ith row, jth column.

By convention, the a_{ij} are labelled down then across.

$i \downarrow \begin{pmatrix} & \\ & \end{pmatrix}$

$\overrightarrow{j}$



For example:

- $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{pmatrix}$ has 2 rows and 3 columns, so is a 2×3 matrix.

  The element 3 is in row 1 and column 2, so $a_{12} = 3$.

  $\begin{pmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \end{pmatrix}$ ← row 1

  ↑ column 2

- $\mathbf{S} = \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$ has 4 rows and 1 column. **S** is a 4×1 **column matrix** or **column vector**.

- $\mathbf{F} = \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$ has 3 rows and 3 columns. **F** is a 3×3 **square matrix**.

- $\mathbf{R} = \begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$ has 1 row and 3 columns. **R** is a 1×3 **row matrix**.

Example 1 — Self Tutor

Lisa goes shopping at store A to buy 2 loaves of bread at \$2.65 each, 3 litres of milk at \$1.55 per litre, and one 500 g tub of butter at \$2.35.

a Represent the quantities purchased in a row matrix **Q**, and the costs in a column matrix **A**.

b When Lisa goes to a different supermarket (store B), she finds that the prices for the same items are \$2.25 for bread, \$1.50 for milk, and \$2.20 for butter.
Write the costs for both stores in a single costs matrix **C**.

a The quantities matrix is $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$

↑ bread ↑ milk ↑ butter

The costs matrix is $\mathbf{A} = \begin{pmatrix} 2.65 \\ 1.55 \\ 2.35 \end{pmatrix}$ ← bread, ← milk, ← butter

b We write the costs for each store in separate columns.

The new costs matrix is $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}$ ← bread, ← milk, ← butter

↑ store A ↑ store B

EXERCISE 12A

1 Write down the order of:

a $\begin{pmatrix} 5 & 1 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$

2 Suppose $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & 2 \\ -3 & 1 \end{pmatrix}$. State the value of:

a a_{12} **b** a_{21} **c** a_{31} **d** a_{32}

3 A grocery list consists of 2 loaves of bread, 1 kg of butter, 6 eggs, and 1 carton of cream. Each loaf of bread costs \$1.95, each kilogram of butter costs \$2.35, each egg costs \$0.45, and each carton of cream costs \$2.95.

a Construct a row matrix $\mathbf{Q}$ showing quantities.

b Construct a column matrix $\mathbf{P}$ showing prices.

c What is the significance of $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.45) + (1 \times 2.95)$?

4 A food processing factory produces cans of beans in three sizes: 200 g, 300 g, and 500 g. In February they produced respectively:

- 1000, 1500, and 1250 cans of each in week 1
- 1500, 1000, and 1000 cans of each in week 2
- 800, 2300, and 1300 cans of each in week 3
- 1200 cans of each in week 4.

a Construct a matrix $\mathbf{P}$ to show February's production levels.

b State the order of $\mathbf{P}$.

5 Over a long weekend holiday, a baker produced the following food items: On Friday he baked 40 dozen pies, 50 dozen pasties, 55 dozen rolls, and 40 dozen buns. On Saturday he baked 25 dozen pies, 65 dozen pasties, 30 dozen buns, and 44 dozen rolls. On Sunday he baked 40 dozen pasties, 40 dozen rolls, and 35 dozen of each of pies and buns. On Monday he baked 40 dozen pasties, 50 dozen buns, and 35 dozen of each of pies and rolls. Represent this information as a matrix, labelling each row and column.



B MATRIX EQUALITY

Two matrices are **equal** if they have the **same order** *and* the elements in corresponding positions are equal.

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i, j.$$

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then $a = w$, $b = x$, $c = y$, and $d = z$.

EXERCISE 12B

1 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$. Explain why:

a $\mathbf{A} \neq \mathbf{B}$ **b** $\mathbf{A} \neq \mathbf{C}$

2 Find x and y if:

a $\begin{pmatrix} x & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & y \end{pmatrix}$ **b** $\begin{pmatrix} -2 & x \\ x-1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ y & y+1 \end{pmatrix}$

c $\begin{pmatrix} x & x^2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} y & 4 \\ 3 & y+1 \end{pmatrix}$ **d** $\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} -y & x \\ x & -y \end{pmatrix}$

C ADDITION AND SUBTRACTION

MATRIX ADDITION

Thao has three stores: A, B, and C. Her stock levels for dresses, skirts, and blouses are given by the matrix:

	Store A	Store B	Store C	
	23	41	68	dresses
	28	39	79	skirts
	46	17	62	blouses

Some newly ordered stock has just arrived. 20 dresses, 30 skirts, and 50 blouses must be added to the stock levels of each store.

The new stock is summarised by the matrix: $\begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix}$

So, the new levels are: $\begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} + \begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix} = \begin{pmatrix} 43 & 61 & 88 \\ 58 & 69 & 109 \\ 96 & 67 & 112 \end{pmatrix}$

To **add** two matrices, they must be of the **same order**, and we **add corresponding elements**.

MATRIX SUBTRACTION

Suppose Thao's stock levels were $\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix}$ and her sales matrix was $\begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix}$.

Thao will be left with her original stock levels less what she has sold.

Clearly, we need to subtract corresponding elements:

$$\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix} - \begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix} = \begin{pmatrix} 14 & 39 & 13 \\ 11 & 12 & 13 \\ 21 & 9 & 15 \end{pmatrix}$$

To **subtract** matrices, they must be of the **same order**, and we **subtract corresponding elements**.

SUMMARY OF MATRIX ADDITION AND SUBTRACTION

- $\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- We can only add or subtract matrices of the same order.
- We add or subtract corresponding elements.
- The result of addition or subtraction is another matrix of the same order.

Example 2 — Self Tutor

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{C}$

a $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1+2 & 2+1 & 3+6 \\ 6+0 & 5+3 & 4+5 \end{pmatrix}$

$= \begin{pmatrix} 3 & 3 & 9 \\ 6 & 8 & 9 \end{pmatrix}$

b $\mathbf{A} + \mathbf{C}$ cannot be found as the matrices do not have the same order.

Example 3 — Self Tutor

If $\mathbf{A} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$,

find $\mathbf{A} - \mathbf{B}$.

$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 3-2 & 4-0 & 8-6 \\ 2-3 & 1-0 & 0-4 \\ 1-5 & 4-2 & 7-3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 & 2 \\ -1 & 1 & -4 \\ -4 & 2 & 4 \end{pmatrix}$

EXERCISE 12C

1 If $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -3 & 7 \\ -4 & -2 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} + \mathbf{B} + \mathbf{C}$ **c** $\mathbf{B} + \mathbf{C}$ **d** $\mathbf{C} + \mathbf{B} - \mathbf{A}$

2 If $\mathbf{P} = \begin{pmatrix} 3 & 5 & -11 \\ 10 & 2 & 6 \\ -2 & -1 & 7 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 17 & -4 & 3 \\ -2 & 8 & -8 \\ 3 & -4 & 11 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} - \mathbf{Q}$ **c** $\mathbf{Q} - \mathbf{P}$

3 A restaurant served 85 men, 92 women, and 52 children on Friday night. On Saturday night they served 102 men, 137 women, and 49 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women, and children served over the two nights.

4 David bought shares in five companies on Monday, and he sold them on Friday. The details are shown in the table alongside.

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

a Write down David's column matrix for:

i cost price **ii** selling price.

b What matrix operation is needed to find David's profit or loss on each type of share?

c Find David's profit or loss matrix.

5 In November, Lou sold 23 fridges, 17 stoves, and 31 microwave ovens. His partner Rose sold 19 fridges, 29 stoves, and 24 microwave ovens.

In December, Lou sold 18 fridges, 7 stoves, and 36 microwaves, and Rose sold 25 fridges, 13 stoves, and 19 microwaves.

a Write their sales for November as a 3×2 matrix.

b Write their sales for December as a 3×2 matrix.

c Write their total sales for November and December as a 3×2 matrix.

6 **a** If $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix}$, find $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$.

b Explain why $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

7 **a** For $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$, find $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.

b Prove that if $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are any 2×2 matrices, then $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$.

Hint: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

D SCALAR MULTIPLICATION

In the pantry there are 6 cans of tomatoes, 4 cans of beans, and 8 cans of fish. We represent this by the column vector $\mathbf{C} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$.

If we doubled the cans in the pantry, we would have $\begin{pmatrix} 12 \\ 8 \\ 16 \end{pmatrix}$ which is $\mathbf{C} + \mathbf{C}$ or $2\mathbf{C}$.

Notice that to get $2\mathbf{C}$ from $\mathbf{C}$ we simply multiply all the matrix elements by 2.

Likewise, trebling the cans in the pantry gives $3\mathbf{C} = \begin{pmatrix} 3 \times 6 \\ 3 \times 4 \\ 3 \times 8 \end{pmatrix} = \begin{pmatrix} 18 \\ 12 \\ 24 \end{pmatrix}$,

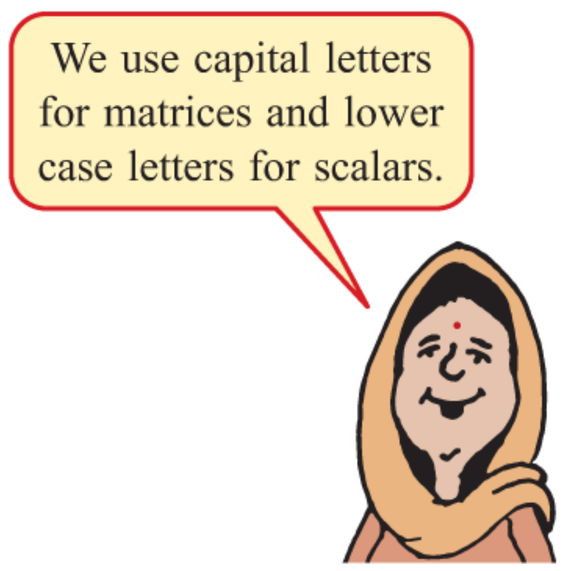
and halving them gives $\frac{1}{2}\mathbf{C} = \begin{pmatrix} \frac{1}{2} \times 6 \\ \frac{1}{2} \times 4 \\ \frac{1}{2} \times 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.



If $\mathbf{A} = (a_{ij})$ has order $m \times n$, and k is a scalar, then $k\mathbf{A} = (ka_{ij})$.

So, to find $k\mathbf{A}$, we multiply each element in $\mathbf{A}$ by k.

The result is another matrix of order $m \times n$.



Example 4 **Self Tutor**

If $\mathbf{A}$ is $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$, find:

a $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$

a
$$3\mathbf{A} = 3\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3\times 1 & 3\times 2 & 3\times 5 \\ 3\times 2 & 3\times 0 & 3\times 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 6 & 15 \\ 6 & 0 & 3 \end{pmatrix}$$

b
$$\tfrac{1}{2}\mathbf{A} = \tfrac{1}{2}\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}\times 1 & \frac{1}{2}\times 2 & \frac{1}{2}\times 5 \\ \frac{1}{2}\times 2 & \frac{1}{2}\times 0 & \frac{1}{2}\times 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} & 1 & 2\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$$

EXERCISE 12D

1 If $\mathbf{B} = \begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$, find:

a $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

2 If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$ **c** $2\mathbf{A} + \mathbf{B}$ **d** $3\mathbf{A} - \mathbf{B}$

3 A builder builds a block of 12 identical apartments. Each apartment is to contain 1 table, 4 chairs, 2 beds, and 1 wardrobe.

Let $\mathbf{A} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$ be the matrix representing the furniture in one apartment.

In terms of $\mathbf{A}$, what is the matrix representing the furniture in *all* apartments? Evaluate this matrix.

4 On weekdays, a video store finds that its average daily rentals are 75 DVD movies, 27 Blu-ray movies, and 102 games. On weekends, the average daily rentals are 43 Blu-ray movies, 136 DVD movies, and 129 games.

a Represent the data using *two* column matrices $\mathbf{A}$ and $\mathbf{B}$ with the structure alongside.

b Find $5\mathbf{A} + 2\mathbf{B}$ and explain what it represents.

$$\begin{pmatrix} \ldots. \\ \ldots. \\ \ldots. \end{pmatrix} \begin{matrix} \leftarrow \text{DVD movies} \\ \leftarrow \text{Blu-ray movies} \\ \leftarrow \text{games} \end{matrix}$$

5 Isabelle sells clothing made by four different companies A, B, C, and D.



Her usual monthly order is:

$$\begin{array}{c} \\ \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix} \end{array}$$

a Find Isabelle's total annual order assuming her usual order is placed each month.

b Find Isabelle's order, to the nearest whole number, if she:

i increases her usual order by 15%

ii decreases her usual order by 15%.

E MATRIX ALGEBRA

ZERO MATRICES

A **zero matrix** is a matrix in which all the elements are zero.

For example, the 2×2 zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and the 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If $\mathbf{A}$ is a matrix of any order and $\mathbf{O}$ is the corresponding **zero matrix**, then $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

For example: $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$.

NEGATIVE MATRICES

The **negative** matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is actually $-1\mathbf{A}$.

$-\mathbf{A}$ is obtained from $\mathbf{A}$ by reversing the sign of each element of $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, then $-\mathbf{A} = \begin{pmatrix} -1 \times 3 & -1 \times -1 \\ -1 \times 2 & -1 \times 4 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix}$

The addition of a matrix and its negative always produces a zero matrix.

$$\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$$

For example: $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

MATRIX ALGEBRA

We now compare our discoveries about matrices so far with ordinary algebra.

In the following table we assume that **A**, **B**, and **C** are matrices of the same order.

Ordinary algebra	*Matrix algebra*
If a and b are real numbers then $a+b$ is also a real number.	If **A** and **B** are matrices then $\mathbf{A}+\mathbf{B}$ is a matrix of the same order.
$a+b=b+a$	$\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ {commutative}
$(a+b)+c=a+(b+c)$	$(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ {associative}
$a+0=0+a=a$	$\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$
$a+(-a)=(-a)+a=0$	$\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}$
a half of a is $\frac{a}{2}$	a half of **A** is $\frac{1}{2}\mathbf{A}$

Example 5 — Self Tutor

Show that:

a if $\mathbf{X}+\mathbf{A}=\mathbf{B}$ then $\mathbf{X}=\mathbf{B}-\mathbf{A}$

b if $3\mathbf{X}=\mathbf{A}$ then $\mathbf{X}=\frac{1}{3}\mathbf{A}$

a
$$\begin{aligned}\mathbf{X}+\mathbf{A}&=\mathbf{B}\\ \therefore\ \mathbf{X}+\mathbf{A}+(-\mathbf{A})&=\mathbf{B}+(-\mathbf{A})\\ \therefore\ \mathbf{X}+\mathbf{O}&=\mathbf{B}-\mathbf{A}\\ \therefore\ \mathbf{X}&=\mathbf{B}-\mathbf{A}\end{aligned}$$

b
$$\begin{aligned}3\mathbf{X}&=\mathbf{A}\\ \therefore\ \tfrac{1}{3}(3\mathbf{X})&=\tfrac{1}{3}\mathbf{A}\\ \therefore\ 1\mathbf{X}&=\tfrac{1}{3}\mathbf{A}\\ \therefore\ \mathbf{X}&=\tfrac{1}{3}\mathbf{A}\end{aligned}$$

EXERCISE 12E

1 Simplify:

a $\mathbf{A}+2\mathbf{A}$ **b** $3\mathbf{B}-3\mathbf{B}$ **c** $\mathbf{C}-2\mathbf{C}$

d $-\mathbf{B}+\mathbf{B}$ **e** $2(\mathbf{A}+\mathbf{B})$ **f** $-(\mathbf{A}+\mathbf{B})$

g $-(2\mathbf{A}-\mathbf{C})$ **h** $3\mathbf{A}-(\mathbf{B}-\mathbf{A})$ **i** $\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})$

2 Find **X** in terms of **A**, **B**, and **C** if:

a $\mathbf{X}+\mathbf{B}=\mathbf{A}$ **b** $\mathbf{B}+\mathbf{X}=\mathbf{C}$ **c** $4\mathbf{B}+\mathbf{X}=2\mathbf{C}$

d $2\mathbf{X}=\mathbf{A}$ **e** $3\mathbf{X}=\mathbf{B}$ **f** $\mathbf{A}-\mathbf{X}=\mathbf{B}$

g $\frac{1}{2}\mathbf{X}=\mathbf{C}$ **h** $2(\mathbf{X}+\mathbf{A})=\mathbf{B}$ **i** $\mathbf{A}-4\mathbf{X}=\mathbf{C}$

3 Find **X** given that:

a $\mathbf{M}=\begin{pmatrix}1&2\\3&6\end{pmatrix}$ and $\frac{1}{3}\mathbf{X}=\mathbf{M}$ **b** $\mathbf{N}=\begin{pmatrix}2&-1\\3&5\end{pmatrix}$ and $4\mathbf{X}=\mathbf{N}$

c $\mathbf{A}=\begin{pmatrix}1&0\\-1&2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1&4\\-1&1\end{pmatrix}$, and $\mathbf{A}-2\mathbf{X}=3\mathbf{B}$.

F MATRIX MULTIPLICATION

THE PRODUCT OF A ROW MATRIX AND A COLUMN MATRIX

Suppose you go to a garden centre and buy:

- 4 cordylines at €13.50 each
- 3 camellias at €22.50 each
- 2 conifers at €26.30 each.



cordyline €13.50 | camellia €22.50 | conifer €26.30

Suppose we write the quantities in the row matrix $\mathbf{A} = \begin{pmatrix} 4 & 3 & 2 \end{pmatrix}$

and the costs in the column matrix $\mathbf{C} = \begin{pmatrix} 13.50 \\ 22.50 \\ 26.30 \end{pmatrix}$.

The total cost of the plants is given by $4 \times €13.50 + 3 \times €22.50 + 2 \times €26.30 = €174.10$.

Notice how each member in the row matrix **A** is multiplied by the corresponding member in the column matrix **C**, and the results are added together. This is how we define the matrix product **AC**.

More generally, for a 1×3 row matrix multiplied by a 3×1 column matrix, we could write:

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = ap + bq + cr$$

EXERCISE 12F.1

1 Determine:

a $\begin{pmatrix} 3 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 6 & -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \end{pmatrix}$

2 **a** Show that the sum of w, x, y, and z is given by $\begin{pmatrix} w & x & y & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

b Represent the *average* of w, x, y, and z in a similar way.

3 Lucy buys 4 shirts, 3 skirts, and 2 blouses costing \$27, \$35, and \$39 each respectively.

a Write down a quantities matrix **Q** and a price matrix **P**.

b Show how to use **P** and **Q** to determine the total cost of Lucy's clothes.

4 In the interschool public speaking competition, a first place is awarded 10 points, second place 6 points, third place 3 points, and fourth place 1 point. One school won 3 first places, 2 seconds, 4 thirds, and 2 fourths.



a Write down this information in terms of a points matrix **P** and a numbers matrix **N**.

b Show how to use **P** and **N** to find the total number of points awarded to the school.

MATRIX MULTIPLICATION

Consider again **Example 1** on page **299**.

Lisa needed 2 loaves of bread, 3 litres of milk, and 1 tub of butter.

We represented this by the quantities matrix $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$.

The prices for each store were summarised in the costs matrix $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}$.

To find the *total cost* of the items in each store, Lisa needs to multiply the number of items by their respective cost.

In Store A, a loaf of bread is \$2.65, a litre of milk is \$1.55, and a tub of butter is \$2.35, so the total cost is $2 \times \$2.65 + 3 \times \$1.55 + 1 \times \$2.35 = \$12.30$.

In Store B, a loaf of bread is \$2.25, a litre of milk is \$1.50, and a tub of butter is \$2.20, so the total cost is $2 \times \$2.25 + 3 \times \$1.50 + 1 \times \$2.20 = \$11.20$.

To do this using matrices, notice that:

$$\mathbf{QC} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \end{pmatrix}$$

total cost in store A: row $\mathbf{Q}$ × column 1; total cost in store B: row $\mathbf{Q}$ × column 2

1×3 ← the same → 3×2 ; resultant matrix: 1×2

Now suppose Lisa's friend Olu needs 1 loaf of bread, 2 litres of milk, and 2 tubs of butter.

The quantities matrix for both Lisa and Olu would be $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ (Lisa, Olu; bread, milk, butter).

Lisa's *total cost* at Store A is \$12.30, and at store B is \$11.20.

Olu's *total cost* at Store A is $1 \times \$2.65 + 2 \times \$1.55 + 2 \times \$2.35 = \$10.45$,
and at Store B is $1 \times \$2.25 + 2 \times \$1.50 + 2 \times \$2.20 = \$9.65$.

So, using matrices we require that

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \\ 10.45 & 9.65 \end{pmatrix}$$

row 1 × column 1; row 1 × column 2; row 2 × column 1; row 2 × column 2

2×3 ← the same → 3×2 ; resultant matrix: 2×2

Having observed the usefulness of multiplying matrices in the contextual examples above, we now define matrix multiplication more formally.

The **product** of an $m \times n$ matrix **A** with an $n \times p$ matrix **B**, is the $m \times p$ matrix **AB** in which the element in the rth row and cth column is the product of the rth row of **A** (as a row matrix) and the cth column of **B** (as a column matrix).

If $\mathbf{C} = \mathbf{AB}$ then $c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$

for each pair i and j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$.

The product **AB** exists *only* if the number of columns of **A** equals the number of rows of **B**.

For example:

If $\mathbf{A} = \underset{2 \times 2}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}$ and $\mathbf{B} = \underset{2 \times 2}{\begin{pmatrix} p & q \\ r & s \end{pmatrix}}$, then $\mathbf{AB} = \underset{2 \times 2}{\begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}}$.

If $\mathbf{P} = \underset{2 \times 3}{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}}$ and $\mathbf{Q} = \underset{3 \times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$, then $\mathbf{PQ} = \underset{2 \times 1}{\begin{pmatrix} ax+by+cz \\ dx+ey+fz \end{pmatrix}}$.

To get the matrix $\mathbf{C} = \mathbf{AB}$ we multiply **rows by columns**. The element c_{ij} is the product of the ith row of **A** and the jth column of **B**.

Example 6 — Self Tutor

For $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$, find:

a **AC** **b** **BC**

a **A** is 1×3 and **C** is 3×2 $\therefore$ **AC** is 1×2

✓

$$\mathbf{AC} = \begin{pmatrix} 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 1 + 3 \times 2 + 5 \times 1 & 1 \times 0 + 3 \times 3 + 5 \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 29 \end{pmatrix}$$

b **B** is 2×3 and **C** is 3×2 $\therefore$ **BC** is 2×2

✓

$$\mathbf{BC} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 1 + 3 \times 2 + 5 \times 1 & 1 \times 0 + 3 \times 3 + 5 \times 4 \\ 2 \times 1 + 1 \times 2 + 3 \times 1 & 2 \times 0 + 1 \times 3 + 3 \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 29 \\ 7 & 15 \end{pmatrix}$$



EXERCISE 12F.2

1 Explain why **AB** cannot be found for $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2 Suppose **A** is $2 \times n$ and **B** is $m \times 3$.

a Under what conditions can we find **AB**?

b If **AB** can be found, what is its order?

c Explain why **BA** cannot be found.

3 For $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 5 & 6 \end{pmatrix}$, find: **a** **AB** **b** **BA**

4 For $\mathbf{A} = \begin{pmatrix} 2 & 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, find: **a** **AB** **b** **BA**

5 Find:

a $\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 2 & -1 \end{pmatrix}$

c $\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

6 Answer the **Opening Problem** on page **298**.

7 At a fair, tickets for the Ferris wheel are £12.50 per adult and £9.50 per child. On the first day of the fair, 2375 adults and 5156 children ride the wheel. On the second day, 2502 adults and 3612 children ride the wheel.

a Write the prices as a 2×1 matrix **P**, and the numbers as a 2×2 matrix **N**.

b Find **NP** and interpret the resulting matrix.

c Find the total income for the two days.

8 You and your friend each go to your local hardware stores A and B to price items you need to purchase. You need to buy 1 hammer, 1 screwdriver, and 2 cans of white paint. Your friend needs 1 hammer, 2 screwdrivers, and 3 cans of white paint. The prices of these goods are:

	Hammer	*Screwdriver*	*Can of paint*
Store A	\$14	\$6	\$38
Store B	\$12	\$4	\$44



a Write the requirements matrix **R** as a 3×2 matrix.

b Write the prices matrix **P** as a 2×3 matrix.

c Find **PR**.

d Find:

i your costs at store A **ii** your friend's costs at store B.

e Do any of the elements of **PR** tell you and your friend the cheapest way to buy all your items? Explain your answer.

PROPERTIES OF MATRIX MULTIPLICATION

INVESTIGATION 1 MATRIX MULTIPLICATION

In this Investigation, we find the properties of 2×2 matrix multiplication which are like those of ordinary number multiplication, and those which are not.

What to do:

1 For ordinary arithmetic $2 \times 3 = 3 \times 2$, and in algebra $ab = ba$.
For matrices, does **AB** always equal **BA**?

Hint: Try $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$.

2 If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, find **AO** and **OA**.

3 Find **AB** for:

a $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

4 For all real numbers a, b, and c, we have the **distributive law** $a(b + c) = ab + ac$.

a Use any three 2×2 matrices **A**, **B**, and **C** to verify that $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

b Now let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.
Prove that in general, $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

c Use the matrices you chose in **a** to verify that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

d Prove that in general, $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

5 **a** If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $w = z = 1$ and $x = y = 0$ is a solution for any values of a, b, c, and d.

b For any real number a, we know that $a \times 1 = 1 \times a = a$.
Is there a matrix **I** such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices **A**?

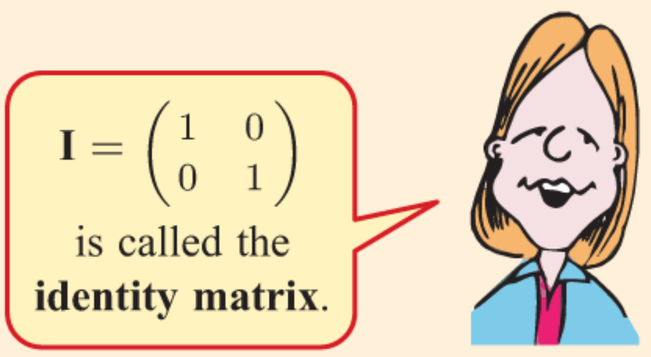
6 Suppose $\mathbf{A}^2 = \mathbf{AA} = \mathbf{A} \times \mathbf{A}$ and that $\mathbf{A}^3 = \mathbf{AAA}$.

a Find $\mathbf{A}^2$ if $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$. **b** Find $\mathbf{A}^3$ if $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$.

c If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ try to find $\mathbf{A}^2$.

d Under what conditions can we square a matrix?

7 Show that if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.



In the **Investigation** you should have found that:

Ordinary algebra	*Matrix algebra*
If a and b are real numbers then so is ab. {closure}	If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure}
$ab = ba$ for all a, b {commutative}	In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative}
$(ab)c = a(bc)$ for all a, b, c {associative}	$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ {associative}
$a0 = 0a = 0$ for all a	If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
$ab = 0 \Leftrightarrow a = 0$ or $b = 0$ {null factor law}	$\mathbf{AB}$ may be $\mathbf{O}$ without requiring $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.
$a(b + c) = ab + ac$ {distributive law}	$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ {distributive law}
$a \times 1 = 1 \times a = a$ {identity law}	If $\mathbf{I}$ is an **identity matrix** then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all $\mathbf{A}$. {identity law}
a^n exists for all $a \geqslant 0$ and $n \in \mathbb{R}$.	$\mathbf{A}^n$ exists provided $\mathbf{A}$ is square and $n \in \mathbb{Z}^+$.

Note that in general, $\mathbf{A}(k\mathbf{B}) = k(\mathbf{AB}) \neq k\mathbf{BA}$. We can change the order in which we multiply by a scalar, but we cannot reverse the order in which we multiply matrices.

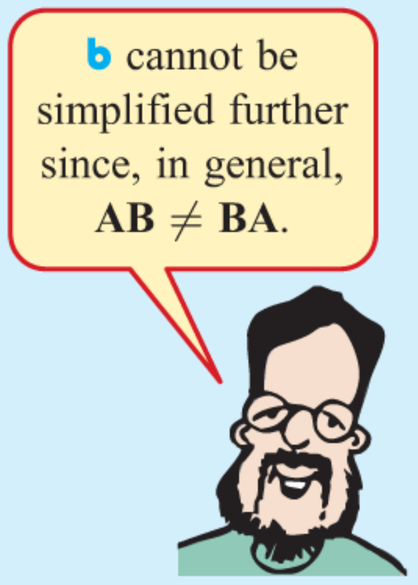
Example 7 — Self Tutor

Suppose $\mathbf{A}$ and $\mathbf{B}$ are square matrices with the same order. Expand and simplify:

a $(\mathbf{A} + 2\mathbf{I})^2$ **b** $(\mathbf{A} - \mathbf{B})^2$

a
$$\begin{aligned}
&(\mathbf{A} + 2\mathbf{I})^2 \\
&= (\mathbf{A} + 2\mathbf{I})(\mathbf{A} + 2\mathbf{I}) && \{\mathbf{A}^2 = \mathbf{AA} \text{ by definition}\} \\
&= (\mathbf{A} + 2\mathbf{I})\mathbf{A} + (\mathbf{A} + 2\mathbf{I})2\mathbf{I} && \{\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}\} \\
&= \mathbf{A}^2 + 2\mathbf{IA} + 2\mathbf{AI} + 4\mathbf{I}^2 && \{(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}\} \\
&= \mathbf{A}^2 + 2\mathbf{A} + 2\mathbf{A} + 4\mathbf{I} && \{\mathbf{AI} = \mathbf{IA} = \mathbf{A} \text{ and } \mathbf{I}^2 = \mathbf{I}\} \\
&= \mathbf{A}^2 + 4\mathbf{A} + 4\mathbf{I}
\end{aligned}$$

b
$$\begin{aligned}
&(\mathbf{A} - \mathbf{B})^2 \\
&= (\mathbf{A} - \mathbf{B})(\mathbf{A} - \mathbf{B}) && \{\mathbf{A}^2 = \mathbf{AA} \text{ by definition}\} \\
&= (\mathbf{A} - \mathbf{B})\mathbf{A} - (\mathbf{A} - \mathbf{B})\mathbf{B} && \{\mathbf{A}(\mathbf{B} - \mathbf{C}) = \mathbf{AB} - \mathbf{AC}\} \\
&= \mathbf{A}^2 - \mathbf{BA} - \mathbf{AB} + \mathbf{B}^2 && \{(\mathbf{B} - \mathbf{C})\mathbf{A} = \mathbf{BA} - \mathbf{CA}\}
\end{aligned}$$



Example 8 — Self Tutor

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A} + l\mathbf{I}$ where k and l are scalars.

$$\begin{aligned}
\mathbf{A}^3 &= \mathbf{A} \times \mathbf{A}^2 \\
&= \mathbf{A}(2\mathbf{A} + 3\mathbf{I}) \\
&= 2\mathbf{A}^2 + 3\mathbf{AI} \\
&= 2(2\mathbf{A} + 3\mathbf{I}) + 3\mathbf{A} \\
&= 7\mathbf{A} + 6\mathbf{I}
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}^4 &= \mathbf{A} \times \mathbf{A}^3 \\
&= \mathbf{A}(7\mathbf{A} + 6\mathbf{I}) \\
&= 7\mathbf{A}^2 + 6\mathbf{AI} \\
&= 7(2\mathbf{A} + 3\mathbf{I}) + 6\mathbf{A} \\
&= 20\mathbf{A} + 21\mathbf{I}
\end{aligned}$$

EXERCISE 12F.3

1 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{A} + \mathbf{I})$ **b** $(\mathbf{B} + 2\mathbf{I})\mathbf{B}$ **c** $\mathbf{A}(\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I})$

d $\mathbf{A}(\mathbf{A}^2 + \mathbf{A} - 2\mathbf{I})$ **e** $(\mathbf{A} + \mathbf{B})(\mathbf{C} + \mathbf{D})$ **f** $(\mathbf{A} + \mathbf{B})^2$

g $(\mathbf{A} + \mathbf{B})(\mathbf{A} - \mathbf{B})$ **h** $(\mathbf{A} + \mathbf{I})^2$ **i** $(3\mathbf{I} - \mathbf{B})^2$

2 **a** If $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A} + l\mathbf{I}$ where k and l are scalars.

b If $\mathbf{B}^2 = 2\mathbf{I} - \mathbf{B}$, find $\mathbf{B}^3$, $\mathbf{B}^4$, and $\mathbf{B}^5$ in linear form.

c If $\mathbf{C}^2 = 4\mathbf{C} - 3\mathbf{I}$, find $\mathbf{C}^3$ and $\mathbf{C}^5$ in linear form.

3 **a** If $\mathbf{A}^2 = \mathbf{I}$, simplify:

i $\mathbf{A}(\mathbf{A} + 2\mathbf{I})$ **ii** $(\mathbf{A} - \mathbf{I})^2$ **iii** $\mathbf{A}(\mathbf{A} + 3\mathbf{I})^2$

b If $\mathbf{A}^3 = \mathbf{I}$, simplify $\mathbf{A}^2(\mathbf{A} + \mathbf{I})^2$.

c If $\mathbf{A}^2 = \mathbf{O}$, simplify:

i $\mathbf{A}(2\mathbf{A} - 3\mathbf{I})$ **ii** $\mathbf{A}(\mathbf{A} + 2\mathbf{I})(\mathbf{A} - \mathbf{I})$ **iii** $\mathbf{A}(\mathbf{A} + \mathbf{I})^3$

4 **a** If $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, determine $\mathbf{A}^2$.

b Consider a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$. Comment on the following argument:

$$\mathbf{A}^2 = \mathbf{A}$$
$$\therefore \mathbf{A}^2 - \mathbf{A} = \mathbf{O}$$
$$\therefore \mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$$
$$\therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O}$$
$$\therefore \mathbf{A} = \mathbf{O} \text{ or } \mathbf{I}$$

c Find *all* 2×2 matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. **Hint:** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5 Give *one* example which shows that "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a *false* statement.

Example 9

Self Tutor

For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$,
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = a\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\therefore \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$
$$\therefore \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ 3a & 4a+b \end{pmatrix}$$

Thus $a + b = 7$ and $2a = 10$
$\therefore a = 5$ and $b = 2$

Checking for consistency: $3a = 3(5) = 15$ ✓ $4a + b = 4(5) + (2) = 22$ ✓

6 For each matrix $\mathbf{A}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

a $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$

7 **a** Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$, write $\mathbf{A}^2$ in the linear form $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$, $p, q \in \mathbb{R}$.

b Hence write $\mathbf{A}^3$ in the linear form $r\mathbf{A} + s\mathbf{I}$, $r, s \in \mathbb{R}$.

c Write $\mathbf{A}^4$ in linear form.

G THE INVERSE OF A MATRIX

The real numbers 5 and $\frac{1}{5}$ are called **multiplicative inverses** because when they are multiplied together, the result is the multiplicative identity 1: $5 \times \frac{1}{5} = \frac{1}{5} \times 5 = 1$

For the matrices $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$, we notice that $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

We say that $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ are *multiplicative inverses* of each other.

The **multiplicative inverse** of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

To find the multiplicative inverse of a matrix $\mathbf{A}$, we need a matrix which, when multiplied by $\mathbf{A}$, gives the identity matrix $\mathbf{I}$.

THE INVERSE OF A 2×2 MATRIX

Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{A}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

$\therefore \quad \mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \mathbf{I}$

$\therefore \quad \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \quad \begin{cases} aw + by = 1 & \ldots. (1) \\ cw + dy = 0 & \ldots. (2) \end{cases}$ and $\begin{cases} ax + bz = 0 & \ldots. (3) \\ cx + dz = 1 & \ldots. (4) \end{cases}$

Solving (1) and (2) simultaneously for w and y gives: $w = \dfrac{d}{ad - bc}$ and $y = \dfrac{-c}{ad - bc}$.

Solving (3) and (4) simultaneously for x and z gives: $x = \dfrac{-b}{ad - bc}$ and $z = \dfrac{a}{ad - bc}$.

So, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$, then $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

In this case $\mathbf{A}^{-1}\mathbf{A} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\therefore \quad \mathbf{A}^{-1}\mathbf{A} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & bd-bd \\ ac-ac & -bc+ad \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \mathbf{I} \quad \text{also,}$$

so $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

Just as the real number 0 does not have a multiplicative inverse, some matrices do not have a multiplicative inverse. This occurs when $ad - bc = 0$.

For the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- The value $ad - bc$ is called the **determinant** of matrix $\mathbf{A}$, denoted $\det \mathbf{A}$ or $|\mathbf{A}|$.
- If $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is **invertible** or **non-singular**, and $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- If $\det \mathbf{A} = 0$, then $\mathbf{A}$ is **singular**, and $\mathbf{A}^{-1}$ does not exist.

Example 10 — **Self Tutor**

Find, if it exists, the inverse matrix of:

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$ b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

$\therefore \det \mathbf{A} = 5(4) - 6(3) = 2$

$\therefore \mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -6 \\ -3 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

$\therefore \det \mathbf{B} = 6(-2) - 3(-4)$

$= -12 + 12$

$= 0$

$\therefore \mathbf{B}^{-1}$ does not exist.

EXERCISE 12G.1

1 Find $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$.

2 Find $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$.

3 Find $|\mathbf{A}|$ for $\mathbf{A}$ equal to:

a $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ b $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ c $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

4 Find $\det \mathbf{B}$ for $\mathbf{B}$ equal to:

a $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$ b $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ c $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ d $\begin{pmatrix} a & -a \\ 1 & a \end{pmatrix}$

5 Suppose $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 4 & 7 \end{pmatrix}$.

a Find $|\mathbf{A}|$.

b Hence find the inverse matrix $\mathbf{A}^{-1}$.

c Check that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

6 Find, if it exists, the inverse matrix of:

a $\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

e $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}$ **g** $\begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix}$ **h** $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$

i $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$ **j** $\begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix}$ **k** $\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ **l** $\begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix}$

Example 11 — Self Tutor

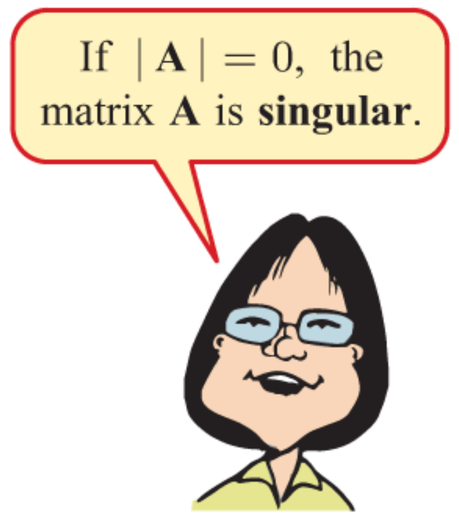
Suppose $\mathbf{A} = \begin{pmatrix} 4 & k \\ 2 & -1 \end{pmatrix}$.

Find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1} = \frac{1}{-4-2k}\begin{pmatrix} -1 & -k \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2k+4} & \frac{k}{2k+4} \\ \frac{2}{2k+4} & \frac{-4}{2k+4} \end{pmatrix}$$

$\mathbf{A}^{-1}$ exists provided that $2k + 4 \neq 0$

$\therefore \quad k \neq -2$



7 For each matrix $\mathbf{A}$, find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

a $\mathbf{A} = \begin{pmatrix} k & 1 \\ -6 & 2 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 0 & k \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ 1 & k \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} k-2 & k \\ -3 & k \end{pmatrix}$ **e** $\mathbf{A} = \begin{pmatrix} k^2 & k-1 \\ 2k & 1 \end{pmatrix}$ **f** $\mathbf{A} = \begin{pmatrix} k+1 & 2 \\ k^2+2 & 3k \end{pmatrix}$

PROPERTIES OF MATRIX DETERMINANTS

INVESTIGATION 2 **PROPERTIES OF MATRIX DETERMINANTS**

In this Investigation we consider the relationships between:

- $\det \mathbf{A}$ and $\det(k\mathbf{A})$ where k is a scalar
- $\det \mathbf{A}$, $\det \mathbf{B}$, and $\det(\mathbf{AB})$.

What to do:

1 For $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$, find:

a $\det \mathbf{A}$ **b** $\det(-\mathbf{A})$ **c** $\det(2\mathbf{A})$ **d** $\det(3\mathbf{A})$

2 Repeat **1** with another 2×2 matrix $\mathbf{A}$ of your choosing.

3 Predict the relationship between $\det \mathbf{A}$ and $\det(k\mathbf{A})$ for any scalar k.

4 Prove your result in **3** for any 2×2 matrix by using $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5 Suppose $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$. Find:

a $\det \mathbf{A}$ **b** $\det \mathbf{B}$ **c** $\det(\mathbf{AB})$

6 Repeat **5** several times with different 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$ of your choosing.

7 Predict the relationship between $\det \mathbf{A}$, $\det \mathbf{B}$, and $\det(\mathbf{AB})$ for square matrices $\mathbf{A}$ and $\mathbf{B}$.

8 Prove your result in **7** for any 2×2 matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

From the **Investigation** you should have found that:

For any 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$:

- $\det(k\mathbf{A}) = k^2 \det \mathbf{A}$ for any $k \in \mathbb{R}$
- $\det(\mathbf{AB}) = \det \mathbf{A} \times \det \mathbf{B}$.

THE INVERSE OF A 3×3 OR LARGER SQUARE MATRIX

For the inverse of a matrix $\mathbf{A}$ to exist, it must satisfy $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ where $\mathbf{A}$, $\mathbf{A}^{-1}$, and $\mathbf{I}$ are square matrices of the same order, and $\mathbf{I}$ is the identity matrix with 1s along its leading diagonal and 0s everywhere else.

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \leftarrow \text{leading diagonal}$$

There is no simple rule for finding the inverse of a 3×3 or larger square matrix, so we instead use technology.

Like 2×2 matrices, the inverse of a larger square matrix $\mathbf{A}$ will only exist if $\det \mathbf{A} \neq 0$. However, we only consider 2×2 determinants in this course.



GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 12G.2

1 Find $\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} -11 & 9 & 15 \\ -1 & 1 & 1 \\ 8 & -6 & -10 \end{pmatrix}$ and hence the inverse of $\begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 2 \\ 1 & -3 & 1 \end{pmatrix}$.

2 Use technology to find the inverse of:

a $\begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 3 & 2 \\ -1 & 5 & -1 \\ 0 & 3 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 13 & 43 & -11 \\ 16 & 9 & 27 \\ -8 & 31 & -13 \end{pmatrix}$

d $\begin{pmatrix} 1.61 & 4.32 & 6.18 \\ 0.37 & 6.02 & 9.41 \\ 7.12 & 5.31 & 2.88 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 1 & 4 & 0 \\ 1 & 2 & 0 & 5 \end{pmatrix}$ **f** $\begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 & 0 \\ 1 & 2 & 0 & 1 & 4 \\ 2 & 1 & 0 & 1 & 5 \\ 3 & 0 & 1 & 2 & 1 \end{pmatrix}$

Check your answers using matrix multiplication.

H SIMULTANEOUS LINEAR EQUATIONS

We can solve $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$ algebraically to get $x = 5,\ y = -2$.

Notice that this system can be written as a matrix equation $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$.

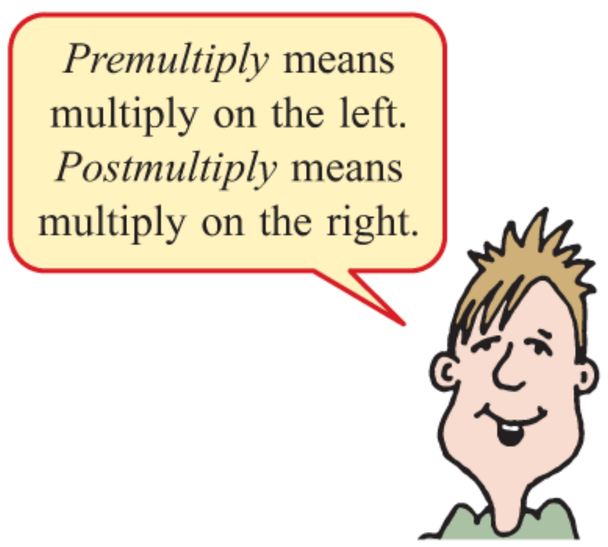
The solution $x = 5,\ y = -2$ is easily checked as

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2(5) + 3(-2) \\ 5(5) + 4(-2) \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix} \quad \checkmark$$

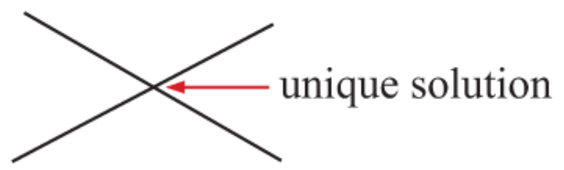
In general, a system of linear equations can be written in the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{x}$ is the vector of unknowns, and $\mathbf{b}$ is a vector of constants.

Provided the inverse matrix $\mathbf{A}^{-1}$ exists, we can solve the matrix equation $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$ by premultiplying each side by $\mathbf{A}^{-1}$. We obtain:

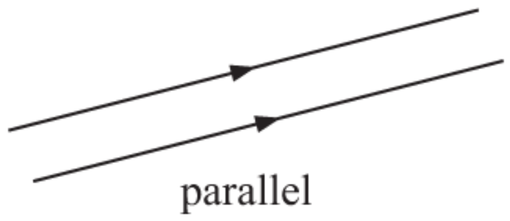
$$\begin{aligned} & \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b} \\ \therefore\ & (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} && \{\text{associativity}\} \\ \therefore\ & \mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b} && \{\text{inverse matrix definition}\} \\ \text{and so}\ & \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} && \{\text{identity law}\} \end{aligned}$$



So, if the matrix of coefficients $\mathbf{A}$ is invertible, then calculating $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ gives a unique solution to the pair of linear equations. This indicates that the lines intersect at a single point.



However, if the matrix of coefficients $\mathbf{A}$ is singular, we cannot calculate $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. This indicates either that the lines are parallel and there are no solutions, or that the lines are coincident and there are infinitely many solutions.



parallel
(no solution)



coincident
(infinitely many solutions)

Example 12 — Self Tutor

a If $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

b Write the system $\begin{cases} 2x + 3y = 4 \\ 5x + 4y = 17 \end{cases}$ in matrix form.

c Hence solve the simultaneous linear equations.

a $\det \mathbf{A} = 2(4) - 3(5)$
$= -7$

$\therefore\ \mathbf{A}^{-1} = -\frac{1}{7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}$

b In matrix form, the system is
$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$.
This has the form $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{x}$ is a vector of unknowns and $\mathbf{b}$ is a vector of constants.

c Premultiplying by $\mathbf{A}^{-1}$, $\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$\therefore \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = -\tfrac{1}{7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 17 \end{pmatrix}$$

$$= -\tfrac{1}{7}\begin{pmatrix} -35 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$\therefore \quad x = 5$ and $y = -2$.

EXERCISE 12H

1 Convert into matrix equations:

a $\begin{cases} 3x - y = 8 \\ 2x + 3y = 6 \end{cases}$ **b** $\begin{cases} 4x - 3y = 11 \\ 3x + 2y = -5 \end{cases}$ **c** $\begin{cases} 3a - b = 6 \\ 2a + 7b = -4 \end{cases}$

2 Use matrix algebra to solve the system:

a $\begin{cases} 2x - y = 6 \\ x + 3y = 14 \end{cases}$ **b** $\begin{cases} 5x - 4y = 5 \\ 2x + 3y = -13 \end{cases}$ **c** $\begin{cases} x - 2y = 7 \\ 5x + 3y = -2 \end{cases}$

d $\begin{cases} 3x + 5y = 4 \\ 2x - y = 11 \end{cases}$ **e** $\begin{cases} 4x - 7y = 8 \\ 3x - 5y = 0 \end{cases}$ **f** $\begin{cases} 7x + 11y = 18 \\ 11x - 7y = -11 \end{cases}$

3 **a** Consider the system $\begin{cases} 2x - 3y = 8 \\ 4x - y = 11 \end{cases}$.

i Write the equations in the form $\mathbf{A}\mathbf{x} = \mathbf{b}$, and find $\det \mathbf{A}$.

ii Does the system have a unique solution? If so, find it.

b Consider the system $\begin{cases} 2x + ky = 8 \\ 4x - y = 11 \end{cases}$.

i Write the system in the form $\mathbf{A}\mathbf{x} = \mathbf{b}$, and find $\det \mathbf{A}$.

ii For what value(s) of k does the system have a unique solution? Find the unique solution.

iii Find k when the system does not have a unique solution. How many solutions does the system have in this case?

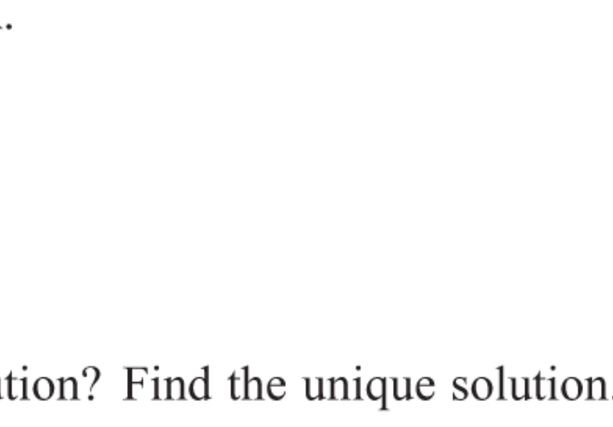
Example 13 **Self Tutor**

A car rental company owns 150 vehicles in total, including x of model X, y of model Y, and z of model Z.

In yard A there are 46 vehicles including 20% of the model X fleet, 40% of the model Y fleet, and 30% of the model Z fleet.

In yard B there are 54 vehicles including 40% of the model X fleet, 20% of the model Y fleet, and 50% of the model Z fleet.

How many vehicles of each type does the rental company have?



There are 150 vehicles in total, so $x + y + z = 150$

Using yard A, $0.2x + 0.4y + 0.3z = 46$

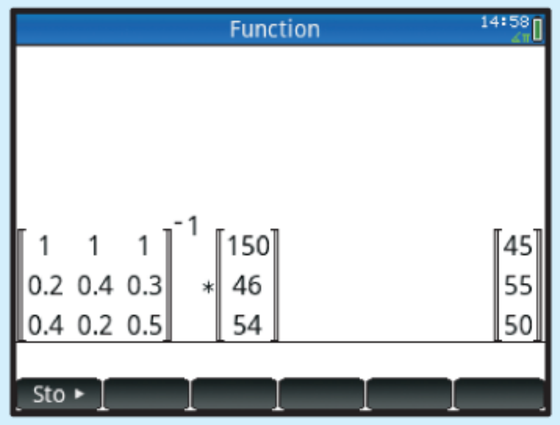
Using yard B, $0.4x + 0.2y + 0.5z = 54$

The system of three equations can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & 1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 150 \\ 46 \\ 54 \end{pmatrix}$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0.2 & 0.4 & 0.3 \\ 0.4 & 0.2 & 0.5 \end{pmatrix}^{-1} \begin{pmatrix} 150 \\ 46 \\ 54 \end{pmatrix}$$

$$= \begin{pmatrix} 45 \\ 55 \\ 50 \end{pmatrix} \quad \{\text{using technology}\}$$



The company has 45 of model X, 55 of model Y, and 50 of model Z.

4 Write as a matrix equation:

a $\begin{cases} x - y - z = 2 \\ x + y + 3z = 7 \\ 9x - y - 3z = -1 \end{cases}$ **b** $\begin{cases} 2x + y - z = 3 \\ y + 2z = 6 \\ x - y + z = 13 \end{cases}$ **c** $\begin{cases} a + b - c = 7 \\ a - b + c = 6 \\ 2a + b - 3c = -2 \end{cases}$

5 Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ -1 & 2 & 1 \\ 0 & 6 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 7 & -3 \\ -1 & -2 & 1 \\ 6 & 12 & -5 \end{pmatrix}$.

a Calculate **AB**.

b *Hence* solve the system $\begin{cases} 4a + 7b - 3c = -8 \\ -a - 2b + c = 3 \\ 6a + 12b - 5c = -15 \end{cases}$.

6 Let $\mathbf{M} = \begin{pmatrix} 5 & 3 & -7 \\ -1 & -3 & 3 \\ -3 & -1 & 5 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 3 & 2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$.

a Calculate **MN**.

b *Hence* solve the system $\begin{cases} 3u + 2v + 3w = 18 \\ u - v + 2w = 6 \\ 2u + v + 3w = 16 \end{cases}$.

7 Use matrix methods and technology to solve:

a $\begin{cases} 3x + 2y - z = 14 \\ x - y + 2z = -8 \\ 2x + 3y - z = 13 \end{cases}$ **b** $\begin{cases} x - y - 2z = 4 \\ 5x + y + 2z = -6 \\ 3x - 4y - z = 17 \end{cases}$ **c** $\begin{cases} x + 3y - z = 15 \\ 2x + y + z = 7 \\ x - y - 2z = 0 \end{cases}$

d $\begin{cases} x + y + z = 6 \\ 2x + 4y + z = 5 \\ 2x + 3y + z = 6 \end{cases}$ **e** $\begin{cases} x + 4y + 11z = 7 \\ x + 6y + 17z = 9 \\ x + 4y + 8z = 4 \end{cases}$ **f** $\begin{cases} 2x - y + 3z = 17 \\ 2x - 2y - 5z = 4 \\ 3x + 2y + 2z = 10 \end{cases}$

g $\begin{cases} x + 2y - z = 23 \\ x - y + 3z = -23 \\ 7x + y - 4z = 62 \end{cases}$ **h** $\begin{cases} 10x - y + 4z = -9 \\ 7x + 3y - 5z = 89 \\ 13x - 17y + 23z = -309 \end{cases}$ **i** $\begin{cases} 1.3x + 2.7y - 3.1z = 8.2 \\ 2.8x - 0.9y + 5.6z = 17.3 \\ 6.1x + 1.4y - 3.2z = -0.6 \end{cases}$

8 Westwood School bought two footballs, one baseball, and three basketballs for a total cost of \$135. Sequoia School bought three footballs, two baseballs, and a basketball for \$121.50. Lamar School bought five footballs and two basketballs for \$156.

a State clearly what the variables x, y, and z must represent if this situation is to be described by the set of equations: $\begin{cases} 2x + y + 3z = 135 \\ 3x + 2y + z = 121.50 \\ 5x + 2z = 156 \end{cases}$.

b East Park International School needs 4 footballs and 5 baseballs, and wishes to order as many basketballs as they can afford. If the total sports budget is \$470, how many basketballs will they be able to purchase?

9 Manager, clerks, and labourers are paid according to an industry standard. The salaries for three companies paying to the industry standard are summarised in the table:

Company	*Managers*	*Clerks*	*Labourers*	*Total salary bill*
Xenon	2	3	8	€903 000
Xanda	1	5	4	€749 000
Xylon	1	2	11	€865 000

a If x, y, and z represent the salaries (in thousands of euros) for managers, clerks, and labourers respectively, show that the above information can be represented by a system of three equations.

b Solve the system of equations in **a**.

c Determine the total salary bill for Xudu which employs 3 managers, 8 clerks, and 37 labourers.

10 A health food company uses cashews, macadamias, and Brazil nuts to make three different mixes. The table below shows the weight in hundreds of grams of each kind of nut required to make one kilogram of mix, and the production cost for each mix.

	Mix A	*Mix B*	*Mix C*
Cashews	5	2	6
Macadamias	3	4	1
Brazil nuts	2	4	3
Cost	\$12.50	\$12.40	\$11.70

a Find the cost per kilogram of each kind of nut.

b Find the cost of producing a mix containing 400 grams of cashews, 200 grams of macadamias, and 400 grams of Brazil nuts.

11 Grade 12 at Klondike High has 76 students in total across three classes. There are p students in class P, q in class Q, and r in class R.

- One third of P, one third of Q, and two fifths of R study Chemistry.
- One half of P, two thirds of Q, and one fifth of R study Physics.
- One quarter of P, one third of Q, and three fifths of R study Geography.
- In total, 27 students study Chemistry, 35 study Physics, and 30 study Geography.

Find the number of students in each class.

12 An asteroid orbits the sun in an elliptical path. If the sun is at the origin of a Cartesian coordinate system, the equation of the path is:

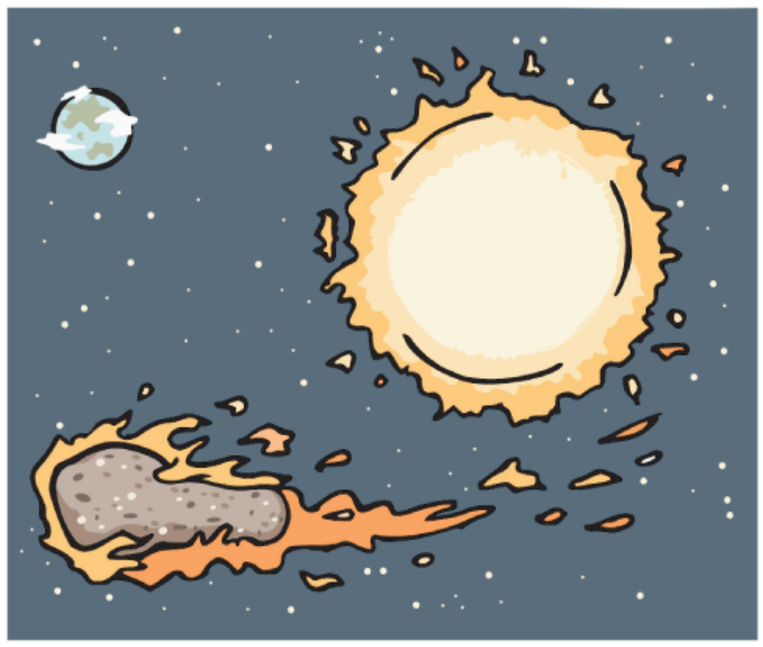
$$x^2 + axy + by^2 + cx + dy + e = 0$$

The distances are measured in astronomical units, where 1 astronomical unit $\approx 1.50 \times 10^8$ km.

An astronomer observes the asteroid at 5 positions:

$(-1.03,\ 2.164)$, $(-0.56,\ -1.868)$, $(0.38,\ -1.668)$, $(1.17,\ 4.876)$, and $(2.89,\ 1.019)$.

Find the equation of the asteroid's orbit.



ACTIVITY CRYPTOGRAPHY

Cryptography is the study of encoding and decoding messages. Cryptography was first developed for the military to send secret messages. Today it is also used to maintain privacy when information is transmitted on public communication services such as the internet.

To send a coded message, it must first be **encrypted** into code called **ciphertext**. When the recipient wishes to read the message, the ciphertext must be **deciphered**.

A simple method for encrypting messages is to use matrix addition or multiplication. The messages are then deciphered using either matrix subtraction or an inverse matrix.

Suppose the letters of the alphabet are assigned integer values, with Z assigned 0 as shown below:

Letter	A	B	C	D	E	F	G	H	I	J	K	L	M
Value	1	2	3	4	5	6	7	8	9	10	11	12	13

Letter	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
Value	14	15	16	17	18	19	20	21	22	23	24	25	0

The word SEND can be written as the string of numbers 19 5 14 4 which we can write in 2×2 matrix form $\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix}$.

Now suppose we encrypt the message by adding the matrix $\begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix}$.

$$\begin{pmatrix} 19 & 5 \\ 14 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 7 \\ 13 & 5 \end{pmatrix} = \begin{pmatrix} 21 & 12 \\ 27 & 9 \end{pmatrix}$$

Before this matrix can be transmitted all of its numbers must be written in **modulo** 26, or mod 26. This means that any number not in the range 0 to 25 is adjusted to be in it by adding or subtracting multiples of 26.

The matrix to be sent is therefore $\begin{pmatrix} 21 & 12 \\ 1 & 9 \end{pmatrix}$, and this is sent as the string 21 12 1 9. By itself, the string says ULAI, which has no apparent meaning.

The message SEND MONEY PLEASE could be broken into groups of four letters, and each group is then encoded.

SEND|MONE|YPLE|ASEE ← repeat the last letter to make group of 4. This is a dummy letter.

For MONE the matrix required is $\begin{pmatrix}13&15\\14&5\end{pmatrix}+\begin{pmatrix}2&7\\13&5\end{pmatrix}=\begin{pmatrix}15&22\\27&10\end{pmatrix}\equiv\begin{pmatrix}15&22\\1&10\end{pmatrix}\pmod{26}$

For YPLE the matrix required is $\begin{pmatrix}25&16\\12&5\end{pmatrix}+\begin{pmatrix}2&7\\13&5\end{pmatrix}=\begin{pmatrix}27&23\\25&10\end{pmatrix}\equiv\begin{pmatrix}1&23\\25&10\end{pmatrix}\pmod{26}$

For ASEE the matrix required is $\begin{pmatrix}1&19\\5&5\end{pmatrix}+\begin{pmatrix}2&7\\13&5\end{pmatrix}=\begin{pmatrix}3&26\\18&10\end{pmatrix}\equiv\begin{pmatrix}3&0\\18&10\end{pmatrix}\pmod{26}$

So, the whole message is 21 12 1 9 15 22 1 10 1 23 25 10 3 0 18 10

The person decoding the message needs to know in advance to subtract $\begin{pmatrix}2&7\\13&5\end{pmatrix}$ from each matrix of numbers, in order to decipher the message.

What to do:

1 Perform the matrix subtractions to check that the original message is obtained.

2 Use the code given to decode the message:

21	12	1	9	22	15	18	25	20	22	2	21	21	1	2	25	10	12
0	20	23	1	21	20	8	1	21	10	15	2	5	23	3	6	12	4

3 Create your own matrix addition code. Encrypt a short message. Supply the decoding matrix to a friend so that he or she can decode it.

4 A code is **broken** when someone discovers how the messages can be decoded. Breaking codes involving matrix addition is relatively easy, but breaking codes involving matrix multiplication is more difficult.

Consider the encryption matrix $\begin{pmatrix}2&3\\1&2\end{pmatrix}$.

The word SEND is encoded as

$$\begin{pmatrix}19&5\\14&4\end{pmatrix}\begin{pmatrix}2&3\\1&2\end{pmatrix}=\begin{pmatrix}43&67\\32&50\end{pmatrix}\equiv\begin{pmatrix}17&15\\6&24\end{pmatrix}\pmod{26}.$$

a What is the coded form of SEND MONEY PLEASE?

b What matrix needs to be supplied to the receiver so that the message can be deciphered? Check your answer by decoding the message.

c Create your own matrix multiplication code using a matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$ where $ad-bc=1$.

d What are the problems in using a 2×2 matrix when $ad-bc\neq1$? How can these problems be overcome?

5 Research **Hill ciphers** and explain how they differ from the methods given previously.

REVIEW SET 12A

1 If $\mathbf{A}=\begin{pmatrix}3&2\\0&-1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1&0\\-2&4\end{pmatrix}$, find:

a $\mathbf{A}+\mathbf{B}$ **b** $3\mathbf{A}$ **c** $-2\mathbf{B}$ **d** $\mathbf{A}-\mathbf{B}$

e $\mathbf{B}-2\mathbf{A}$ **f** $3\mathbf{A}-2\mathbf{B}$ **g** $\mathbf{AB}$ **h** $\mathbf{BA}$

i $\mathbf{A}^{-1}$ **j** $\mathbf{A}^2$ **k** $\mathbf{ABA}$ **l** $(\mathbf{AB})^{-1}$

2 Find a, b, c, and d if:

a $\begin{pmatrix} a & b-2 \\ c & d \end{pmatrix} = \begin{pmatrix} -a & 3 \\ 2-c & -4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 2a \\ b & -2 \end{pmatrix} + \begin{pmatrix} b & -a \\ c & d \end{pmatrix} = \begin{pmatrix} a & 2 \\ 2 & 6 \end{pmatrix}$

3 Write **Y** in terms of **A**, **B**, and **C**:

a $\mathbf{B} - \mathbf{Y} = \mathbf{A}$ **b** $2\mathbf{Y} + \mathbf{C} = \mathbf{A}$ **c** $\mathbf{AY} = \mathbf{B}$

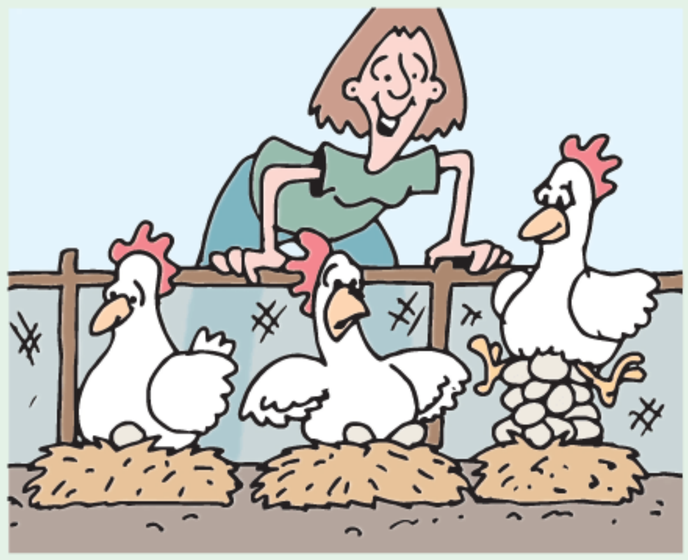
d $\mathbf{YB} = \mathbf{C}$ **e** $\mathbf{C} - \mathbf{AY} = \mathbf{B}$ **f** $\mathbf{AY}^{-1} = \mathbf{B}$

4 Susan keeps 3 hens in a pen. She calls them Anya, Betsy, and Charise. Each week the hens lay eggs according to the matrix **L**.

Write, in terms of **L**, a matrix to describe:

a the eggs laid by the hens over a 4 week period

b the eggs each hen loses each fortnight when Susan collects the eggs.



5 Let $\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 4 & -1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -7 & 9 \\ 9 & -3 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 1 \end{pmatrix}$.

Evaluate, if possible:

a $2\mathbf{A} - 2\mathbf{B}$ **b** $\mathbf{AC}$ **c** $\mathbf{CB}$

6 Given that all matrices are 2×2 and **I** is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{I} - \mathbf{A})$ **b** $(\mathbf{A} - \mathbf{B})(\mathbf{B} + \mathbf{A})$ **c** $(2\mathbf{A} - \mathbf{I})^2$

7 If $\mathbf{A}^2 = 5\mathbf{A} + 2\mathbf{I}$, write $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $r\mathbf{A} + s\mathbf{I}$.

8 If $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

9 Find, if possible, the inverse matrix of:

a $\begin{pmatrix} 6 & 8 \\ 5 & 7 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$ **c** $\begin{pmatrix} 11 & 5 \\ -6 & -3 \end{pmatrix}$

10 For what values of k does $\begin{cases} x + 4y = 2 \\ kx + 3y = -6 \end{cases}$ have a unique solution?

11 Solve using an inverse matrix:

a $\begin{cases} 3x - 4y = 2 \\ 5x + 2y = -1 \end{cases}$ **b** $\begin{cases} 4x - y = 5 \\ 2x + 3y = 9 \end{cases}$ **c** $\begin{cases} 3x - y + 2z = 3 \\ 2x + 3y - z = -3 \\ x - 2y + 3z = 2 \end{cases}$

12 When a rock is thrown from the top of a cliff, its distance above sea level after t seconds is given by $s(t) = at^2 + bt + c$. After 1 second the rock is 63 m above sea level, after 2 seconds 72 m, and after 7 seconds 27 m.

a Find a, b, and c.

b Find the height of the cliff.

c Find the time taken for the rock to reach the water below.

13 Hung, Quan, and Ariel bought tickets for three separate performances. The table below shows the number of tickets bought by each person, and the total cost of the tickets.

	Opera	*Play*	*Concert*	*Total cost*
Hung	3	2	5	€267
Quan	2	3	1	€145
Ariel	1	5	4	€230

a Represent this information as a system of linear equations.

b Find the cost per ticket for each of the performances.

c How much would it cost Phuong to purchase 4 opera, 1 play, and 2 concert tickets?

REVIEW SET 12B

1 For $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{Q} - \mathbf{P}$ **c** $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

2 A library owns several copies of a popular trilogy of novels, according to the matrix:

paperback hard cover

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 6 & 3 \end{pmatrix} \begin{matrix} \leftarrow \text{book 1} \\ \leftarrow \text{book 2} \\ \leftarrow \text{book 3} \end{matrix}$$



a At present, the books on loan are described by the matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$. Write a matrix to describe the books currently on the shelves.

b The values of the books (in dollars) are described by the matrix $\mathbf{C} = \begin{pmatrix} 7 & 7 & 8 \\ 15 & 16 & 20 \end{pmatrix}$.

i Which book has value \$16?

ii Find the total value of the books currently on loan.

3 Prove that for any square matrix $\mathbf{A}$, $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$.

4 Write $\mathbf{X}$ in terms of $\mathbf{A}$ and $\mathbf{B}$ if:

a $2\mathbf{X} = \mathbf{B} - \mathbf{A}$ **b** $3(\mathbf{A} + \mathbf{X}) = 2\mathbf{B}$ **c** $\mathbf{B} - 4\mathbf{X} = \mathbf{A}$

5 Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$, and $\mathbf{A} + 2\mathbf{X} = -\mathbf{B}$. Find $\mathbf{X}$.

6 If $\mathbf{A}$ is $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{B}$ is $\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}$, find, if possible:

a $2\mathbf{B}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{BA}$

7 If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, under what conditions are the following true?

a If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$.

b $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

8 For $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, find:

a $\det \mathbf{A}$

b $\det(-2\mathbf{A})$

c $\det(\mathbf{A}^2)$

9 Solve using an inverse matrix:

a $\begin{cases} x + y = 5 \\ x - 2y = 4 \end{cases}$

b $\begin{cases} 3x + 2y = 3 \\ 5x + 3y = 4 \end{cases}$

c $\begin{cases} 2x + y + z = 8 \\ 4x - 7y + 3z = 10 \\ 3x - 2y - z = 1 \end{cases}$

10 If $\mathbf{M} = \begin{pmatrix} k & 2 \\ 2 & k \end{pmatrix}\begin{pmatrix} k-1 & -2 \\ -3 & k \end{pmatrix}$ has an inverse $\mathbf{M}^{-1}$, what values can k have?

11 **a** For what values of k does the system $\begin{cases} kx + 3y = -6 \\ x + (k+2)y = 2 \end{cases}$ have a unique solution?

b State the solution in this case.

12 The points $(2, 2)$, $(-2, 4)$, and $(1, 3)$ lie on a circle $x^2 + y^2 + ax + by + c = 0$.

a Write down a system of linear equations for a, b, and c.

b Find the equation of the circle.

13 The table below summarises the last four purchases at the theatre box office.

Customer	*Adult*	*Student*	*Senior*	*Child*	*Total cost*
A	2	0	2	1	\$294
B	1	4	0	2	\$368
C	2	0	4	0	\$360
D	0	3	2	1	\$314

a Find the cost of each ticket type.

b Find the total price paid by the next customer, who buys 2 adult and 4 child tickets.

Chapter 13

Eigenvalues and eigenvectors

Contents:

A Eigenvalues and eigenvectors
B Matrix diagonalisation
C Matrix powers
D Markov chains

OPENING PROBLEM

In springtime in a particular seaside town, it has been observed that:

- Each day, the weather is one of two states: sunny or rainy.
- Tomorrow's weather *only* depends on today's weather.
- If it is sunny today, the probability that it will be sunny tomorrow is 0.8.
- If it is rainy today, the probability that it will be rainy tomorrow is 0.4.

Things to think about:

a The matrix $\mathbf{T} = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$ is called a *transition matrix*.

$t_{11} = 0.8$ is the probability that it will be sunny tomorrow given that it is sunny today.

What is the meaning of:

i $t_{22} = 0.4$ **ii** $t_{21} = 0.2$ **iii** $t_{12} = 0.6$?



b The column matrix $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a *state matrix* which indicates that it is sunny today.

The state matrix for *tomorrow* is $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$.

i What does $\mathbf{s}_1$ *mean*? **ii** What do you think $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0$ will represent?

c Can you use the matrices $\mathbf{T}$ and $\mathbf{s}_0$ to find the probability that it will be sunny:

i in 4 day's time **ii** in a fortnight's time?

d **i** Is there a column matrix $\mathbf{s}$ such that $\mathbf{T}\mathbf{s} = \mathbf{s}$?

ii Assuming that such a matrix $\mathbf{s}$ exists, what might it mean for the weather in the town in the long term?

The weather model described in the **Opening Problem** using state and transition matrices is an example of a Markov process.

In this Chapter we study **eigenvalues** and **eigenvectors**, which are properties of square matrices. We will explore how these properties provide understanding to the behaviour of Markov processes over time.

We will see further applications of eigenvalues and eigenvectors later in the course, including geometric transformations, and in understanding the behaviour of physical systems.

A EIGENVALUES AND EIGENVECTORS

Suppose $\mathbf{A}$ is a square matrix.

If $\mathbf{x}$ is a non-zero vector and λ is a constant such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then λ is an **eigenvalue** of $\mathbf{A}$ and $\mathbf{x}$ is its corresponding **eigenvector**.

FINDING EIGENVALUES

If $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ then $\lambda\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{0}$

$\therefore\ (\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ $\quad\{\text{since } \mathbf{I}\mathbf{x} = \mathbf{x}\}$

Now if $\det(\lambda\mathbf{I} - \mathbf{A}) \neq 0$, there is a unique solution to the matrix equation, but it is the solution $\mathbf{x} = \mathbf{0}$.

For $\mathbf{x}$ to be a non-zero vector, we therefore require $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. We conclude that:

> The eigenvalues of $\mathbf{A}$ are the solutions to $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$.

The equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ is a polynomial equation to be solved for λ. We therefore define:

> The **characteristic polynomial** of an $n \times n$ matrix $\mathbf{A}$ is $p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$.
> The eigenvalues of $\mathbf{A}$ are the solutions to $p(\lambda) = 0$.

In this course, we only consider the eigenvalues of 2×2 matrices. The characteristic polynomials for 2×2 matrices are always quadratic, so there may be two real and distinct eigenvalues, a single repeated eigenvalue, or else eigenvalues which are complex conjugates.

Example 1 — Self Tutor

For the matrix $\mathbf{A} = \begin{pmatrix} 0 & -3 \\ -4 & -1 \end{pmatrix}$, find:

a the characteristic polynomial **b** the eigenvalues.

a
$$p(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$$
$$= \begin{vmatrix} \lambda & 3 \\ 4 & \lambda + 1 \end{vmatrix}$$
$$= \lambda(\lambda + 1) - 12$$
$$= \lambda^2 + \lambda - 12$$

b
$$p(\lambda) = 0$$
$$\therefore \quad \lambda^2 + \lambda - 12 = 0$$
$$\therefore \quad (\lambda - 3)(\lambda + 4) = 0$$
$$\therefore \quad \lambda = 3 \text{ or } -4$$
$$\therefore \quad \text{the eigenvalues are } 3 \text{ and } -4.$$

Example 2 — Self Tutor

Find the eigenvalues of:

a $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$

a
$$\text{If} \quad \det(\lambda\mathbf{I} - \mathbf{A}) = 0$$
$$\text{then} \quad \begin{vmatrix} \lambda + 3 & 1 \\ -1 & \lambda + 1 \end{vmatrix} = 0$$
$$\therefore \quad (\lambda + 3)(\lambda + 1) - (-1) = 0$$
$$\therefore \quad \lambda^2 + 4\lambda + 3 + 1 = 0$$
$$\therefore \quad \lambda^2 + 4\lambda + 4 = 0$$
$$\therefore \quad (\lambda + 2)^2 = 0$$
$$\therefore \quad \lambda = -2$$
$$\therefore \quad \text{the eigenvalue is } -2.$$

b
$$\text{If} \quad \det(\lambda\mathbf{I} - \mathbf{A}) = 0$$
$$\text{then} \quad \begin{vmatrix} \lambda - 2 & -5 \\ 1 & \lambda \end{vmatrix} = 0$$
$$\therefore \quad \lambda(\lambda - 2) - (-5) = 0$$
$$\therefore \quad \lambda^2 - 2\lambda + 5 = 0$$
$$\therefore \quad \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2}$$
$$= \frac{2 \pm \sqrt{-16}}{2}$$
$$= \frac{2 \pm 4i}{2}$$
$$= 1 \pm 2i$$
$$\therefore \quad \text{the eigenvalues are } 1 + 2i \text{ and } 1 - 2i.$$

EXERCISE 13A.1

1 For the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$, find:

a the characteristic polynomial **b** the eigenvalues.

2 Find the eigenvalues of:

a $\mathbf{A} = \begin{pmatrix} 2 & -3 \\ -8 & 0 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} -3 & 5 \\ 2 & -1 \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} -5 & -6 \\ 1 & -1 \end{pmatrix}$ **e** $\mathbf{A} = \begin{pmatrix} -3 & 7 \\ -7 & 11 \end{pmatrix}$ **f** $\mathbf{A} = \begin{pmatrix} 2 & 8 \\ -1 & 7 \end{pmatrix}$

3 **a** Use the definition of eigenvalues to explain why $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ must only have the single eigenvalue 1.

b Perform calculations to verify this result.

4 $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 7 \end{pmatrix}$ are eigenvectors of $\mathbf{A} = \begin{pmatrix} 8 & 4 \\ 7 & 5 \end{pmatrix}$.

a Use the definition of eigenvalues and eigenvectors to find the eigenvalues of $\mathbf{A}$.

b *Hence* write down the characteristic polynomial of $\mathbf{A}$.

5 Consider $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ -5 & 3 \end{pmatrix}$.

a Find $\mathbf{A}^2$ and $\mathbf{A}^{-1}$. **b** Find the eigenvalues of $\mathbf{A}$, $\mathbf{A}^2$, and $\mathbf{A}^{-1}$.

6 Use the definition of eigenvalues and eigenvectors to explain the connection between the eigenvalues of:

a $\mathbf{A}$ and $-\mathbf{A}$ **b** $\mathbf{A}$ and $\mathbf{A}^2$ **c** $\mathbf{A}$ and $\mathbf{A}^{-1}$.

FINDING EIGENVECTORS

Having found the eigenvalue(s) for the matrix $\mathbf{A}$, we return to the matrix equation $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ to find the corresponding eigenvectors $\mathbf{x}$.

If we find an eigenvector $\mathbf{x}$ which satisfies the equation, then any scalar multiple of $\mathbf{x}$ will also satisfy the equation. Therefore, there are infinitely many eigenvectors corresponding to a particular eigenvalue.

Example 3 **Self Tutor**

Find the eigenvalues and corresponding eigenvectors for $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$.

If $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ then $\begin{vmatrix} \lambda & -3 \\ -1 & \lambda + 2 \end{vmatrix} = 0$

$\therefore \lambda(\lambda + 2) - (-1)(-3) = 0$

$\therefore \lambda^2 + 2\lambda - 3 = 0$

$\therefore (\lambda - 1)(\lambda + 3) = 0$

$\therefore \lambda = 1$ or -3

The eigenvalues are 1 and -3.

For $\lambda = 1$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} 1 & -3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad a - 3b = 0$$

Letting $b = t$, $t \neq 0$, then $a = 3t$

$$\therefore \quad \mathbf{x} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} 3 \\ 1 \end{pmatrix}t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue 1.

For $\lambda = -3$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\therefore \begin{pmatrix} -3 & -3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad -a - b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -t$

$$\therefore \quad \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -1 \\ 1 \end{pmatrix}t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue -3.

For any given eigenvalue, the system $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ gives equations which are scalar multiples of one another.



EXERCISE 13A.2

1 Find the eigenvalues and corresponding eigenvectors of each matrix:

a $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 2 & 5 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$

2 **a** Show that $\mathbf{A} = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$ has a single eigenvalue λ.

b Find the corresponding eigenvector $\mathbf{x}$.

c Verify that $\mathbf{Ax} = \lambda\mathbf{x}$.

3 Let $\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ where $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$.

Show that the eigenvalues of $\mathbf{A}$ are λ_1 and λ_2, and find the corresponding eigenvectors.

4 Let $\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a \neq 0$.

a Show that the only eigenvalue of $\mathbf{A}$ is a.

b Show that any non-zero vector $\mathbf{x} = \begin{pmatrix} s \\ t \end{pmatrix}$ is an eigenvector of $\mathbf{A}$.

c Explain the result in **b** using the definition of eigenvalues and eigenvectors.

5 **a** Find the eigenvalues and corresponding eigenvectors of $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -4 & 2 \end{pmatrix}$.

b *Hence* find the eigenvalues and corresponding eigenvectors of $\mathbf{A}^2$.

6 Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}$.

a Show that $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues.

b Find the corresponding eigenvectors for matrix $\mathbf{A}$.

c Are the eigenvectors of $\mathbf{A}$ also eigenvectors of $\mathbf{B}$?

7 Show that the eigenvectors of $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are mutually perpendicular.

Example 4 **Self Tutor**

Consider $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & 0 \end{pmatrix}$ which has eigenvalues $1+2i$ and $1-2i$.

Find the corresponding eigenvectors of $\mathbf{A}$.

For $\lambda = 1+2i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\therefore \begin{pmatrix} 1+2i-2 & 0-5 \\ 0-(-1) & 1+2i-0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1+2i & -5 \\ 1 & 1+2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad a + (1+2i)b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -(1+2i)t = (-1-2i)t$

$$\therefore \quad \mathbf{x} = \begin{pmatrix} -1-2i \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -1-2i \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue $1+2i$.

For $\lambda = 1-2i$, consider $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ with $\mathbf{x} = \begin{pmatrix} a \\ b \end{pmatrix}$.

$$\therefore \begin{pmatrix} 1-2i-2 & 0-5 \\ 0-(-1) & 1-2i-0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -1-2i & -5 \\ 1 & 1-2i \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \quad a + (1-2i)b = 0$$

Letting $b = t$, $t \neq 0$, then $a = -(1-2i)t = (-1+2i)t$

$$\therefore \quad \mathbf{x} = \begin{pmatrix} -1+2i \\ 1 \end{pmatrix} t, \quad t \neq 0$$

Any vector of the form $\begin{pmatrix} -1+2i \\ 1 \end{pmatrix} t$, $t \neq 0$ is an eigenvector corresponding to the eigenvalue $1-2i$.

8 Let $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

a Show that $\begin{pmatrix} 1 \\ -i \end{pmatrix} t, \ t \neq 0$ are eigenvectors of $\mathbf{A}$. State the corresponding eigenvalue.

b State the other eigenvalue of $\mathbf{A}$.

c Hence find the remaining eigenvectors of $\mathbf{A}$.

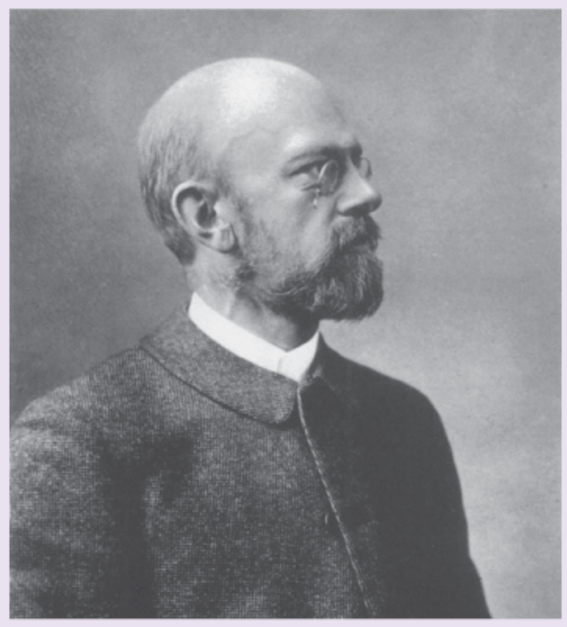
9 Find the eigenvalues and corresponding eigenvectors of:

a $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 5 & -5 \\ 4 & 1 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} -1 & -3 \\ 2 & 1 \end{pmatrix}$

HISTORICAL NOTE

One of the first known applications of eigenvectors came from **Leonhard Euler's** study of the rotational motion of rigid bodies. The Italian mathematician **Joseph-Louis Lagrange** recognised that the principal axes for the rotation correspond to the eigenvectors of the inertia matrix.

In the 19th century, numerous mathematicians and physicists considered the properties of eigenvalues and eigenvectors in their studies. However, it was not until 1904 that the Prussian mathematician **David Hilbert** gave them the German description *eigen*, meaning "own".



David Hilbert

B MATRIX DIAGONALISATION

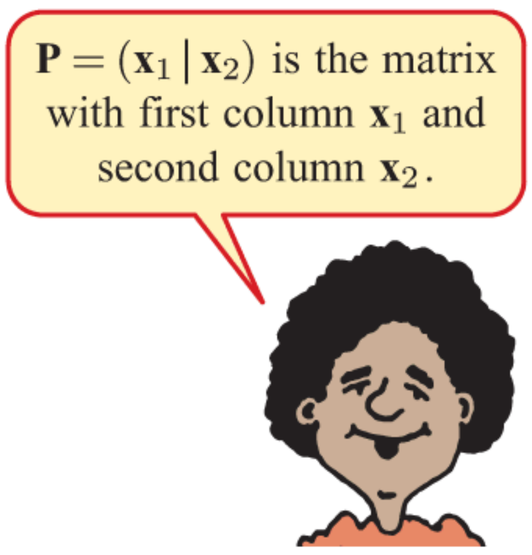
A non-zero square matrix is said to be **diagonal** if the elements *not* on its leading diagonal are zero.

So, a 2×2 diagonal matrix has the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ where $a, b \in \mathbb{R}$ and are not both zero.

A square matrix $\mathbf{A}$ is **diagonalisable** if there exists a matrix $\mathbf{P}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. We say that $\mathbf{P}$ diagonalises $\mathbf{A}$.

In this course we only consider the diagonalisation of 2×2 matrices.

If $\mathbf{A}$ is a 2×2 matrix with *distinct* eigenvalues λ_1, λ_2 and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, then $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ diagonalises $\mathbf{A}$, and $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.



If $\mathbf{A}$ only has a single eigenvalue, then either:

- $\mathbf{A} = a\mathbf{I}, \ a \neq 0$, in which case $\mathbf{A}$ is already diagonalised, or
- the matrix $\mathbf{P} = (\mathbf{x} \,|\, \mathbf{x})$ formed from its eigenvector is singular, and so $\mathbf{A}$ cannot be diagonalised.

Example 5 **Self Tutor**

The matrix $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively.

Show that $\mathbf{P}_1 = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ and $\mathbf{P}_2 = (\mathbf{x}_2 \,|\, \mathbf{x}_1)$ both diagonalise $\mathbf{A}$.

$$\mathbf{P}_1 = (\mathbf{x}_1 \,|\, \mathbf{x}_2) = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$\therefore \det \mathbf{P}_1 = 3 - (-1) = 4$

$\therefore \mathbf{P}_1^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

$$\begin{aligned}\therefore \quad & \mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1 \\ &= \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 1 & -3 \end{pmatrix} \\ &= \tfrac{1}{4}\begin{pmatrix} 4 & 0 \\ 0 & -12 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\end{aligned}$$

$$\mathbf{P}_2 = (\mathbf{x}_2 \,|\, \mathbf{x}_1) = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

$\therefore \det \mathbf{P}_2 = -1 - 3 = -4$

$\therefore \mathbf{P}_2^{-1} = -\frac{1}{4}\begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix}$

$$\begin{aligned}\therefore \quad & \mathbf{P}_2^{-1}\mathbf{A}\mathbf{P}_2 \\ &= -\tfrac{1}{4}\begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= -\tfrac{1}{4}\begin{pmatrix} 1 & -3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ -3 & 1 \end{pmatrix} \\ &= -\tfrac{1}{4}\begin{pmatrix} 12 & 0 \\ 0 & -4 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}\end{aligned}$$

EXERCISE 13B

1 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ respectively.

Show that $\mathbf{P}_1 = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ and $\mathbf{P}_2 = (\mathbf{x}_2 \,|\, \mathbf{x}_1)$ both diagonalise $\mathbf{A}$.

2 The matrix $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 2 & -2 \end{pmatrix}$ has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 6$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ respectively.

Show that $\mathbf{P}_1 = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ and $\mathbf{P}_2 = (\mathbf{x}_2 \,|\, \mathbf{x}_1)$ both diagonalise $\mathbf{A}$.

Example 6 **Self Tutor**

Let $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 0 \end{pmatrix}$.

a Find the eigenvalues λ_1, λ_2 $(\lambda_1 > \lambda_2)$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$.

b Verify that $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

a If $\det(\lambda\mathbf{I}-\mathbf{A})=0$ then $\begin{vmatrix}\lambda-2 & -5\\ -1 & \lambda\end{vmatrix}=0$

$$\therefore\ \lambda^2-2\lambda-5=0$$

$$\therefore\ \lambda=\frac{2\pm\sqrt{4+20}}{2}$$

$$\therefore\ \lambda=1\pm\sqrt{6}$$

For $\lambda_1=1+\sqrt{6}$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\ b\end{pmatrix}$

gives $\begin{pmatrix}\sqrt{6}-1 & -5\\ -1 & 1+\sqrt{6}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\therefore\ -a+(1+\sqrt{6})b=0$$

Letting $b=t$, $t\neq0$, then $a=(1+\sqrt{6})t$

$$\therefore\ \begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1+\sqrt{6}\\ 1\end{pmatrix}t,\ t\neq0$$

$\therefore$ choosing $t=1$, $\mathbf{x}_1=\begin{pmatrix}1+\sqrt{6}\\ 1\end{pmatrix}$ is an eigenvector corresponding to $\lambda_1=1+\sqrt{6}$.



For $\lambda_2=1-\sqrt{6}$, $(\lambda\mathbf{I}-\mathbf{A})\mathbf{x}=\mathbf{0}$ with $\mathbf{x}=\begin{pmatrix}a\\ b\end{pmatrix}$

gives $\begin{pmatrix}-1-\sqrt{6} & -5\\ -1 & 1-\sqrt{6}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$$\therefore\ -a+b(1-\sqrt{6})=0$$

Letting $b=t$, $t\neq0$, then $a=(1-\sqrt{6})t$

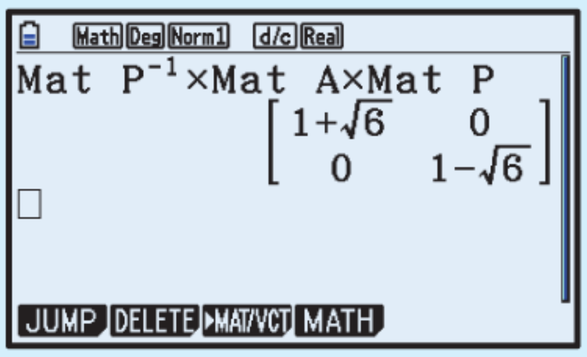
$$\therefore\ \begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}1-\sqrt{6}\\ 1\end{pmatrix}t,\ t\neq0$$

$\therefore$ choosing $t=1$, $\mathbf{x}_2=\begin{pmatrix}1-\sqrt{6}\\ 1\end{pmatrix}$ is an eigenvector corresponding to $\lambda_2=1-\sqrt{6}$.

b Let $\mathbf{P}=(\mathbf{x}_1\,|\,\mathbf{x}_2)=\begin{pmatrix}1+\sqrt{6} & 1-\sqrt{6}\\ 1 & 1\end{pmatrix}$

$$\therefore\ \mathbf{P}^{-1}\mathbf{AP}=\begin{pmatrix}1+\sqrt{6} & 0\\ 0 & 1-\sqrt{6}\end{pmatrix}\quad\{\text{using technology}\}$$

$\therefore$ $\mathbf{P}$ diagonalises $\mathbf{A}$.



3 For each matrix $\mathbf{A}$:

i Find the eigenvalues λ_1, λ_2 $(\lambda_1>\lambda_2)$ and the corresponding eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$.

ii Verify that $\mathbf{P}=(\mathbf{x}_1\,|\,\mathbf{x}_2)$ diagonalises $\mathbf{A}$.

a $\mathbf{A}=\begin{pmatrix}1 & 4\\ 1 & -2\end{pmatrix}$

b $\mathbf{A}=\begin{pmatrix}5 & 6\\ -1 & -1\end{pmatrix}$

4 Consider the matrix $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 2 & 7 \end{pmatrix}$.

a Find the eigenvalues λ_1, λ_2 $(\lambda_1 > \lambda_2)$ and their corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$.

b If $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$, verify that: **i** $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ **ii** $\mathbf{P}^{-1}\mathbf{A}^2\mathbf{P} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$

C MATRIX POWERS

We have already seen that we can calculate powers of a matrix $\mathbf{A}$, provided that $\mathbf{A}$ is square. For larger powers, this is tedious to calculate by hand.

However, if matrix $\mathbf{A}$ with eigenvalues λ_1, λ_2 is diagonalisable by $\mathbf{P}$, then

$$\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \mathbf{D}$$

$$\therefore \quad \mathbf{PP}^{-1}\mathbf{APP}^{-1} = \mathbf{PDP}^{-1} \qquad \{\text{pre-multiplying by } \mathbf{P} \text{ and post-multiplying by } \mathbf{P}^{-1}\}$$

$$\therefore \quad \mathbf{A} = \mathbf{PDP}^{-1} \qquad \{\mathbf{PP}^{-1} = \mathbf{I}\}$$

$$\therefore \quad \mathbf{A}^n = \underbrace{(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1})....(\mathbf{PDP}^{-1})}_{n \text{ times}}$$

$$= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1}\mathbf{PDP}^{-1}....\mathbf{PDP}^{-1}$$

$$= \mathbf{PD} \quad \mathbf{D} \quad \mathbf{D} \quad .... \quad \mathbf{DP}^{-1} \qquad \{\mathbf{P}^{-1}\mathbf{P} = \mathbf{I}\}$$

$$= \mathbf{PD}^n\mathbf{P}^{-1}$$

Now $\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\therefore \quad \mathbf{D}^2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\therefore \quad \mathbf{D}^3 = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}$$

$$\vdots$$

$$\therefore \quad \mathbf{D}^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$$

If $\mathbf{A}$ is diagonalisable by $\mathbf{P}$ then $\mathbf{A}^n = \mathbf{P}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\mathbf{P}^{-1}$.

Example 7 **Self Tutor**

The matrix $\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 1 & -2 \end{pmatrix}$ has eigenvalues $1, -3$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Use the diagonalisation of $\mathbf{A}$ to find $\mathbf{A}^6$.

The matrix $\mathbf{P} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$ diagonalises $\mathbf{A}$ with $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}$.

Now $\mathbf{A}^n = \mathbf{P}\begin{pmatrix} 1^n & 0 \\ 0 & (-3)^n \end{pmatrix}\mathbf{P}^{-1}$

$$\therefore \quad \mathbf{A}^6 = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (-3)^6 \end{pmatrix}\tfrac{1}{4}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

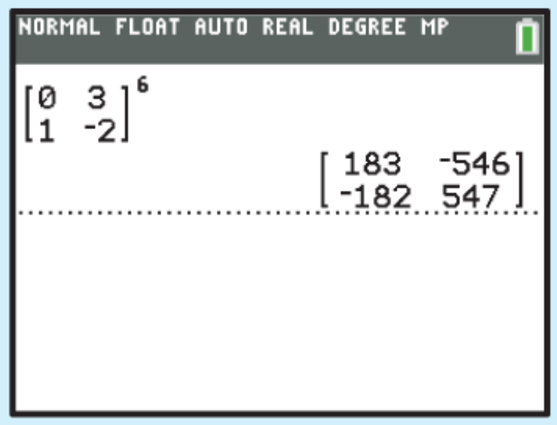
$$= \tfrac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3^6 & 3^7 \end{pmatrix}$$

$$= \tfrac{1}{4}\begin{pmatrix} 3+3^6 & 3-3^7 \\ 1-3^6 & 1+3^7 \end{pmatrix}$$

$$= \begin{pmatrix} 183 & -546 \\ -182 & 547 \end{pmatrix}$$

Check:



EXERCISE 13C

1 The matrix $\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$ has eigenvalues 1, 4 with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Use the diagonalisation of $\mathbf{A}$ to find $\mathbf{A}^5$.

2 Consider the matrix $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$.

a Find the eigenvalues of $\mathbf{A}$ and their corresponding eigenvectors.

b State a matrix $\mathbf{P}$ which will diagonalise $\mathbf{A}$.

c Write $\mathbf{A}^n$ in the form $\mathbf{PD}^n\mathbf{P}^{-1}$.

d Hence find the matrix $\mathbf{P}^{-1}\mathbf{A}^{13}\mathbf{P}$.

3 If $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$, find the matrix $\mathbf{A}^{60}$ exactly.

4 If $\mathbf{C} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}$, find the matrix $\mathbf{C}^{2015}$ exactly.

DISCUSSION

We have just seen that if a matrix $\mathbf{A}$ is diagonalisable, with eigenvalues λ_1, λ_2, then its powers can be written in the form $\mathbf{A}^n = \mathbf{P}\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}\mathbf{P}^{-1}$ for $n \in \mathbb{Z}^+$.

We can readily extend this to *fractional* powers.

For example, we can define the "square root" of $\mathbf{A}$ as $\sqrt{\mathbf{A}} = \mathbf{A}^{\frac{1}{2}} = \mathbf{P}\begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 \\ 0 & \lambda_2^{\frac{1}{2}} \end{pmatrix}\mathbf{P}^{-1}$.

1 Would all matrices necessarily have a "square root"?

2 Use the above definition to verify that $\left(\mathbf{A}^{\frac{1}{2}}\right)^2 = \mathbf{A}$.

3 $\mathbf{B} = \mathbf{A}^{\frac{1}{2}}$ is a solution to the matrix equation $\mathbf{B}^2 = \mathbf{A}$. Is it the *only* solution?

D MARKOV CHAINS

In the real-world models we have considered in the course so far, we have based our model on whatever data was available and used it to generate a function or other exact representation of the system. We know that our model is not exact, but we have made no attempt to include the random variation in the system within the model.

A **stochastic model** includes the random variation in a system by using probability to predict the outcomes of future events.

A **Markov chain** is a stochastic model which describes how a sequence of random events evolves over time.

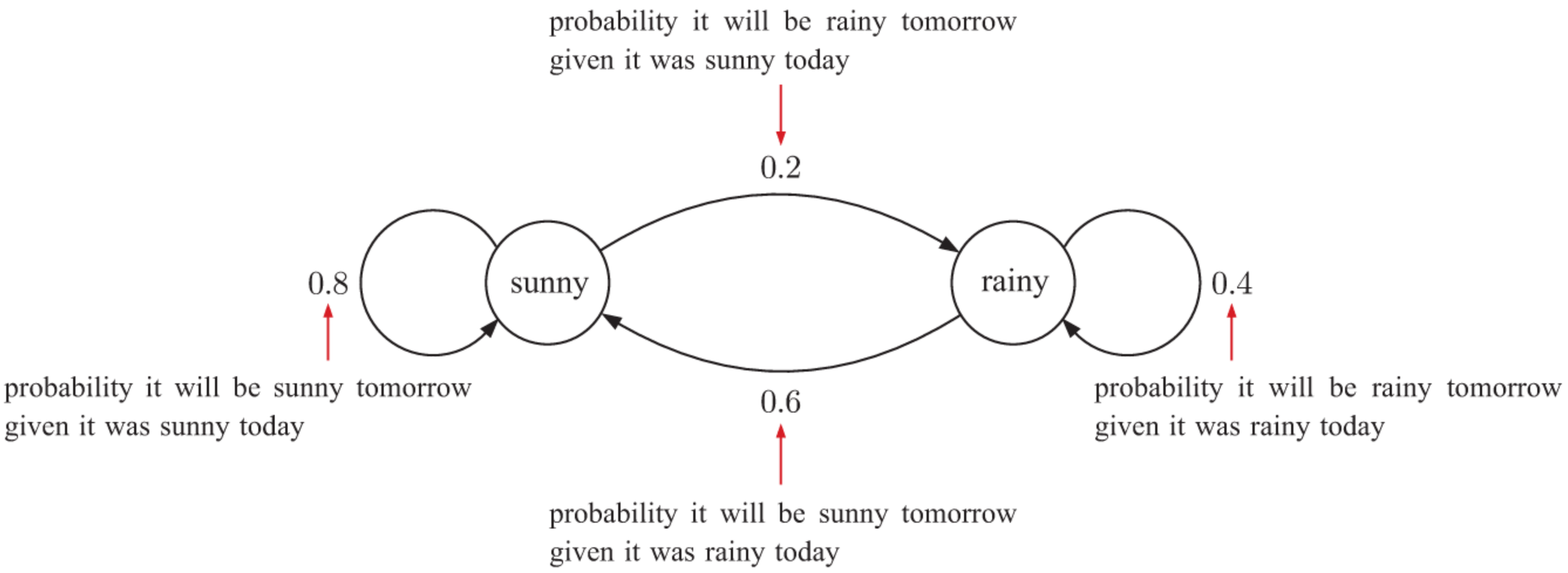
- Time is measured in **discrete** steps.
- Each time step, the possible outcomes of each event are the **states** that the system could be in.
- The probability distribution for the possible states of the system at the next time step depends *only* on the state at the current time step. This is known as the **memoryless** or **Markov property**.

For example, the **Opening Problem** describes a Markov chain.

- Time is measured in days.
- Each day, the weather can take two possible states: "sunny" or "rainy".
- Tomorrow's weather depends only on today's weather.

TRANSITION DIAGRAMS

We can illustrate how the weather changes or **transitions** from one day to the next using a **transition diagram**:



Notice that:

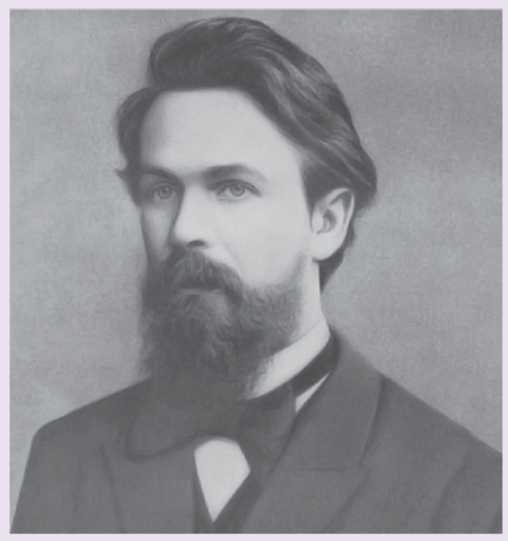
- Each state is represented by a circle.
- Arrows show movement between states.
- Each arrow is labelled with the corresponding **transition probability**.

HISTORICAL NOTE

The word "stochastic" comes from the Greek word *stokhastikós* which means "acting based on guesswork".

Stochastic processes were the main research focus of the Russian mathematician **Andrey Markov** (1856 - 1922), after whom Markov chains were named.

Today, Markov chains have many applications in genetics, economics, and telecommunications.



Andrey Markov

MATRIX REPRESENTATION

One method for studying Markov chains is to use matrices.

STATE MATRICES

The **state matrix** $\mathbf{s}_n$ shows the state of the system at time n.

The state matrix may represent either:

- the probability or proportion of the population that are in each state, or
- the actual number from the population that are in the given state.

The **initial state matrix** is $\mathbf{s}_0$.

For example, in the **Opening Problem** we suppose today represents time step $n = 0$. If it is sunny today then the initial state matrix $\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ← It is sunny today with probability 1.
← It is rainy today with probability 0.

TRANSITION MATRICES

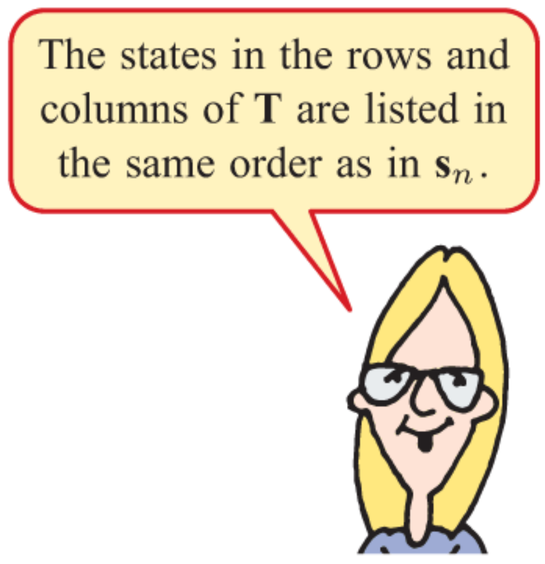
The **transition matrix** $\mathbf{T}$ organises the transition probabilities.

In a transition matrix $\mathbf{T} = (t_{ij})$:

- The columns represent the current state.
- The rows represent the next state.
- t_{ij} is the probability of moving to state i from state j.

$$t_{ij} = \text{P(next state is } i \,|\, \text{current state is } j)$$

- The sum of each column is 1.

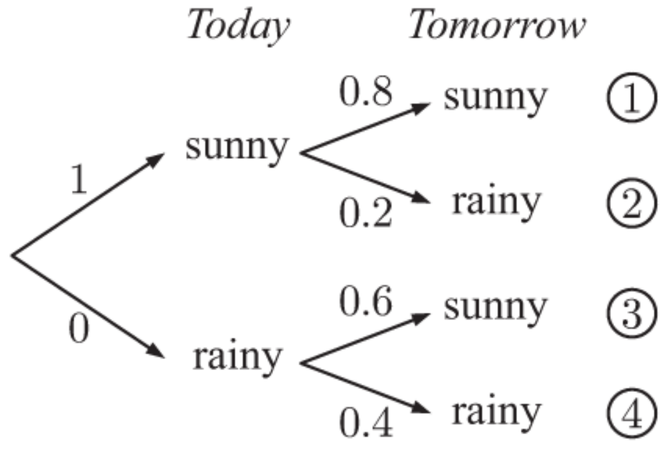


The transition matrix for the Markov chain in the **Opening Problem** is shown alongside.

$$\mathbf{T} = \begin{array}{c} \\ \text{sunny} \\ \text{rainy} \end{array} \begin{array}{c} \textit{Today} \\ \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix} \end{array} \begin{array}{l} \\ \text{sunny} \\ \text{rainy} \end{array} \quad \textit{Tomorrow}$$

USING MATRICES TO CALCULATE FUTURE PROBABILITIES

When we studied conditional probability, we used tree diagrams to help us perform calculations.



For example:

$$\begin{aligned} \text{P(sunny tomorrow)} &= 1 \times 0.8 + 0 \times 0.6 \quad \{\text{paths ① and ③}\} \\ &= 0.8 \end{aligned}$$

$$\begin{aligned} \text{and P(rainy tomorrow)} &= 1 \times 0.2 + 0 \times 0.4 \quad \{\text{paths ② and ④}\} \\ &= 0.2 \end{aligned}$$

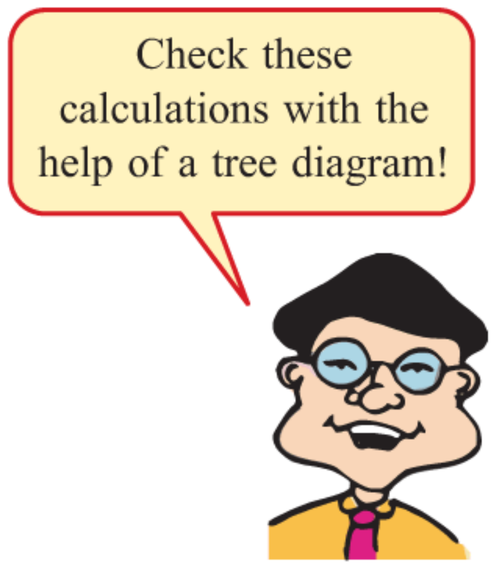
Notice that $\mathbf{Ts}_0 = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times 0.8 + 0 \times 0.6 \\ 1 \times 0.2 + 0 \times 0.4 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$

So, the probabilities of each state after 1 day are given by $\mathbf{s}_1 = \mathbf{Ts}_0$.

After 2 days, $\mathbf{s}_2 = \mathbf{Ts}_1$

$$= \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$$

$$= \begin{pmatrix} 0.76 \\ 0.24 \end{pmatrix}$$



So, given that it is sunny today, in 2 days from now there is a 76% chance that it will be sunny and a 24% chance that it will be rainy.

In general, $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$.

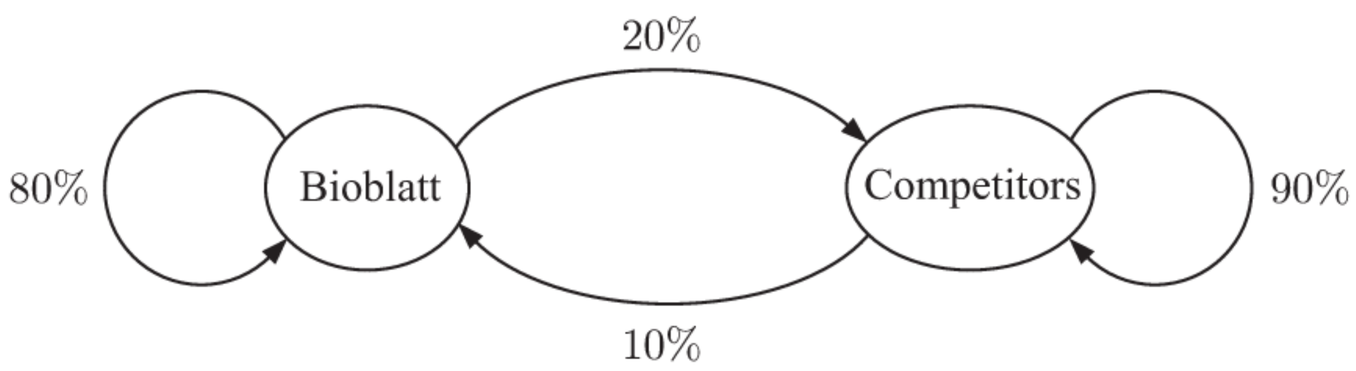
DISCUSSION

In the formula $\mathbf{s}_n = \mathbf{T}^n \mathbf{s}_0$, what assumption do we make about $\mathbf{T}$?

Discuss whether this assumption is necessarily realistic.

EXERCISE 13D.1

1 Bioblatt is a new paper supplier. The transition diagram below shows the percentages of customers who choose each week between Bioblatt and its competitors.



a Construct a transition matrix $\mathbf{T}$ with elements in decimal form.

b In the matrix $\mathbf{T}$, state the meaning of the element:

i 0.2 **ii** 0.9

c Initially Bioblatt has no share of the market, so the initial market share column matrix is $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Calculate $\mathbf{Ts}_0$ and interpret your answer.

d Find Bioblatt's market share after:

i two weeks **ii** four weeks.

2 A country town of 1000 residents has two main grocery stores named Arrows and Bells.

Of those who shop at Arrows one week, 90% shop at Arrows again the next week, and 10% shop at Bells the next week.

Of those who shop at Bells one week, 70% shop at Bells again the next week, and 30% shop at Arrows the next week.

a Display this information using:

i a transition diagram **ii** a transition matrix $\mathbf{T}$.

b In the current week, 600 people shopped at Arrows, and 400 people shopped at Bells. State the initial state matrix $\mathbf{s}_0$.

c Find the expected number of people who will shop at Arrows:

i 1 week later **ii** 3 weeks later.

3 At a local high school, 30% of students who bring their lunch from home on one day will buy their lunch at the canteen the next day. 50% of students who buy their lunch at the canteen on one day will bring their lunch from home the next.

a Display this information using:

i a transition diagram **ii** a transition matrix $\mathbf{T}$.

b On Monday, 400 students from the school were surveyed. 243 brought their lunch from home, while the remainder bought their lunch at the canteen. Construct an initial state matrix $\mathbf{s}_0$ for this situation.

c How many students do you expect to buy their lunch at the canteen on:

i Tuesday **ii** Friday?

4 Two brands of sheep's milk cheese are available. The current transition patterns for sales of the cheeses are shown alongside.

Brand bought this week

Brand bought next week	Baaah	Sheez
Baaah	0.7	0.4
Sheez	0.3	0.6

a State the meaning of the value 0.4 in the table.

b Construct a transition matrix $\mathbf{T}$.

c Suppose the market shares for the first week are given by the state matrix $\mathbf{s}_1 = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$.

i Find $\mathbf{Ts}_1$ and explain what it represents.

ii Find Baaah's market share in the third week.

iii Find Sheez's market share for the sixth week.



5 100 smokers are attempting to quit. They meet for a barbecue on the first Saturday of each month to discuss their progress and encourage each other. In January, 62 members of the group were still smoking, while the rest of the group had gone all of the previous month without smoking.

Statistics indicate that in such situations, some of the smokers will give up in the next month, while some of those who had gone without smoking will return to the habit. The proportions are described below.

This month

Next month	Smoking	Not Smoking
Smoking	80%	10%
Not Smoking	20%	90%

a Write a matrix $\mathbf{s}_0$ to describe the state of the membership at the end of January.

b Suppose $\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$.

- **i** Evaluate $\mathbf{T}\mathbf{s}_0$.
- **ii** Estimate how many members will *not* have smoked between the January and February meetings.
- **iii** Evaluate $\mathbf{T}^2\mathbf{s}_0$, and interpret the result.

c How many members do you expect to be smoking in January the following year?

Example 8

Self Tutor

Scientists are attempting to reverse the effect of salt damage on land adjacent to the Hudson River. Data was collected and examined from a property near the river over several years. This allowed the scientists to construct the following table which compares the proportions of very good, usable, and very poor land between one year and the next.

		Situation this year		
		Very good	Usable	Very poor
Situation next year	Very good	0.91	0.30	0.02
	Usable	0.06	0.65	0.28
	Very poor	0.03	0.05	0.70

a Draw a transition diagram for this situation.

b Write down the transition matrix $\mathbf{T}$.

c In the matrix $\mathbf{T}$, state the meaning of the value:

- **i** 0.91
- **ii** 0.03
- **iii** 0.28

d Find the matrix $\mathbf{T}^2$, and explain the meaning of the value in:

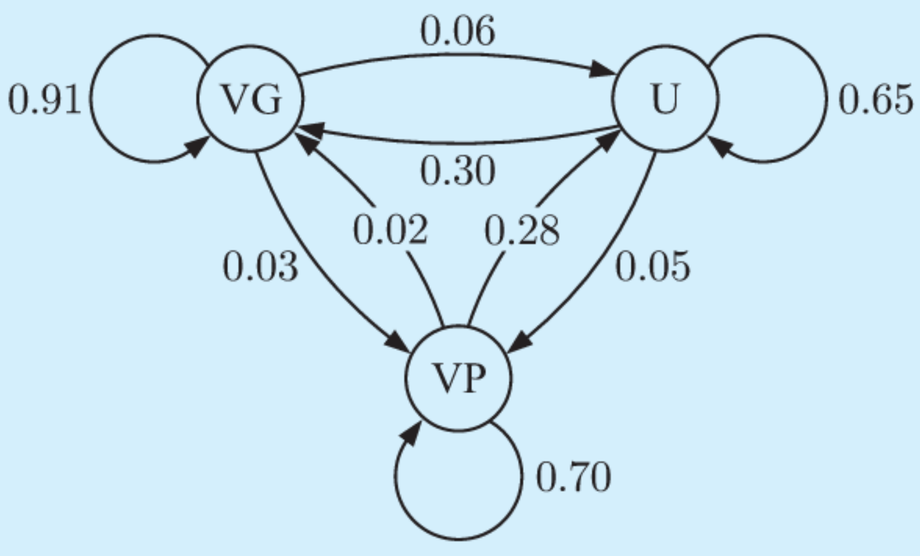
- **i** row 1, column 2
- **ii** row 3, column 3.

e The initial land quality matrix is $\begin{pmatrix} 0 \\ 0.1 \\ 0.9 \end{pmatrix}$. Explain what this means.

f Find the land quality after:

- **i** one year
- **ii** two years
- **iii** four years.

a



b $\mathbf{T} = \begin{pmatrix} 0.91 & 0.30 & 0.02 \\ 0.06 & 0.65 & 0.28 \\ 0.03 & 0.05 & 0.70 \end{pmatrix}$

c **i** The value 0.91 means that 91% of the land which is very good this year will be very good next year.

ii The value 0.03 means that 3% of the land which is very good this year will be very poor next year.

iii The value 0.28 means that 28% of the land which is very poor this year will be usable next year.

d $\mathbf{T}^2 = \begin{pmatrix} 0.91 & 0.30 & 0.02 \\ 0.06 & 0.65 & 0.28 \\ 0.03 & 0.05 & 0.70 \end{pmatrix}^2 = \begin{pmatrix} 0.8467 & 0.469 & 0.1162 \\ 0.102 & 0.4545 & 0.3792 \\ 0.0513 & 0.0765 & 0.5046 \end{pmatrix}$

i In row 1, column 2 we have 0.469. This means that 46.9% of land which is usable now will be very good in two years' time.

ii In row 3, column 3 we have 0.5046. This means that 50.46% of land which is very poor now will be very poor in two years' time.

e $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 0.1 \\ 0.9 \end{pmatrix}$ ← very good, ← usable, ← very poor

$\mathbf{s}_0$ means that there is no very good land, 10% is usable, and 90% is very poor this year.

f **i** $\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0 = \begin{pmatrix} 0.91 & 0.30 & 0.02 \\ 0.06 & 0.65 & 0.28 \\ 0.03 & 0.05 & 0.70 \end{pmatrix} \begin{pmatrix} 0 \\ 0.1 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 0.048 \\ 0.317 \\ 0.635 \end{pmatrix}$

After one year, 4.8% is very good, 31.7% is usable, and 63.5% is very poor.

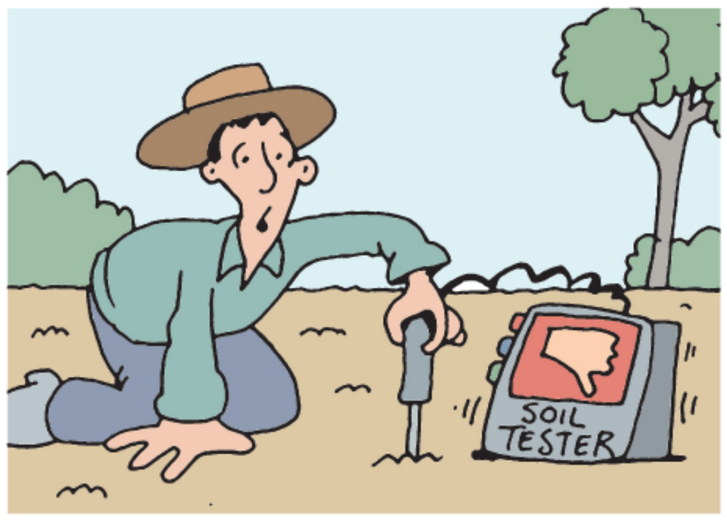
ii $\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0 \approx \begin{pmatrix} 0.151 \\ 0.387 \\ 0.462 \end{pmatrix}$ After two years, 15.1% is very good, 38.7% is usable, and 46.2% is very poor.

iii $\mathbf{s}_4 = \mathbf{T}^4\mathbf{s}_0 \approx \begin{pmatrix} 0.363 \\ 0.366 \\ 0.270 \end{pmatrix}$ After four years, 36.3% is very good, 36.6% is usable, and 27.0% is very poor.

6 A farmer is starting a soil restoration programme. For the soil type and conditions of his land, the transition matrix from one year to the next is:

$$\mathbf{T} = \begin{array}{c} \\ \text{VG} \\ \text{U} \\ \text{VP} \end{array} \begin{array}{c} \begin{array}{ccc} \text{VG} & \text{U} & \text{VP} \end{array} \\ \begin{pmatrix} 0.8 & 0.2 & 0.0 \\ 0.1 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.7 \end{pmatrix} \end{array}$$

VG = Very good
U = Usable
VP = Very poor



a Draw a transition diagram for this situation.

b State the meaning of the number in:

i row 3, column 1 **ii** row 1, column 3.

c Find $\mathbf{T}^2$, and state the meaning of the number in:

i row 3, column 2 **ii** row 2, column 3.

d The present land is all very poor. Write down the initial land quality matrix.

e Describe the land quality after 3 years.

7 A hockey team's fitness coach has observed the following trends over a long period:

		Condition this week		
		Fully fit	Getting treatment	Cannot play
Condition next week	Fully fit	0.88	0.75	0.08
	Getting treatment	0.06	0.17	0.42
	Cannot play	0.06	0.08	0.50

a Draw a transition diagram for this situation.

b Write down the transition matrix $\mathbf{T}$ for the condition of the team's players.

c State the meaning of the number in:

i row 1, column 1 **ii** row 2, column 3.

d Find $\mathbf{T}^2$, and state the meaning of the number in row 1, column 2.

e The club has 30 fully fit players, 6 getting treatment, and 2 who cannot play due to injury. How many players would the coach expect in each category:

i next week **ii** in two weeks' time?

ACTIVITY 1 HIDDEN MARKOV CHAINS



Markov chains are not always directly observable. A Markov chain will sometimes generate a sequence of events that we can see, but we know nothing about the underlying chain itself. In this case we call it a **hidden Markov chain**.

In this online Activity we study a hidden Markov chain based on the **Opening Problem**.

STEADY STATES

INVESTIGATION STEADY STATES



Click on the icon to access a simulation of the weather in the town in the **Opening Problem**.

What to do:

1 Run the simulation for:

a a month **b** 100 days **c** 1000 days.

2 Describe how the *proportion* of sunny days changes during the simulation.

3 Run the simulation again with different initial state matrices $\mathbf{s}_0$. Comment on your results.

4 Use the formula $\mathbf{s}_n = \mathbf{T}^n\mathbf{s}_0$ to find:

a $\mathbf{s}_{30}$ **b** $\mathbf{s}_{100}$ **c** $\mathbf{s}_{1000}$

Comment on your results.

5 Repeat **4** using different initial state matrices $\mathbf{s}_0$. What do you notice?

From the **Investigation**, you should have found that the proportions of sunny and rainy days *converge* to $0.75 = \frac{3}{4}$ and $0.25 = \frac{1}{4}$ respectively, regardless of the initial state matrix $\mathbf{s}_0$.

When there is very little or no change in the values of the state matrices from one time step to the next, we say the system has reached a **steady state**.

FINDING THE STEADY STATE OF A MARKOV CHAIN

In a steady state, $\mathbf{s}_{n+1} = \mathbf{s}_n$

$\therefore \quad \mathbf{T}\mathbf{s}_n = \mathbf{s}_n$

$\therefore \quad \mathbf{T}\mathbf{s} = \mathbf{s} \qquad \{\text{letting } \mathbf{s} = \mathbf{s}_n\}$

We can therefore define:

The **steady state matrix s** of a Markov chain is an **eigenvector** of the transition matrix **T** with **eigenvalue** 1.

From **Section A**, we know there are infinitely many eigenvectors corresponding to a particular eigenvalue. The eigenvector we need is the one whose elements have the correct sum.

If the state matrix is a matrix of probabilities, its elements sum to 1. Otherwise, the elements sum to the total population being considered.

For example, for the weather scenario in the **Opening Problem**, $\mathbf{T} = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$.

If $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ is the steady state matrix then $\mathbf{T}\mathbf{s} = \mathbf{s}$

$$\therefore \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 0.8a + 0.6b \\ 0.2a + 0.4b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Using either row, $0.2a + 0.4b = b$

$\therefore \quad 0.2a = 0.6b$

$\therefore \quad a = 3b$

In this case, **s** is a matrix of probabilities, so $a + b = 1$

$\therefore \quad 3b + b = 1$

$\therefore \quad b = \frac{1}{4}$ and $a = \frac{3}{4}$

Hence $\mathbf{s} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}$ which agrees with our observations in the **Investigation**.

SUMMARY

We can find the steady state of a Markov chain with transition matrix **T** using either:

Method 1: Calculate $\mathbf{s}_n = \mathbf{T}^n\mathbf{s}_0$ for large values of n.

Method 2: Find **s** such that $\mathbf{T}\mathbf{s} = \mathbf{s}$.

Note that in general, the steady state does not depend on $\mathbf{s}_0$. So, if using *Method 1* and $\mathbf{s}_0$ is not given, choose any $\mathbf{s}_0 \neq \mathbf{0}$.

EXERCISE 13D.2

1 Jada has noticed that some people at her work keep leaving the fridge door partly open. She estimates that the door is currently left open 10% of the time, and she suspects the problem is getting worse. Jada has noticed that the state in which a person leaves the fridge depends on the state in which they found it, as described in the table alongside:

State when they leave \ *State when they arrive*	Open	Closed
Open	30%	5%
Closed	70%	95%

Suppose $\mathbf{s}_0 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$ and $\mathbf{T} = \begin{pmatrix} 0.3 & 0.05 \\ 0.7 & 0.95 \end{pmatrix}$.

a Explain the significance of $\mathbf{s}_0$ and $\mathbf{T}$.

b Find algebraically, the steady state of the system, $\mathbf{s}$.

c Hence predict the proportion of time the fridge will be left open in the long term. Is Jada's concern that the problem is getting worse justified?

d Use $\mathbf{T}^{10}\mathbf{s}_0$ to check your answer to **b**.

2 Two bus services, Clydes and Roos, operate between two cities. The transition matrix alongside shows the preferences of passengers who use these services on a monthly basis. This month Clydes carried 425 passengers and Roos carried 716 passengers.

$$\mathbf{T} = \begin{array}{c} \\ \text{C} \\ \text{R} \end{array} \begin{array}{c} \begin{array}{cc} \textit{Now} & \\ \text{C} & \text{R} \end{array} \\ \begin{pmatrix} 0.84 & 0.21 \\ 0.16 & 0.79 \end{pmatrix} \end{array} \begin{array}{l} \\ \textit{Next} \\ \textit{month} \end{array}$$

a Write down the initial state matrix $\mathbf{s}_0$.

b Predict the monthly number of passengers for each bus service in the long term by considering $\mathbf{T}^n\mathbf{s}_0$ for large n.

c List two assumptions you made in **b**. Do you think these assumptions are reasonable?

d Use **b** to estimate the proportion of passengers that use each bus service in the long term.

e Find the steady state proportions for the system algebraically.

3 Suppose $\mathbf{T} = \begin{pmatrix} 0.1 & 0.4 \\ 0.9 & 0.6 \end{pmatrix}$ is a transition matrix for a discrete dynamical system. The state matrix of the system gives the proportions of the population with a particular characteristic.

a Use the characteristic polynomial of $\mathbf{T}$ to find its eigenvalues.

b Find the corresponding eigenvectors of $\mathbf{T}$.

c Hence find the steady state matrix $\mathbf{s}$ of the system.

4 A netball squad has 40 players. 30 are currently fully fit, 5 are getting treatment, and 5 are unavailable due to injury. The weekly transition matrix for the fitness status of the players is:

$$\mathbf{T} = \begin{pmatrix} 0.86 & 0.68 & 0.00 \\ 0.12 & 0.24 & 0.32 \\ 0.02 & 0.08 & 0.68 \end{pmatrix} \begin{array}{l} \leftarrow \text{fully fit} \\ \leftarrow \text{getting treatment} \\ \leftarrow \text{injured} \end{array}$$



a How many players would be expected in each group:

i next week **ii** after two weeks

iii in the long term?

b If the squad fields three teams of 9 each week, discuss the implications of your answer to **a iii**.

c Under these transition conditions, how many members would the squad require to ensure they would have a fit team for each of the three games? Explain your answer.

5 Each year there is considerable bird migration between the islands Paua, Manu, and Chalk. Over time, a researcher has observed the trends shown in the table.

Next year \ *This year*	Paua	Manu	Chalk
Paua	75%	20%	15%
Manu	15%	60%	20%
Chalk	10%	20%	65%

a Find the transition matrix $\mathbf{T}$ which shows the migration pattern of the birds from one year to the next.

b Find $\mathbf{T}^2$, and explain what the numbers in its third column represent.

c In 2019, 26% of the birds lived on Paua, 39% lived on Manu, and 35% lived on Chalk. Estimate the populations on each island in:

i 2020 **ii** 2022.

d Find the steady state proportions of birds living on each island.

e Once the steady state proportions have been reached, do the birds stop migrating between the islands? If not, what percentage of birds move between islands each year?

6 A legal firm has four city offices A, B, C, and D. A courier delivers original documents between the offices. After each delivery he waits for a new delivery from that location. The probabilities for the courier's movement are summarised in the transition matrix $\mathbf{T} = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 \\ 0.5 & 0 & 0.5 & 0.3 \\ 0.3 & 0.1 & 0 & 0.6 \\ 0.2 & 0.5 & 0.3 & 0 \end{pmatrix} \begin{matrix} \leftarrow \text{A} \\ \leftarrow \text{B} \\ \leftarrow \text{C} \\ \leftarrow \text{D} \end{matrix}$.

a If the courier is now at B, what is the chance that his:

i first delivery goes to C

ii second delivery goes to A

iii third delivery goes to D?

b If the courier is now at B, where is he most likely to be:

i after 2 deliveries **ii** after 3 deliveries?

c In the long run, which office does the courier visit the most?



7 Consider a Markov chain with transition matrix $\mathbf{T} = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$ and a state matrix which represents proportions of the population.

a Show that the eigenvalues of $\mathbf{T}$ are 1 and $1-p-q$.

b Show that the steady state matrix for this system is $\begin{pmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{pmatrix}$.

c Interpret the physical significance of the steady state proportions.

ACTIVITY 2 LESLIE MATRICES

In the transition matrices we have studied so far, the elements have all been probabilities. In the case of populations, they describe how the members of the population change from one state to another. Each member always takes a state, so the sum of the elements in each column is 1.

By contrast, if we wish to study the age distribution of a population, we need to include births and deaths. This means that new members are added to the population, while existing members are removed. The elements of the transition matrix are not necessarily probabilities, but are *rates*.

The transition matrix for a model of this form is called a **Leslie matrix**, named after its inventor, the Scottish and later Australian mathematician, **Patrick Leslie** (1815 - 1881).

A Leslie matrix has the structure

$$\mathbf{L} = \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ s_1 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \end{pmatrix}$$



where f_i is the average number of female offspring born to each female in the ith age group
s_i is the proportion of female offspring from the ith age group which survive to the next.

The difference between a Leslie population model and a Markov chain is that $f_i + s_i$ is not necessarily 1.

What to do:

1 Each year, a female adult penguin lays an average of 4 eggs, each of which has a 50% chance of containing a female chick. Each year, 80% of the female chicks die, and each female adult has a 60% chance of surviving to the next year.

a The Leslie matrix for this system is $\mathbf{L} = \begin{pmatrix} 0 & 2 \\ 0.2 & 0.6 \end{pmatrix}$. Explain what each element of $\mathbf{L}$ represents.

b Construct a transition diagram for the system.

c Suppose that initially there are 250 female chicks and 300 adult female penguins in the colony, so $\mathbf{s}_0 = \begin{pmatrix} 250 \\ 300 \end{pmatrix}$.

i Calculate $\mathbf{L}\mathbf{s}_0$ and $\mathbf{L}^2\mathbf{s}_0$, and interpret their meaning.

ii By calculating $\mathbf{L}^5\mathbf{s}_0$ and $\mathbf{L}^{10}\mathbf{s}_0$, predict what will happen to the population over time.

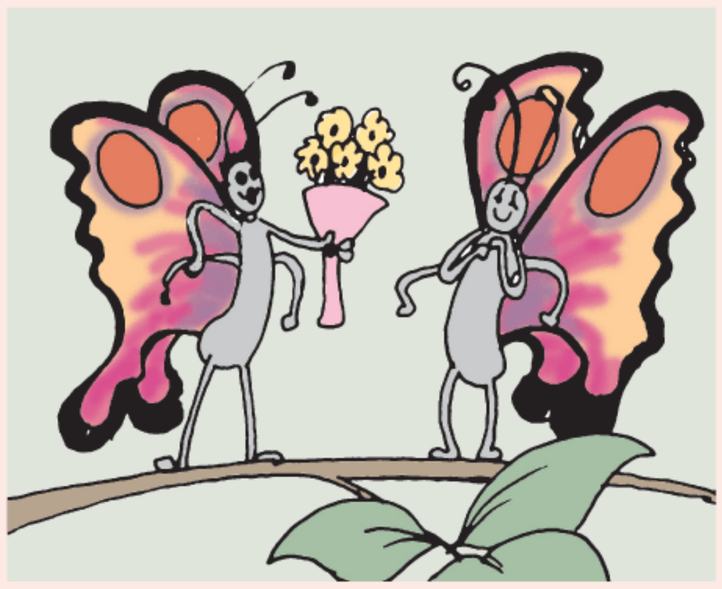
d A particularly harsh winter means that in one year only a small number of chicks survive. The population is reduced to $\mathbf{s}_0 = \begin{pmatrix} 20 \\ 250 \end{pmatrix}$. Is the population likely to recover? Explain your answer.

e Let $\mathbf{s} = \begin{pmatrix} a \\ b \end{pmatrix}$ be a matrix of the *proportions* of chicks and adults in the steady state female population. Find $\mathbf{s}$ algebraically and interpret its meaning.

2 Research into a rare butterfly has revealed that after 24 hours, caterpillars will turn into butterflies, and after 48 hours, the butterflies are ready to mate. Any female butterflies that survive this long will give birth and die after 72 hours.

$\frac{3}{4}$ of the female caterpillars die before pupating into a butterfly, and $\frac{2}{3}$ of the adult female butterflies die before mating.

On average, the female butterflies that survive to mate produce 12 female caterpillar offspring.



At a particular time there are 120 female caterpillars, 15 adolescent female butterflies, and 8 adult female butterflies in a colony.

a The Leslie matrix for this system is $\mathbf{L} = \begin{pmatrix} 0 & 0 & 12 \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$.

Explain what each element of $\mathbf{L}$ represents.

b Construct a transition diagram for the system.

c According to the model, how many females will there be in each category after:

i 24 hours **ii** 48 hours **iii** 72 hours?

d Predict what will happen to the butterfly population in the future using $\mathbf{s}_n = \mathbf{L}^n\mathbf{s}_0$ for large values of n.

REVIEW SET 13A

1 For the matrix $\mathbf{A} = \begin{pmatrix} -4 & 8 \\ -1 & 3 \end{pmatrix}$, find:

a the characteristic polynomial **b** the eigenvalues.

2 $\mathbf{x} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$ is an eigenvector of $\mathbf{A} = \begin{pmatrix} 3 & -7 \\ -3 & -1 \end{pmatrix}$.

a Find $\mathbf{Ax}$, and hence state the eigenvalue corresponding to $\mathbf{x}$.

b Find the remaining eigenvalue.

3 Consider the matrix $\mathbf{A} = \begin{pmatrix} 8 & 3 \\ 8 & 6 \end{pmatrix}$.

a Find the eigenvalues λ_1, λ_2 $(\lambda_1 > \lambda_2)$ and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ of $\mathbf{A}$.

b Verify that $\mathbf{Ax}_1 = \lambda_1\mathbf{x}_1$ and $\mathbf{Ax}_2 = \lambda_2\mathbf{x}_2$.

4 Find the eigenvalue(s) and corresponding eigenvector(s) of:

a $\mathbf{A} = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} -3 & 2 \\ 1 & -5 \end{pmatrix}$

5 Let $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 5 & 3 \end{pmatrix}$.

a Find the eigenvalues λ_1, λ_2 $(\lambda_1 > \lambda_2)$ and the corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$.

b Verify that $\mathbf{P} = (\mathbf{x}_1 \,|\, \mathbf{x}_2)$ will diagonalise $\mathbf{A}$.

6 The matrix $\mathbf{A} = \begin{pmatrix} -2 & 10 \\ -1 & -9 \end{pmatrix}$ has eigenvalues -7, -4 with corresponding eigenvectors $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$. Use the diagonalisation of $\mathbf{A}$ to find $\mathbf{A}^6$.

7 If $\mathbf{C} = \begin{pmatrix} 2 & -3 \\ -8 & 4 \end{pmatrix}$, find the matrix $\mathbf{C}^{10}$ exactly.

8 Each night, the 500 passengers on a cruise ship can eat at restaurant A or restaurant B. 70% of passengers who eat at A one night will eat at A again the next night. 80% of passengers who eat at B one night will eat at B again the next night.

a Display this information using:

i a transition diagram **ii** a transition matrix.

b On Sunday, 250 passengers ate at each restaurant. How many passengers will eat at:

i restaurant A on Monday **ii** restaurant B on Tuesday?

9 A university student is researching the connection between adults who smoke and the smoking habits of their children. Her results are presented in the table below.

Next generation \ *This generation*	Smoker	Non-smoker
Smoker	55%	5%
Non-smoker	45%	95%

a Construct a transition matrix $\mathbf{T}$ from the table.

b In the matrix $\mathbf{T}$, state the meaning of the value in row 2, column 1.

c For the generation in which data was first collected, 60% of adults smoked, so $\mathbf{s}_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$. Calculate $\mathbf{T}^2\mathbf{s}_0$ and interpret your answer.

d Find algebraically the steady state of the system $\mathbf{s}$. Hence predict the proportion of adult smokers in the long term.

10 Bill is a taxi driver. He collects customers from taxi rank A, B, or C. After driving a customer to their destination, he returns to a taxi rank to collect his next customer.
The table alongside shows the probabilities with which he returns to each taxi rank.

Next time \ *This time*	A	B	C
A	0.5	0.3	0.2
B	0.2	0.4	0.2
C	0.3	0.3	0.6

a Display this information using:

i a transition diagram **ii** a transition matrix $\mathbf{T}$.

b Interpret the element in row 1, column 3 of $\mathbf{T}$.

c Bill started his shift at taxi rank B. He is now returning to a taxi rank to collect his third customer. Which taxi rank is he most likely to return to?

d In the long run, which taxi rank does Bill use most often?

REVIEW SET 13B

1 Find the eigenvalue(s) of:

a $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & -4 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 3 & 18 \\ -2 & -9 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ -3 & 2 \end{pmatrix}$

2 The matrix $\mathbf{B} = \begin{pmatrix} 4 & -1 \\ k & 2 \end{pmatrix}$ has eigenvalues $3 \pm \sqrt{8}$. Find the value of k.

3 Consider the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$.

a Find the eigenvalues and corresponding eigenvectors of $\mathbf{A}$.

b Without performing any calculations, find the eigenvalues and corresponding eigenvectors of $3\mathbf{A}$.

4 Let $\mathbf{A} = \begin{pmatrix} 3 & -5 \\ 2 & -3 \end{pmatrix}$.

a Show that $\begin{pmatrix} 3+i \\ 2 \end{pmatrix} t$, $t \neq 0$ are eigenvectors of $\mathbf{A}$. State the corresponding eigenvalue.

b State the other eigenvalue of $\mathbf{A}$.

c Hence find the remaining eigenvectors of $\mathbf{A}$.

5 Find the eigenvalue(s) and corresponding eigenvector(s) of:

a $\mathbf{A} = \begin{pmatrix} 2 & -9 \\ 4 & -10 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 1 & -1 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$

6 Consider the matrix $\mathbf{B} = \begin{pmatrix} 5 & 0 \\ -5 & 4 \end{pmatrix}$.

a Find the eigenvalues and corresponding eigenvectors of $\mathbf{B}$.

b Write down a matrix $\mathbf{P}$ that diagonalises $\mathbf{B}$, and find $\mathbf{P}^{-1}\mathbf{BP}$.

c Hence calculate: **i** $\mathbf{B}^3$ **ii** $\mathbf{B}^6$.

7 Consider the matrix $\mathbf{A} = \begin{pmatrix} 7 & 6 \\ 8 & 9 \end{pmatrix}$.

a Find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

b Hence find the exact value of $\mathbf{A}^{10}$.

8 Chris and Suzie are candidates seeking to be elected mayor at the next council elections. Telephone polls of people living in their electorate have revealed the trends in voting patterns shown in the table.

		This week	
		Chris	Suzie
Next week	Chris	80%	15%
	Suzie	20%	85%

a Display this information using:

i a transition diagram **ii** a transition matrix $\mathbf{T}$.

b Calculate $\mathbf{T}^2$ and $\mathbf{T}^3$.

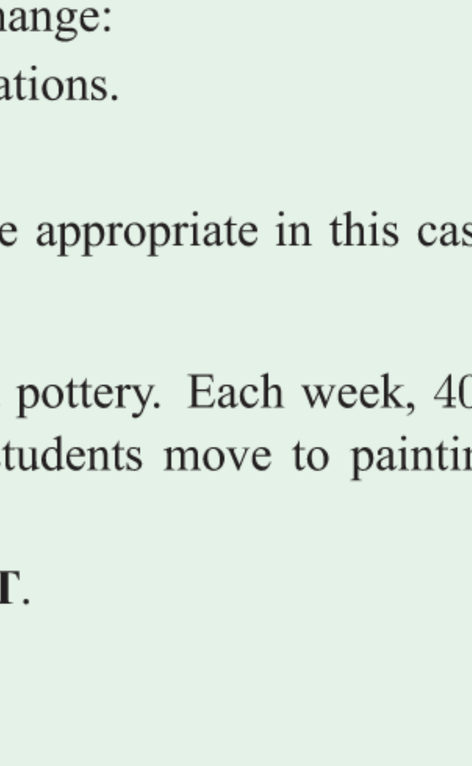
c Explain what column 2 of $\mathbf{T}^2$ means.

d The polls currently suggest that Chris is winning with 60% of the vote. Who do you expect will win if the election is held in:

i 2 weeks **ii** 4 weeks?

9 Doctors classify people as underweight, healthy weight, or overweight. The following table shows the connection between the weight classifications of people and their children.

Next generation \ *This generation*	Underweight	Healthy	Overweight
Underweight	0.43	0.18	0.09
Healthy	0.47	0.56	0.55
Overweight	0.10	0.26	0.36



a State the transition matrix $\mathbf{T}$, and calculate $\mathbf{T}^2$.

b What percentage of underweight people of the current generation are expected to have:

i healthy weight children **ii** overweight grandchildren?

c Currently the weight status of people is 15% underweight, 56% healthy, and 29% overweight. Find how these proportions are expected to change:

i in the next generation **ii** in two generations.

d Find the steady state for this model, and interpret your answer.

e Do you think the underlying assumptions of a Markov chain are appropriate in this case? Explain your answer.

10 In a weekly art class, 120 students can choose between painting and pottery. Each week, 40% of the painting students move to pottery, and 20% of the pottery students move to painting. Initially, 60 students chose painting and 60 students chose pottery.

a Write down the initial state matrix $\mathbf{s}_0$ and the transition matrix $\mathbf{T}$.

b Find $\mathbf{T}\mathbf{s}_0$, and interpret your answer.

c Find the eigenvalues and corresponding eigenvectors of $\mathbf{T}$.

d Write an expression for the number of painting students after n weeks, $n \in \mathbb{N}$.

e Hence find the number of painting students each week in the long term.

Chapter 14

Affine transformations

Contents:

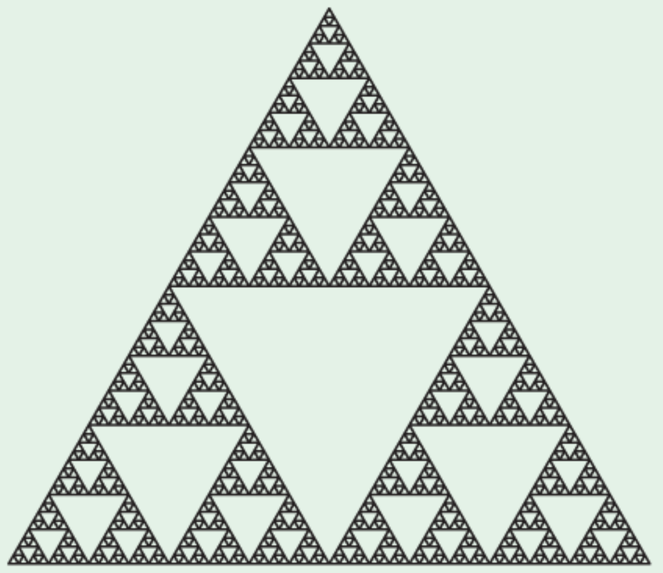
A Translations
B Rotations about the origin
C Reflections
D Stretches
E Enlargements
F Composite transformations
G Area



OPENING PROBLEM

Fractals are geometric constructions that are **self-similar**. This means that any part of a fractal "looks like" the overall fractal.

For example, the figure alongside shows a fractal called **Sierpiński's triangle**.



Most fractals can be generated by repeating a simple set of rules.

Things to think about:

a What transformation is needed to map △OBD onto △OAE?

b What transformation is needed to map △OAE onto:

i △ECD **ii** △ABC?

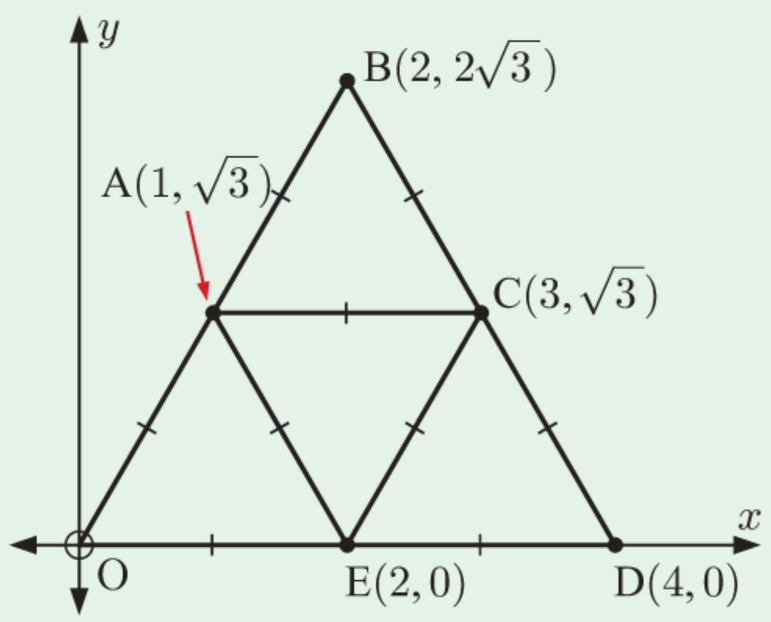
c Can you describe the transformations in **b** using vectors?

d What *sequence* of transformations is needed to map △OBD onto:

i △ECD **ii** △ABC?

e Can we describe the transformations in **a** and **d** using matrices?

f How can we repeat the transformations in **e** to produce Sierpiński's triangle?



When we studied transformations of functions, we saw how algebraic changes to the function resulted in particular transformations. In general, we were interested in the transformation of the whole curve, rather than the movement of a particular point on the curve.

In this Chapter we consider how an **object** point (x, y), specified by the vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, is moved to an **image** point (x', y'), specified by the vector $\mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$, using matrices.

A **linear transformation** is a geometric transformation which can be described by the matrix equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Linear transformations include **stretches**, **rotations**, **reflections**, and any combination or **composition** of these.

An **affine transformation** is a geometric transformation which can be described by the matrix equation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$.

In addition to the linear transformations, affine transformations include **translations**.

A TRANSLATIONS

We have seen previously that when an object is translated, every point on the object moves the same distance in the same direction. This movement is described by a **translation vector**, which in the Cartesian plane gives x and y components of the translation.

Suppose the point (x, y) is translated through $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$ to the image point (x', y').

$x' = x + e$ and $y' = y + f$, so the image is given by $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$.

EXERCISE 14A

1 Find the image point when the point $(3, 2)$ is translated through:

a $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **d** $\begin{pmatrix} -3 \\ -5 \end{pmatrix}$ **e** $\begin{pmatrix} h \\ k \end{pmatrix}$

2 Find the image of each point under the translation $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$:

a $(0, 0)$ **b** $(4, 1)$ **c** $(2, -2)$ **d** $(-3, -4)$ **e** (x, y)

3 Find the object point whose image is $(-1, 3)$ under the translation:

a $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **c** $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$ **e** $\begin{pmatrix} h \\ k \end{pmatrix}$

4 Find the object point which produces each image under the translation $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$:

a $(0, 0)$ **b** $(3, -2)$ **c** $(7, 4)$ **d** $(-1, -5)$ **e** (x, y)

5 Find the translation vector which maps:

a $(2, 3)$ onto $(4, 7)$ **b** $(1, -2)$ onto $(-2, 0)$

c $(-3, 1)$ onto $(1, -4)$ **d** $(5, -2)$ onto $(0, 1)$

e (x, y) onto $(x - 1, y + 3)$ **f** (x, y) onto $(2, -3)$.

6 The point (x, y) is translated through $\mathbf{b}_1 = \begin{pmatrix} e_1 \\ f_1 \end{pmatrix}$ and the result is translated through $\mathbf{b}_2 = \begin{pmatrix} e_2 \\ f_2 \end{pmatrix}$.

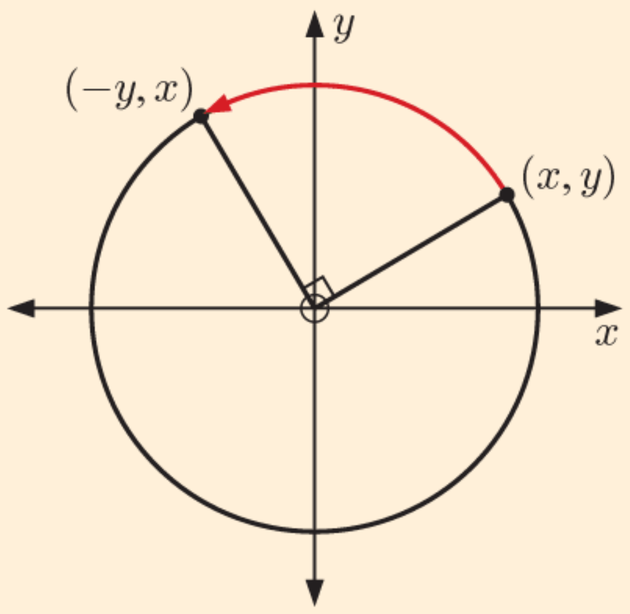
a Describe the image point using vector addition.

b Describe the overall translation using a single translation vector.

7 If the translation of (x, y) through $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$ is written in the form $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$, find the matrix $\mathbf{A}$ and state its determinant.

B ROTATIONS ABOUT THE ORIGIN

INVESTIGATION 1 ROTATIONS ABOUT THE ORIGIN



Under an anticlockwise rotation about $O(0,\ 0)$ through $\frac{\pi}{2}$, the point $(x,\ y)$ has image $(-y,\ x)$.

$\therefore \quad x' = -y$ and $y' = x$

In matrix form, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

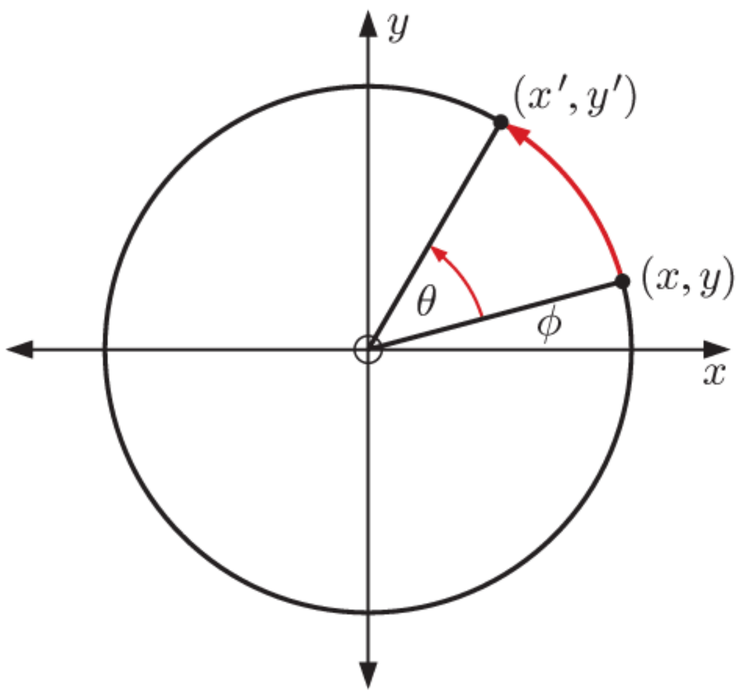
with transformation matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

What to do:

Copy and complete the table alongside, then summarise your observations.

Transformation	**A**	$\lvert\mathbf{A}\rvert$
Anticlockwise rotation about O through $\frac{\pi}{2}$.	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	
Clockwise rotation about O through $\frac{\pi}{2}$.		
Rotation about O through π.		
Rotation about O through 0.		

Now consider an anticlockwise rotation about O through an arbitrary angle θ.



Let $(x,\ y)$ be a point which makes an angle ϕ with the positive x-axis.

$\therefore$ the image point $(x',\ y')$ makes the angle $\phi + \theta$ with the positive x-axis.

The situation can be compared with the complex number $x + iy$, written in polar form as $r\,\mathrm{cis}\,\phi$, which is rotated in the complex plane to the image

$$x' + iy' = r\,\mathrm{cis}(\theta + \phi)$$

$$\begin{aligned}
\text{Now} \quad r\,\mathrm{cis}(\theta + \phi) &= re^{i(\theta+\phi)} \\
&= re^{i\theta}e^{i\phi} \\
&= r(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= r(\cos\theta\cos\phi + i\cos\theta\sin\phi \\
&\qquad + i\sin\theta\cos\phi - \sin\theta\sin\phi) \\
&= \cos\theta(r\cos\phi) - \sin\theta(r\sin\phi) \\
&\qquad + i\,[\sin\theta(r\cos\phi) + \cos\theta(r\sin\phi)] \\
&= (\cos\theta)x - (\sin\theta)y \\
&\qquad + i\,[(\sin\theta)x + (\cos\theta)y]
\end{aligned}$$

In **Chapter 11** we saw that multiplication by $e^{i\theta}$ produces an anticlockwise rotation through θ about the origin in the complex plane.



Equating real and imaginary parts, $x' = (\cos\theta)x - (\sin\theta)y$ and $y' = (\sin\theta)x + (\cos\theta)y$.

$$\therefore \quad \mathbf{x}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}$$

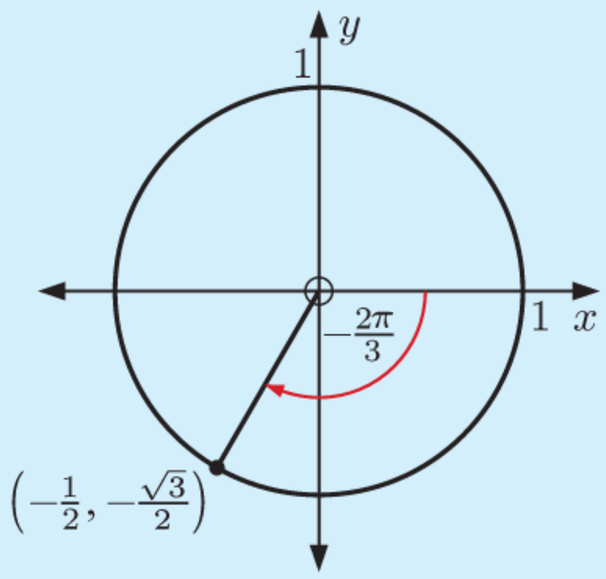
Notice that $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1.$

For a rotation anticlockwise about $O(0, 0)$ through θ, the transformation matrix

$$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ with } |\mathbf{A}| = 1.$$

Example 1 Self Tutor

Find the transformation matrix $\mathbf{A}$ for a clockwise rotation about O through $\frac{2\pi}{3}$.



A clockwise rotation through $\frac{2\pi}{3}$ is equivalent to an anticlockwise rotation through $-\frac{2\pi}{3}$.

For $\theta = -\frac{2\pi}{3}$, $\cos\theta = -\frac{1}{2}$, $\sin\theta = -\frac{\sqrt{3}}{2}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

Example 2 Self Tutor

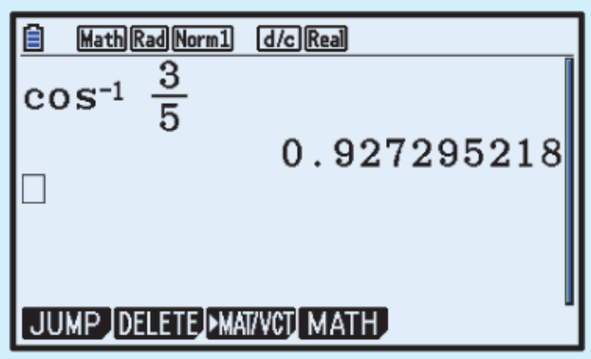
Find the nature of the transformation with equations $\begin{cases} x' = \dfrac{3x - 4y}{5} \\ y' = \dfrac{4x + 3y}{5} \end{cases}$

$$\mathbf{A} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \text{ where } |\mathbf{A}| = \left(\tfrac{3}{5}\right)^2 + \left(\tfrac{4}{5}\right)^2 = 1.$$

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $|\mathbf{A}| = 1$, $\mathbf{A}$ is a rotation matrix.

If the angle of rotation is θ, $\cos\theta = \frac{3}{5}$ and $\sin\theta = \frac{4}{5}$, so θ is in quadrant 1.

$$\therefore \quad \theta = \cos^{-1}\left(\tfrac{3}{5}\right)$$
$$\approx 0.927^c$$



$\therefore$ the transformation is an anticlockwise rotation about $O(0, 0)$ through 0.927^c.

EXERCISE 14B

1 Find the transformation matrix $\mathbf{A}$ for a rotation about O:

a anticlockwise through $\frac{\pi}{2}$

b anticlockwise through $\frac{\pi}{3}$

c clockwise through $\frac{\pi}{4}$

d clockwise through $\frac{5\pi}{6}$

e anticlockwise through $\frac{5\pi}{4}$.

2 Find the nature of the transformation with equations:

a $\begin{cases} x' = \frac{1}{\sqrt{2}}(x+y) \\ y' = \frac{1}{\sqrt{2}}(y-x) \end{cases}$

b $\begin{cases} x' = \dfrac{-5x+12y}{13} \\ y' = \dfrac{-12x-5y}{13} \end{cases}$

3 We have seen that $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the transformation matrix for an anticlockwise rotation about O through angle θ.

a Write down the transformation matrix for an anticlockwise rotation about O through angle $-\theta$.

b Find $\mathbf{A}^{-1}$.

c Discuss the significance of your results.

Example 3 Self Tutor

Find the image of $(5, 1)$ under an anticlockwise rotation about O through $-\frac{3\pi}{4}$.

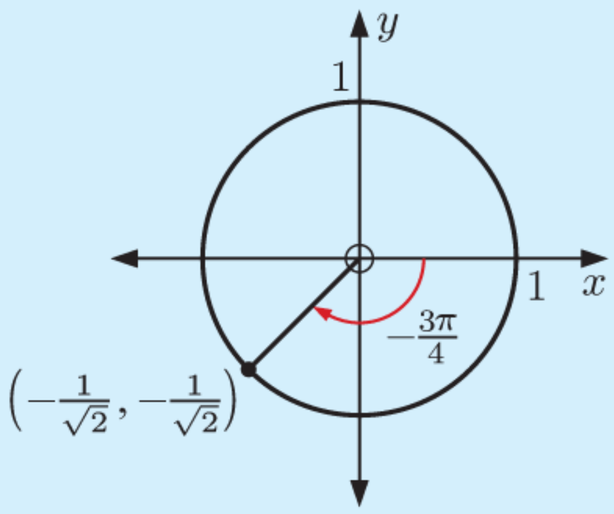
For $\theta = -\frac{3\pi}{4}$, $\cos\theta = -\frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$.

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{x}' = \mathbf{A}\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4}{\sqrt{2}} \\ -\frac{6}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -2\sqrt{2} \\ -3\sqrt{2} \end{pmatrix}$$

$\therefore$ the image is $(-2\sqrt{2}, -3\sqrt{2})$.



4 Find the image of:

a $(5, -1)$ under an anticlockwise rotation about O through $-\frac{2\pi}{3}$

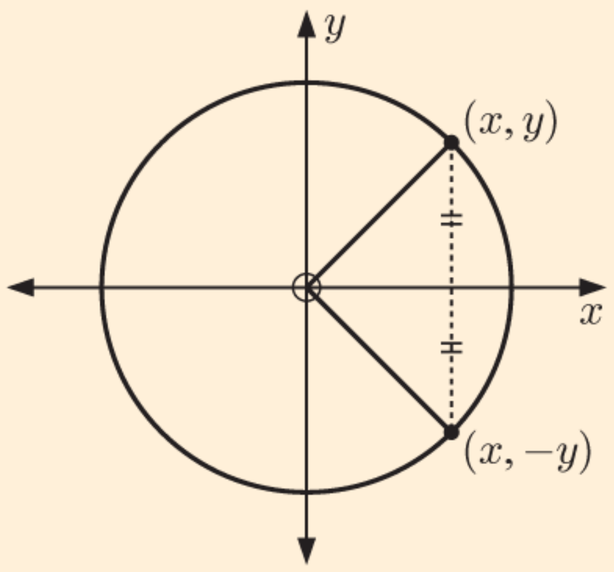
b $(-3, 1)$ under an anticlockwise rotation about O through $\frac{5\pi}{6}$

c $(2, 3)$ under an anticlockwise rotation about O through $\frac{\pi}{4}$

d $(-1, -4)$ under an anticlockwise rotation about O through $-\frac{3\pi}{4}$.

C REFLECTIONS

INVESTIGATION 2 REFLECTIONS



If the point (x, y) is reflected in the x-axis, the image point is $(x, -y)$.

$\therefore \quad x' = x$ and $y' = -y$.

In matrix form, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

with transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

What to do:

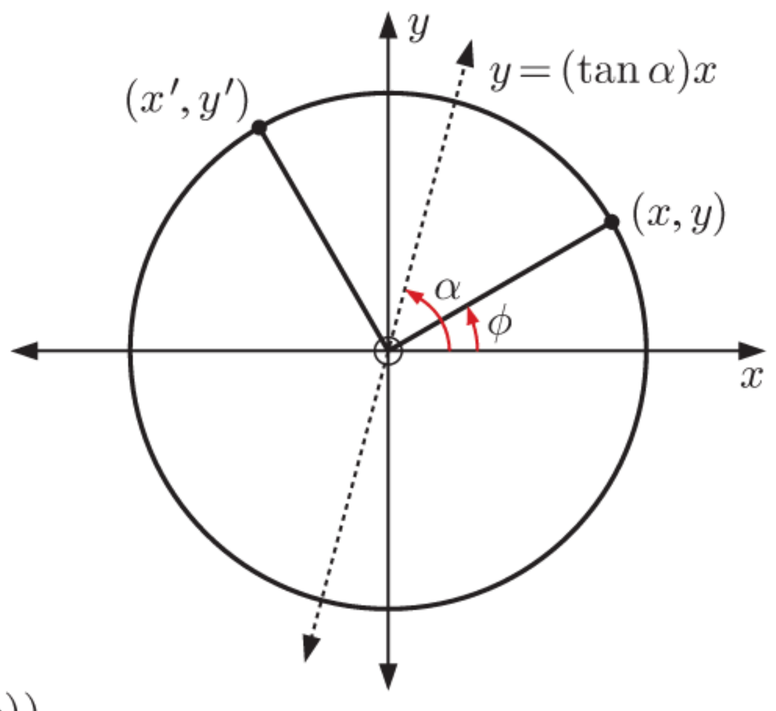
Copy and complete the table alongside, then summarise your observations.

Transformation	**A**	$\lvert\mathbf{A}\rvert$
Reflection in the x-axis.	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	
Reflection in the y-axis.		
Reflection in the line $y = x$.		
Reflection in the line $y = -x$.		

Now consider a reflection in the line $y = (\tan\alpha)x$, so the mirror line passes through the origin and makes an angle α with the x-axis.

Let (x, y) be a point which makes an angle ϕ with the positive x-axis.

$\therefore$ the image point (x', y') makes the angle $\alpha + (\alpha - \phi) = 2\alpha - \phi$ with the positive x-axis.



Comparing with the complex number $x + iy = r\,\text{cis}\,\phi$, the reflected point in the complex plane is

$$x' + iy' = r\,\text{cis}(2\alpha - \phi).$$

Now
$$\begin{aligned} r\,\text{cis}(2\alpha - \phi) &= re^{i(2\alpha - \phi)} \\ &= re^{i2\alpha}e^{-i\phi} \\ &= r(\cos 2\alpha + i\sin 2\alpha)(\cos(-\phi) + i\sin(-\phi)) \\ &= r(\cos 2\alpha\cos\phi - i\cos 2\alpha\sin\phi + i\sin 2\alpha\cos\phi + \sin 2\alpha\sin\phi) \\ &= \cos 2\alpha(r\cos\phi) + \sin 2\alpha(r\sin\phi) + i\,[\sin 2\alpha(r\cos\phi) - \cos 2\alpha(r\sin\phi)] \\ &= \cos 2\alpha x + \sin 2\alpha y + i(\sin 2\alpha x - \cos 2\alpha y) \end{aligned}$$

Equating real and imaginary parts, $x' = \cos 2\alpha x + \sin 2\alpha y$ and $y' = \sin 2\alpha x - \cos 2\alpha y$.

$$\therefore \quad \mathbf{x}' = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \mathbf{x}$$

Notice that $\begin{vmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{vmatrix} = -\cos^2 2\alpha - \sin^2 2\alpha = -1$

For a reflection in the mirror line $y = (\tan \alpha)x$, the transformation matrix is

$$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} \text{ with } |\mathbf{A}| = -1.$$

Example 4 Self Tutor

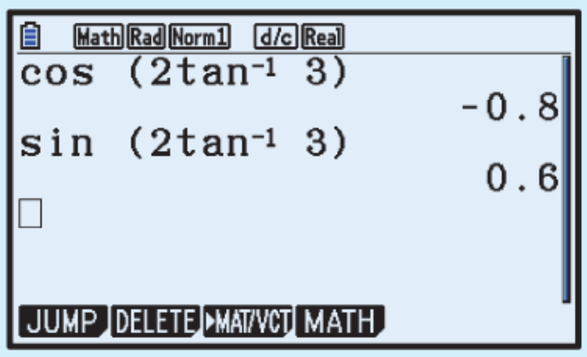
Find the transformation matrix $\mathbf{A}$ for a reflection in the line $y = 3x$.

$m = \tan \alpha = 3$

$\therefore \quad \alpha = \tan^{-1} 3$

$\therefore \quad \cos 2\alpha = -\frac{4}{5}$ and $\sin 2\alpha = \frac{3}{5}$

$$\therefore \quad \mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$



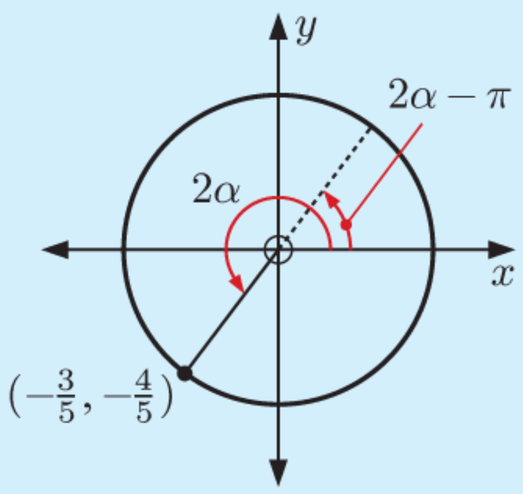
Example 5 Self Tutor

Find the nature of the transformation with equations: $\begin{cases} x' = \dfrac{-3x - 4y}{5} \\ y' = \dfrac{-4x + 3y}{5} \end{cases}$

$\mathbf{A} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$ where $|\mathbf{A}| = -\left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = -1$.

Since $\mathbf{A}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ and $|\mathbf{A}| = -1$,

$\mathbf{A}$ is a reflection matrix where $\cos 2\alpha = -\frac{3}{5}$ and $\sin 2\alpha = -\frac{4}{5}$.



$\therefore \quad \tan 2\alpha = \frac{4}{3}, \quad \pi < 2\alpha < \frac{3\pi}{2}$

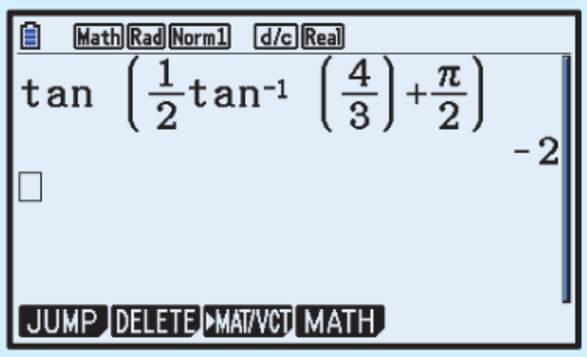
$\therefore \quad \tan(2\alpha - \pi) = \frac{4}{3}, \quad 0 < 2\pi - \alpha < \frac{\pi}{2}$

$\therefore \quad 2\alpha - \pi = \tan^{-1}\left(\frac{4}{3}\right)$

$\therefore \quad \alpha = \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) + \frac{\pi}{2}$

$\therefore \quad \tan \alpha = -2$

$\therefore$ the transformation is a reflection in the line $y = -2x$.



EXERCISE 14C

1 Find the transformation matrix $\mathbf{A}$ for:

a a reflection in the y-axis

b a reflection in the line $y = -x$

c a reflection in the line $y = 5x$

d a reflection in the line $y = \sqrt{3}x$.

2 Find the nature of the transformation with equations:

a $\begin{cases} x' = \frac{1}{\sqrt{2}}(x + y) \\ y' = \frac{1}{\sqrt{2}}(x - y) \end{cases}$

b $\begin{cases} x' = \dfrac{8y - 15x}{17} \\ y' = \dfrac{8x + 15y}{17} \end{cases}$

3 The transformation $\mathbf{x}' = \begin{pmatrix} s & s \\ t + \sqrt{2} & t \end{pmatrix}\mathbf{x}$ maps $(\sqrt{2}, -\sqrt{2})$ onto $(0, 2)$ for some $s, t \in \mathbb{R}$.

Discuss whether this transformation could be:

a a rotation about O

b a reflection in the line $y = (\tan\alpha)x$.

4 Let $\mathbf{A}$ be the transformation matrix for a reflection in the line $y = (\tan\alpha)x$. Show that $\mathbf{A}^{-1} = \mathbf{A}$, and explain the significance of this result.

5 Find the image of each point under a reflection in the line $y = -2x$:

a $(-4, 2)$

b $(3, -1)$

c $(-2, 4)$

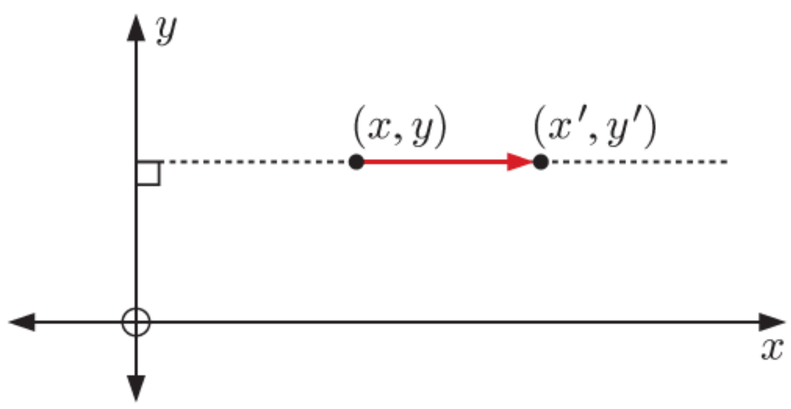
D STRETCHES

HORIZONTAL STRETCHES

In a horizontal stretch with scale factor k, the point (x, y) is moved to the image (x', y') which is k times the distance from the y-axis.

This means that $x' = kx$ and $y' = y$

so $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.



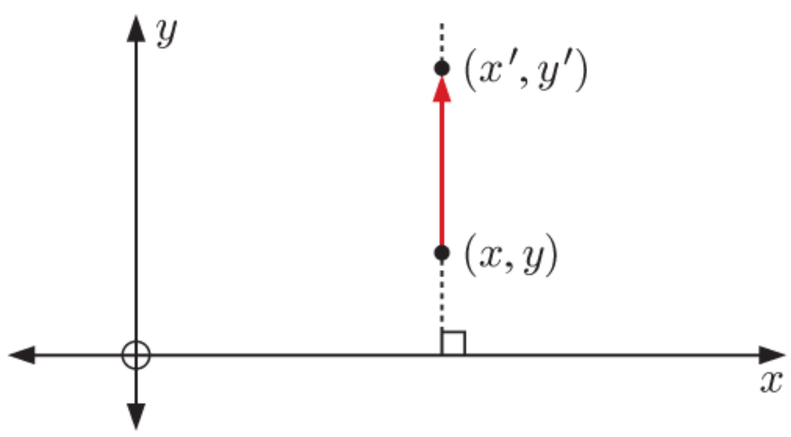
For a **horizontal stretch** with **scale factor** k, the transformation matrix $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

VERTICAL STRETCHES

In a vertical stretch with scale factor k, the point (x, y) is moved to the image (x', y') which is k times the distance from the y-axis.

This means that $x' = x$ and $y' = ky$

so $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.



For a **vertical stretch** with **scale factor** k, the transformation matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$.

EXERCISE 14D

1 Describe the transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which $\mathbf{A}$ is:

a $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

2 Consider a horizontal stretch with scale factor $\frac{5}{2}$.

a Write down the transformation matrix $\mathbf{A}$.

b Hence find the image of $(3, 4)$ under the transformation.

c Find $\mathbf{A}^{-1}$ and describe the transformation $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$.

3 Consider a vertical stretch with scale factor $\frac{3}{4}$.

a Write down the transformation matrix $\mathbf{A}$.

b Hence find the image of $(-1, 2)$ under the transformation.

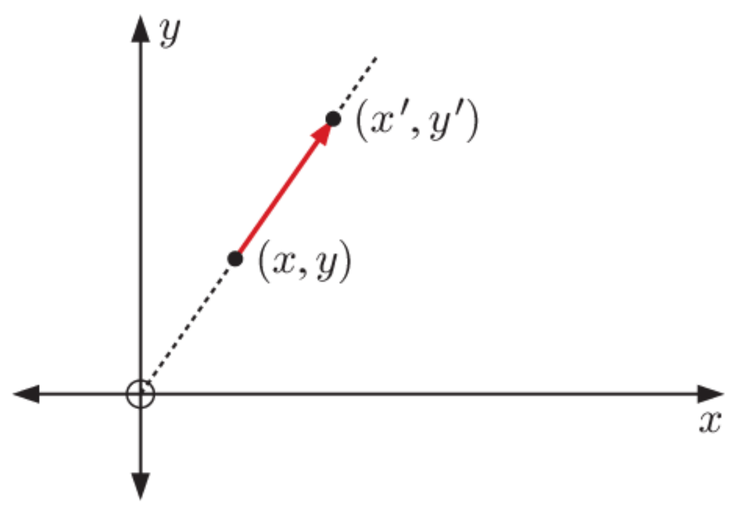
c Find $\mathbf{A}^{-1}$ and describe the transformation $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$.

E ENLARGEMENTS

In an enlargement with scale factor k, the point (x, y) is moved directly towards or away from the origin. The image is k times the distance from the origin that the object was.

This means that $x' = kx$ and $y' = ky$

so $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.



For an **enlargement** with scale factor k, the transformation matrix is $\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.

EXERCISE 14E

1 Describe the transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which $\mathbf{A}$ is:

a $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

If the image is closer to the origin than the object, we can call the transformation a **reduction**.



2 Consider an enlargement with scale factor $\frac{4}{3}$.

a Write down the transformation matrix $\mathbf{A}$.

b Hence find the image of $(3, 6)$ under the transformation.

c Find $\mathbf{A}^{-1}$ and describe the transformation $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$.

3 **a** Find the image point $\mathbf{x}'$ when $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ is stretched horizontally with scale factor k.

b Find the image point $\mathbf{x}''$ when $\mathbf{x}'$ is stretched vertically with scale factor k.

c Explain your answer by considering the product $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$.

F COMPOSITE TRANSFORMATIONS

When we apply one transformation after another, the result is a **composite transformation**.

For example, the enlargements we considered in the previous Section are composite transformations. The enlargement with scale factor k is the composition of a horizontal stretch with scale factor k and a vertical stretch with scale factor k.

If we perform a linear transformation with transformation matrix $\mathbf{A}$, the result is $\mathbf{Ax}$.

If we perform a second linear transformation with transformation matrix $\mathbf{B}$ on the result, we obtain the image $\mathbf{B}(\mathbf{Ax}) = \mathbf{BAx}$.

So, the composition of these linear transformations is itself a linear transformation, and has the transformation matrix $\mathbf{BA}$.

Example 6 **Self Tutor**

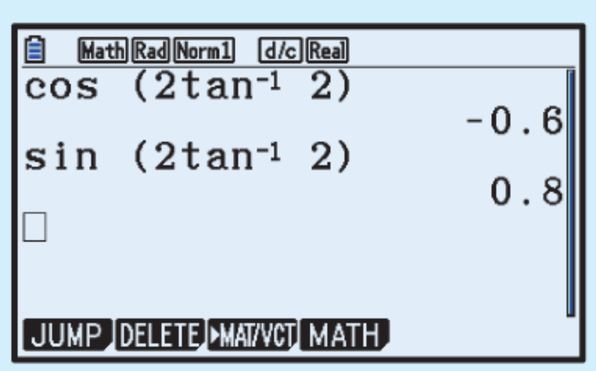
Find the single linear transformation equivalent to an anticlockwise rotation about O through $\frac{\pi}{2}$, followed by a reflection in the line $y = 2x$.

An anticlockwise rotation of $\frac{\pi}{2}$ about O has the transformation matrix $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

For the reflection in the line $y = 2x$, $\tan\alpha = 2$

$\therefore \cos 2\alpha = \cos(2\tan^{-1} 2) = -\frac{3}{5}$

and $\sin 2\alpha = \sin(2\tan^{-1} 2) = \frac{4}{5}$



$\therefore$ the reflection has transformation matrix $\mathbf{B} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$.

Now $\mathbf{BA} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$

$\therefore$ $\mathbf{BA}$ has the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ with $|\mathbf{BA}| = |\mathbf{B}||\mathbf{A}| = (-1)(1) = -1$

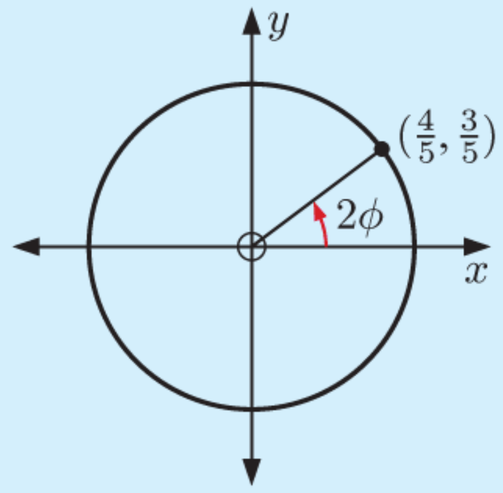
$\therefore$ the composition of these transformations is a reflection.

If the reflection is in the line $y = \tan\phi x$ then $\cos 2\phi = \frac{4}{5}$ and $\sin 2\phi = \frac{3}{5}$

$\therefore \tan 2\phi = \frac{3}{4}$ and $0 < 2\phi < \pi$

$\therefore \phi = \frac{1}{2}\tan^{-1}\left(\frac{3}{4}\right)$

$\therefore \tan\phi = \tan\left(\frac{1}{2}\tan^{-1}\left(\frac{3}{4}\right)\right) = \frac{1}{3}$



$\therefore$ the composite transformation is a reflection in the line $y = \frac{1}{3}x$.

The affine transformation $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$ is the composition of the linear transformation with transformation matrix $\mathbf{A}$, followed by a translation through $\mathbf{b}$.

EXERCISE 14F

1 Let $\mathbf{A}$ be the transformation matrix for a reflection in the line $y = x$, and $\mathbf{B}$ be the transformation matrix for an anticlockwise rotation about O through $\frac{\pi}{3}$.

a Find the transformation matrices $\mathbf{A}$ and $\mathbf{B}$.

b Determine the nature of the composite transformation:

i $\mathbf{x}' = \mathbf{ABx}$ **ii** $\mathbf{x}' = \mathbf{BAx}$

c Use matrices to explain why the order is generally important in the composition of two linear transformations.

2 Find the single linear transformation equivalent to:

a a reflection in the x-axis followed by an anticlockwise rotation of $\frac{\pi}{2}$ about O

b an anticlockwise rotation through $\frac{2\pi}{3}$ about O followed by a reflection in the line $y = -x$

c a reflection in the line $y = \sqrt{3}x$ followed by a reflection in the y-axis

d a reflection in the line $y = x$ followed by a reflection in the line $y = 3x$.

3 Find the single transformation equivalent to the following composite transformations. In each case, discuss whether the composite transformation would be different if the order of transformations was reversed.

a A horizontal stretch with scale factor k followed by a vertical stretch with scale factor k.

b A rotation about O through θ followed by a rotation about O through ϕ.

c A reflection in the line $y = (\tan\alpha)x$ followed by a reflection in the line $y = (\tan\beta)x$.

4 Write down the matrix equation for each affine transformation:

a A reflection in the line $y = \frac{1}{2}x$ followed by a translation through $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

b An anticlockwise rotation through $\frac{\pi}{6}$ about O followed by a translation through $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

c A horizontal stretch with scale factor $\frac{1}{3}$ followed by a translation through $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

5 Show that a reflection in a line through O, followed by a rotation about O, is equivalent to a reflection in another line through O.

6 **a** Write down the matrix equation for a translation through $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ *followed by* an enlargement with scale factor 2.

b Show that this composite transformation is an affine transformation.

c Show that the composition of a translation through $\mathbf{c}$ followed by a linear transformation with transformation matrix $\mathbf{A}$, is an affine transformation of the form $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$.

7 What transformation is needed before a clockwise rotation through $\frac{\pi}{2}$ about O, in order to give a reflection in the line $y = \frac{1}{2}x$?

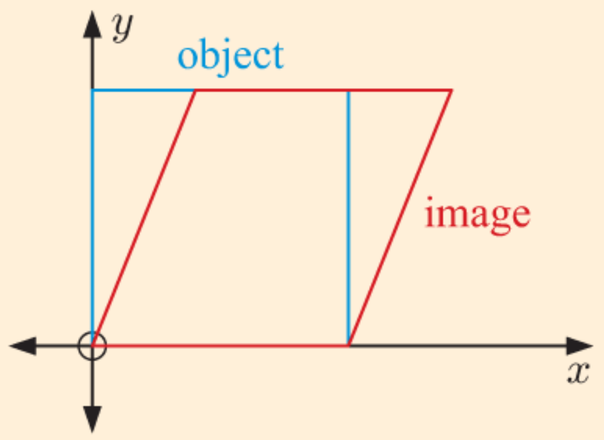
INVESTIGATION 3 PROPERTIES OF LINEAR TRANSFORMATIONS

We have seen that linear transformations are a subset of affine transformations. They have a number of useful properties which are not shared by affine transformations in general.

What to do:

1 There is a unique point which is invariant under *any* linear transformation. Identify this point and explain why it is always invariant.

2 The linear transformation $\mathbf{x}' = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\mathbf{x}$ is called a **shear** parallel to the x-axis.



a The square OABC with vertices $O(0, 0)$, $A(1, 0)$, $B(1, 1)$, and $C(0, 1)$ is called the **unit square**.

Perform a shear parallel to the x-axis with $k = \frac{1}{2}$ on each vertex of the unit square. Illustrate your answer.

b Show that the transformation matrix has eigenvalue 1 with corresponding eigenvectors $t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $t \in \mathbb{R}$.

c What is special about object points $\mathbf{x}$ which are themselves eigenvectors of the transformation matrix? Explain your answer using the definitions of eigenvalues and eigenvectors.

3 A shear parallel to the y-axis has the transformation matrix $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$.

a Perform a shear parallel to the y-axis and with $k = \frac{3}{2}$ on each vertex of the unit square.

b Show that the transformation matrix has eigenvalue 1 with corresponding eigenvectors $t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

c What is special about object points $\mathbf{x}$ which are themselves eigenvectors of the transformation matrix? Compare your answer to **2 c**.

4 **a** For each of the following linear transformations, find the eigenvalues and corresponding eigenvectors of the transformation matrix:

i $\mathbf{x}' = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\mathbf{x}$ **ii** $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\mathbf{x}$

b Apply each transformation in **a** to its eigenvectors. Comment on your observations.

5 Suppose the linear transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ moves the point P to its image point P′.

Explain why P′ lies on the line (OP) if and only if $\overrightarrow{OP}$ is an eigenvector of $\mathbf{A}$.

6 Consider an anticlockwise rotation about the origin through angle θ.
Suppose an object point is located by a position vector which is an eigenvector $\mathbf{v}$ of the transition matrix $\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

a For what values of θ will the image point be located by a scalar multiple of $\mathbf{v}$?

b Show that the eigenvalues of $\mathbf{A}$ are $e^{\pm i\theta}$.

c Explain the result in **a** by considering values of θ for which the eigenvalues of $\mathbf{A}$ are real. Show that there is a repeated eigenvalue which is either 1 or -1.

d For either of the possible repeated eigenvalues in **c**, use the rotation to explain how *every* non-zero vector in the plane is a corresponding eigenvector. Prove your result algebraically in each case.

ACTIVITY 1 REPEATED AFFINE TRANSFORMATIONS

In this Activity, we will consider what happens when we repeatedly apply an affine transformation to a point and its successive images.

What to do:

1 Suppose the transformation $\mathbf{x}' = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix}\mathbf{x}$ is repeatedly applied to the point $\mathrm{P}(1, 0)$ and its successive images.

a Explain why the transformation is a rotation, and find the angle of rotation.

b Draw a diagram to illustrate how, under the transformation:

i P is mapped onto its image P'

ii P' is mapped onto its image P''

iii P'' is mapped onto its image P'''.

c Predict what will happen if we continue applying the transformation to successive image points.

d Click on the icon to obtain a spreadsheet that applies the transformation and plots the image points.
Use the spreadsheet to check your prediction.

SPREADSHEET



e Experiment with different initial object points. Do you observe the same behaviour?

2 Let $\mathbf{A}_1 = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ and $\mathbf{A}_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}$.

a Calculate $\mathbf{A}_1\mathbf{A}_2$.

b Describe the composite transformation $\mathbf{x}' = \mathbf{A}_1\mathbf{A}_2\mathbf{x}$.

c Predict what will happen if we continue applying the transformation in **b** to the point $\mathrm{P}(1, 0)$ and its successive images.

d Use the spreadsheet to check your prediction.

e **i** Predict what would happen if $\mathbf{A}_2$ was replaced by $\mathbf{A}_2 = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}$.

ii Recalculate $\mathbf{A}_1\mathbf{A}_2$ and use the spreadsheet to check your prediction.

3 Repeat **2** but with $\mathbf{A}_1 = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{pmatrix}$. Compare and contrast your observations.

4 Consider the transformation $\mathbf{x}' = \begin{pmatrix} 0.6 & 0.8 \\ -0.8 & k \end{pmatrix}\mathbf{x} + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$.

a Use your spreadsheet with $k = 0.6$ to observe what happens when we repeatedly apply this transformation. Can you explain this behaviour?

b Vary the value of k slightly and discuss your results. For example, try $k = 0.55$ and $k = 0.65$.

5 Experiment with repeating different types of transformations (or composites thereof). You may want to consider stretches and shears, both with and without translations.

G AREA

INVESTIGATION 4 AREA

When the affine transformation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ is performed on an object, the *size* or *shape* of the object may be altered.

In this Investigation we will explore the relationship between the ratio $\dfrac{\text{area of image}}{\text{area of object}}$ and $|\det \mathbf{A}|$.

We will do this by considering transformations of the unit square OABC with vertices $\text{O}(0, 0)$, $\text{A}(1, 0)$, $\text{B}(1, 1)$, and $\text{C}(0, 1)$.

What to do:

1 The basic transformations we have studied are written in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$ as:

Translation through $\begin{pmatrix} e \\ f \end{pmatrix}$	$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}$
Rotation about O through θ	$\mathbf{A} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Reflection in $y = (\tan\alpha)x$	$\mathbf{A} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Horizontal stretch with scale factor k	$\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Vertical stretch with scale factor k	$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
Enlargement with scale factor k	$\mathbf{A} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

a For each transformation, decide whether it will change the *size* of an object. If it will, find the ratio $\dfrac{\text{area of image}}{\text{area of object}}$.

b Find $\det \mathbf{A}$ for each transformation matrix. Comment on your answers.

2 For each transformation matrix **A** below:

i Apply the transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ to the unit square OABC.

ii Illustrate OABC and its image O′A′B′C′.

iii Calculate $\det \mathbf{A}$.

iv Calculate $\dfrac{\text{area of O}'\text{A}'\text{B}'\text{C}'}{\text{area of OABC}}$.

a $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ **d** $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}$

3 Summarise your observations.

From the **Investigation**, you should have found that:

For the transformation $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$, area of image $= |\det \mathbf{A}| \times$ area of object.

EXERCISE 14G

1 For each transformation, find the ratio $\dfrac{\text{area of image}}{\text{area of object}}$:

a $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$ **b** $\mathbf{x}' = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\mathbf{x}$

c $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ **d** $\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ -2 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

2 For each transformation:

i Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

ii Find the ratio $\dfrac{\text{area of image}}{\text{area of object}}$.

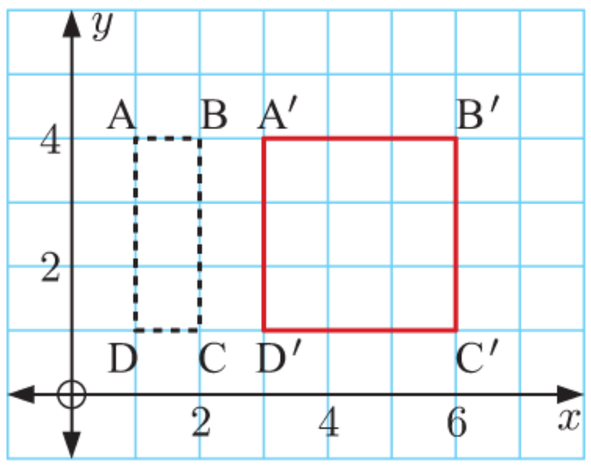
a a translation through $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ **b** a rotation about O through $\frac{\pi}{3}$

c a reflection in the x-axis **d** a horizontal stretch with scale factor 2

e a vertical stretch with scale factor $\frac{1}{3}$ **f** an enlargement with scale factor $\frac{4}{3}$.

3 A **shear** parallel to the x-axis with shear factor k has equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.
Determine whether this transformation will change the area of an object.

4



An affine transformation maps rectangle ABCD onto square A′B′C′D′.

a Identify the transformation.

b Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

c Verify that $\dfrac{\text{area A}'\text{B}'\text{C}'\text{D}'}{\text{area ABCD}} = |\det \mathbf{A}|$.

5 A$(-1, 0)$, B$(2, -2)$, C$(5, 1)$, and D$(2, 7)$ are the vertices of quadrilateral ABCD. A vertical stretch with scale factor $1\frac{1}{2}$ is applied to ABCD.

a Illustrate the object and image.

b Find the area of ABCD.

c Write down the transformation equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$.

d *Hence* find the area of the image.

6 The circle $x^2 + y^2 = 9$ is transformed by a vertical stretch with scale factor $\frac{5}{3}$.

a Illustrate the object and image.

b Find the area of the image.

7 The circle $x^2 + y^2 = a^2$ is transformed by a horizontal stretch with scale factor $\frac{b}{a}$, $b > 0$.

a Illustrate the object and image.

b Prove that the image has area πab.

8 P$(1, 1)$, Q$(4, 1)$, and R$(2, 2)$ are vertices of a triangle. The triangle is transformed by $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

a Find the image points P′, Q′, and R′.

b Illustrate the object and image on the same set of axes.

c Find the area of P′Q′R′.

9 **a** Find the transformation matrix for a vertical stretch with scale factor k.

b Find the single transformation matrix for a vertical stretch with scale factor 2, followed by a reflection in the line $y = -2x$.

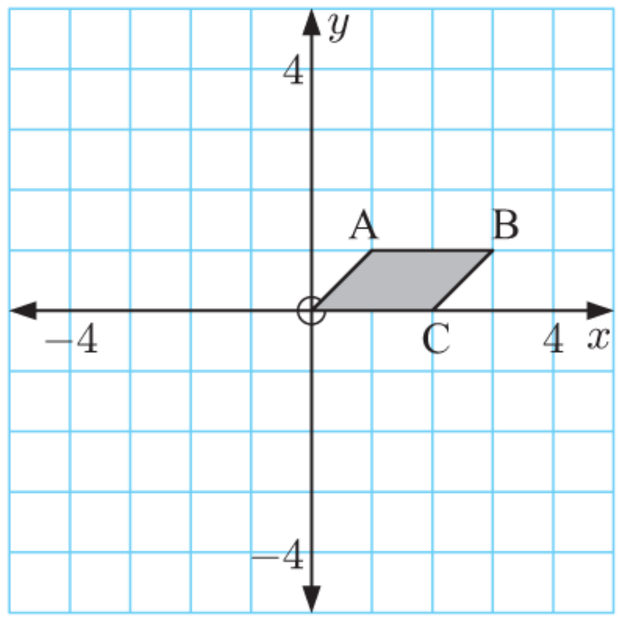
c What effect does the composition of transformations in **b** have on area?

10 The parallelogram OABC is vertically stretched with scale factor $\frac{1}{2}$, reflected in the line $y = -x$, then enlarged with scale factor 3.

a Find the single transformation matrix for this sequence of transformations.

b Find the image coordinates O′, A′, B′, and C′.

c Find the area of the image.



11 A sequence of linear transformations is to be performed on an object. Does the *order* in which the transformations are performed affect the area of the image? Explain your answer.

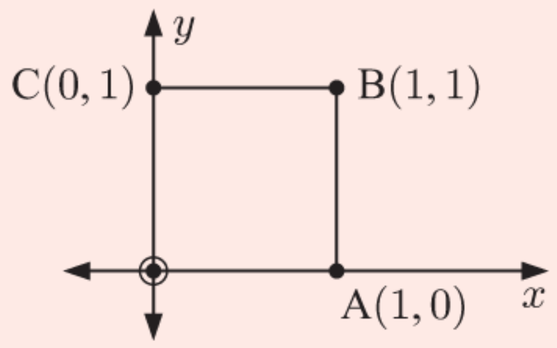
DISCUSSION

Under a linear transformation, will the shape of an object be preserved in its image?

For example, will a line always remain a line, a circle always remain a circle, and so on?

ACTIVITY 2 — SENSE

The labelling of the unit square OABC is anticlockwise. When OABC is transformed to O′A′B′C′, if the labelling of O′A′B′C′ is also anticlockwise, we say that **sense** has been **preserved** by the transformation. Otherwise, we say that sense has been **reversed**.



What to do:

1 For each matrix **A** below:

i Illustrate OABC and its image O′A′B′C′ under the transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

ii Find $\det \mathbf{A}$. **iii** Determine whether sense has been preserved or reversed.

a $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **c** $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ **d** $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ -2 & 0 \end{pmatrix}$

2 Deduce the connection between $\det \mathbf{A}$ and sense for a linear transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

3 Discuss the effect of the following transformations on sense:

a a rotation **b** a reflection

c an enlargement **d** a horizontal or vertical stretch

e a rotation followed by a reflection **f** an even number of reflections

g an odd number of reflections

ACTIVITY 3 — FRACTAL GEOMETRY

In *The Fractal Geometry of Nature* (1982), **Benoit B. Mandelbrot** (1924 - 2010) argued that many natural structures and processes are modelled by the geometry of fractals rather than the objects of classical geometry.

> "*Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.*"

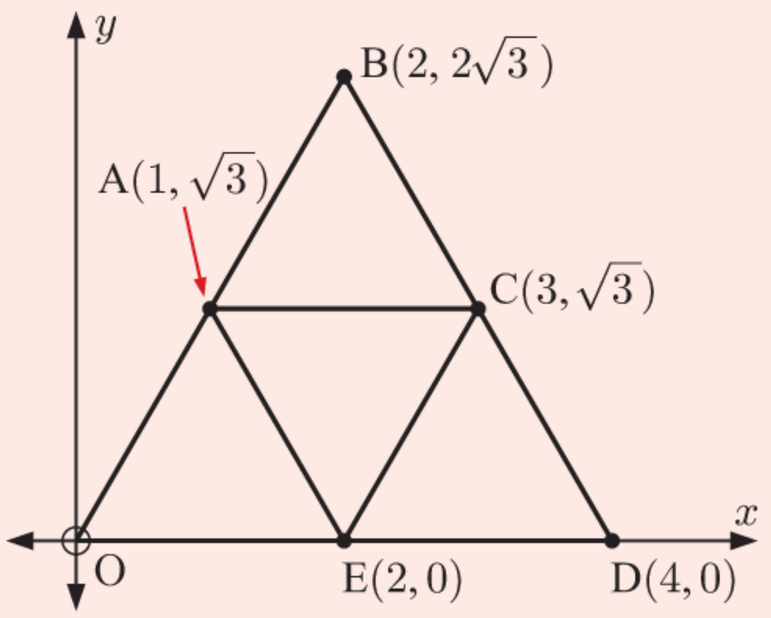
Inspired by the work of Mandelbrot, **John Hutchinson** sought to formalise the study of fractals through **iterated function systems** (IFS) in 1981. The name "iterated function systems", and their subsequent popularity was largely due to **Michael Barnsley**, author of *Fractals Everywhere* (1988).

Some fractals, including the Sierpiński triangle in the **Opening Problem**, may be generated by an iterated function system of affine transformations.

PART 1: SIERPIŃSKI'S TRIANGLE

The diagram alongside shows the first iteration Sierpiński triangle. It is generated from the triangle OBD using the affine IFS $\{f_1, f_2, f_3\}$ where:

- f_1 is the transformation mapping $\triangle$OBD to $\triangle$OAE
- f_2 is the transformation mapping $\triangle$OBD to $\triangle$ECD
- f_3 is the transformation mapping $\triangle$OBD to $\triangle$ABC.



What to do:

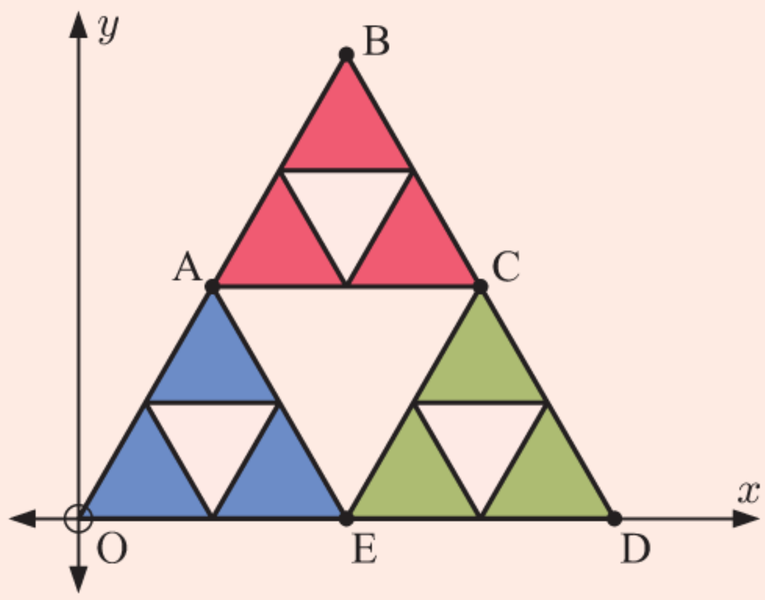
1 Describe in words, the transformation:

a $\triangle$OBD to $\triangle$OAE **b** $\triangle$OAE to $\triangle$ECD **c** $\triangle$OAE to $\triangle$ABC.

2 Hence, or otherwise, use an affine matrix equation to define the transformation:

a f_1 **b** f_2 **c** f_3

3 The *second iteration* Sierpiński triangle is shown alongside.



a Describe the transformation(s) mapping the first iteration to:

i the blue region

ii the green region

iii the red region.

b Compare these transformations with $\{f_1, f_2, f_3\}$. What do you notice?

4 Click on the icon to see higher iterations of Sierpiński's triangle.

SIERPIŃSKI'S TRIANGLE



5 **a** Show that the area of $\triangle$OBD is $4\sqrt{3}$ units^2.

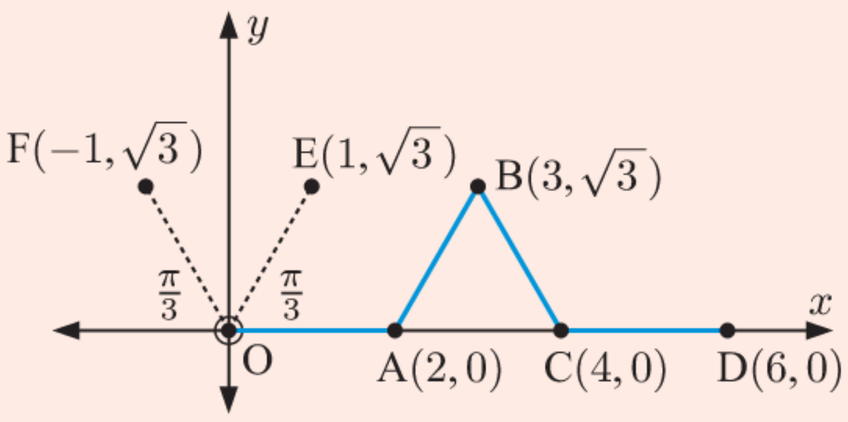
b Explain why the areas of the iterations of the Sierpiński triangle form a geometric sequence with common ratio $\frac{3}{4}$.

c The complete **Sierpiński triangle** is the object when $n \to \infty$. Explain why the complete Sierpiński triangle has no area.

PART 2: VON KOCH'S CURVE

The von Koch curve begins with the line segment [OD] shown. Its first iteration, shown in blue, is generated by the affine IFS $\{f_1, f_2, f_3, f_4\}$ where:

- f_1 is the transformation mapping [OD] to [OA]
- f_2 is the transformation mapping [OD] to [AB]
- f_3 is the transformation mapping [OD] to [CB]
- f_4 is the transformation mapping [OD] to [CD].



What to do:

1 Describe in words, the transformation:

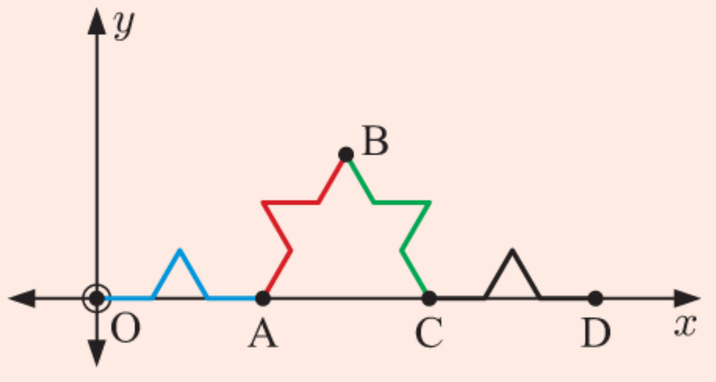
a [OD] to [OA] **b** [OA] to [OE] **c** [OE] to [AB]

d [OA] to [OF] **e** [OF] to [CB] **f** [OA] to [CD].

2 Hence, or otherwise, use an affine matrix equation to define the transformation:

a f_1 **b** f_2 **c** f_3 **d** f_4

3 The *second iteration* of the von Koch curve is shown alongside.



a Describe the transformation(s) mapping the first iteration to:

i the blue curve **ii** the red curve

iii the green curve **iv** the black curve.

b Compare these transformations with $\{f_1, f_2, f_3, f_4\}$. What do you notice?

4 Click on the icon to see higher iterations of the von Koch curve.



5 **a** Explain why the length of the nth iteration is given by $6\left(\frac{4}{3}\right)^n$.

b The complete **von Koch curve** is the object as $n \to \infty$. Discuss the length of the complete von Koch curve.

PART 3: BARNSLEY FERN

In the IFS for the Sierpiński triangle and von Koch curve, every transformation is applied at each iteration. However, fractals may also be generated from an IFS in a different way.

The **Barnsley fern** is a fractal generated by an affine IFS $\{f_1, f_2, f_3, f_4\}$ where each affine transformation is assigned a probability, p_1, p_2, p_3, and p_4, respectively.

The *first iteration* consists of the origin $(0, 0)$, and the image of $(0, 0)$ under *one* of the maps f_1, f_2, f_3, or f_4, chosen at random. So, the first iteration consists of 2 points.

The *second iteration* consists of the first iteration, and the image of the first iteration under one of the maps f_1, f_2, f_3, or f_4, again chosen at random. So, the second iteration consists of 4 points.



Click on the icon to see higher iterations of this process. The complete Barnsley fern is the object as $n \to \infty$.

REVIEW SET 14A

1 Find the image when the point $(-4, 1)$ is translated through:

a $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ **d** $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$

2 Find the object point which produces each image under the translation $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$:

a $(1, 5)$ **b** $(0, -1)$ **c** $(-6, -3)$

3 Find the transformation matrix **A** for:

a an anticlockwise rotation about O through $\frac{4\pi}{3}$

b a reflection in the line $y = -3x$

c a vertical stretch with scale factor $\frac{4}{3}$.

4 Find the nature of the linear transformation with equations:

a $\begin{cases} x' = \dfrac{-3x + 4y}{5} \\ y' = \dfrac{4x + 3y}{5} \end{cases}$ **b** $\begin{cases} x' = \dfrac{12x - 5y}{13} \\ y' = \dfrac{5x + 12y}{13} \end{cases}$

5 Find the image of $(3, -4)$ under an anticlockwise rotation about O through $\frac{7\pi}{6}$.

6 Describe the transformation $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for which **A** is:

a $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$

7 Consider a horizontal stretch with scale factor $\frac{7}{2}$.

a Write down the transformation matrix $\mathbf{A}$.

b Hence find the image of $(6, -4)$ under the transformation.

c Find $\mathbf{A}^{-1}$ and describe the transformation $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$.

8 Find the image of $(-2, 4)$ under:

a a reflection in the line $y = -\frac{1}{\sqrt{3}}x$ **b** an enlargement with scale factor $\frac{9}{4}$.

9 Find the single linear transformation equivalent to:

a an anticlockwise rotation through $\frac{\pi}{3}$ followed by a reflection in the line $y = x$

b a reflection in the line $y = -\sqrt{3}x$ followed by a reflection in the x-axis.

10 **a** Find the matrix equation for a vertical stretch with scale factor 4, followed by a reflection in the line $y = -x$.

b Find the image of $(-3, 5)$ under this composite transformation.

11 For each transformation:

i Write down the transformation matrix. **ii** Find the ratio $\dfrac{\text{area of image}}{\text{area of object}}$.

a a horizontal stretch with scale factor 5 **b** a reflection in the y-axis

c a reduction with scale factor $\frac{1}{4}$.

12 A$(-1, -2)$, B$(3, -2)$, and C$(1, 0)$ are the vertices of a triangle. The triangle is transformed by $\mathbf{x}' = \begin{pmatrix} 3 & -1 \\ 2 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -4 \\ -6 \end{pmatrix}$.

a Find the image points A′, B′, and C′.

b Illustrate the object and image on the same set of axes.

c Find the area of A′B′C′.

REVIEW SET 14B

1 Find the image of each point under the translation $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$:

a $(2, 6)$ **b** $(-1, 1)$ **c** $(-7, -2)$

2 Find the translation vector which maps:

a $(1, 3)$ onto $(4, 5)$ **b** $(-2, -1)$ onto $(4, -2)$ **c** $(8, -1)$ onto $(-1, -4)$.

3 Find the transformation matrix $\mathbf{A}$ for:

a a clockwise rotation about O through $\frac{\pi}{6}$ **b** a horizontal stretch with scale factor $\frac{5}{3}$

c a reflection in the line $y = -\frac{1}{2}x$.

4 Find the nature of the linear transformation(s) with transformation matrix:

a $\begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$ **c** $\begin{pmatrix} -\frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$

5 Find the image of $(-1, 5)$ under a reflection in the line $y = -3x$.

6 Consider a vertical stretch with scale factor $\frac{3}{5}$.

a Write down the transformation matrix $\mathbf{A}$.

b Hence find the image of $(-4, -2)$ under the transformation.

c Find $\mathbf{A}^{-1}$ and describe the transformation $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$.

7 **a** The point $\mathrm{A}(3, 1)$ is reflected in the line $y = 2x$.

i Find the coordinates of the image point B.

ii Show that $\mathrm{OA} = \mathrm{OB}$.

b Show that if any point P is reflected in the line $y = (\tan\alpha)x$ to P′, then $\mathrm{OP} = \mathrm{OP}'$.

8 Find the single transformation equivalent to:

a a reflection in the line $y = \frac{1}{\sqrt{3}}x$, then an anticlockwise rotation about O through $-\frac{\pi}{6}$

b a reflection in the x-axis, then a reflection in the y-axis.

9 Write down the matrix equation for the affine transformation:

a an enlargement with scale factor 3 followed by a translation through $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b a translation through $\begin{pmatrix} 2 \\ -7 \end{pmatrix}$ followed by a horizontal stretch with scale factor 6.

10 Let $\mathbf{A}$ be the transformation matrix for an anticlockwise rotation about O through $\frac{3\pi}{4}$, and $\mathbf{B}$ be the transformation matrix for an enlargement with scale factor $\frac{3}{2}$.

a Find the transformation matrices $\mathbf{A}$ and $\mathbf{B}$.

b Determine the nature of the composite transformation $\mathbf{x}' = \mathbf{ABx}$.

c Find the image of $(-4, 5)$ under this transformation.

d Does $\mathbf{AB} = \mathbf{BA}$? Interpret this result in the context of the transformations.

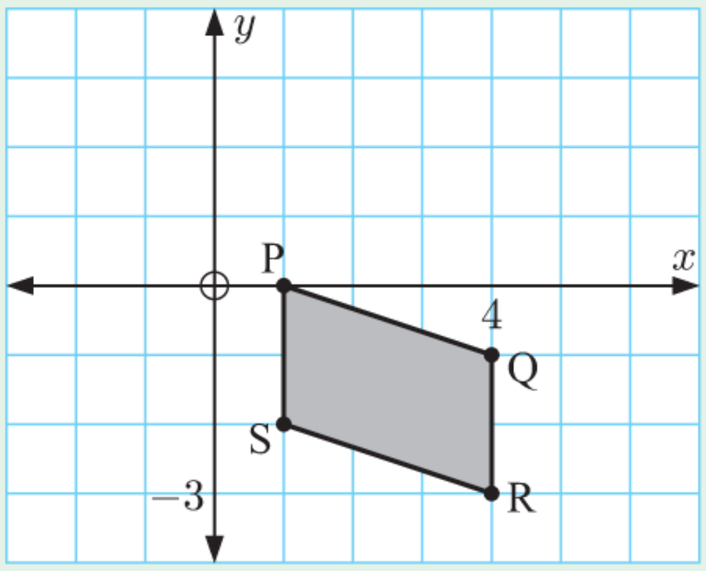
11 A circle with area 9π cm^2 is transformed by a vertical stretch with scale factor $\frac{5}{3}$.

a Describe the shape of the image. **b** Find the area of the image.

12 The quadrilateral PQRS is rotated anticlockwise about O through $\frac{2\pi}{3}$, then horizontally stretched with scale factor 4.

a Find the exact image coordinates of P′, Q′, R′, and S′.

b Find the area of P′Q′R′S′.



Chapter 15

Graph theory

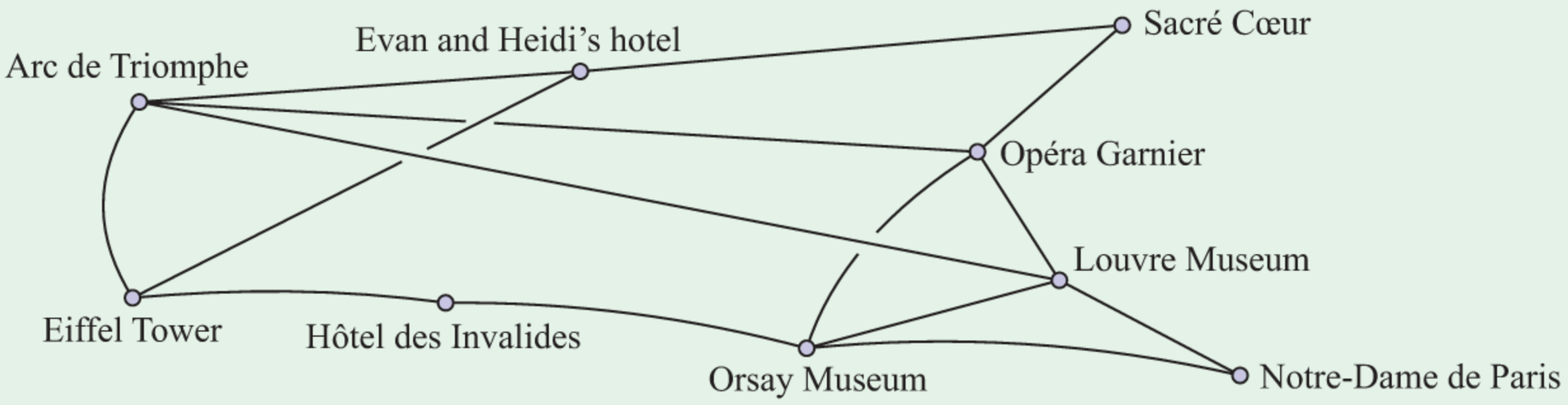
Contents:

A Graphs
B Properties of graphs
C Routes on graphs
D Adjacency matrices
E Transition matrices for graphs
F Trees
G Minimum spanning trees
H Eulerian graphs
I The Chinese Postman Problem
J Hamiltonian graphs
K The Travelling Salesman Problem



OPENING PROBLEM

Evan and Heidi are visiting Paris. This map shows their hotel, the landmarks they would like to visit, and the roads connecting them.



Things to think about:

a Evan would like to visit each landmark exactly once, finishing back at their hotel. Can you find a route suitable for Evan?

b Heidi is more interested in the sights and scenery between the landmarks. She wishes to walk along each road on the map exactly once, finishing back at their hotel.

i Is there a route suitable for Heidi? Explain your answer.

ii Can you find a suitable route for Heidi which starts at their hotel but finishes at the Eiffel Tower?



Graphs are used to show how things are connected. They can be used to help solve problems in train scheduling, traffic flow, bed usage in hospitals, and project management.

The theory of graphs was developed centuries ago, but the application of the theoretical ideas is relatively recent. Real progress was made during and after the Second World War as mathematicians, industrial technicians, and members of the armed services worked together to improve military operations.

Businesses and governments use graphs to find *optimum solutions* to practical problems. These may be the solutions which cost the least, take the least time, or require the smallest amount of material.

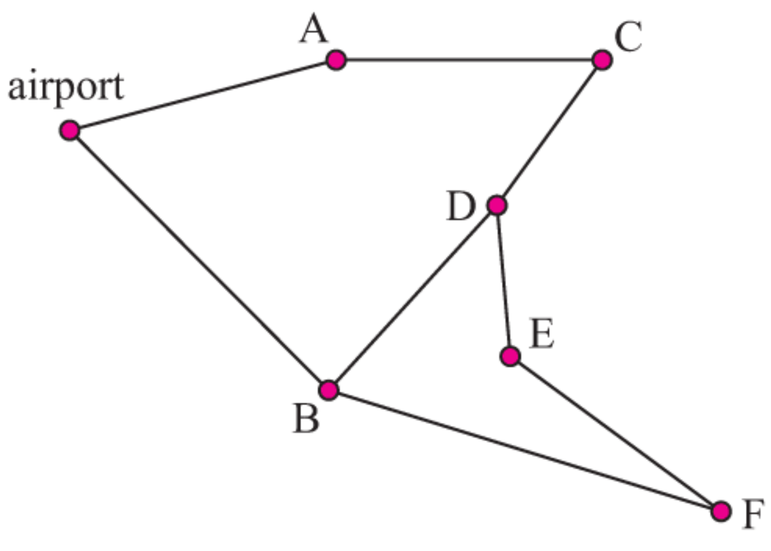
A GRAPHS

A **graph** or **network** is a structure which shows the physical connections or relationships between things of interest.

The things of interest are represented by **dots** or **points**, and the connections or relationships are represented by **lines**.

For example, this graph shows the road connections between several towns and the airport which serves them.

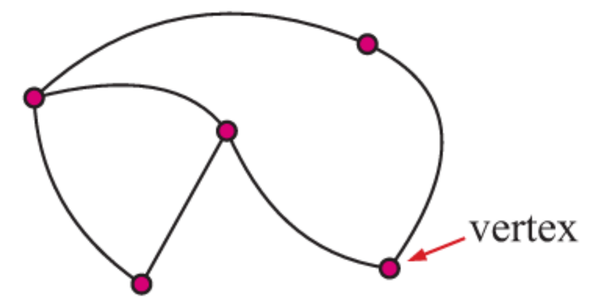
We can see that towns A and C are directly connected by road, but towns C and F are not.



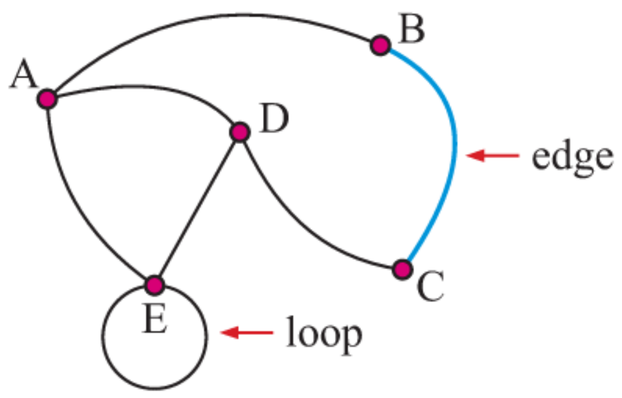
The lines on a graph do not always represent physical connections. They can also represent relationships between objects or people, such as teams who play one another in a sporting tournament, or who is directly related to whom in a family.

TERMINOLOGY

- The points on a graph are known as **vertices** (singular vertex).



- We can move from one vertex to another along lines called **edges**.
  Edges are named according to the vertices at either end. For example, the blue edge is named BC.
  An edge from a vertex to itself is called a **loop**.

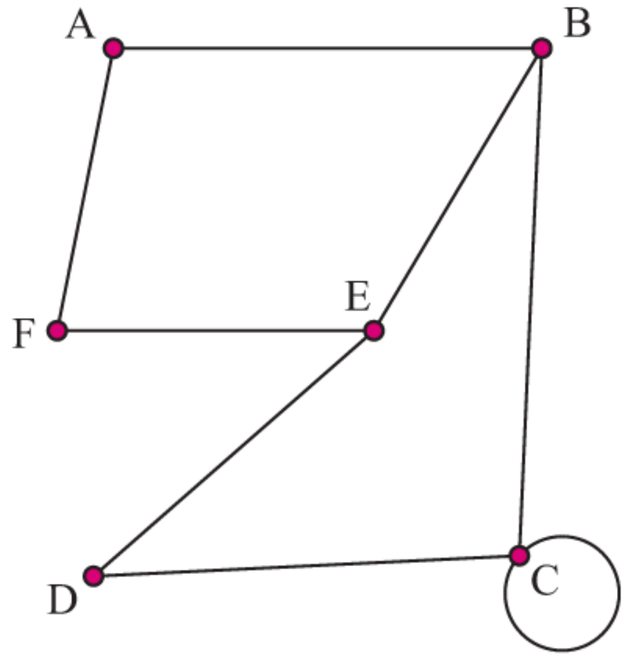


UNDIRECTED GRAPHS

If we are allowed to move in either direction along the edges of a graph, the graph is **undirected**.

In an undirected graph:

- **Adjacent vertices** are vertices which are connected by an edge.
  For example, A and B are adjacent vertices, but D and F are not.
- **Adjacent edges** are edges which share a common vertex.
  For example, AB and BE are adjacent edges, but AB and CD are not.
- The **degree** of a vertex is the number of edges connected to it.
  For example, A has degree 2 and E has degree 3. A loop contributes 2 to the degree of a vertex, so C has degree 4.



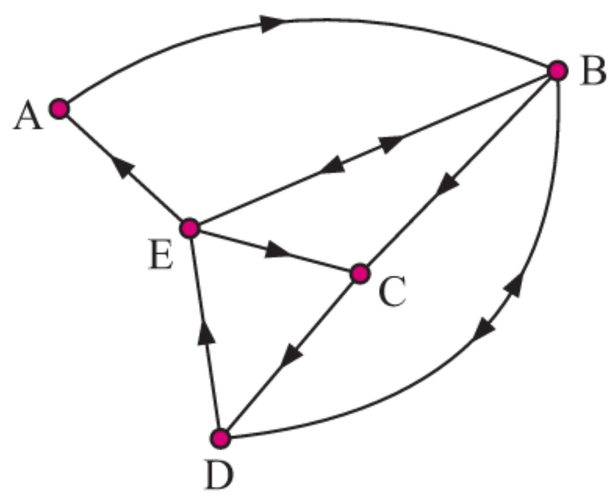
DIRECTED GRAPHS

A **directed graph** contains arrows indicating the direction we can move along the edges.

In a directed graph:

- The **in degree** of a vertex is the number of edges coming in to the vertex.
- The **out degree** of a vertex is the number of edges going out from the vertex.

For example, vertex E has in degree 2 and out degree 3.



EXERCISE 15A

1 **a** State the number of vertices and edges in this graph.

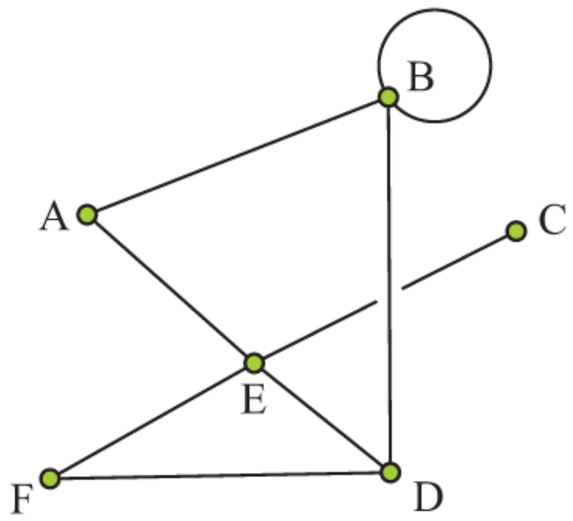
b Name the vertices adjacent to:

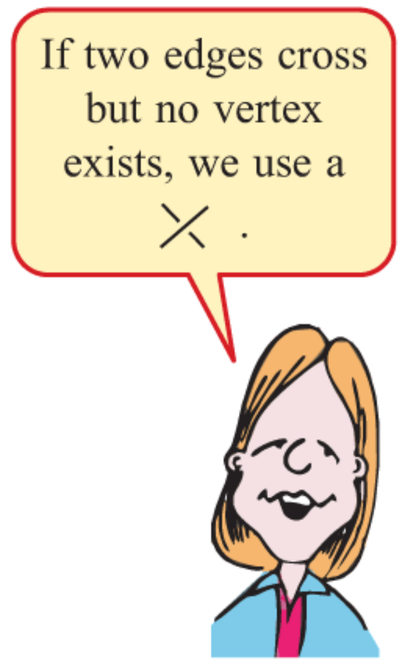
i A **ii** D

c Name the edges adjacent to:

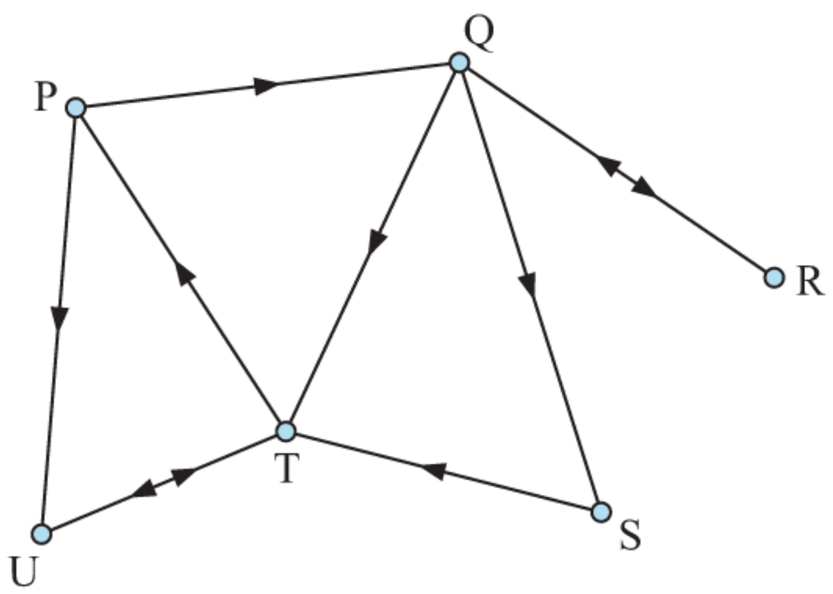
i CE **ii** DF

d State the degree of each vertex.





2



For the directed graph alongside:

a Copy and complete:

Vertex	*in degree*	*out degree*
P		
Q		
R		
S		
T		
U		

b Describe how you could get from S to R.

3 A computer network consists of seven computers connected to a server. The server is also connected to a modem and two printers.

Draw a graph to illustrate this network, using:

a a vertex for each individual component

b a vertex for each group of common components.



4 A network of roads connects the towns P, Q, R, S, and T.

- P is connected to Q, S, and T.
- R is connected to Q.
- T is connected to P and S.
- Q is connected to P, R, and S.
- S is connected to P, Q, R, and T.

a How can you tell from the given information that the graph is directed?

b Draw a graph to display this information.

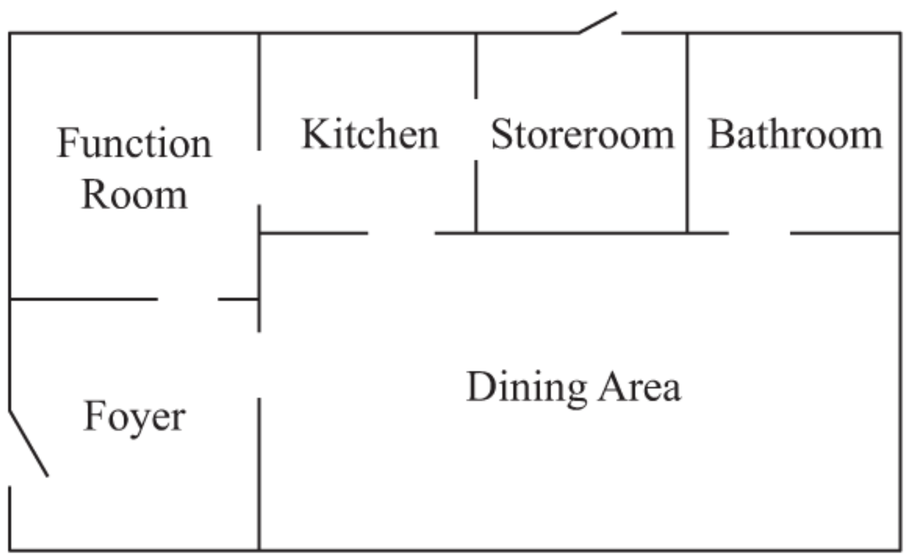
5 A meeting consists of two people (A and B) who can only speak English, three people (C, D, and E) who can only speak Chinese, and an interpreter (I) who can speak both languages.

a Draw a graph to illustrate the people who can communicate directly.

b How many people can communicate directly with:

i B **ii** E **iii** I?

6



The layout of a restaurant is shown alongside.

a Draw a graph to describe the layout. Use a single vertex to represent the exterior of the restaurant.

b How many rooms are connected to:

i the kitchen

ii the dining area?

7 The weights of members of a judo club are shown alongside.

Members are only allowed to compete against each other if the difference between their weights is less than 10 kg.

a Draw a graph showing which members can compete against each other.

b Find the degree of the vertex corresponding to Warren. Interpret your answer.

c Copy your graph in **a**. Make the copy a directed graph by indicating which member is heavier in each contest. For example, D → M indicates that Declan is heavier than Matt.

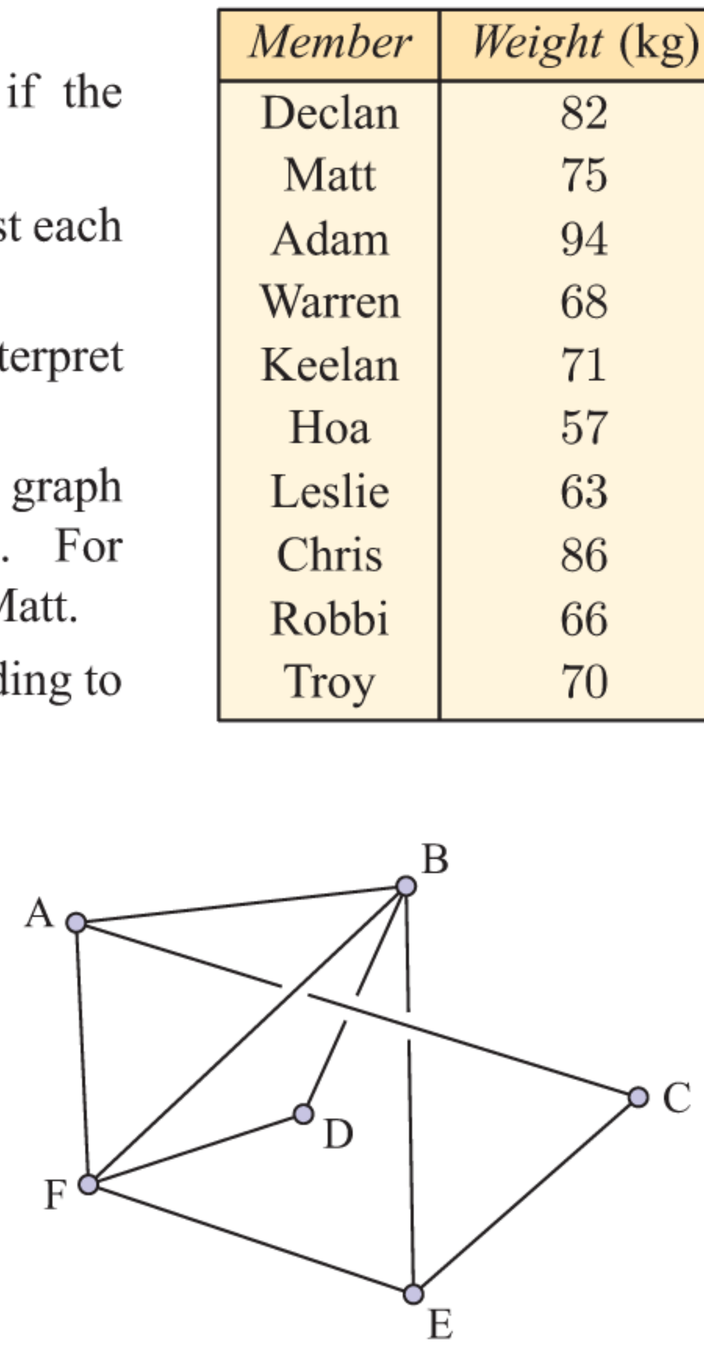
d Find the in degree and out degree of the vertex corresponding to Keelan. Interpret your answers.

Member	*Weight* (kg)
Declan	82
Matt	75
Adam	94
Warren	68
Keelan	71
Hoa	57
Leslie	63
Chris	86
Robbi	66
Troy	70

8 **a** For the graph alongside, find the:

i number of edges

ii sum of the degree of the vertices.



b Establish a general formula between the number of edges e on an undirected graph, and the sum of the degrees of the vertices d. Explain your formula.

c An undirected graph has 7 vertices, with degrees 1, 2, 2, 3, 4, 5, and 5. How many edges does the graph have?

d Explain why an undirected graph must have an even number of vertices of odd degree.

9 In an undirected graph, is the degree of a vertex necessarily equal to the number of adjacent vertices? Explain your answer.

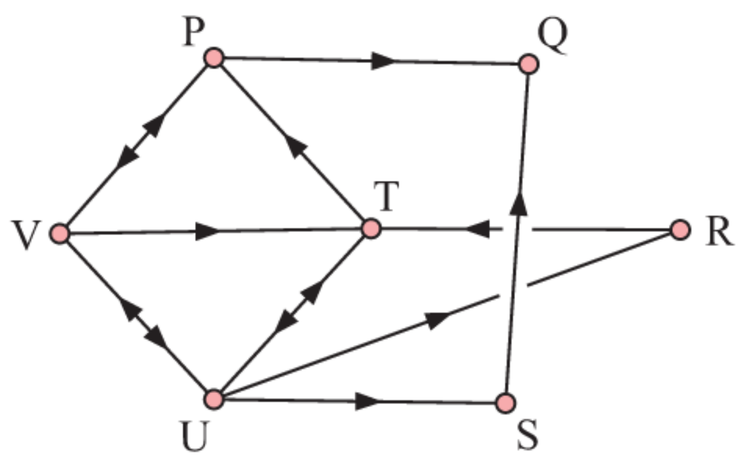
10 **a** Find the in degree and out degree for each vertex of this directed graph.

b Find the:

i sum of the in degrees of the vertices

ii sum of the out degrees of the vertices.

Explain your answers.



11 Suppose a directed graph has 5 vertices. Each vertex has the same in degree k, but the out degrees of the vertices are all distinct.

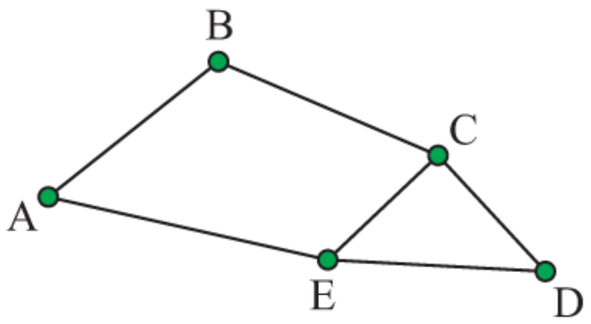
a Find the smallest possible value for k.

b Draw an example of such a graph.

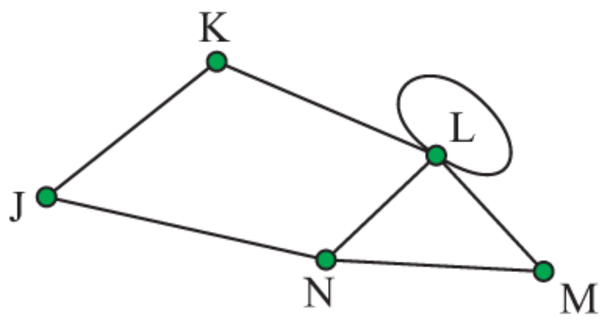
B PROPERTIES OF GRAPHS

The following terms are used to describe graphs:

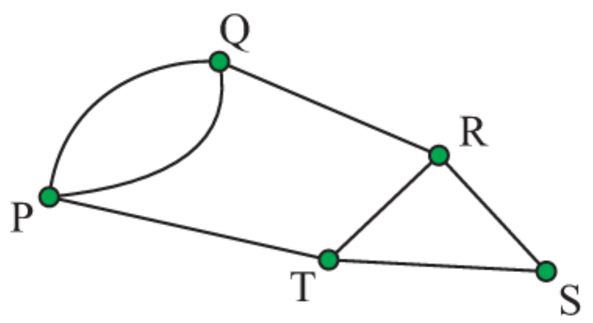
- A graph is called **simple** if it contains no loops, and if there is a maximum of one edge joining any pair of distinct vertices.



simple graph

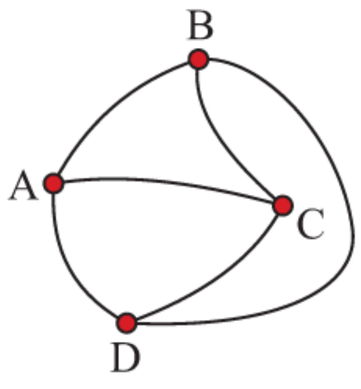


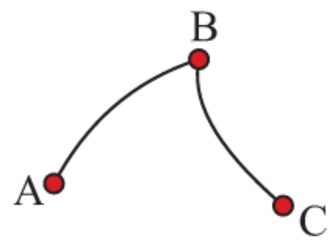
not a simple graph as there is a loop at L



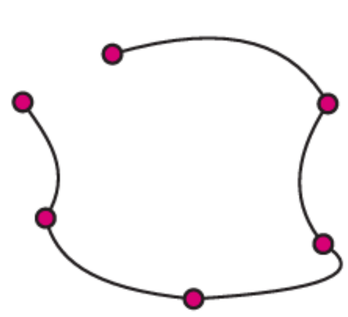
not a simple graph as there are two edges joining P and Q

- A **subgraph** is a graph that is made from a subset of the vertices and edges of another graph.

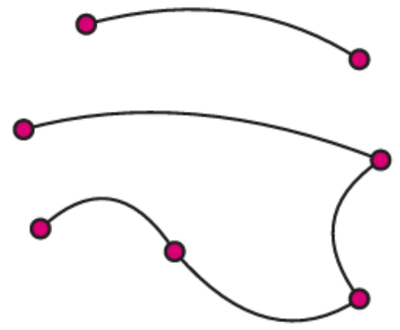
For example, a subgraph of 

is 

- An undirected graph is **connected** if it is possible to travel from every vertex to every other vertex by following edges.
  Otherwise, the graph is **disconnected**.

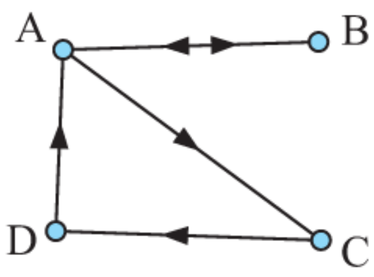


a connected graph

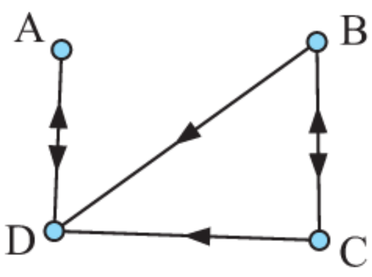


a disconnected graph

- A directed graph is **strongly connected** if it is possible to travel from every vertex to every other vertex by following edges in the correct direction.

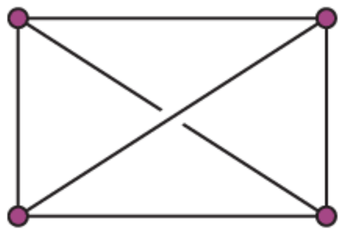


a strongly connected graph

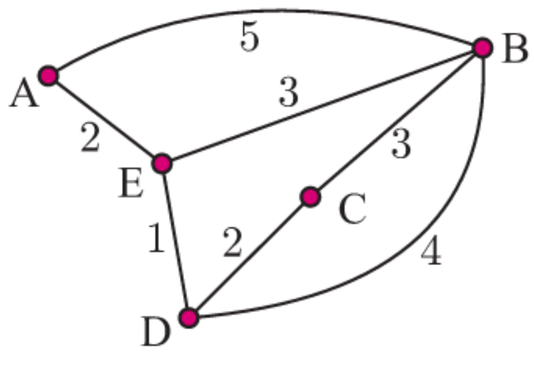


not strongly connected as we cannot travel from A to B

- A **complete graph** is a simple undirected graph in which every vertex is connected by an edge to every other vertex.
  The notation k_n is used to describe the complete graph with n vertices.
  For example, the complete graph with 4 vertices k_4 is shown alongside.



- A **weighted graph** has numbers assigned to its edges. These numbers may represent the cost, time, or distance required to travel along that edge.
  The lengths of the edges are not drawn in proportion to their weight.



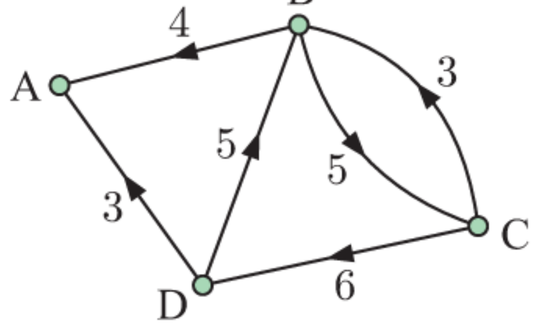
A **weighted adjacency table** is used to summarise the weights on a graph.

The symbol – indicates that the two vertices are not adjacent.

For a weighted undirected graph, the weighted adjacency table is symmetric about the leading diagonal.

	A	B	C	D	E
A	–	5	–	–	2
B	5	–	3	–	3
C	–	3	–	2	–
D	–	–	2	–	1
E	2	3	–	1	–

For a weighted directed graph, the weighted adjacency table will not necessarily be symmetrical.



From \ *To*

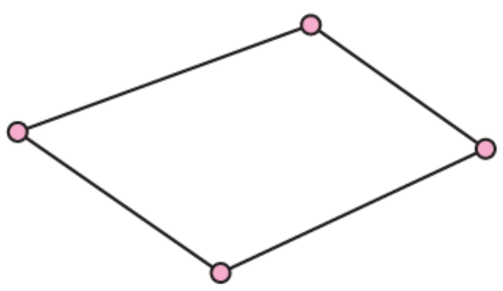
	A	B	C	D
A	–	–	–	–
B	4	–	5	–
C	–	3	–	6
D	3	5	–	–

EXERCISE 15B

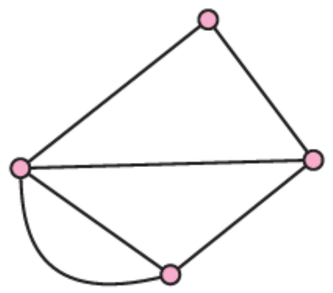
1 For each of the following, determine whether the graph is:

i simple **ii** connected **iii** complete.

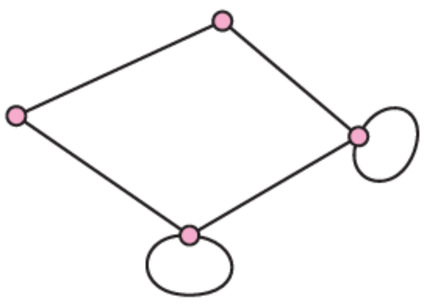
a



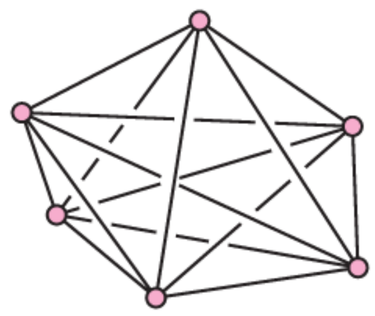
b



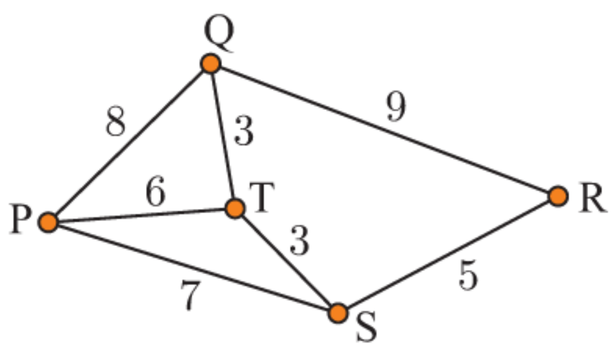
c



d



2



The time in minutes to run various tracks in an orienteering course are shown alongside.

a How long does it take to run from:

i T to S **ii** P to R via Q?

b Draw a weighted adjacency table for this graph.

3 This table shows the cost, in pounds, of travelling directly between railway stations.

a Represent these direct trips on a weighted graph.

b Find the degree of vertex B, and interpret your answer.

c Determine whether the graph is:

i connected **ii** complete.

d Find the cheapest way to travel from A to D.

	A	B	C	D	E	F
A	–	20	–	–	–	15
B	20	–	18	40	–	–
C	–	18	–	25	–	22
D	–	40	25	–	30	–
E	–	–	–	30	–	20
F	15	–	22	–	20	–

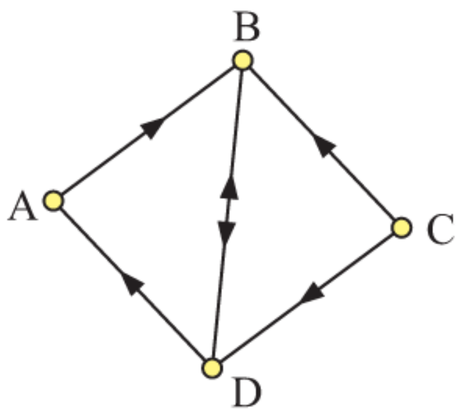
4 Suppose graph A is a subgraph of graph B.

Discuss whether the following statements are true or false:

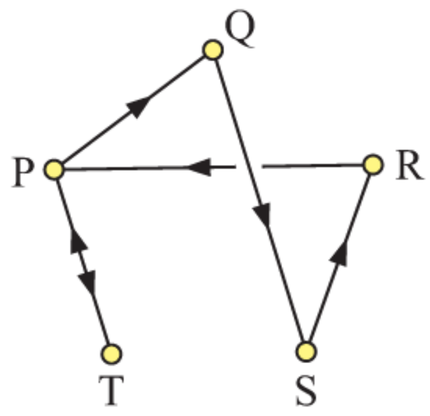
a If A is connected then B must be connected. **b** If B is complete then A must be simple.

5 Determine whether each directed graph is strongly connected:

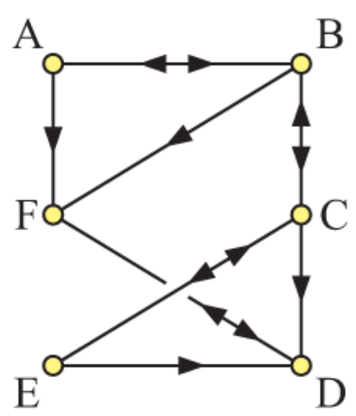
a



b



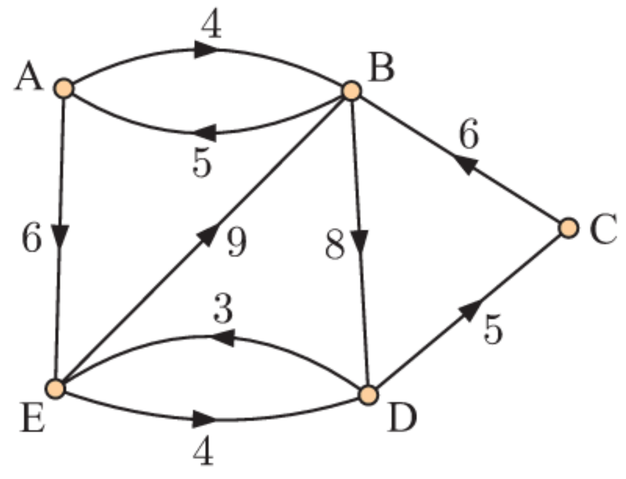
c



6 Consider the weighted directed graph alongside.

a Is the graph strongly connected?

b Construct a weighted adjacency table for the graph.



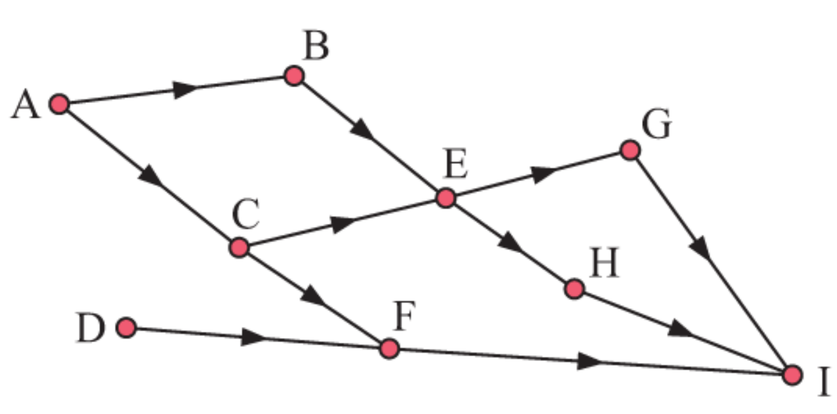
7 Construct the graph with the weighted adjacency table:

	P	Q	R	S
P	–	3	–	–
Q	2	–	2	–
R	4	1	–	2
S	3	–	1	–

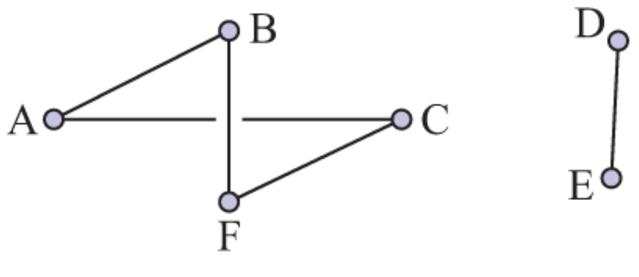
8 This directed graph shows ski runs down a mountain.

a Explain why the graph is not strongly connected.

b Find the minimum number of chairlifts which must be added to the graph so that it is strongly connected. Explain your answer.



9



a Explain why this graph is disconnected.

b Draw a subgraph of this graph which *is* connected.

10 **a** State the minimum number of edges that a simple connected graph with n vertices can have.

b How many edges does a complete graph with n vertices have?

c Hence show that a simple connected graph with n vertices and e edges satisfies the inequality $2n - 2 \leqslant 2e \leqslant n^2 - n$.

ACTIVITY 1 THE PAGERANK ALGORITHM

The **PageRank algorithm** is one of the methods used by the search engine Google to determine a web page's relevance or importance.

In this Activity we will see how the PageRank algorithm makes use of weighted directed graphs.

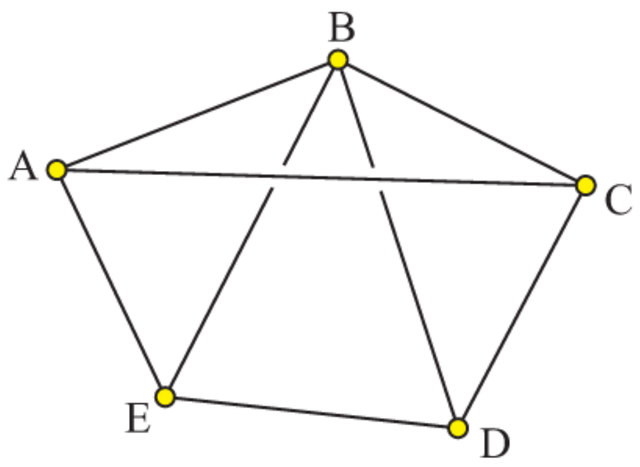


C ROUTES ON GRAPHS

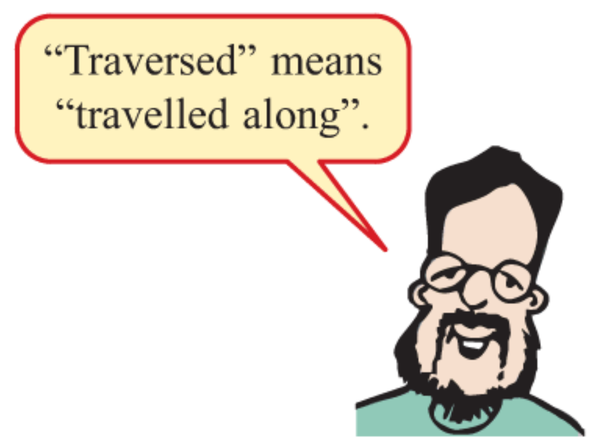
We will now consider various ways of moving from vertex to vertex along the edges of a graph.

- A **walk** is a sequence of vertices such that each successive pair of vertices is adjacent.
- A **trail** is a walk in which no *edge* is repeated.
- A **path** is a walk in which no *edge* or *vertex* is repeated.
- A **circuit** is a trail which starts and finishes at the same vertex.
- A **cycle** is a circuit in which no vertex is repeated (except the finish coinciding with the start).

In the graph alongside:



- $A \to C \to E$ is *not* a walk, as there is no edge between C and E.
- $D \to C \to B \to C$ is a walk, but it is not a trail as edge BC is repeated.
- $E \to B \to A \to C \to B$ is a trail as no edges are repeated, but it is not a path as vertex B is repeated.
- $A \to C \to D$ and $C \to B \to E \to A$ are both paths.
- $E \to B \to A \to C \to B \to D \to E$ is a circuit, but it is not a cycle as vertex B is repeated.
- $A \to B \to C \to A$ is a cycle.

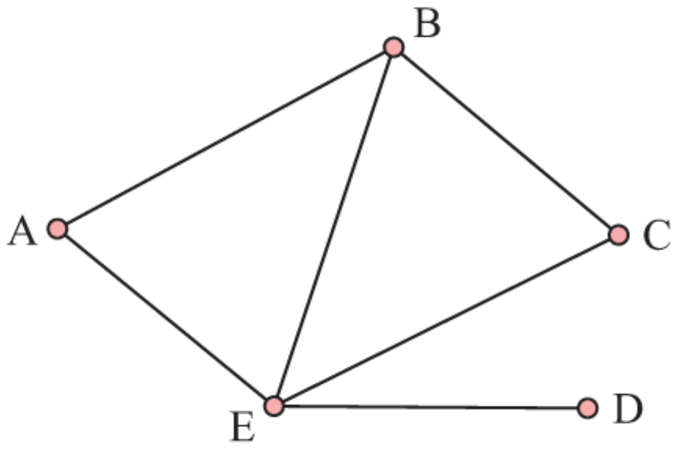


The **length** of a route is the number of edges *traversed.*

For example, $E \to B \to A \to C \to B$ is a trail of length 4.

EXERCISE 15C

1 Consider the graph alongside.



a Is $B \to E \to C \to B$ a walk?

b Is $A \to E \to B \to C \to E$ a trail?

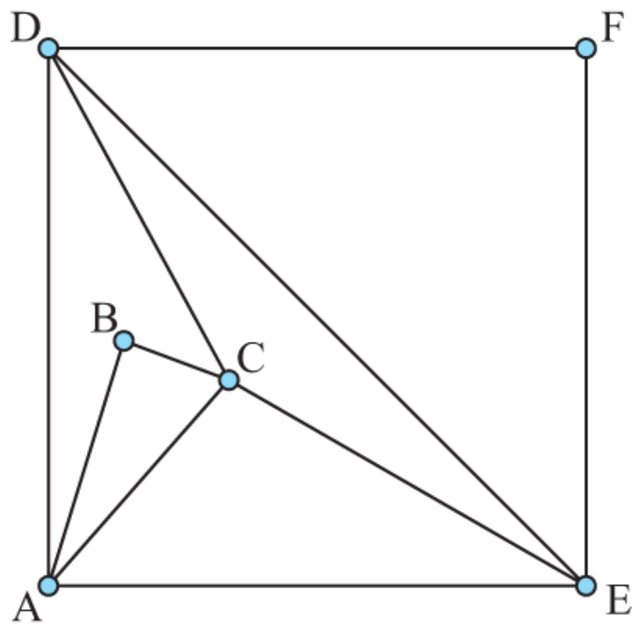
c Is $D \to E \to C \to B \to E$ a path?

d Is $E \to C \to B \to E$ a circuit?

e Is $C \to B \to A \to E$ a cycle?

Give reasons for your answers.

2 Consider the graph alongside. Find, if possible:



a a path of length 2 from A to D

b a path of length 3 from A to D

c a path of length 4 from A to D

d a trail which is not a path, of length 5 from B to D

e a cycle of length 5

f a cycle of length 7

g a circuit which is not a cycle, of length 7

h a circuit of length 10.

3 Explain whether each statement is true or false:

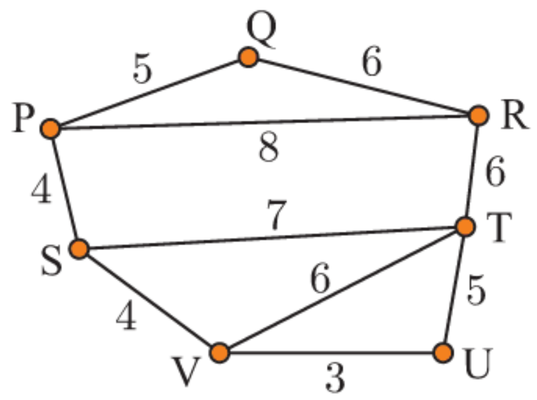
a Every path is a trail.

b Every trail is a path.

c Every cycle is a circuit.

d No circuit is a path.

4 Consider the weighted graph alongside.

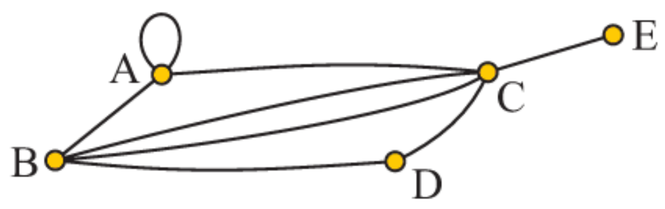


a Find all paths which start at P and end at T.

b Find the total weight of each path in **a**.

c Find the trail of lowest total weight which starts at Q and ends at U.

5 Consider the graph alongside. Find, if possible:



a a trail which includes every edge of the graph

b a circuit which includes every edge of the graph.

THEORY OF KNOWLEDGE

The map alongside is the original map of London's underground train network, used from 1908 to 1933.



In 1933, it was redesigned by the draughtsman **Henry Beck** (1902 - 1974), who said:

"*If you're going underground, why do you need to bother about geography? It's not so important. Connections are the thing.*" [1]

Beck's 1933 graphical representation can be found at:

www.ltmuseum.co.uk/collections/collections-online/maps/item/1993-100

1 What advantages and disadvantages does Beck's graph have compared with the original map?

2 Discuss Beck's analysis. Do you agree with him?

3 Describe the graphs corresponding to the underground maps for:

a Sydney **b** Glasgow **c** Namma.

4 Compare the London Tube map with the underground maps for the Moscow Metro and the Copenhagen Metro. What design features of each graph are most effective in helping travellers to navigate the system?

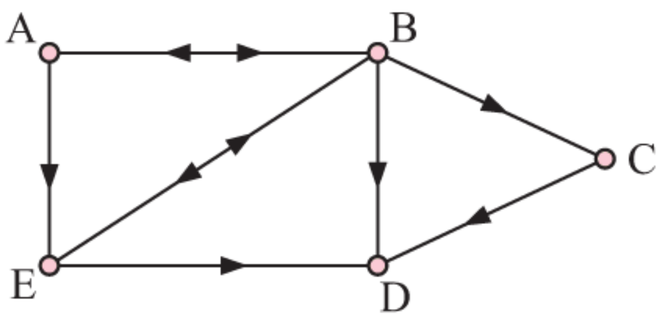
5 Does GPS navigation software remove the need for people to learn how to read a map?

6 Is the convenience provided by technology always beneficial to us as *people*?

[1] "The London Underground Map: Imagining Modern Time and Space", *Design Issues* 19, no. 1 (2003): pp 25 - 35.

D ADJACENCY MATRICES

An **adjacency matrix** is a matrix which shows how the nodes of a network are connected.



In the graph alongside, the vertices A, B, C, D, and E represent road intersections, and the edges represent roads which are either one-way or two-way.

The table below shows the number of 1-step routes between the vertices:

Starting vertex \ *Finishing vertex*	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	1
C	0	0	0	1	0
D	0	0	0	0	0
E	0	1	0	1	0

There is one 1-step route from B to D, so we write a 1 here.

There are no 1-step routes from C to E, so we write a 0 here.

The adjacency matrix for the graph is therefore $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

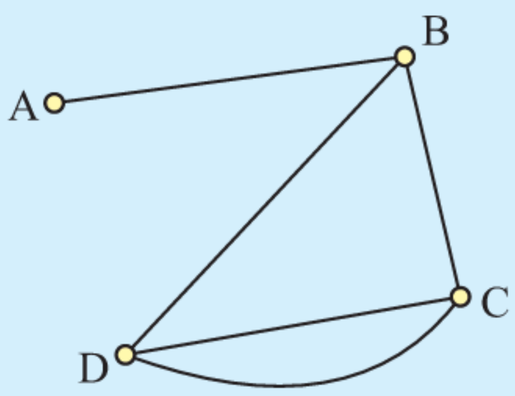
Since we do not usually write the labels for the vertices on the adjacency matrix, it is important to remember what the rows and columns refer to. To make this easier, we put them in alphabetical order.

Example 1

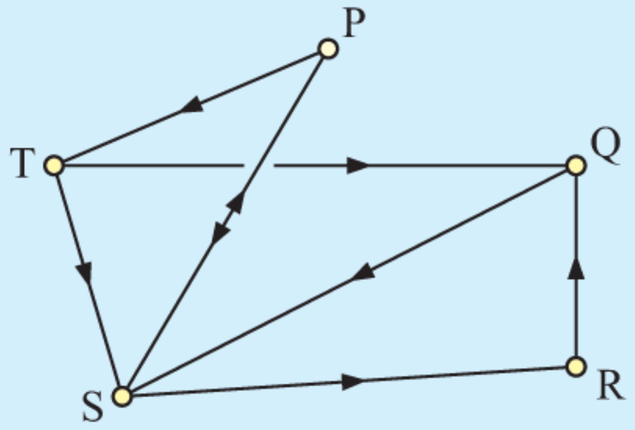
Self Tutor

Find the adjacency matrix for each graph:

a



b



a There are 4 vertices, so the adjacency matrix has order 4×4. The vertices are in order A, B, C, D.

The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

b There are 5 vertices, so the adjacency matrix has order 5×5. The vertices are in order P, Q, R, S, T.

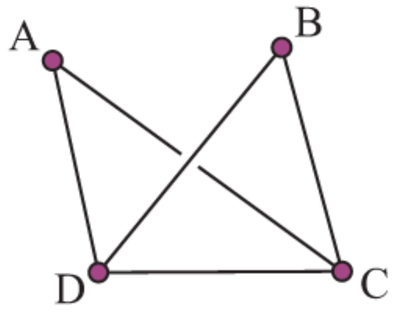
The adjacency matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

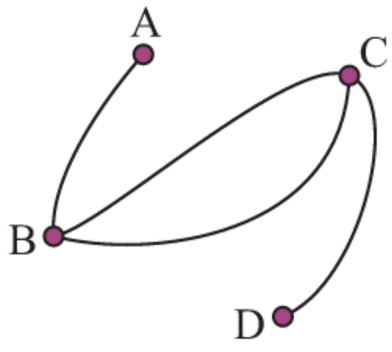
EXERCISE 15D.1

1 Construct an adjacency matrix for each graph:

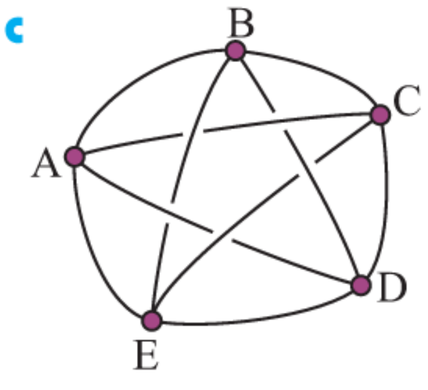
a



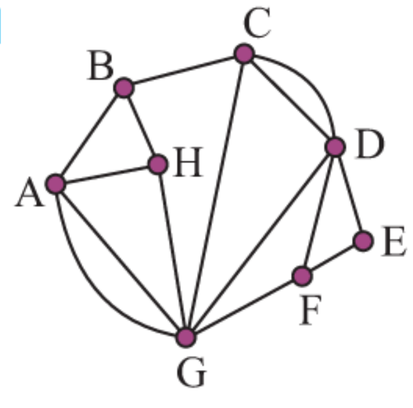
b



c



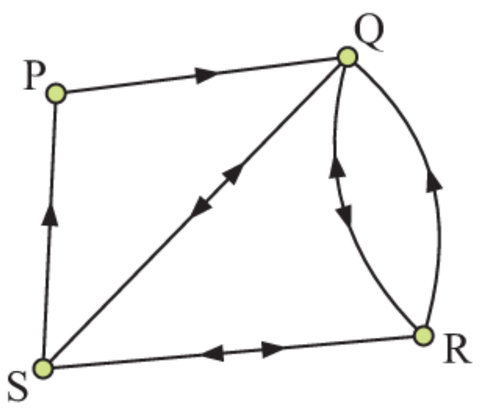
d



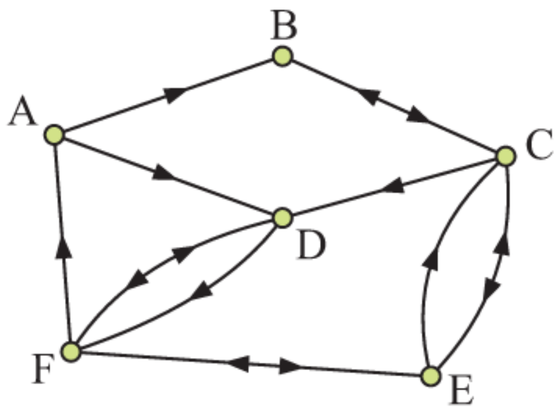
2 Explain why the adjacency matrix for an undirected graph is always symmetric about the leading diagonal.

3 Construct an adjacency matrix for each directed graph:

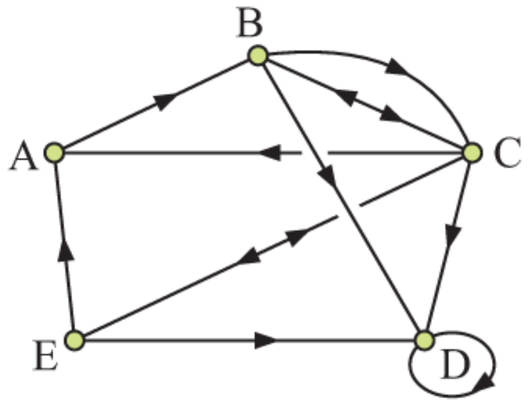
a



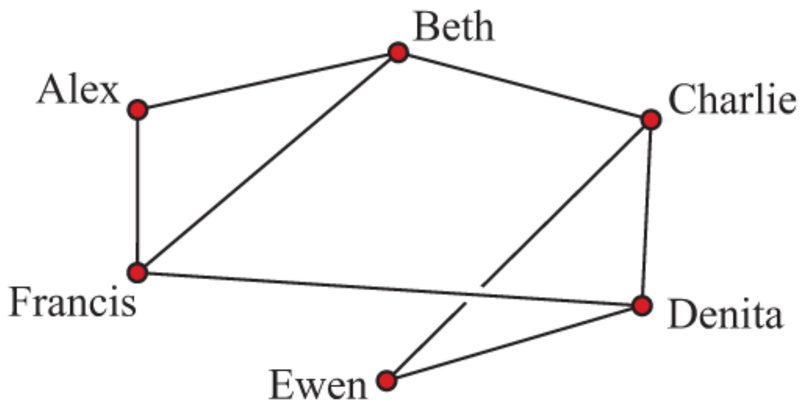
b



c



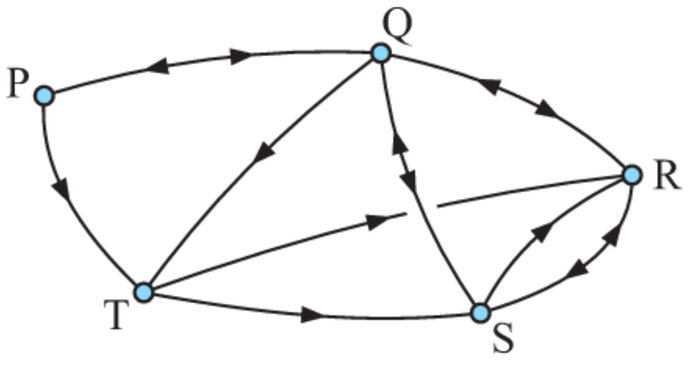
4 A volleyball team of six players is about to start a new season. The graph alongside shows which players have played together before.



a Construct an adjacency matrix for the graph.

b What feature of the adjacency matrix indicates that the graph is undirected?

5



a Construct an adjacency matrix for the graph alongside.

b Find the sum of the 2nd row of the adjacency matrix, and interpret your answer.

c Find the sum of the 3rd column of the adjacency matrix, and interpret your answer.

6 Construct a graph for each adjacency matrix:

a

	A	B	C	D
A	0	1	1	1
B	1	0	1	0
C	1	1	0	1
D	1	0	1	0

b

	A	B	C
A	0	1	1
B	1	0	0
C	0	1	0

c

	A	B	C	D	E
A	0	1	2	1	1
B	1	0	1	0	1
C	1	0	0	2	0
D	0	0	1	0	0
E	1	0	0	1	0

7 The adjacency matrix for a network of streets is shown alongside.

a Identify the one-way street in the network.

b Draw the graph of the network.

	A	B	C	D	E	F
A	0	1	1	0	0	1
B	1	0	1	0	0	0
C	1	1	0	1	0	1
D	0	0	1	0	1	0
E	0	0	1	1	0	1
F	1	0	1	0	1	0

8

	P	Q	R	S	T
P	0	2	1	0	1
Q	2	0	1	0	1
R	1	1	0	0	0
S	0	0	0	0	1
T	1	1	0	1	0

Consider the adjacency matrix alongside.

a Is there a direct route from Q to S? Explain your answer.

b Use the matrix to identify a 2-step route from Q to S.

c Draw the graph for the matrix.

MULTI-STEP ROUTES

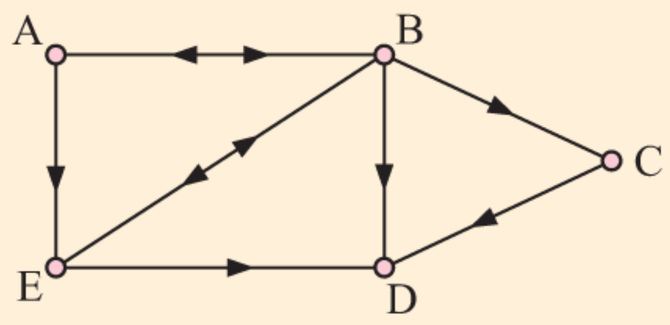
An adjacency matrix gives a clear picture of the direct 1-step connections between vertices.

We now consider longer, multi-step routes.

INVESTIGATION MULTI-STEP ROUTES

On page **385** we saw the graph alongside, which has adjacency matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$



What to do:

1 Consider a table for the 2-step routes between vertices.

There are two 2-step routes from A to D $\{A \to B \to D$ and $A \to E \to D\}$, so we write a 2 here.

a Find the 2-step route(s) from E to A. Hence find the value of the shaded entry.

b Complete the rest of the table.

Finishing vertex

Starting vertex	A	B	C	D	E
A				2	
B					
C					
D					
E					

2 Find the matrix $\mathbf{A}^2$ by hand or using technology. Comment on your answer.

3 Predict the meaning of $\mathbf{A}^3$ and $\mathbf{A}^4$. Check your prediction by finding appropriate routes on the graph.

MATRIX POWERS

GRAPHICS CALCULATOR INSTRUCTIONS

From the **Investigation** you should have discovered that:

If a graph has 1-step adjacency matrix **A**, then:

- $\mathbf{A}^2$ is the adjacency matrix for the **2-step routes**
- $\mathbf{A}^3$ is the adjacency matrix for the **3-step routes**

  ⋮
- $\mathbf{A}^n$ is the adjacency matrix for the ***n*-step routes**.

To understand *why* the powers of **A** give the multi-step adjacency matrices, consider the 2-step matrix:

$$\mathbf{A}^2 = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}\!\!\begin{array}{c}\begin{array}{ccccc}\text{A} & \text{B} & \text{C} & \text{D} & \text{E}\end{array}\\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}\end{array} \times \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}\!\!\begin{array}{c}\begin{array}{ccccc}\text{A} & \text{B} & \text{C} & \text{D} & \text{E}\end{array}\\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}\end{array} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array}\!\!\begin{array}{c}\begin{array}{ccccc}\text{A} & \text{B} & \text{C} & \text{D} & \text{E}\end{array}\\ \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}\end{array}$$

The element in row 2 column 4 of $\mathbf{A}^2$ shows there are two 2-step routes from B to D.

This value comes from the multiplication row 2 $\times$ column 4 as follows:

$1 \times 0 = 1$ route from B to A $\times$ 0 routes from A to D
$0 \times 1 = 0$ routes from B to B $\times$ 1 route from B to D
$1 \times 1 = 1$ route from B to C $\times$ 1 route from C to D
$1 \times 0 = 1$ route from B to D $\times$ 0 routes from D to D
$1 \times 1 = 1$ route from B to E $\times$ 1 route from E to D.

Adding these terms gives the two 2-step routes from B to D. The routes are B $\rightarrow$ C $\rightarrow$ D and B $\rightarrow$ E $\rightarrow$ D.

DISCUSSION

The matrix $\mathbf{A}^2$ tells us the number of 2-step *walks* between vertices.

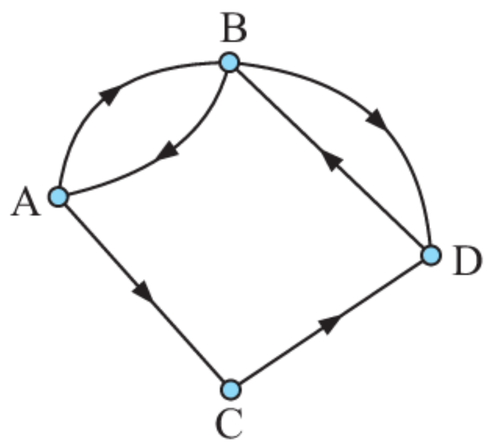
What does $\mathbf{A}^2$ tell us about the number of *paths* between vertices?

LIMITATIONS OF ADJACENCY MATRICES

Adjacency matrices are very good at telling us how many routes exist, but they do not tell us whether the routes are sensible or efficient.

For example, the graph of plane flights alongside has

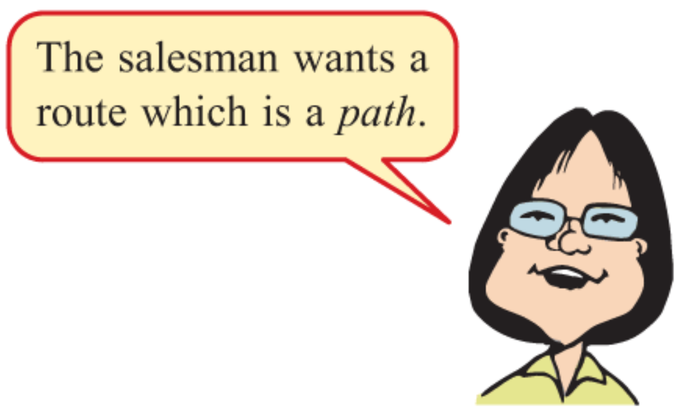
adjacency matrix $\mathbf{A} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\!\!\begin{array}{c}\begin{array}{cccc}\text{A} & \text{B} & \text{C} & \text{D}\end{array}\\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}\end{array}$.



$\begin{array}{cccc}\text{A} & \text{B} & \text{C} & \text{D}\end{array}$

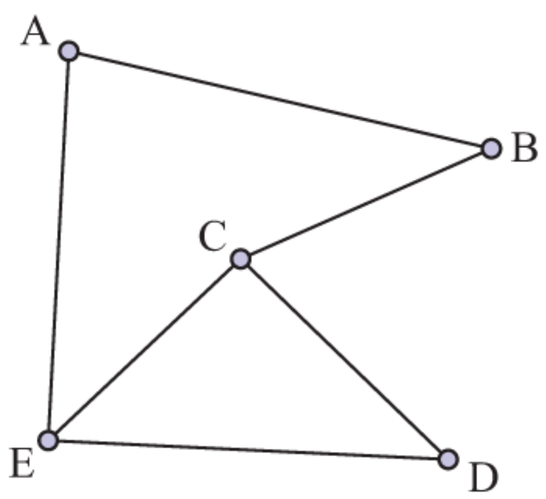
Since $\mathbf{A}^3 = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array}\!\!\begin{array}{c}\\ \begin{pmatrix} 0 & 3 & 1 & 0 \\ 2 & 0 & 0 & 3 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{pmatrix}\end{array}$, there are three 3-step routes from A to B: A $\rightarrow$ B $\rightarrow$ A $\rightarrow$ B
A $\rightarrow$ B $\rightarrow$ D $\rightarrow$ B
A $\rightarrow$ C $\rightarrow$ D $\rightarrow$ B

Suppose a salesman wants to travel to different places on his way to B, and has enough time for two stopovers. The most sensible route for him to use is A $\rightarrow$ C $\rightarrow$ D $\rightarrow$ B, since otherwise he is back-tracking to a place he has already been.



EXERCISE 15D.2

1 Consider the graph alongside.



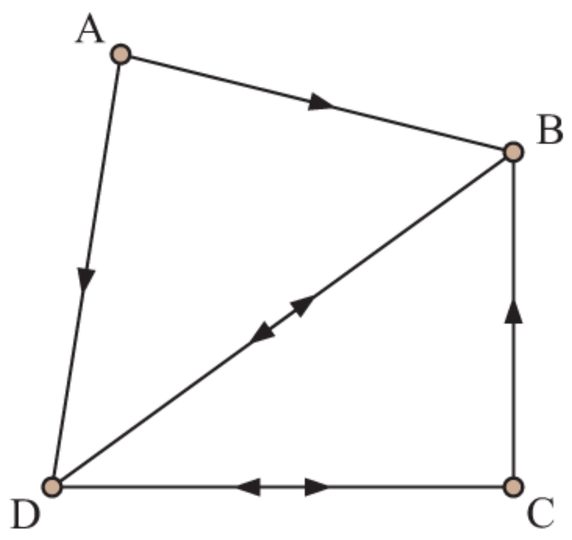
a Find the adjacency matrix $\mathbf{A}$.

b Calculate $\mathbf{A}^2$.

c *Hence* find the number of 2-step routes from E to B.

d List the routes in **c**.

2



Consider the graph alongside.

a Find the adjacency matrix $\mathbf{A}$.

b Hence find $\mathbf{A}^2$.

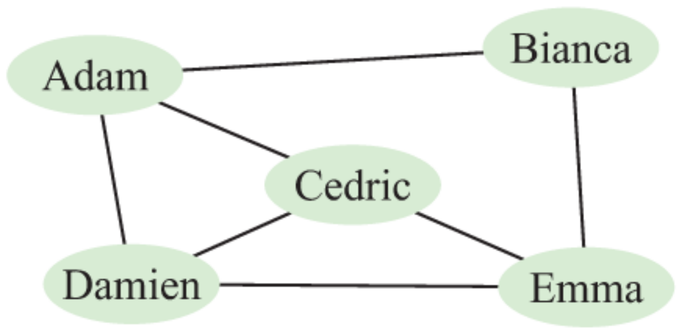
c For your matrix $\mathbf{A}^2$:

i State the significance of the value in row 4 column 4.

ii Explain why the value in row 1, column 2 is different from the value in row 2, column 1.

d Calculate $\mathbf{A}^3$ using technology, and explain its meaning.

3 The graph alongside shows the friendships between students in a class. Assume that friendships are mutual, so if Adam is friends with Bianca, then Bianca is friends with Adam.

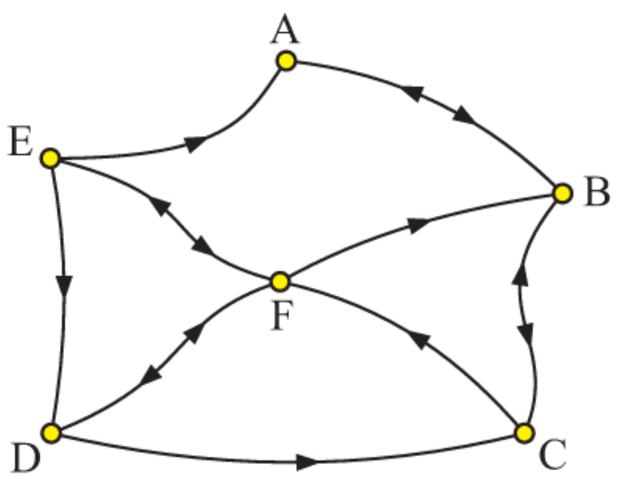


a Find the adjacency matrix $\mathbf{A}$.

b Find $\mathbf{A}^2$ and $\mathbf{A}^3$. What information is conveyed by these matrices?

c A teacher notices that Adam is friends with both Bianca and Cedric, but Bianca and Cedric do not like each other. How do we observe this in the adjacency matrices?

4 Consider the graph alongside.



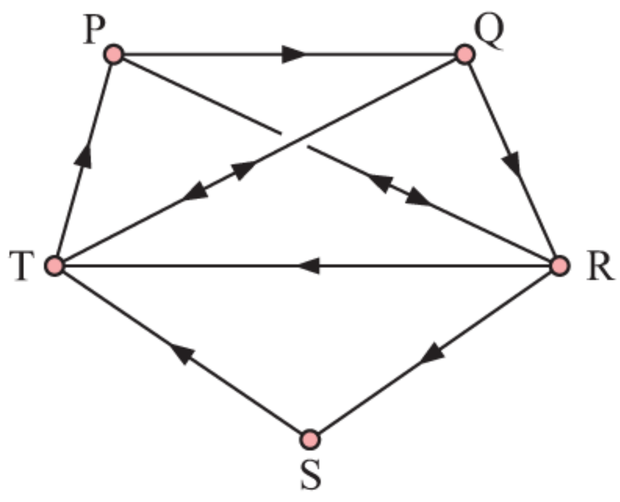
a Find the adjacency matrix $\mathbf{A}$ for the graph.

b Use technology to find $\mathbf{A}^3$.

c State the number of 3-step routes from E to C, and list these routes.

d State the number of 3-step routes from F to B, and of these routes, list those which are paths.

5 The graph alongside shows a gossip network between five friends: Peter, Quentin, Robert, Sam, and Terry.



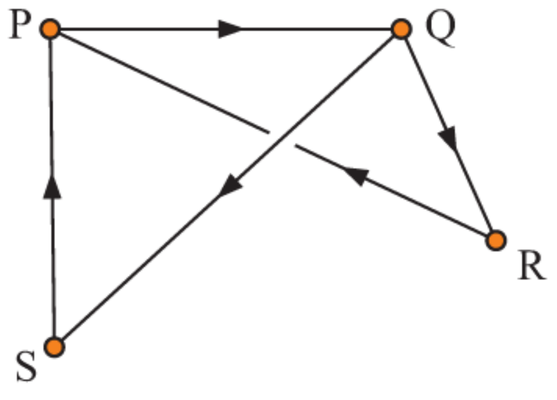
a Find the adjacency matrix $\mathbf{A}$ for the graph.

b Use technology to calculate:

i $\mathbf{A}^2$ **ii** $\mathbf{A}^3$ **iii** $\mathbf{A}^4$

c If Sam starts a rumour, through how many people must it pass before it gets back to him? Explain how this answer can be found from the adjacency matrices you have calculated.

6 The graph alongside shows the mini-train lines between four locations at a theme park.



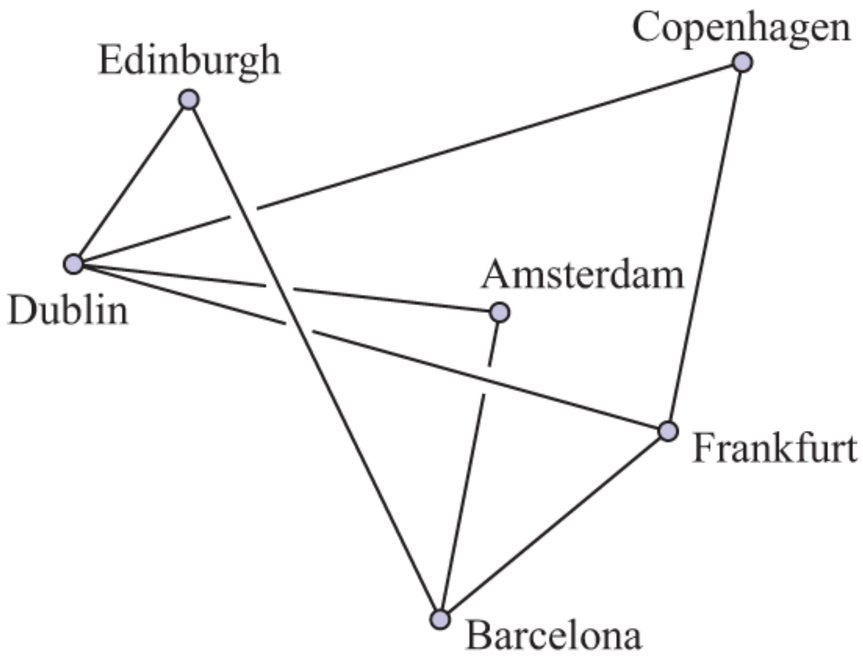
a Find the adjacency matrix **A** for the graph.

b Find $\mathbf{A}^2$ and $\mathbf{A}^3$. Explain what information is conveyed by these matrices.

c Find the least number of train rides which you will need to travel from:

i S to R **ii** R to S **iii** S to R and back to S.

7 The graph alongside shows the flights that Cheapair offers in western Europe.



a Find the adjacency matrix **A**.

b Use technology to calculate:

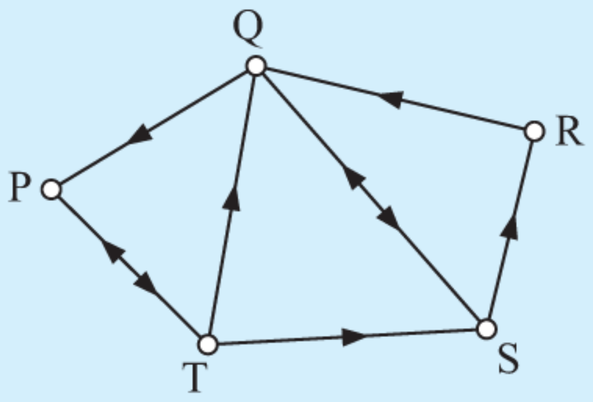
i $\mathbf{A}^2$ **ii** $\mathbf{A}^4$

c Arabella lives in Amsterdam. She has a ticket which allows her to make four flights. How many combinations of three different places could she visit before returning home?

d Describe the limitations of using adjacency matrices when answering **c**.

Example 2 — Self Tutor

Consider the graph alongside.



a Find the adjacency matrix **A**.

b Find $\mathbf{A} + \mathbf{A}^2$.

c Explain the significance of $\mathbf{A} + \mathbf{A}^2$.

d For this graph, how many steps do you need, to be able to travel from any vertex to any other vertex? Explain your answer.

a
$$\mathbf{A} = \begin{array}{c} \\ \text{P} \\ \text{Q} \\ \text{R} \\ \text{S} \\ \text{T} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{P} & \text{Q} & \text{R} & \text{S} & \text{T} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

b Using technology,
$$\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 2 & 2 & 1 & 2 & 1 \end{pmatrix}$$

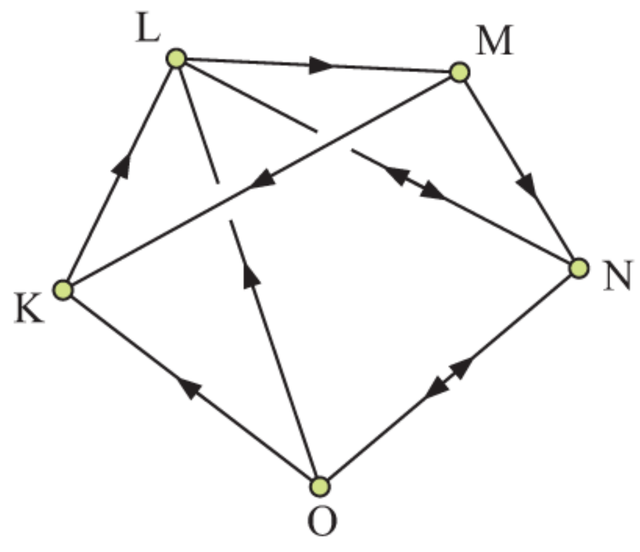
c $\mathbf{A} + \mathbf{A}^2$ tells us the total number of routes of at most 2 steps between each pair of vertices.

d The zeros in the matrix $\mathbf{A} + \mathbf{A}^2$ show cases where it is not possible to travel between vertices in 2 steps or less.

Using technology, $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 2 & 2 & 1 & 2 & 2 \\ 3 & 3 & 1 & 3 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 3 & 2 & 2 & 1 \\ 4 & 5 & 2 & 4 & 2 \end{pmatrix}$.

There are no zeros in the matrix $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$, so we can travel between any two vertices in 3 steps or less.

8 This graph shows the streets connecting locations K, L, M, N, and O.



a Find the adjacency matrix $\mathbf{A}$.

b Calculate $\mathbf{A} + \mathbf{A}^2$.

c What do the zeros in $\mathbf{A} + \mathbf{A}^2$ tell us about the street system?

d If the street connecting O and N is blocked by an accident, how does this affect the connectivity of the system?

9

A	B	C
D	E	F

In a board game, a counter can be moved horizontally or vertically to an adjacent square on the board shown.

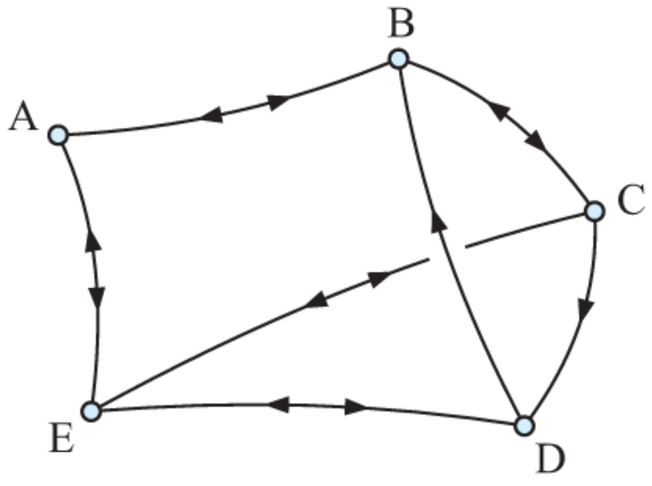
a Construct an adjacency matrix $\mathbf{A}$ for the game.

b Calculate $\mathbf{A} + \mathbf{A}^2$, and hence show that it is not possible to move between any two squares in at most two moves.

c Calculate $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$, and hence show that it *is* possible to move between any two squares in at most three moves.

d Suppose the rules of the game change so the counter may now be moved diagonally as well. Use matrices to show that it is now possible to move between any two squares in at most two moves.

10 A bus service is used to connect a group of nearby towns. The graph alongside shows the services which currently operate.



a Find the adjacency matrix $\mathbf{A}$ for the graph.

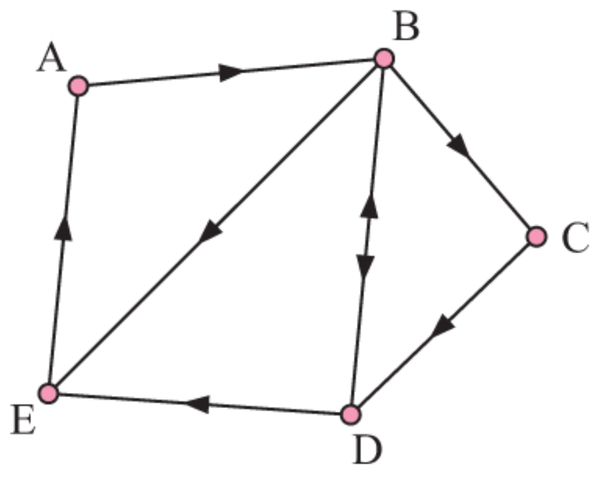
b The bus company aims to ensure that people can travel between any two towns in at most 2 trips. Use the matrix $\mathbf{A} + \mathbf{A}^2$ to show that this has been achieved.

c Due to increasing costs, the bus company needs to cancel one of its services.

i Show that if the service from C to D is cancelled, the company will fail to achieve its aim.

ii Find a service which should be cancelled instead.

11 The graph alongside shows how electrical signals are transmitted between stations A, B, C, D, and E.



a Find the adjacency matrix $\mathbf{A}$ for the graph.

b Calculate $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$ and explain its meaning.

c In practice, the strength and clarity of signals decreases the more times they bounce off other stations.

Calculate $\mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{4}\mathbf{A}^3$, and suggest what it might represent.

d What is the effect on the system if:

i BE is blocked by a new high-rise building

ii D is no longer able to send signals to B?

e Using the matrix $\mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{4}\mathbf{A}^3$, which scenario in **d** would create the more serious problems?

MATRIX POWERS

12 The floor plan of a museum can be described by the adjacency matrix alongside. The letters A to F represent particular areas of the museum.

$$\mathbf{A} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \\ \text{F} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

a Is $\mathbf{A}$ symmetric about the leading diagonal? What does this tell you?

b Draw a graph to represent the museum.

c Which letter do you think represents the museum's:

i entrance **ii** shop **iii** exit?

d From the entrance, one must pass through two other rooms to get to the dinosaur room.

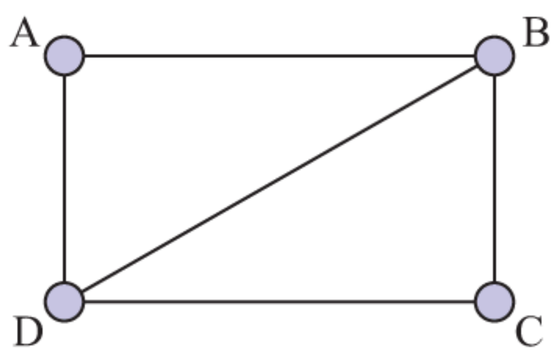
i Calculate $\mathbf{A} + \mathbf{A}^2$ and $\mathbf{A}^3$.

ii Explain how a comparison between $\mathbf{A} + \mathbf{A}^2$ and $\mathbf{A}^3$ reveals possible positions of the dinosaur room.

iii Where do you think the dinosaur room is? Explain your answer.

E TRANSITION MATRICES FOR GRAPHS

The graph alongside shows intercity bus routes between cities A, B, C, and D.

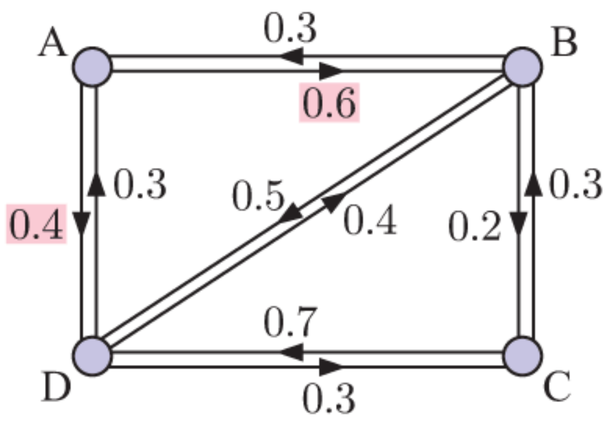


However, this graph does not tell us which cities are most frequently visited by the buses.

Instead, we could use a directed weighted graph to show the movement of buses between cities.

The weights of the edges indicate probabilities.

For example, from city A, 60% of the buses travel to city B, and 40% travel to city D.



We can construct a **transition matrix** from this graph. Like the adjacency matrices, the rows are the starting vertices and the columns are the finishing vertices.

$$\mathbf{T} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 0 & 0.6 & 0 & 0.4 \\ 0.3 & 0 & 0.2 & 0.5 \\ 0 & 0.3 & 0 & 0.7 \\ 0.3 & 0.4 & 0.3 & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \textit{Starting} \\ \textit{vertex} \\ \\ \end{array}$$

Finishing vertex

DISCUSSION

The transition matrices for Markov chains had the "from" and "to" around the other way. So, the rows and columns are essentially switched. We could say the transition matrices are *transposed*.

The transition matrices we see here are more conventional, but in this book we have aimed to be consistent with the course.

1 Which orientation of "from" and "to" is more intuitive?

2 What do we now need to do to our state matrix in order to calculate future states?

The powers of the transition matrix allow us to study the movement of the buses in the future. For example, in $\mathbf{T}^3 = \begin{pmatrix} 0.138 & 0.372 & 0.122 & 0.368 \\ 0.21 & 0.239 & 0.166 & 0.385 \\ 0.129 & 0.321 & 0.101 & 0.449 \\ 0.24 & 0.308 & 0.213 & 0.239 \end{pmatrix}$, the shaded element tells us that a bus currently in B has probability 0.385 of being in D in three stops' time.

The initial state matrix $\mathbf{s}_0$ must now be written as a *row matrix* rather than a column matrix, and we must *premultiply* $\mathbf{T}^n$ by $\mathbf{s}_0$ to find $\mathbf{s}_n$.

For example, if a bus is initially at B, then $\mathbf{s}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$ and

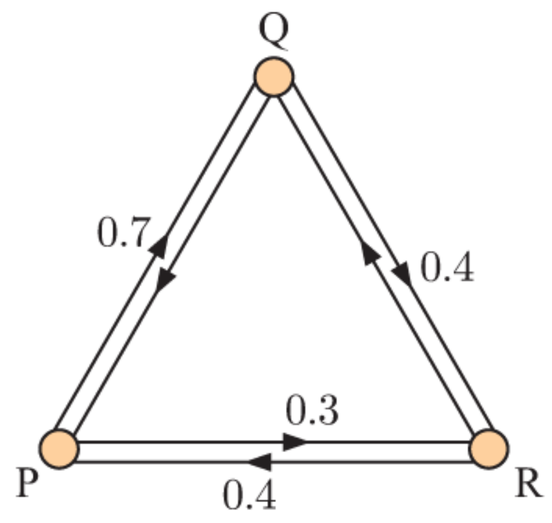
$$\mathbf{s}_1 = \mathbf{s}_0\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0.6 & 0 & 0.4 \\ 0.3 & 0 & 0.2 & 0.5 \\ 0 & 0.3 & 0 & 0.7 \\ 0.3 & 0.4 & 0.3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 & 0 & 0.2 & 0.5 \end{pmatrix}$$

So, one stop from now, a bus currently at B has probability 0.3 of being at A, probability 0.2 of being at C, and probability 0.5 of being at D.

EXERCISE 15E

1 This graph shows the probabilities with which trains move between the stations P, Q, and R.



a Copy and complete the graph.

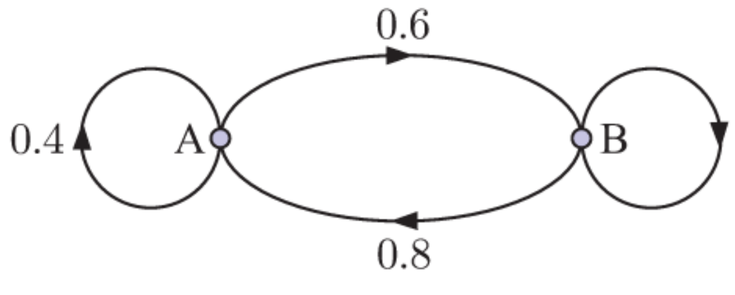
b Construct a transition matrix $\mathbf{T}$ for the graph.

c Interpret the value in row 1, column 3 of $\mathbf{T}$.

d A train is currently at P, so the initial state matrix is $\mathbf{s}_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$.

Find $\mathbf{s}_0\mathbf{T}^3$ and interpret you answer.

2 This graph shows the jogging tracks connecting two rest stations A and B, and the probabilities with which Maria jogs along them.



a Copy and complete the graph.

b Construct a transition matrix $\mathbf{T}$ for the graph.

c Maria starts at station A. Find the probability that, after jogging along 4 tracks, Maria is once again at station A.

3 This directed graph shows the trails on a ropes course between the platforms A, B, C, and D.

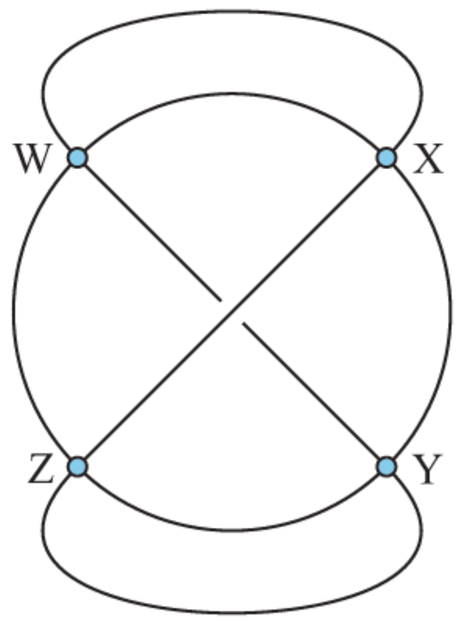
At each platform, Prisha chooses her next trail at random.

a Using fractions, write the probabilities on each edge of the graph.

b Construct the transition matrix for the graph.

c Prisha starts the ropes course at A. After Prisha has travelled along 3 trails, what is the probability that she will be on platform D? Find the routes that Prisha could have taken for this to occur.

4 This undirected graph shows the tunnels a mouse can travel through between junctions W, X, Y, and Z. At each junction, the mouse chooses his next tunnel at random.



a Construct a transition matrix $\mathbf{T}$.

b The mouse is initially at junction X, so $\mathbf{s}_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$.

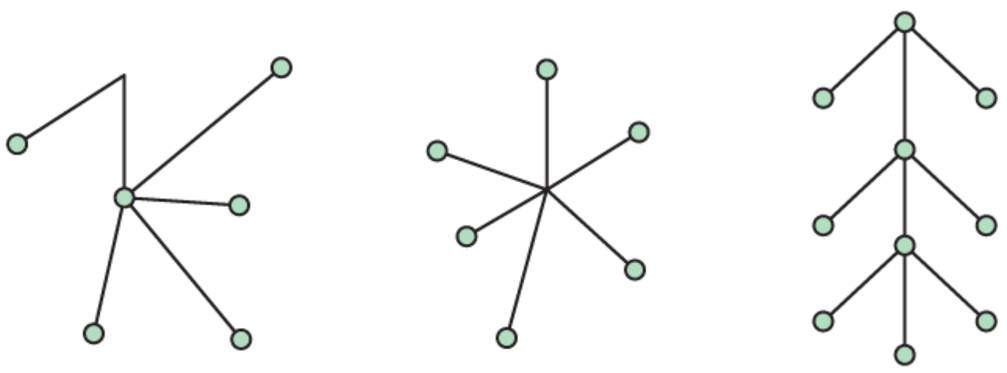
i Which junction is he most likely to be at after running through 2 tunnels? Explain this result.

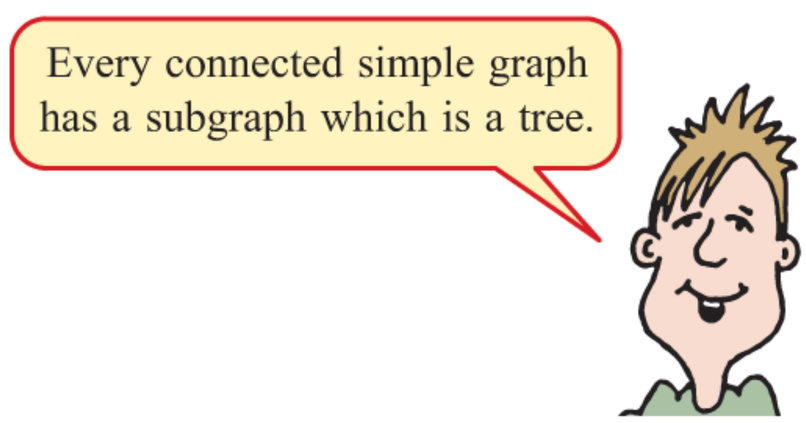
ii By considering $\mathbf{s}_0\mathbf{T}^n$ for large values of n, show that in the long term, the mouse will visit each junction equally often.

F TREES

A **tree** is a connected, simple graph with no cycles.

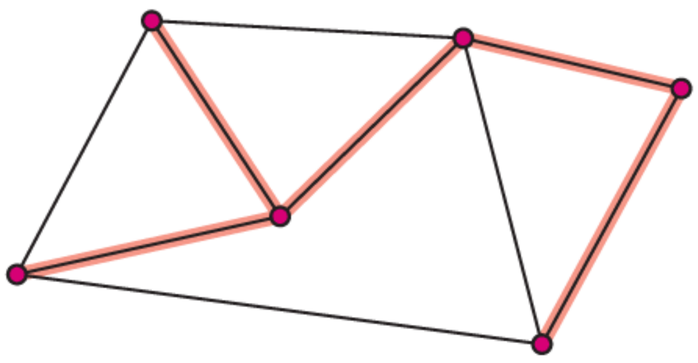
Some examples of trees are shown below:





A **spanning tree** of a graph is a tree which contains all the vertices of the graph.

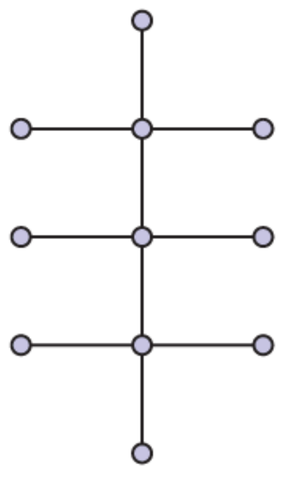
For example, a spanning tree of the graph alongside is highlighted in red.



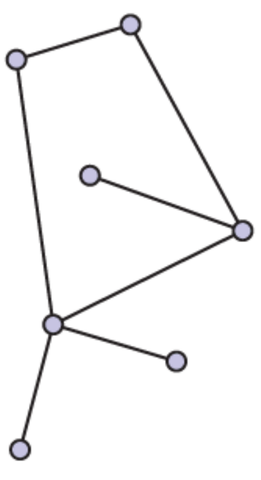
EXERCISE 15F

1 State whether each graph is a tree:

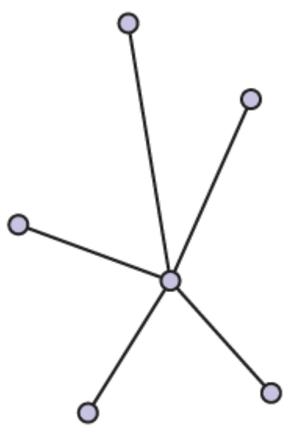
a



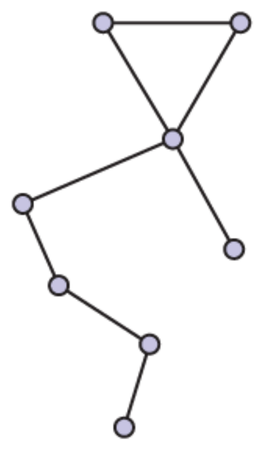
b



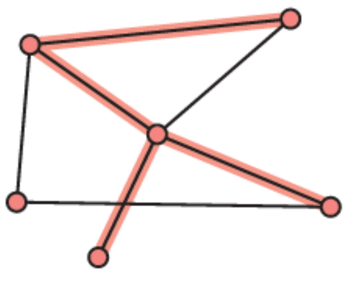
c



d



2 For the graph alongside, determine whether the shaded subgraph is:

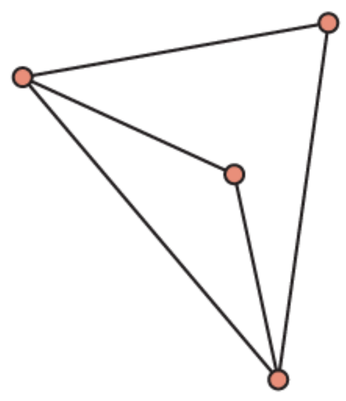


a a tree **b** a spanning tree.

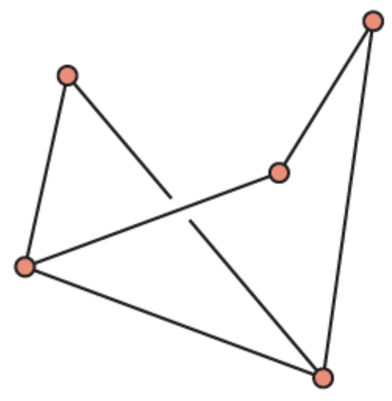
Explain your answers.

3 **a** Draw a spanning tree for each graph:

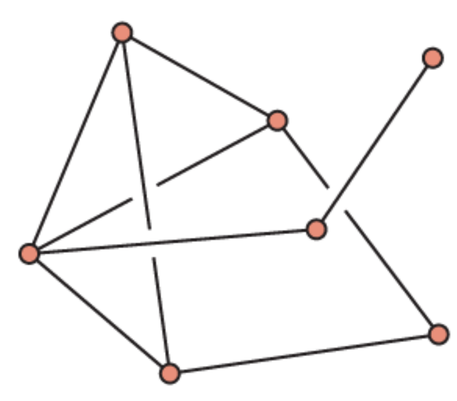
i



ii



iii



b Use the graphs in **a** to copy and complete this table:

	Number of vertices	*Number of edges in spanning tree*
i		
ii		
iii		

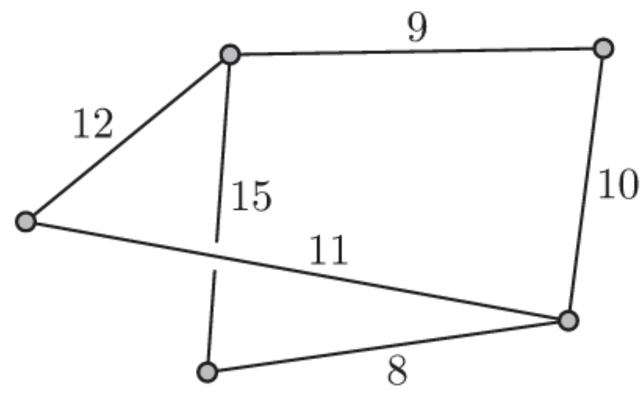
c Copy and complete:

If a graph has n vertices, its spanning tree has ..... edges.

Explain this result.

4 Draw as many spanning trees as you can for the graph alongside. Find the total weight of each spanning tree.

PRINTABLE DIAGRAM

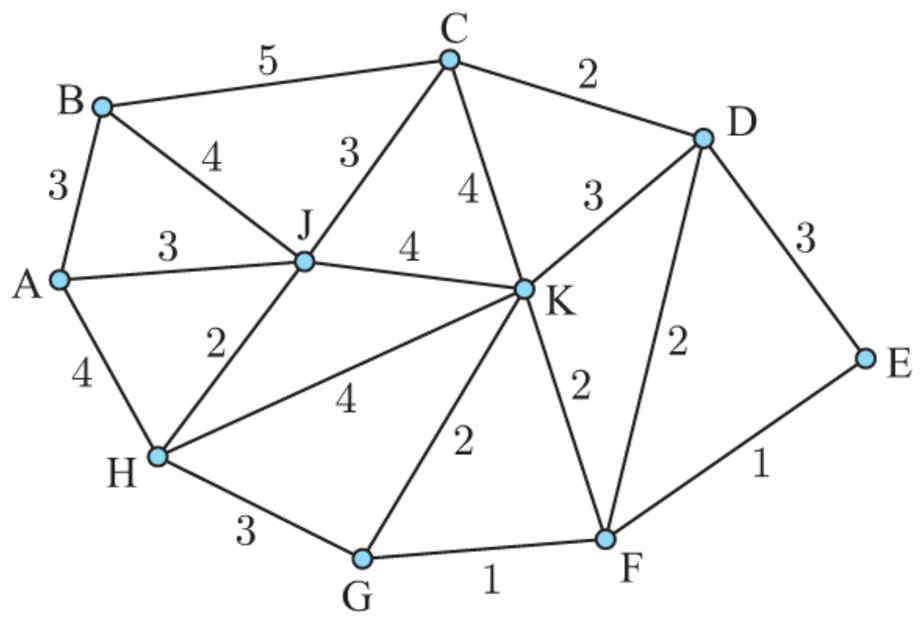


5 Does every graph have a spanning tree? Explain your answer.

G MINIMUM SPANNING TREES

This graph represents the walking trails between landmarks in a national park. The weights are the lengths of the trails in kilometres.

The park's planning group wants to pave sections of the trails to enable wheelchair access. The paved trails must connect all of the landmarks in the park in a way that minimises the paving required.



We must therefore find the spanning tree which has the minimum weight. This is known as the **minimum spanning tree**.

We will consider two algorithms for finding minimum spanning trees:

- **Kruskal's algorithm**
- **Prim's algorithm**

KRUSKAL'S ALGORITHM

In **Kruskal's algorithm**, we choose edges one at a time, taking the edge of least weight at every stage while ensuring that no cycles are formed. For a graph with n vertices, the minimum spanning tree is obtained after $n-1$ successful choices of edge.

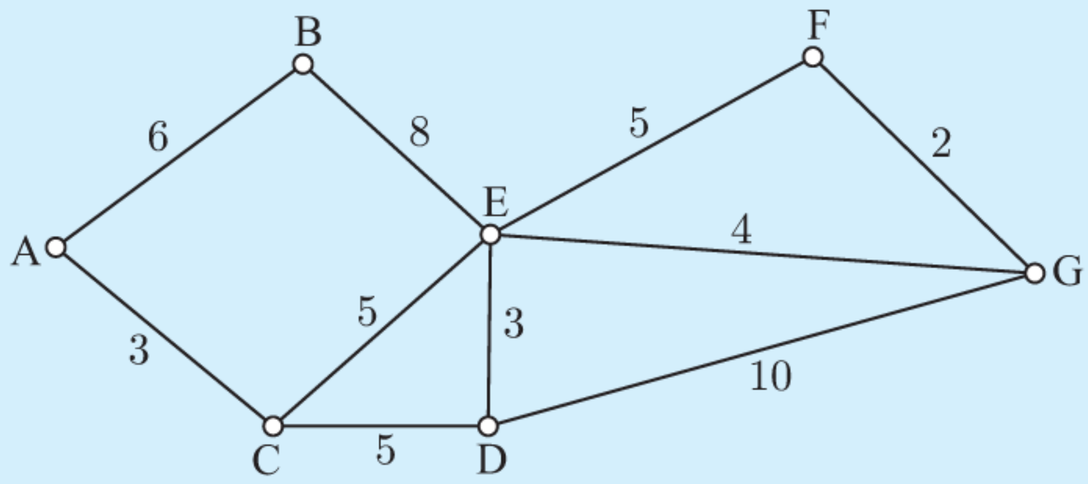
Step 1: Start with the shortest (or **least weight**) edge. If there are several, choose one at random.

Step 2: Choose the shortest edge remaining that does not complete a cycle. If there is more than one possible choice, pick one at random.

Step 3: Repeat *Step 2* until $n-1$ edges have been chosen.

Example 3 **Self Tutor**

Use Kruskal's algorithm to find the minimum spanning tree of the graph given.

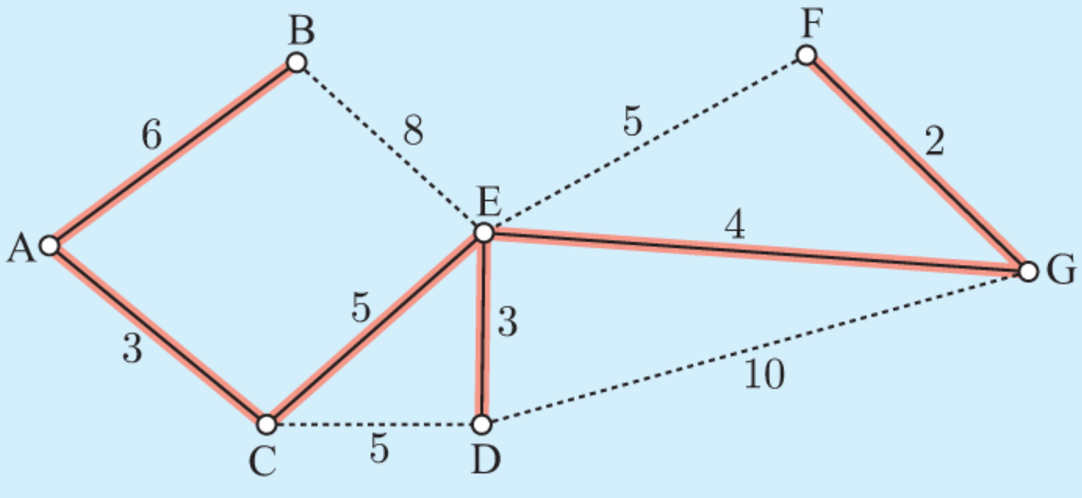


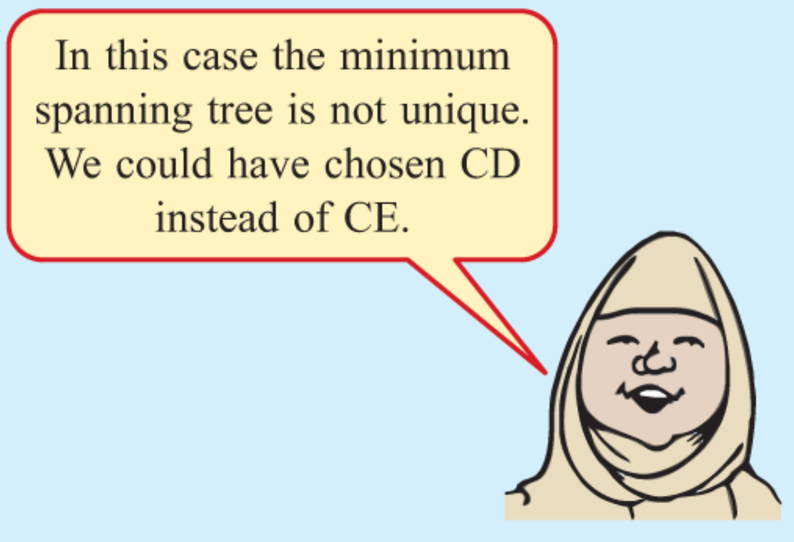
There are 7 vertices, so we require 6 edges. Edge FG has the least weight.

Edge	*Weight*	*Circuit*	*Edge List*
FG	2	No	FG
DE	3	No	FG, DE
AC	3	No	FG, DE, AC
EG	4	No	FG, DE, AC, EG
EF	5	Yes - reject	FG, DE, AC, EG
CE	5	No	FG, DE, AC, EG, CE
CD	5	Yes - reject	FG, DE, AC, EG, CE
AB	6	No	FG, DE, AC, EG, CE, AB

We have 6 edges, so we stop the algorithm.

The minimum spanning tree has total weight $2+3+3+4+5+6=23$, and is shown below.





EXERCISE 15G.1

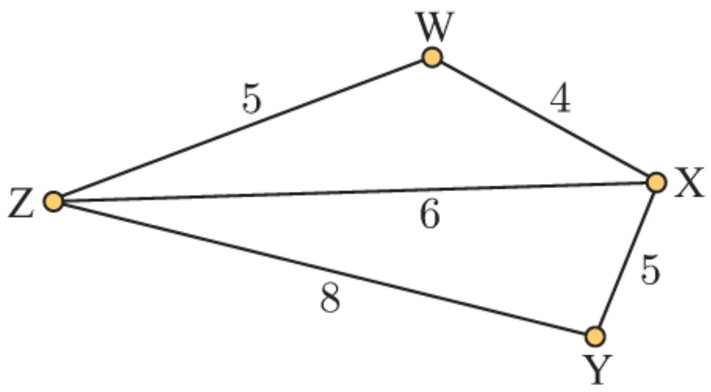
PRINTABLE DIAGRAMS

1 For each graph:

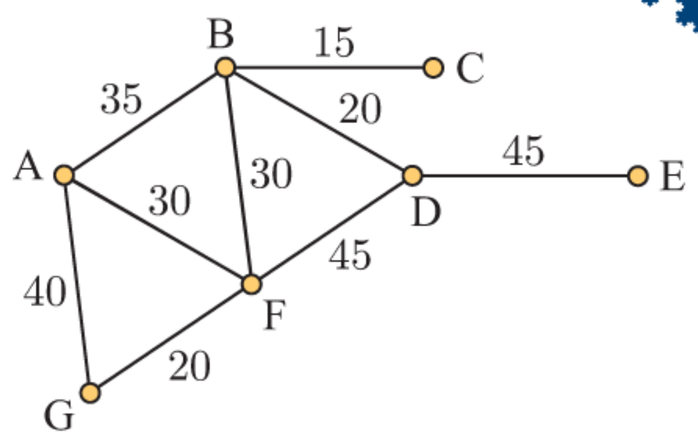
i Use Kruskal's algorithm to find the minimum spanning tree.

ii State the total weight of the minimum spanning tree.

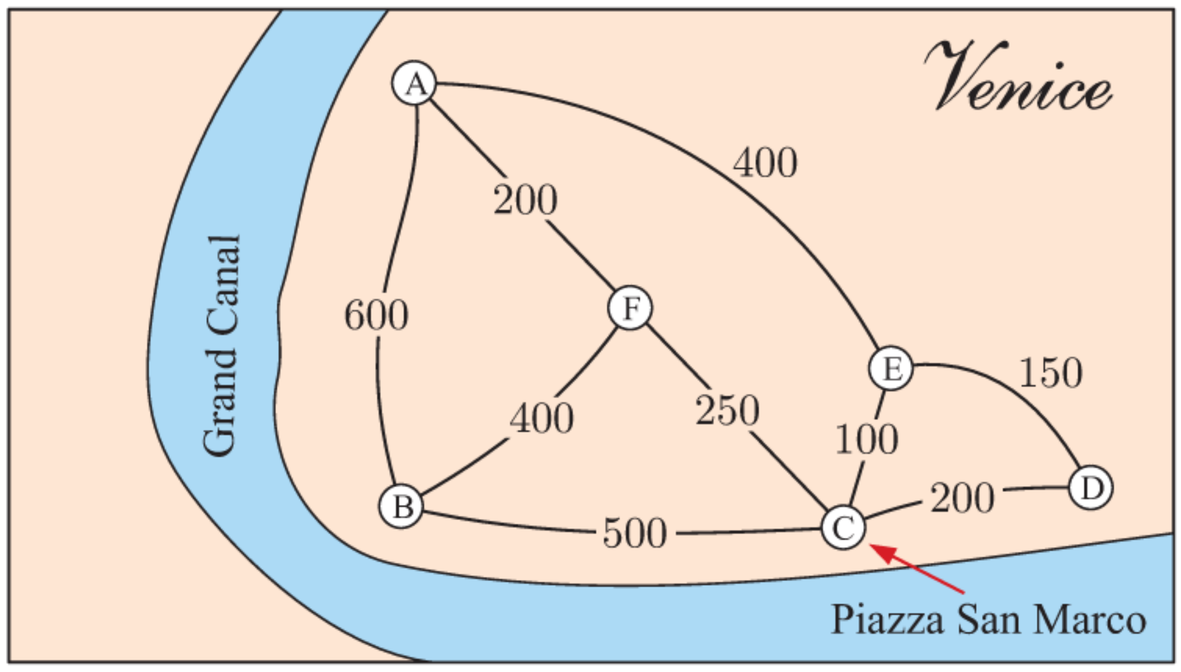
a



b



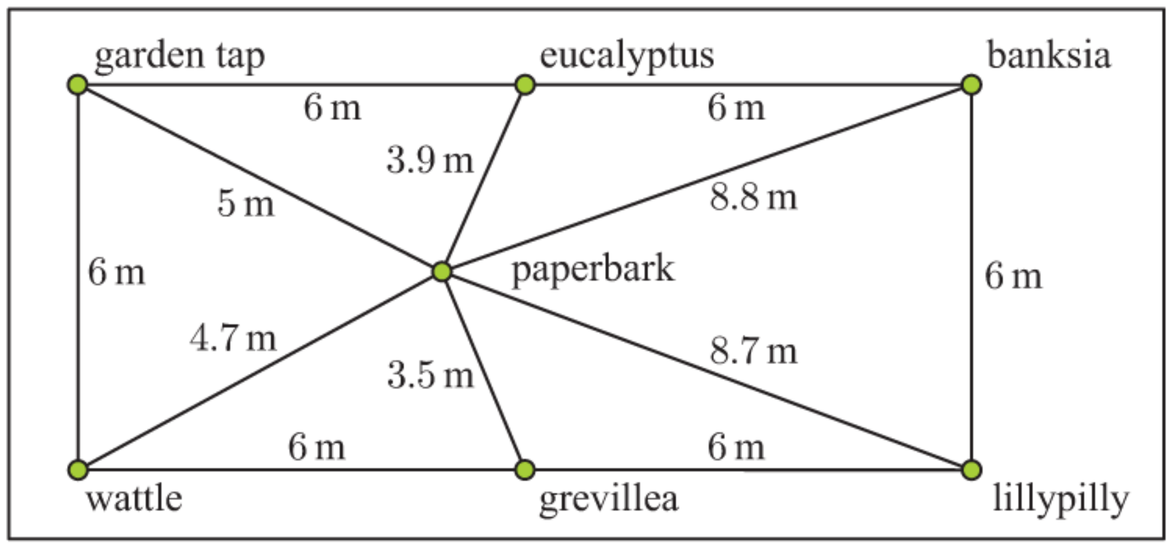
2 When Venice is flooded, the authorities set up a network of raised platforms for pedestrians to walk on. If they connect the tourist attractions shown with the minimum length of platform possible, which walkways do they raise platforms over? Distances are given in metres.



3 An irrigation system is being installed to connect each tree in a garden with the tap.

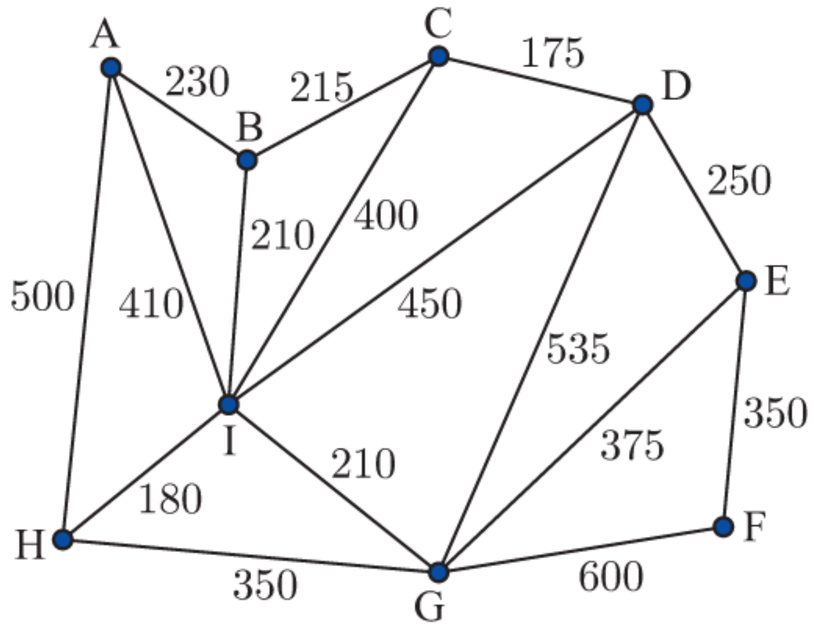
a Find the minimum length of pipe needed.

b Is there a unique network of pipes that will give this length?



c Water pressure drops the further you are from the tap. Which network should be chosen to maximise the water pressure?

4



A new university is being constructed.

The new faculty buildings are shown alongside, including the distances in metres between buildings.

a Draw a graph which ensures all buildings are connected to the university computer network, but which minimises the amount of cable used.

b What is this length of cable?

5 Consider the national park trails problem on page **395**.

a Find the trails which should be paved.

b State the total length of paving needed.

6 **a** Draw the graph for this weighted adjacency table.

b Find a minimum spanning tree for the graph.

	A	B	C	D	E	F	G
A	–	–	30	–	–	50	45
B	–	–	70	35	40	–	–
C	30	70	–	50	–	–	20
D	–	35	50	–	10	–	15
E	–	40	–	10	–	15	–
F	50	–	–	–	15	–	10
G	45	–	20	15	–	10	–

PRIM'S ALGORITHM

In **Prim's algorithm**, we choose edges one at a time, at each stage connecting an additional vertex to the tree until all the vertices are connected.

Step 1: Start with any vertex.

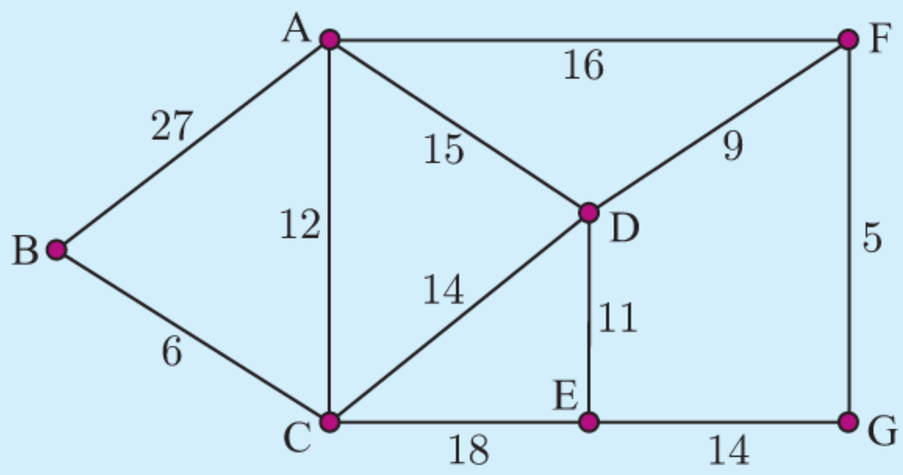
Step 2: Join this vertex to the nearest vertex. If two or more vertices are an equal distance away, choose one at random.

Step 3: Join the vertex which is nearest to *either* of those already connected.

Step 4: Continue joining connected vertices to the nearest unconnected vertex until all vertices are connected.

Example 4 **Self Tutor**

Find the minimum spanning tree for this graph.



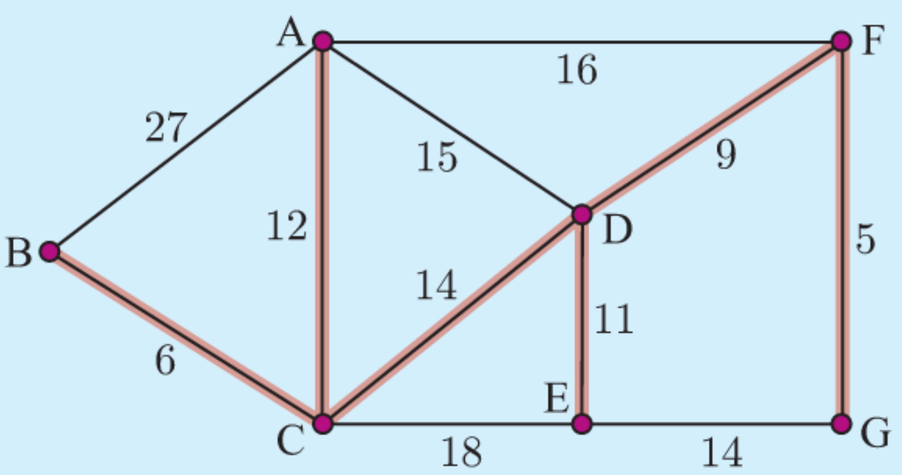
Step 1: Select any vertex at random, say vertex E.

Step 2: The vertex nearest E is D, so we include the edge DE.

Step 3: Of the other edges from D and E, the edge DF is shortest, so we include this edge.

Step 4: We continue joining connected vertices to the nearest unconnected vertex until all vertices are connected. The following connections are made: FG (weight 5), DC (weight 14), CB (weight 6), and CA (weight 12).

The minimum spanning tree is shown alongside. Its weight is
$11 + 9 + 5 + 14 + 6 + 12 = 57$



Alternatively, we can apply Prim's algorithm to a weighted adjacency table using the following steps. Alongside we see the procedure applied to the weighted adjacency table for the graph in **Example 4**.

Step 1: Select any vertex at random, say vertex E.

Step 2: Delete the row of the chosen vertex.

Step 3: Number the column of the chosen vertex.

Step 4: Circle the lowest undeleted entry in any of the numbered columns. If there is more than one possible choice, pick one at random.
The row of the circled entry gives us the new chosen vertex.

Step 5: Repeat *Steps 2* to *4* until all rows are deleted.
The circled entries give the edges of the minimum spanning tree.

	A	B	C	D	E	F	G
					(1)		
A	–	27	12	15	–	16	–
B	27	–	6	–	–	–	–
C	12	6	–	14	18	–	–
D	15	–	14	–	(11)	9	–
~~E~~	~~–~~	~~–~~	~~18~~	~~11~~	~~–~~	~~–~~	~~14~~
F	16	–	–	9	–	–	5
G	–	–	–	–	14	5	–

	A	B	C	D	E	F	G
		(6)	(5)	(2)	(1)	(3)	(4)
~~A~~	~~–~~	~~27~~	~~(12)~~	~~15~~	~~–~~	~~16~~	~~–~~
~~B~~	~~27~~	~~–~~	~~6~~	~~–~~	~~–~~	~~–~~	~~–~~
~~C~~	~~12~~	~~6~~	~~–~~	~~(14)~~	~~18~~	~~–~~	~~–~~
~~D~~	~~15~~	~~–~~	~~14~~	~~–~~	~~(11)~~	~~9~~	~~–~~
~~E~~	~~–~~	~~–~~	~~18~~	~~11~~	~~–~~	~~–~~	~~14~~
~~F~~	~~16~~	~~–~~	~~–~~	~~(9)~~	~~–~~	~~–~~	~~5~~
~~G~~	~~–~~	~~–~~	~~–~~	~~–~~	~~14~~	~~(5)~~	~~–~~

The minimum spanning tree has total weight $12 + 6 + 14 + 11 + 9 + 5 = 57$.

EXERCISE 15G.2

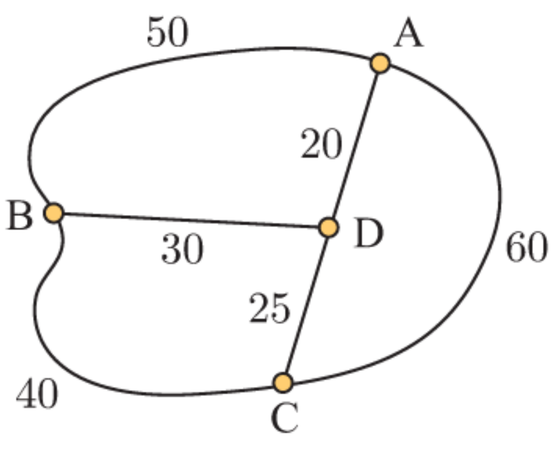
1 For each graph:

i Use Prim's algorithm to find the minimum spanning tree.

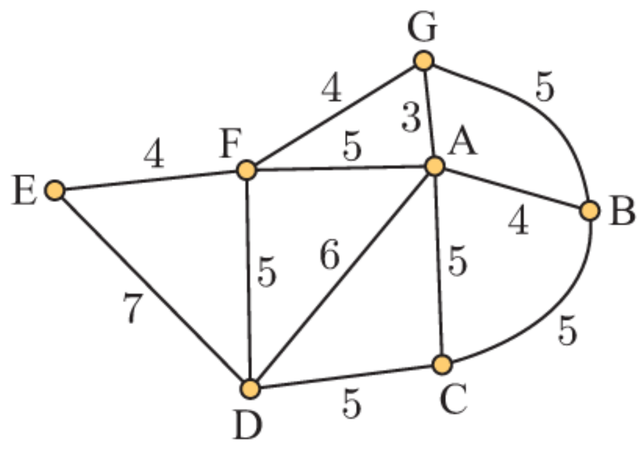
ii State the total weight of the minimum spanning tree.

PRINTABLE DIAGRAMS

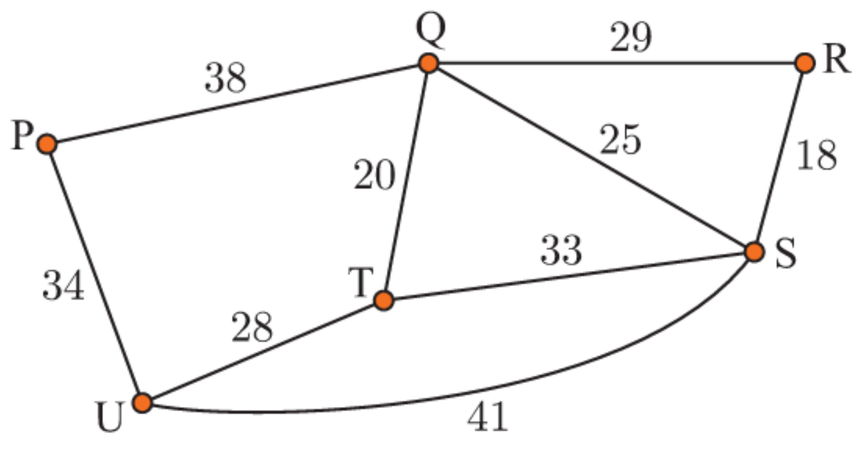
a



b



2

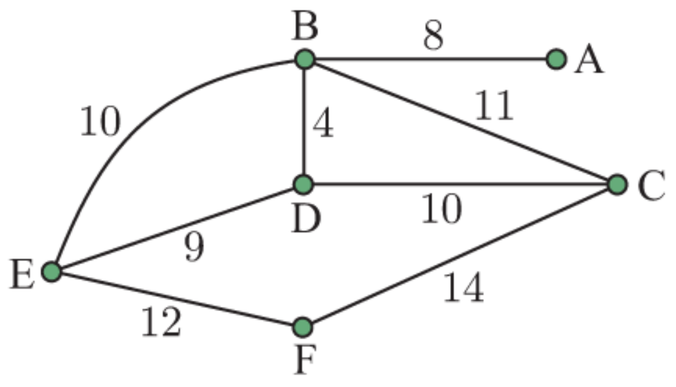


Broadband cabling is to be installed to six locations. The distances between each of these locations are given in metres alongside.

a Use Prim's algorithm to connect the locations using the least amount of cable.

b State the length of cable needed in this case.

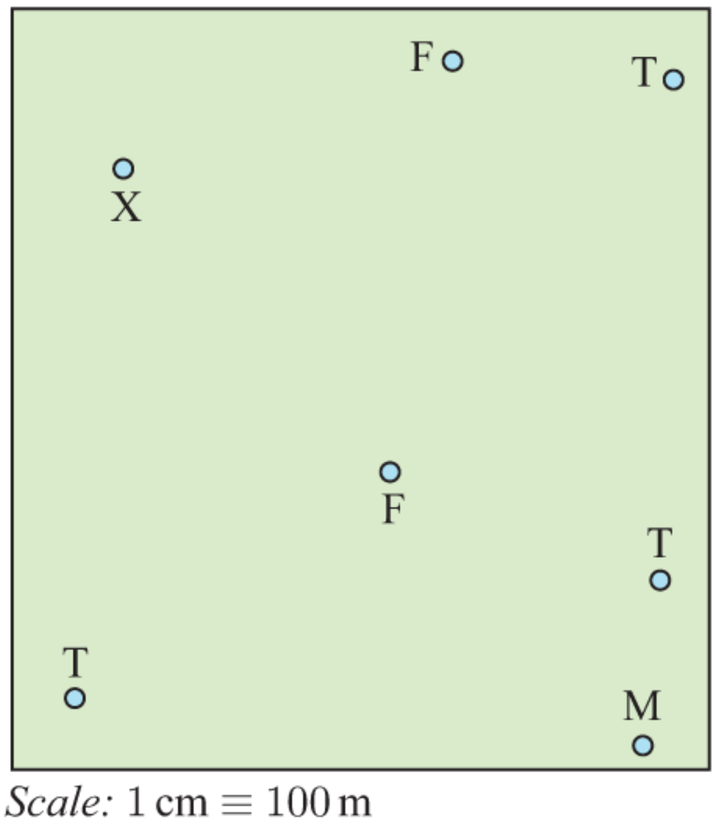
3 The diagram shows the distances, in kilometres, between towns in an alpine area. After heavy snowfalls the road authorities wish to reconnect towns as quickly as possible by clearing the minimum length of road.



a Find the roads that should be cleared and the minimum total length.

b The road from D to E is very steep, and the authorities realise that just clearing it of snow will not make it safe in icy conditions. Which roads should be cleared in this case?

4



Scale: 1 cm ≡ 100 m

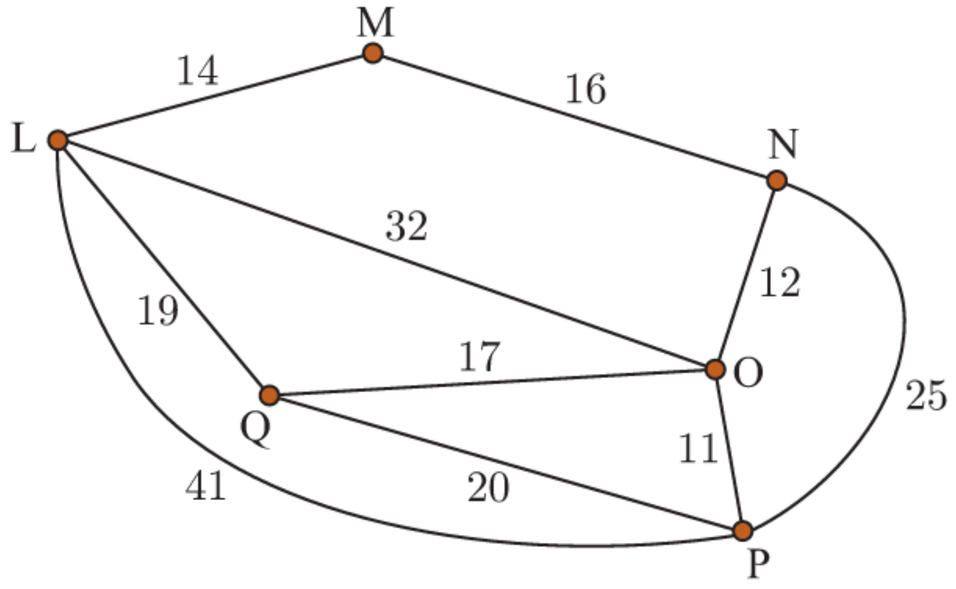
The scale diagram alongside shows a plan of a park. The positions of proposed drinking fountains (F) and taps (T) are indicated with letters.

a Find the minimum length of pipe needed to provide water at each of the drinking fountains and taps from the meter at M. Draw a plan to help the workers install them.

b The council is placing an ornamental fountain at point X. Find the minimum length of pipe needed to supply the water to all of the outlets.

c What assumptions have you made in your answers?

5 Five towns are to be connected to a new hydroelectic power station at L. The diagram alongside shows the cost in millions of dollars for connections between them.



a Find the minimum cost of connection.

b Just before the project begins, it is found that the connection costs for LM and OP were poorly estimated. Extra transformers and cabling needed will double the estimated cost. Find the most cost effective solution now.

6 This weighted adjacency table represents a graph.

	A	B	C	D	E
A	–	10	8	7	10
B	10	–	5	4	9
C	8	5	–	7	10
D	7	4	7	–	8
E	10	9	10	8	–

a How do we know the graph is complete?

b Apply Prim's algorithm to the table to construct a minimum spanning tree for the graph.

7 A fibre optic network is to be created between five suburbs. The table alongside shows the costs, in thousands of dollars, of connecting the suburbs.

	A	B	C	D	E
A	–	43	25	40	16
B	43	–	10	27	30
C	25	10	–	62	20
D	40	27	62	–	50
E	16	30	20	50	–

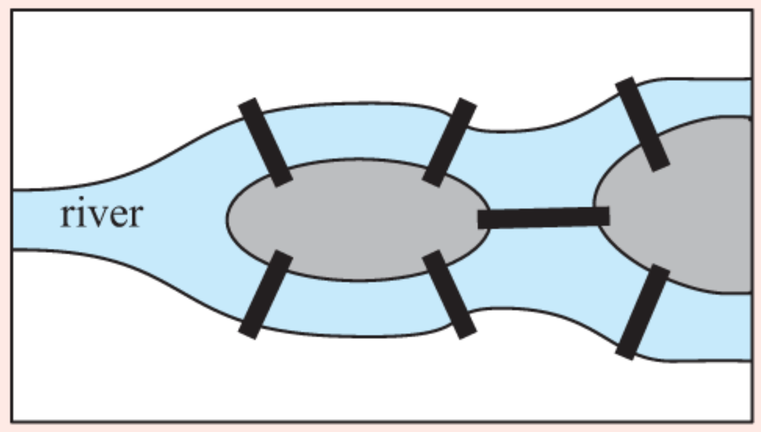
a Apply Prim's algorithm to the table to find the edges in the minimum spanning tree for the network.

b Find the minimum cost of creating the connections.

H EULERIAN GRAPHS

ACTIVITY 2 THE BRIDGES OF KÖNIGSBERG

Leonhard Euler was one of the founders of graph theory. One of his most famous contributions to mathematics concerned the town of Kaliningrad, or Königsberg as it was then known. The town was situated on the river Pregel in what was then Prussia, and had seven bridges linking two islands and the north and south banks of the river. A simplified map is shown alongside.

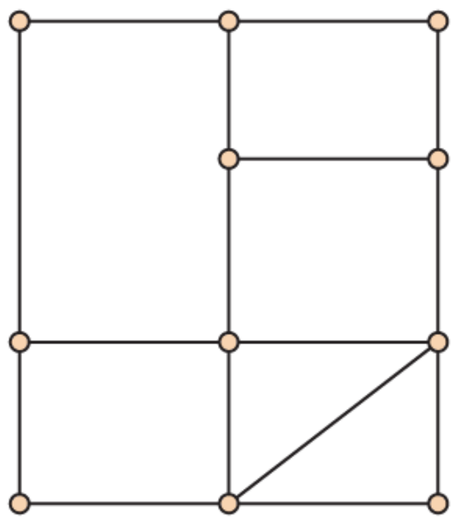


What to do:

1. Can a tour be made of the town which returns to the original point and which crosses each bridge exactly once?
2. Euler determined that such a tour would be possible if either one bridge was removed, or one was added.
   a. Which bridge would you remove?
   b. Where would you add a bridge?

It is often necessary to traverse *every edge* of a graph.

For example, a person delivering pamphlets to the houses in their neighbourhood must traverse every street. To minimise the time the job takes, it is preferable to traverse each street only once.

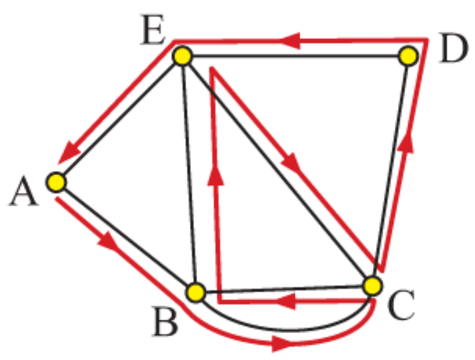


An **Eulerian circuit** is a circuit which traverses every edge exactly once.

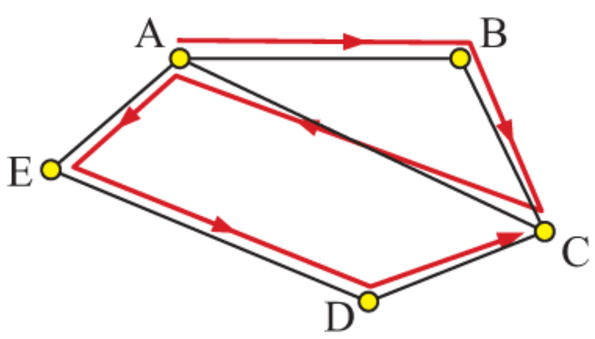
An **Eulerian trail** is a trail which traverses every edge exactly once, but does not start and end at the same vertex.

For example:

- A → B → C → B → E → C → D → E → A is an Eulerian circuit.



- A → B → C → A → E → D → C is an Eulerian trail.



A graph is **Eulerian** if it contains an Eulerian circuit.

A graph is **semi-Eulerian** if it contains an Eulerian trail but not an Eulerian circuit.

TESTING FOR EULERIAN GRAPHS

For a large graph, it can be time-consuming to identify Eulerian circuits and trails. We can instead classify a graph using the degree of its vertices.

If a connected graph contains *any* vertices of odd degree, it is not Eulerian.

Proof: For a graph to contain an Eulerian circuit, each vertex must be entered by an edge and left by another edge which is not a loop.

Therefore, if there is a vertex of odd degree, then at least one edge from the vertex is unused, and the graph is not Eulerian.

The converse of this statement is also true. We can therefore state that:

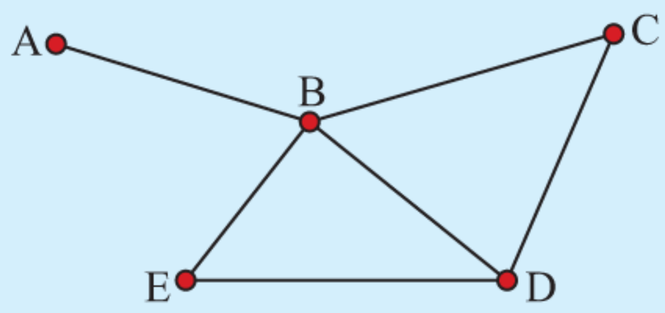
- A connected graph is **Eulerian** if there are *no* vertices of odd degree.
- A connected graph is **semi-Eulerian** if there are *exactly two* vertices of odd degree. The Eulerian trail starts at one of these vertices and ends at the other.

Example 5 **Self Tutor**

a Show that this graph is semi-Eulerian.

b Find an Eulerian trail.

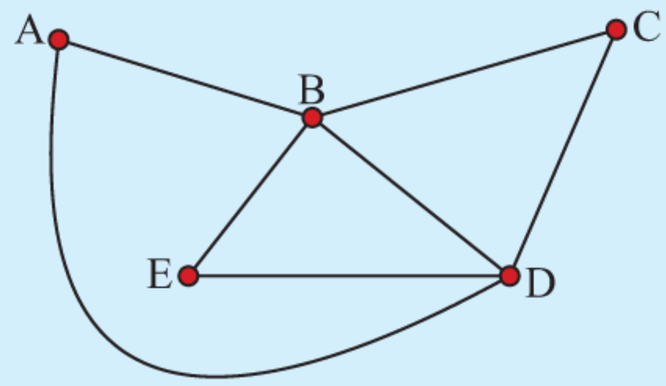
c Add an edge so that the resulting graph is Eulerian. Find an Eulerian circuit in this case.



a Exactly two vertices (A and D) have odd degree.
$\therefore$ the graph is semi-Eulerian.

b The Eulerian trail must start and end at the vertices of odd degree.
One such trail is A → B → C → D → B → E → D.

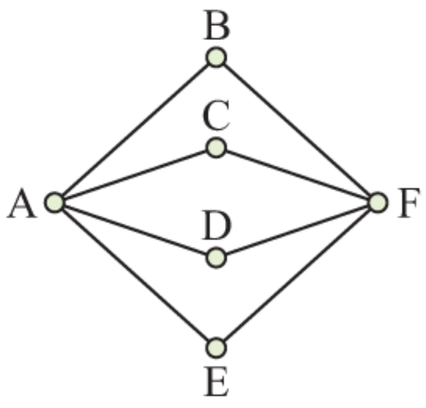
c We add an edge connecting A to D, so all vertices now have even degree and the graph is Eulerian.
An Eulerian circuit is A → B → C → D → B → E → D → A.



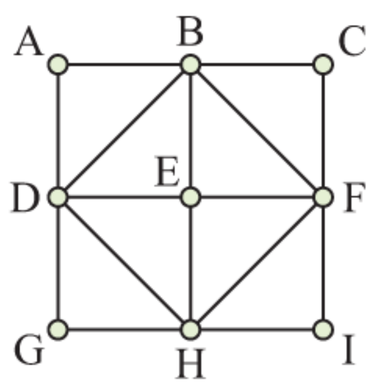
EXERCISE 15H

1 Classify the following graphs as Eulerian, semi-Eulerian, or neither. If the graph is Eulerian, find an Eulerian circuit. If the graph is semi-Eulerian, find an Eulerian trail.

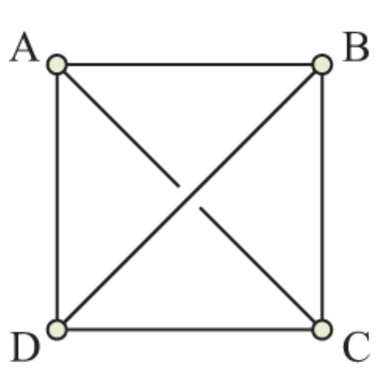
a



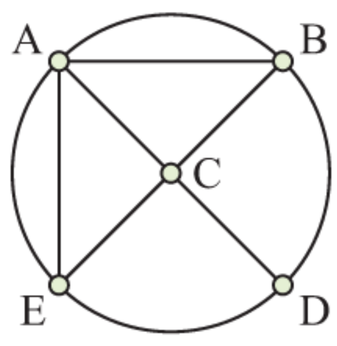
b



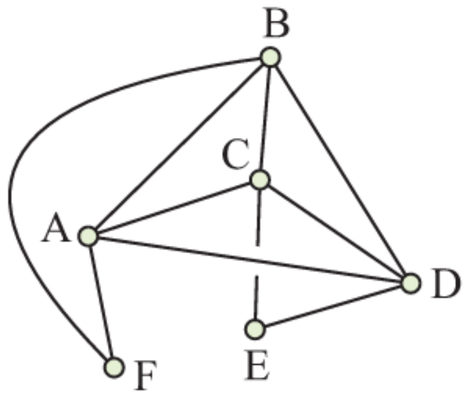
c



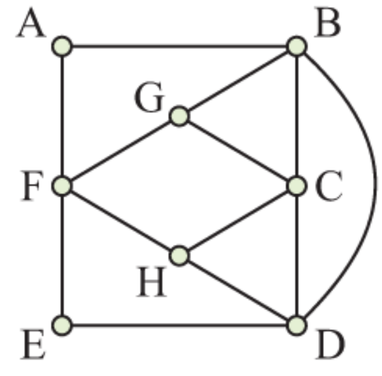
d



e



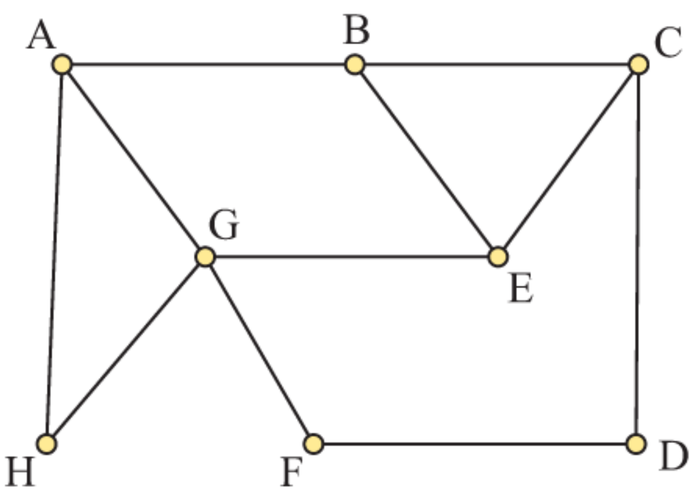
f



2 Give an example of a graph with 7 vertices which is:

a Eulerian **b** semi-Eulerian **c** neither.

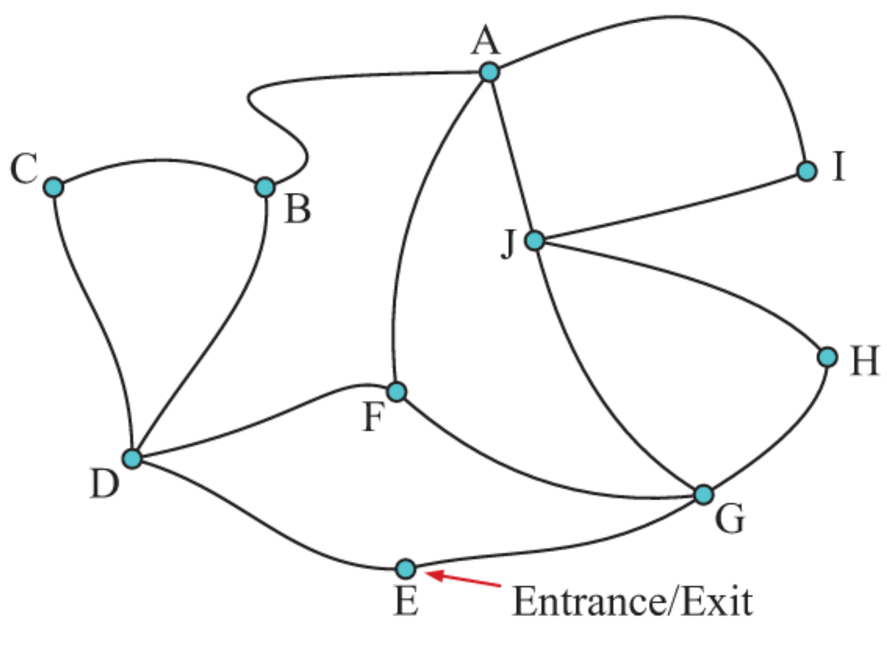
3 **a** Show that the graph alongside is neither Eulerian nor semi-Eulerian.



b Show that the subgraph formed by removing the edge CE is semi-Eulerian. Find an Eulerian trail in this case.

c Show that the subgraph formed by removing the edges CE *and* AB is Eulerian. Find an Eulerian circuit in this case.

4

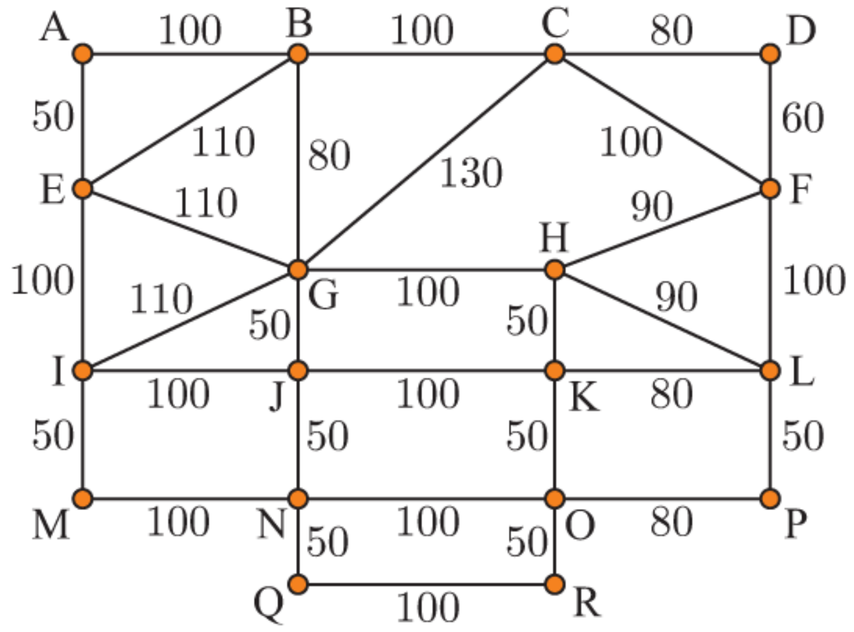


The map of a zoo is shown alongside. The edges represent tracks that visitors can walk along.

a Is it possible for visitors to perform a tour of the zoo which starts at E, traverses every track exactly once, and returns to E? Explain your answer.

b Name a track that could be built to make such a tour possible. Describe the tour in this case.

5 Suppose you have a job as a road cleaner. The roads to be cleaned are shown alongside. The numbers indicate the distances in metres.

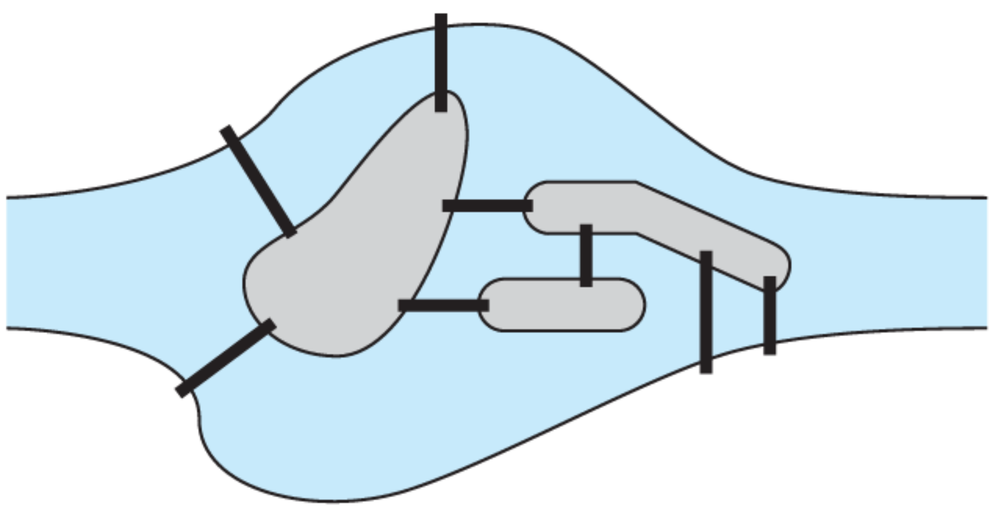


a Show that you can start at A, clean every street exactly once, and return to A.

b Explain why you can determine the length of such a route without actually specifying what the order of the vertices will be.

c Hence find the length of the route.

6



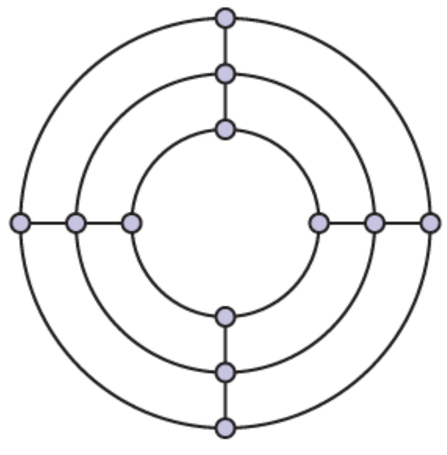
A simplified map of a small town is shown alongside.

a Does the graph contain an Eulerian circuit?

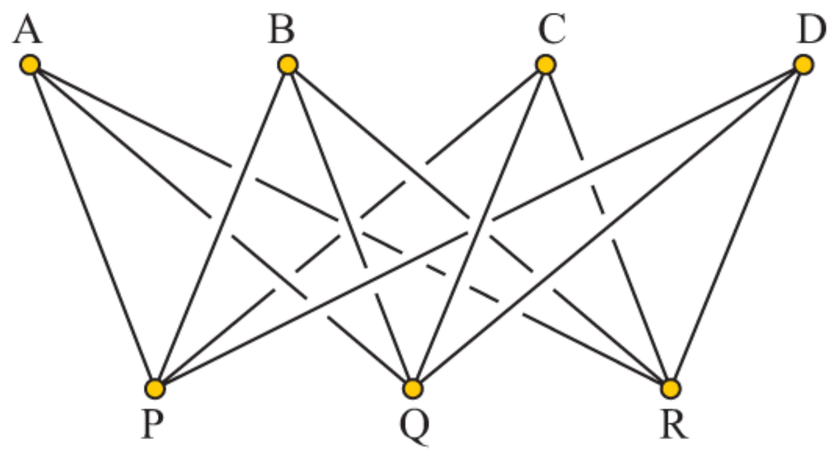
b Would either the addition or removal of one bridge create an Eulerian circuit?

7 **a** How many continuous pen strokes are needed to draw the diagram alongside, if no line segment is repeated?

b How is this problem related to Eulerian and semi-Eulerian graphs?



8 A **complete bipartite graph** contains two disjoint sets of vertices. Each vertex in one set is adjacent to every vertex in the other set, but *not* adjacent to any vertex in the same vertex set.



The complete bipartite graph alongside is denoted $k_{4,\,3}$ since there are 4 vertices in one set and 3 vertices in the other.

a Is the graph $k_{4,\,3}$ Eulerian, semi-Eulerian, or neither?

b For what values of m and n is the graph $k_{m,\,n}$ Eulerian? State the length of the Eulerian circuit in this case.

c For what values of m and n is the graph $k_{m,\,n}$ semi-Eulerian?

I THE CHINESE POSTMAN PROBLEM

The **Chinese Postman Problem** was posed by the Chinese mathematician **Kwan Mei-Ko**:

> *A postman has to leave from a depot, travel along a set number of roads to deliver the mail, then return to the depot. How should he do this so he travels the minimum distance?*

In effect, the postman has to travel along every edge of a weighted graph, and return to the starting vertex.



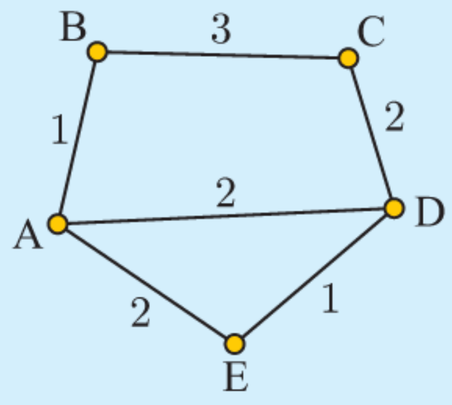
If all the vertices of the graph have even degree, the graph is Eulerian and any Eulerian circuit is a solution to the Chinese Postman Problem.

If a graph is not Eulerian, some of the edges must be walked twice to solve the Chinese Postman Problem. The task is to minimise the total weight of the edges we walk twice.

For semi-Eulerian graphs, there are exactly two vertices with odd degree. We need to walk twice along the edges forming the shortest path between the odd vertices. For small graphs, we can do this efficiently by inspection.

Example 6 — **Self Tutor**

Solve the Chinese Postman Problem for this weighted graph.



The vertices A and D have odd degree, so the graph is semi-Eulerian.

We need to walk twice between these vertices. The most efficient way to achieve this is to traverse the edge AD twice.

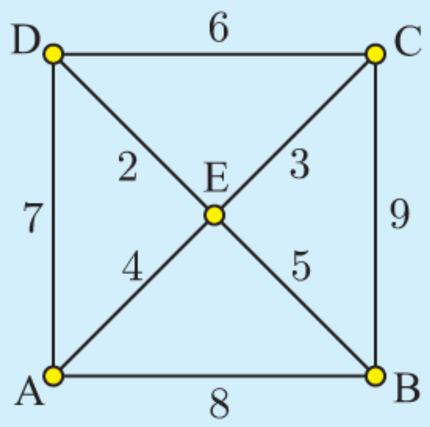
A route that traverses every edge at least once will have weight equal to the sum of the weights of all the edges, plus the weight of edge AD. The total weight is $11 + 2 = 13$.

An example solution is $\text{A} \to \text{B} \to \text{C} \to \text{D} \to \text{A} \to \text{E} \to \text{D} \to \text{A}$.

If there are more than two vertices of odd degree, we must consider each possible pairing of the odd vertices.

Example 7 **Self Tutor**

Solve the Chinese Postman Problem for the weighted graph:



The graph is neither Eulerian nor semi-Eulerian, since vertices A, B, C, and D are all odd.

There are three possible pairings of these vertices: AB and CD, AC and BD, AD and BC.

For each case we find the shortest path between the vertices:

Pairing	*Shortest path*	*Weight*	*Total weight*
AB	A → B	8	13
CD	C → E → D	5	
AC	A → E → C	7	14
BD	B → E → D	7	
AD	A → E → D	6	14
BC	B → E → C	8	

The combination of pairs with the lowest overall weight is AB and CD. The most efficient route traverses routes A → B and C → E → D twice each.

An example solution is

A → E → B → A → B → C → E → D → C → E → D → A with total weight 57.

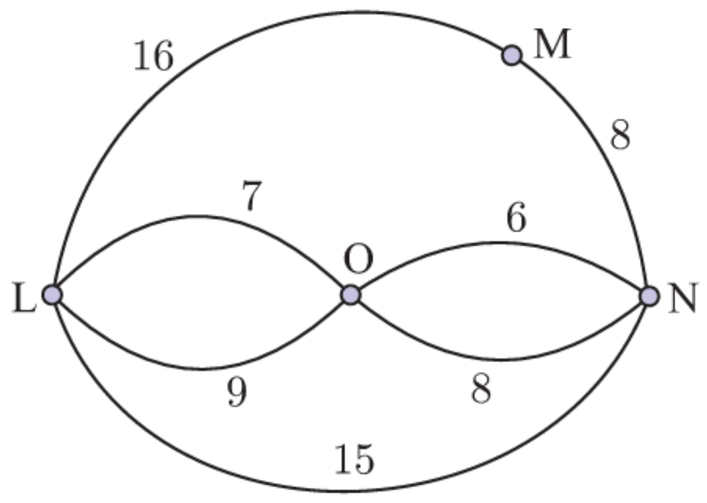
EXERCISE 15I

1 This graph shows the roads joining towns L, M, N, and O. The distances are shown in kilometres.

The district council depot is in town L. The council wishes to inspect all the roads linking the towns, returning to L in the shortest possible distance.

a Explain why this graph is Eulerian.

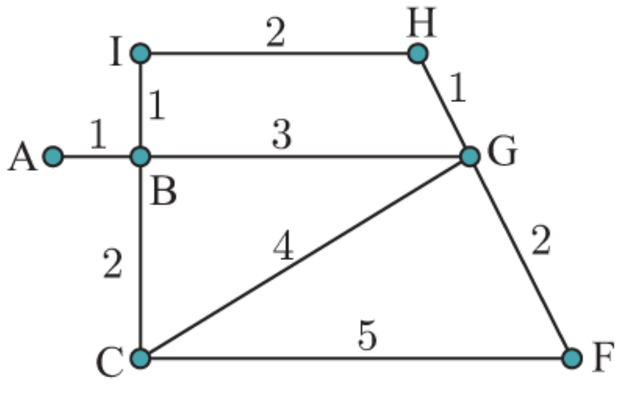
b Find the minimum distance to be travelled during the inspection, and a possible route.



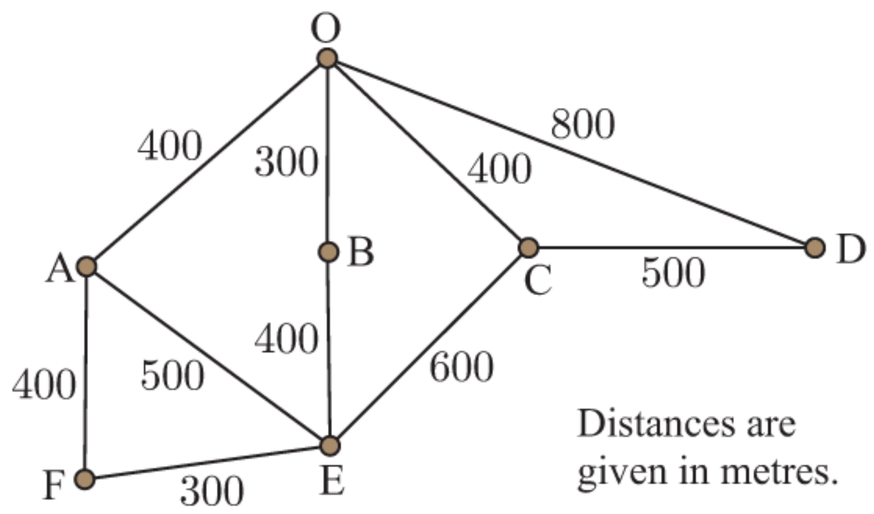
2 A snowplough must clear snow by driving along all of the roads shown, starting and finishing at its garage A. All distances shown are in kilometres.

a Is the graph Eulerian, semi-Eulerian, or neither?

b Show that the shortest distance the snowplough must travel is 24 km.



3

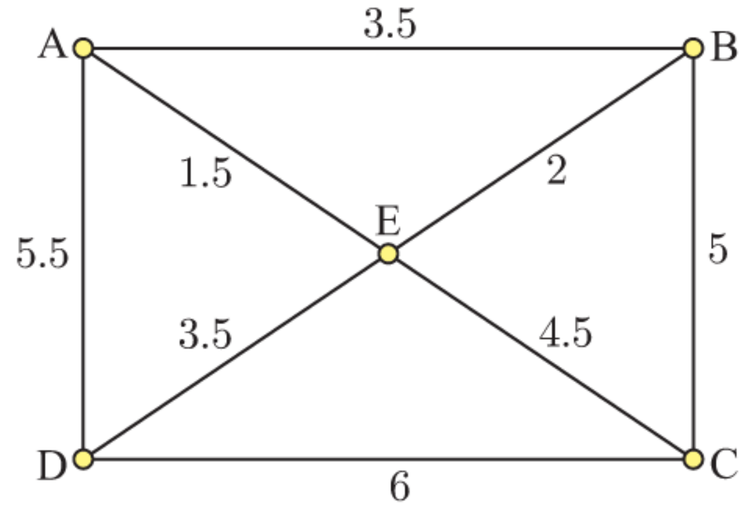


A worker needs to inspect all of the tunnels in this sewerage network. He will enter and leave through the manhole at O. Find the shortest distance that he can walk, and state a possible route for this shortest distance.

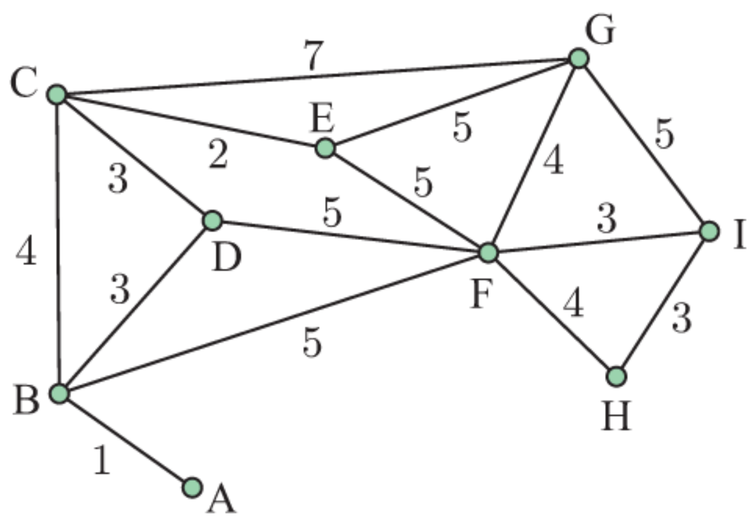
4 A carnival procession wishes to march down each of the roads shown, and return to its starting point E. All lengths are shown in kilometres.

a List the three different ways in which the four vertices of odd degree can be paired.

b Find the shortest distance that the procession has to march.



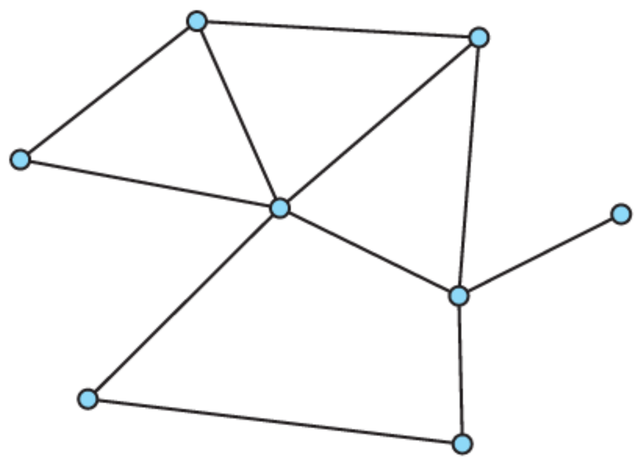
5



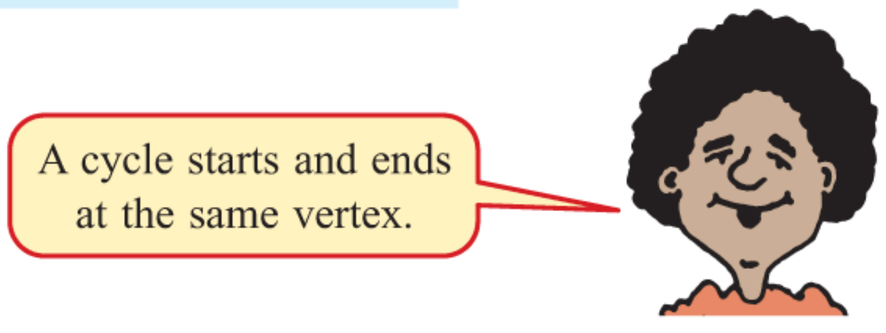
The graph opposite shows the roads in Postman Peter's mailing route. He starts and finishes his round at A.
Find the shortest distance Peter needs to walk, and state a possible route with this shortest distance.

J HAMILTONIAN GRAPHS

The railway lines connecting a city's tourist attractions are shown alongside. Kerry is a tourist to the city, and would like to visit each of the attractions. She is not interested in traversing every *edge* of the graph, but rather wishes to visit every *vertex*.

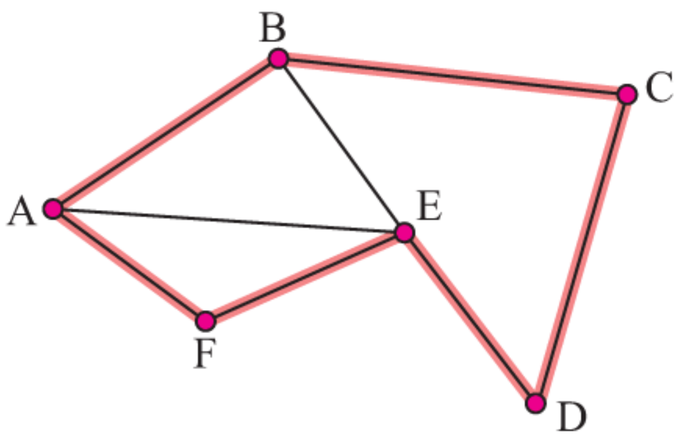


- A **Hamiltonian cycle** is a cycle which visits each vertex (except the starting and ending vertex) exactly once.
- A **Hamiltonian path** is a path which visits each vertex exactly once, but does not start and end at the same vertex.

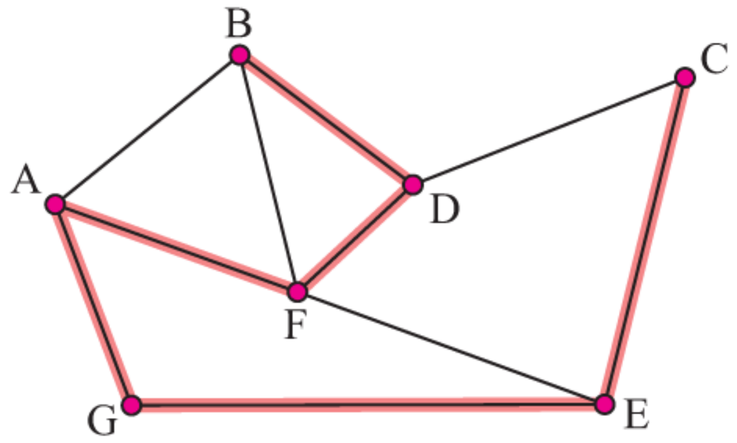


For example:

- A → B → C → D → E → F → A is a Hamiltonian cycle.



- B → D → F → A → G → E → C is a Hamiltonian path.



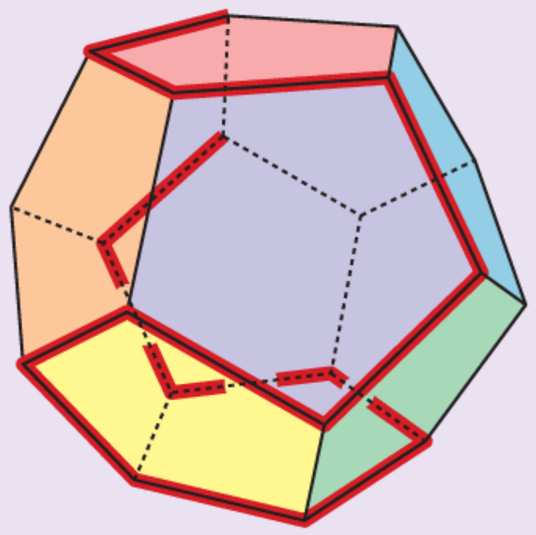
- A graph is **Hamiltonian** if it contains a Hamiltonian cycle.
- A graph is **semi-Hamiltonian** if it contains a Hamiltonian path, but not a Hamiltonian cycle.

HISTORICAL NOTE

Hamiltonian graphs are named after the Irish mathematician **William Rowan Hamilton** (1805 - 1865).

Hamilton invented a game called **The Icosian Game**. The object of the game is to find a way around the edges of a dodecahedron so that each vertex is visited exactly once.

Hamilton sold the game to a London toy company for £25. However, the game did not prove to be popular, primarily because it was too easy.



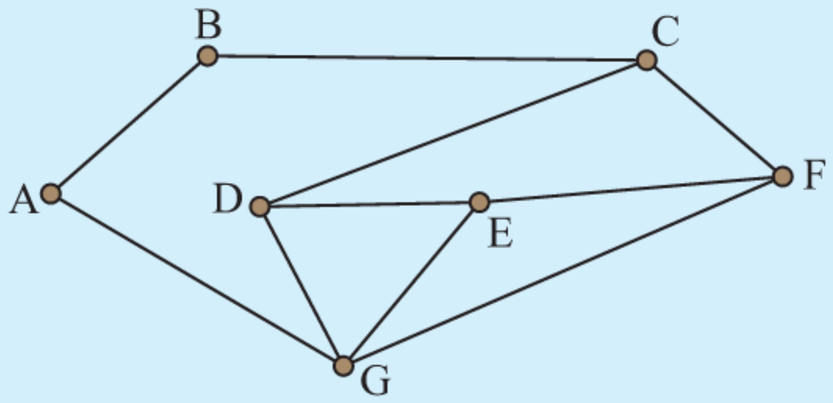
In contrast to Eulerian graphs, there is no easy way to test whether a graph is Hamiltonian, semi-Hamiltonian, or neither. We instead try to find Hamiltonian cycles and paths by inspection.

Example 8 Self Tutor

For the graph alongside:

a Find a Hamiltonian cycle which starts and ends at A.

b Find a Hamiltonian path which starts at E and ends at B.

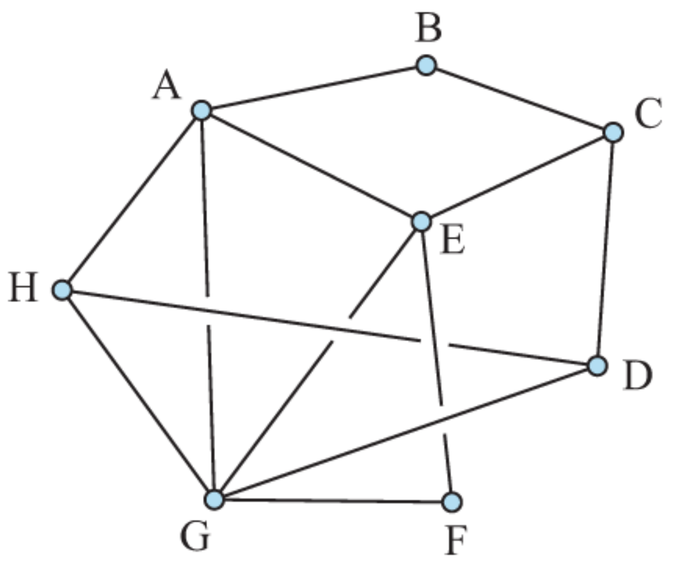


a A → B → C → D → E → F → G → A

b E → D → C → F → G → A → B

EXERCISE 15J

1 Consider the graph alongside.



a Find a Hamiltonian cycle starting and finishing at A.

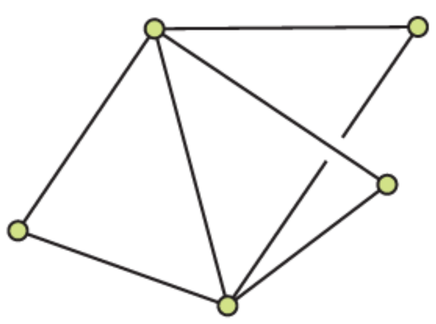
b Find a Hamiltonian path which starts at E and finishes at:

i H **ii** B.

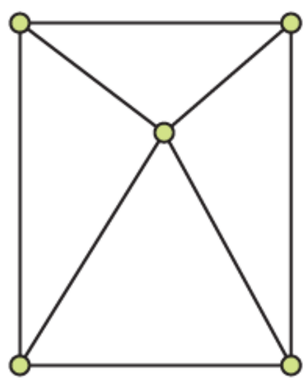
c Is the graph Hamiltonian? Explain your answer.

2 Decide whether each graph is Hamiltonian, semi-Hamiltonian, or neither:

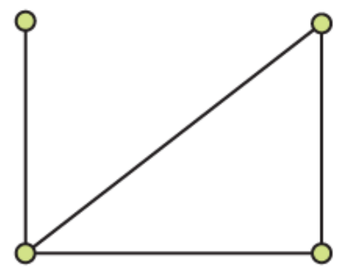
a



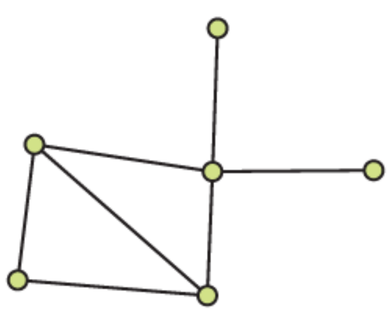
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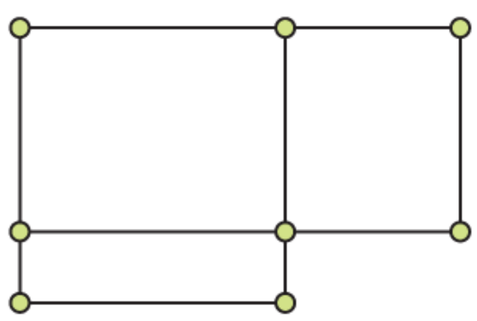
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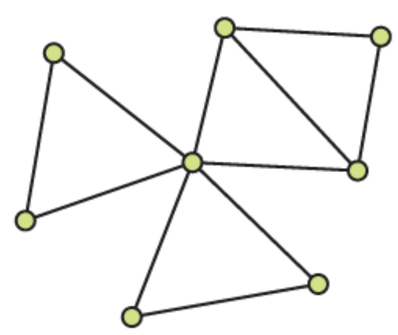
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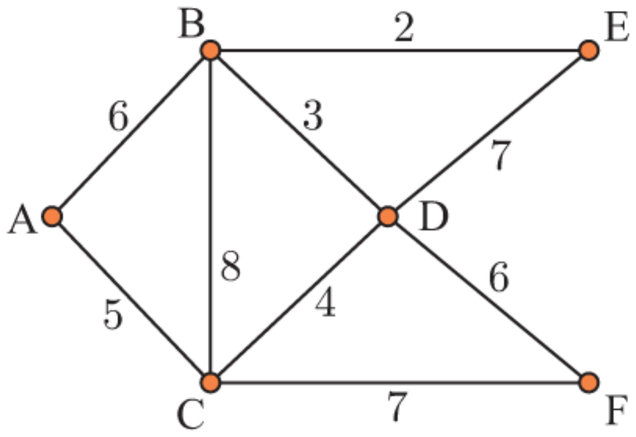
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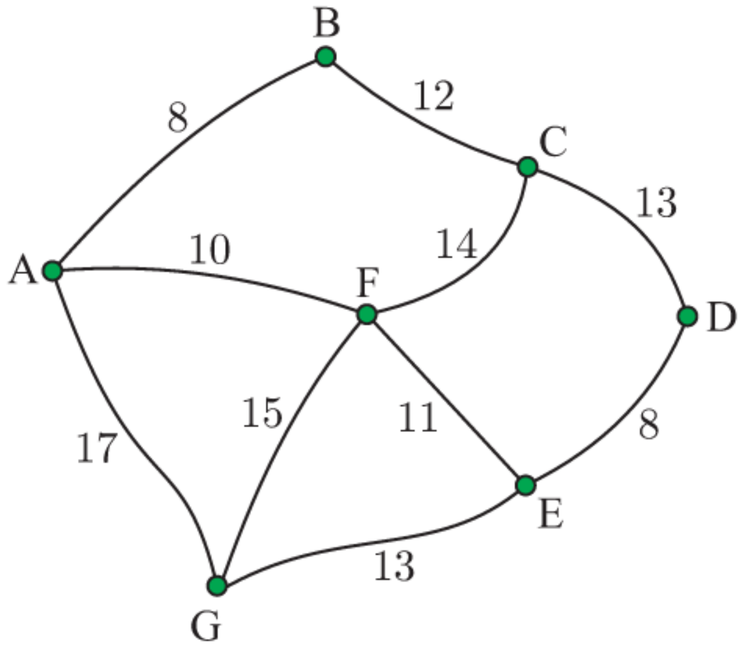
3 **a** For the given graph, list *all* Hamiltonian paths which start at A.

b Which of the Hamiltonian paths in **a** has the lowest total weight?

c Is the graph Hamiltonian, semi-Hamiltonian, or neither? Explain your answer.



4



The campsites at a national park are shown alongside. The numbers indicate distances between them in kilometres.

A park ranger is currently at campsite A, and must inspect each of the other campsites.

a One possible path is

$$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G.$$

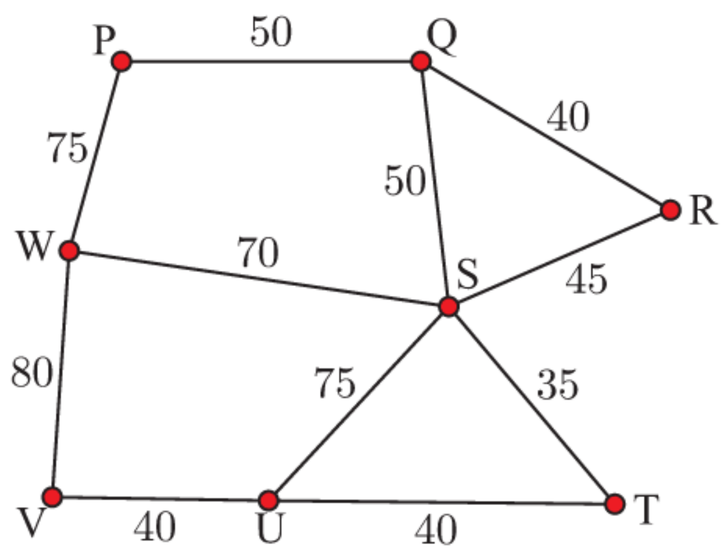
Find the total length of this path.

b Find an alternative Hamiltonian path the ranger can take, which ends at:

i F **ii** D.

In each case, state the length of the path.

5 The roads linking a group of rural towns are shown alongside. The numbers indicate distances in kilometres.



A milk supplier is based in town P. He must visit each of the other towns, before returning to P.

a If the supplier visits town Q first, find the Hamiltonian cycle the supplier can traverse. Calculate the total distance of this cycle.

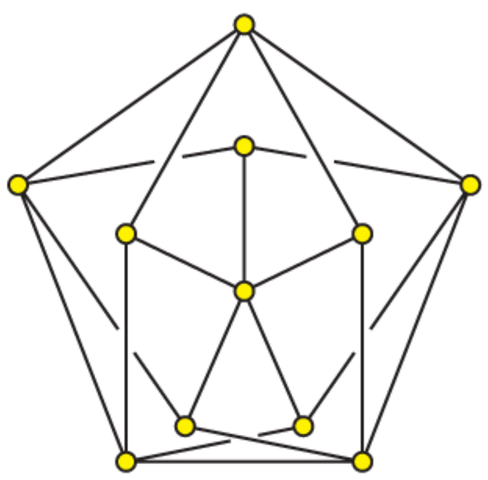
b Suppose the road between S and R is inaccessible due to flood.

i Is it still possible for the supplier to complete a Hamiltonian cycle?

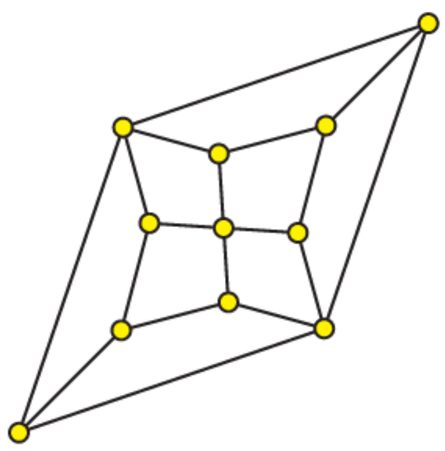
ii Find a Hamiltonian path the supplier can take.

6 Find, if possible, a Hamiltonian cycle in each graph:

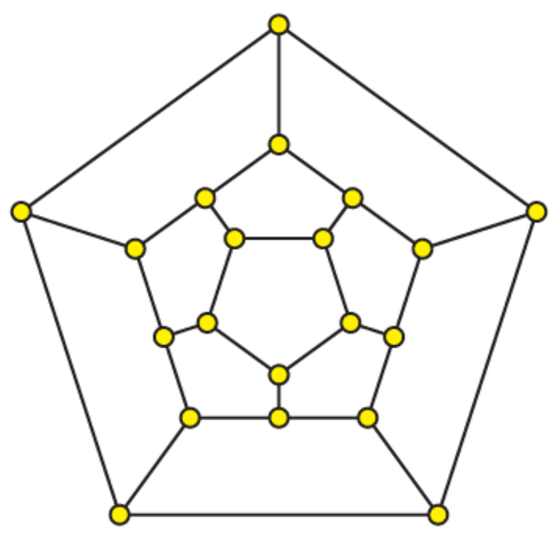
a



b



c



7 Give an example of a graph which is:

a both Hamiltonian and Eulerian

b Hamiltonian but not Eulerian

c Eulerian but not Hamiltonian

d semi-Hamiltonian and semi-Eulerian.

8 **a** Explain why the complete graph k_n contains a Hamiltonian cycle for all $n \geqslant 3$.

b Explain why k_2 does *not* contain a Hamiltonian cycle.

K THE TRAVELLING SALESMAN PROBLEM

The **Travelling Salesman Problem** (TSP) is described by:

> *A travelling salesman leaves his home town, visits all of the towns in the district, then returns home. How should he do this so that the distance he travels is minimised?*

The salesman has to travel along the edges of a graph, starting and finishing at the same vertex, and visiting every other vertex in between.

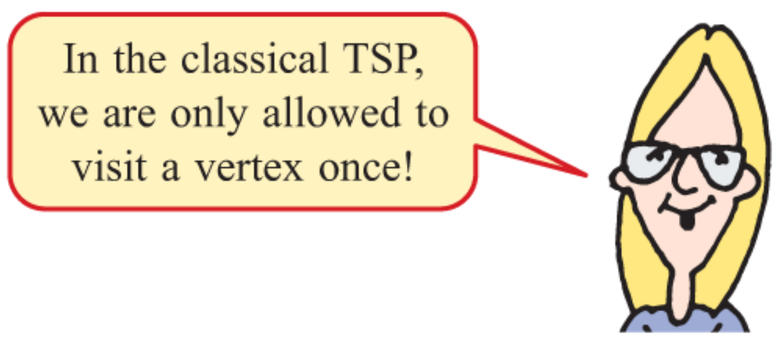


THE CLASSICAL TRAVELLING SALESMAN PROBLEM

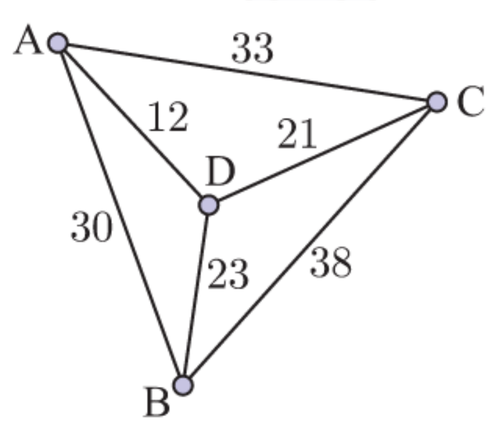
We will first consider *complete* graphs which have the **Euclidean property** that the direct route between any two vertices is the shortest route between them.

For graphs of this type, we solve the **classical Travelling Salesman Problem**:

> For a given weighted complete graph, find a Hamiltonian cycle of least weight.



For example, suppose the salesman lives in town A. He wants to visit each of the towns exactly once, before returning to A. The distances are given in kilometres.



The Hamiltonian cycles that start and finish at A are:

$A \to B \to C \to D \to A$:
$30 + 38 + 21 + 12 = 101$

$A \to B \to D \to C \to A$:
$30 + 23 + 21 + 33 = 107$

$A \to C \to B \to D \to A$:
$33 + 38 + 23 + 12 = 106$

$A \to D \to C \to B \to A$:
$12 + 21 + 38 + 30 = 101$

$A \to C \to D \to B \to A$:
$33 + 21 + 23 + 30 = 107$

$A \to D \to B \to C \to A$:
$12 + 23 + 38 + 33 = 106$

Notice that the three cycles on the right are simply the reverse of those on the left, so we did not need to show these calculations.

The salesman should take the route $A \to B \to C \to D \to A$ or its reverse. These routes have the minimum length of 101 km.

This method of solving the TSP can be used for small graphs, but it becomes impractical as the size increases. However, there are algorithms we can apply to determine the upper and lower bounds for m, the weight of the minimium weight Hamiltonian cycle.

FINDING AN UPPER BOUND

For a *complete* graph with at least 3 vertices, we can visit any unvisited vertex whenever we like, and once all the vertices have been visited, we can return to the starting vertex immediately.

We are therefore guaranteed that a Hamiltonian cycle exists for all complete graphs k_n, $n \geqslant 3$.

For this special case, a Hamiltonian cycle can be found using the **nearest neighbour algorithm**:

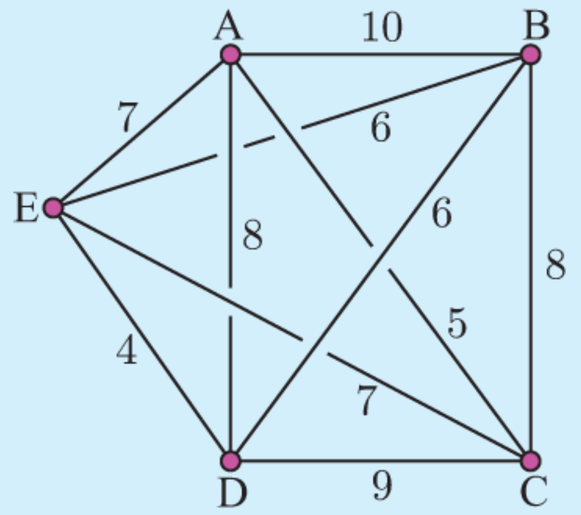
Step 1: Choose a starting vertex.
Step 2: Follow the edge of least weight to an unvisited vertex. If there is more than one such edge, choose one at random.
Step 3: Repeat *Step 2* until all vertices have been visited.
Step 4: Return to the starting vertex by adding the corresponding edge.
The sequence of visited vertices is a Hamiltonian cycle.

This algorithm delivers a Hamiltonian cycle of low, but not necessarily minimum, weight. The weight of the resulting Hamiltonian cycle therefore provides an upper bound for m.

Example 9 **Self Tutor**

Consider the weighted complete graph alongside.

Use the nearest neighbour algorithm starting at A to find a Hamiltonian cycle for the graph. Hence find an upper bound for the TSP.



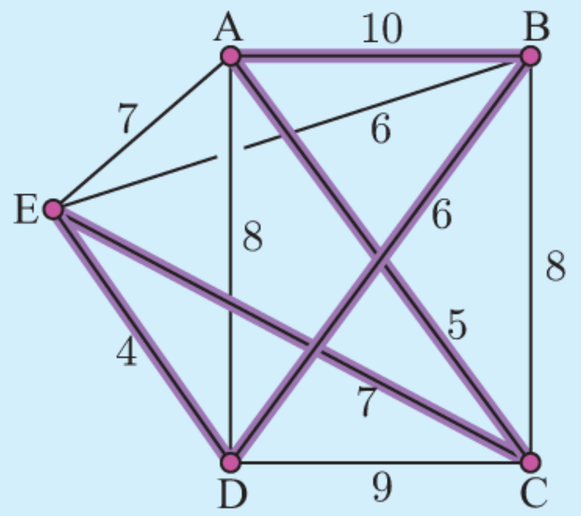
From A, the nearest vertex is C, so we add edge AC

The nearest unvisited vertex to C is E, so we add edge CE.

Continuing this process, we choose ED, then DB.

All vertices have now been visited, so we add BA to complete the Hamiltonian cycle $A \to C \to E \to D \to B \to A$.

So, the upper bound for the TSP is $5+7+4+6+10=32$.



FINDING A LOWER BOUND

Let G be a weighted complete graph on the n vertices $V_1, V_2, ..., V_n$.

The vertices are labelled so that $V_1 \to V_2 \to .... \to V_n \to V_1$ is the Hamiltonian cycle with minimum weight m, and is therefore the solution to the TSP for G.

$\therefore$ $m =$ weight of V_1V_2 + weight of V_1V_n + weight of path $V_2 \to V_3 \to .... \to V_n$.

Now consider the graph H which is G with vertex V_1 removed and all edges connected to V_1 removed. H is the complete graph on the $n-1$ vertices $V_2, V_3, ..., V_n$.

The path $V_2 \to V_3 \to .... \to V_n$ is a spanning tree of H, but not necessarily a minimum spanning tree.

$\therefore$ weight of path $V_2 \to V_3 \to .... \to V_n \geqslant$ weight of a minimum spanning tree of H

$\therefore$ $m \geqslant$ weight of V_1V_2 + weight of V_1V_n + weight of a minimum spanning tree of H

This observation is the basis for the **deleted vertex algorithm**, which gives a lower bound for the TSP.

Step 1: Delete a vertex, together with all edges connected to it, from the original graph.

Step 2: Find the minimum spanning tree for the remaining graph.

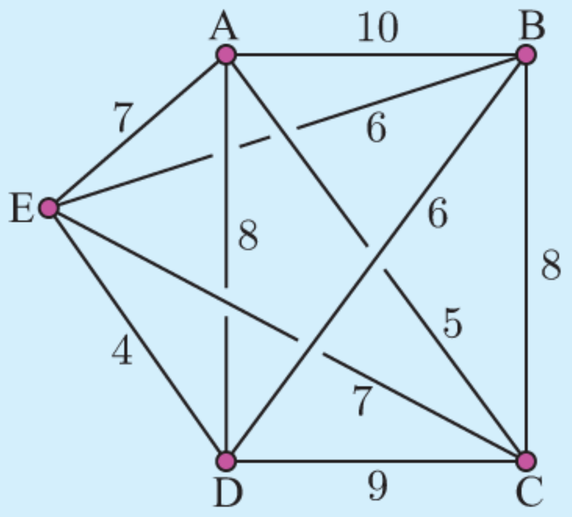
Step 3: Add to the length of the minimum spanning tree, the lengths of the two shortest deleted edges.

The resulting value is a lower bound for m, the minimum weight for the solution to the TSP.

Example 10 **Self Tutor**

The graph alongside is copied from **Example 9**.

Apply the deleted vertex algorithm by deleting vertex A. Hence obtain a lower bound for the TSP.

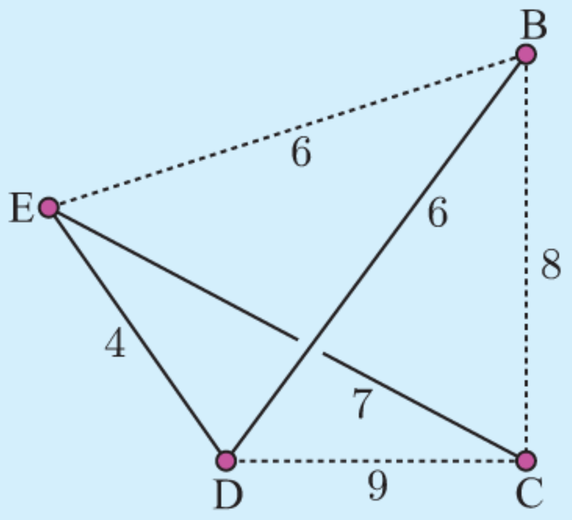


We use Kruskal's algorithm to find a minimum spanning tree for the graph with vertex A and all edges connected to it deleted.

The minimum spanning tree has weight 17.

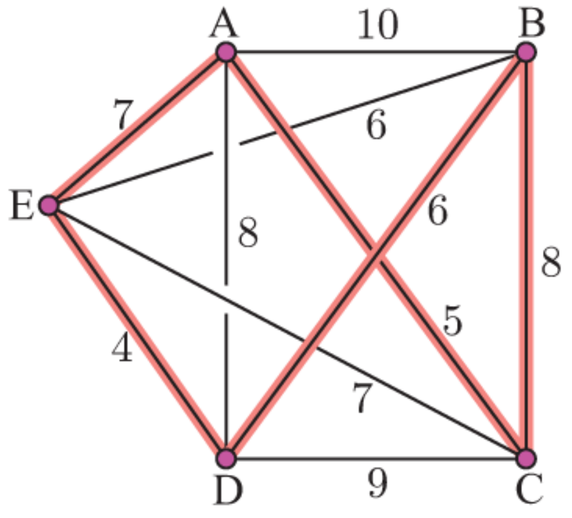
The two shortest deleted edges have weights 5 and 7.

So, the lower bound for the TSP is $17 + 5 + 7 = 29$.



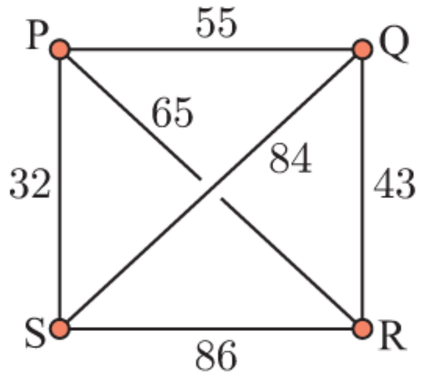
From **Examples 9** and **10**, the solution to the TSP for this graph satisfies $29 \leqslant m \leqslant 32$.

In fact, the Hamiltonian cycle of minimum weight is
A → C → B → D → E → A, with weight 30.



EXERCISE 15K.1

1 Consider the weighted complete graph alongside.



a Use the nearest neighbour algorithm starting at P to find a Hamiltonian cycle for the graph. Hence find an upper bound for the TSP.

b Apply the deleted vertex algorithm by deleting vertex P. Hence find a lower bound for the TSP.

c The minimum weight Hamiltonian cycle for this graph has weight 216.

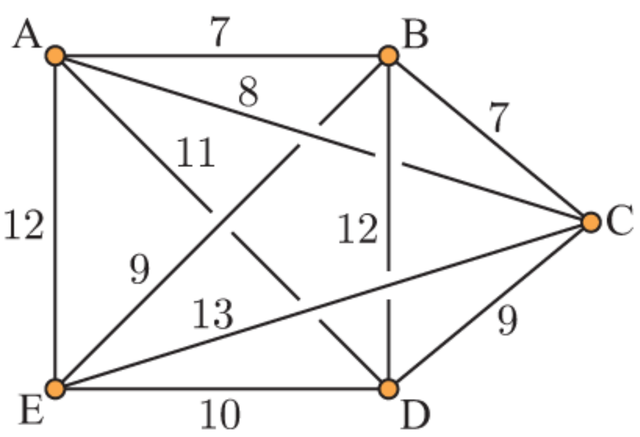
i Verify that this is consistent with the bounds you have obtained.

ii Find the Hamiltonian cycle of minimum weight which starts and finishes at P.

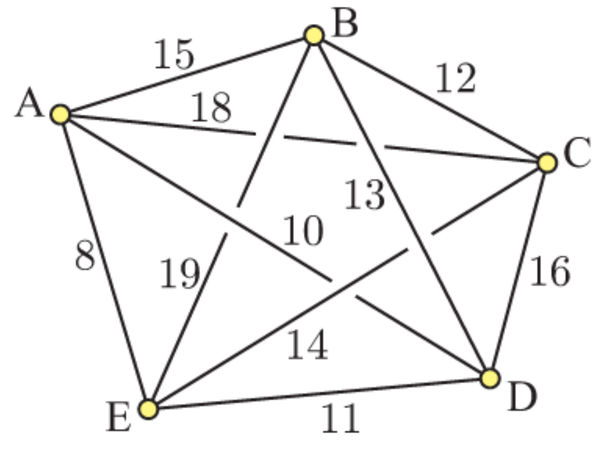
2 **a** Apply the nearest neighbour algorithm starting at vertex C. Hence find an upper bound for the TSP.

b Apply the deleted vertex algorithm by deleting vertex C. Hence find a lower bound for the TSP.

c Solve the TSP for this graph.



3 Consider the graph alongside.



a Find an upper bound for the TSP using the nearest neighbour algorithm beginning at:

i vertex A **ii** vertex B **iii** vertex C
iv vertex D **v** vertex E.

Which is the best bound found?

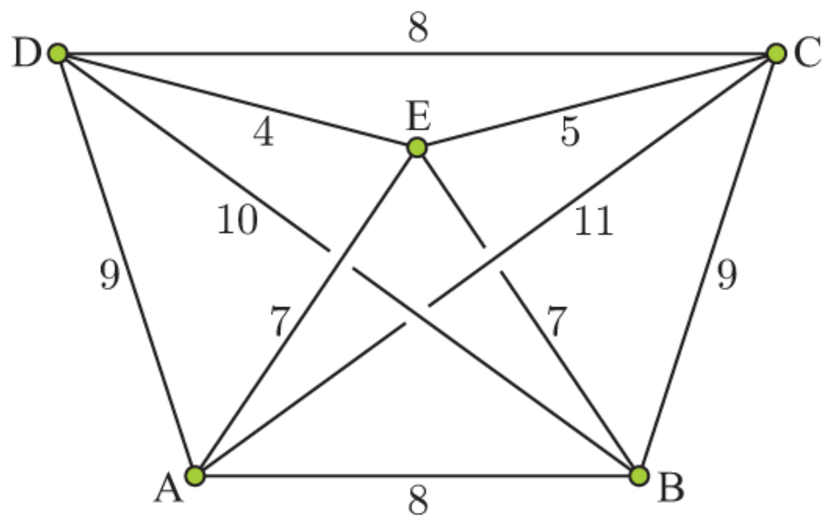
b Find a lower bound for the TSP using the deleted vertex algorithm and deleting:

i vertex A **ii** vertex B **iii** vertex C **iv** vertex D **v** vertex E.

Which is the best bound found?

c Solve the TSP for this graph.

4 Consider the TSP for the given graph.



a Apply the deleted vertex algorithm with A, B, C, D, and E in turn to find a lower bound.

b Apply the nearest neighbour algorithm beginning with vertex A to find an upper bound and a Hamiltonian cycle.

c Suppose the network represents five towns.

i At which town would you choose to base yourself to minimise your travel between towns? Explain your answer.

ii Apply the nearest neighbour algorithm beginning with your chosen vertex to find an upper bound and a Hamiltonian cycle.

5 A hygiene inspector lives in Town A and has to visit hawkers centres in towns B, C, D, E, and F. The distances between towns are shown alongside in kilometres. Use the nearest neighbour algorithm to recommend a route for the inspector.

	A	B	C	D	E	F
A	–	16	13	11	7	8
B	16	–	10	5	12	10
C	13	10	–	4	7	9
D	11	5	4	–	6	7
E	7	12	7	6	–	9
F	8	10	9	7	9	–

6 The table below shows the distances, in kilometres, between towns in France. Twice per year, a company representative must visit each town in turn, and then return home.

Bordeaux
870 Calais
641 543 Dijon
550 751 192 Lyons
649 1067 507 316 Marseille
457 421 297 445 761 Orléans
247 625 515 431 733 212 Poitiers
519 803 244 59 309 392 421 St-Etienne
244 996 726 535 405 582 435 582 Toulouse

a Use the nearest neighbour algorithm to find a Hamiltonian cycle visiting each town, and beginning and ending at:

i Toulouse **ii** Calais.

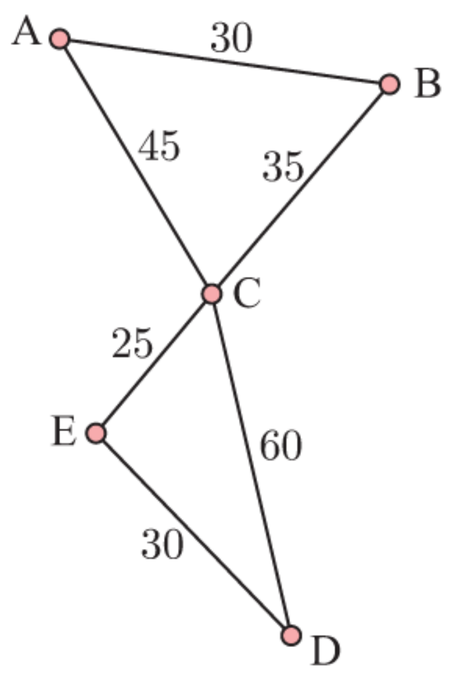
b Which town, Toulouse or Calais, would be the preferred home town for the company representative to minimise travel? Explain your answer.

THE PRACTICAL TSP

So far we have only considered complete graphs in which the direct route between two vertices is the shortest. In this case it is always possible to find a Hamiltonian cycle, and it is never advantageous to visit a vertex more than once.

However, in practice these conditions may not be realistic.

For example, consider this graph which shows the roads between towns, and the time it takes in minutes to travel along each road.



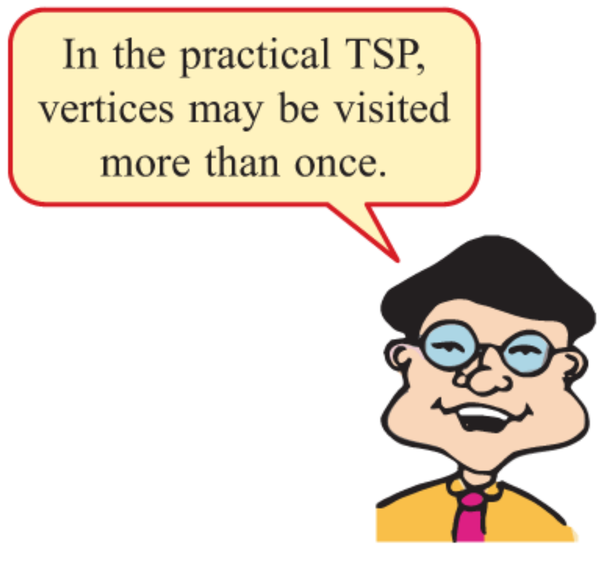
Notice that:

- The graph is not complete, nor is it Hamiltonian. It is impossible to visit every town and return to the starting point without visiting some towns more than once.
- It is faster to travel from C to D via E than to travel directly from C to D. This may be because the road connecting C and D is very windy, or of poor quality. In this case it may be preferable to take the fastest route through E, even if this means visiting E more than once.

We therefore consider the **practical TSP**:

> For a given weighted graph, find the route of least weight which starts and finishes at the same vertex, and visits each vertex *at least* once.

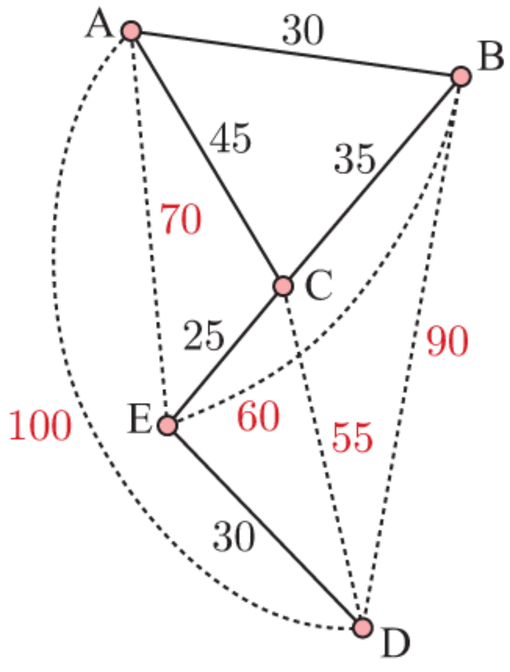
In the graph above, the solution to the practical TSP is the route A → B → C → E → D → E → C → A, which takes 220 minutes. Notice that C and E are visited twice.



We can convert a practical TSP into a classical TSP. To do this, we turn the graph into a complete Euclidean graph by adding or replacing edges to show the least weight between vertices.

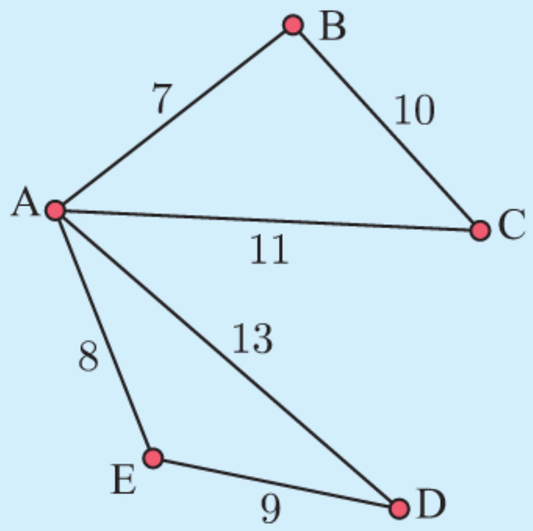
For example, the fastest route between A and E is $45 + 25 = 70$ minutes, and the fastest route between C and D is $25 + 30 = 55$ minutes.

Solving the classical TSP for the transformed graph is equivalent to solving the practical TSP for the original graph.



Example 11 **Self Tutor**

Consider the graph alongside.

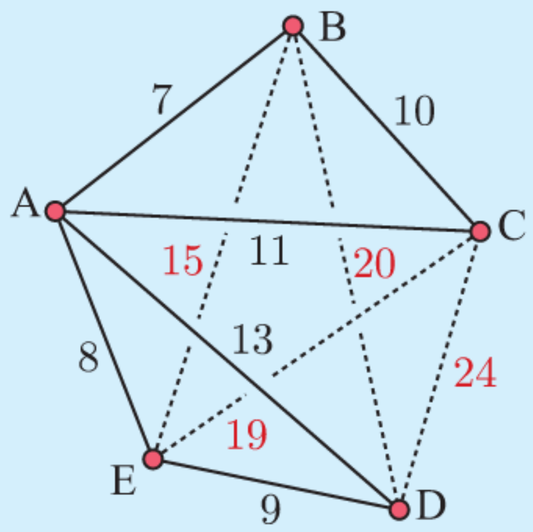


a Complete the graph with edges showing the shortest distance between vertices.

b Use the nearest neighbour algorithm starting at vertex A to find an upper bound for the TSP.

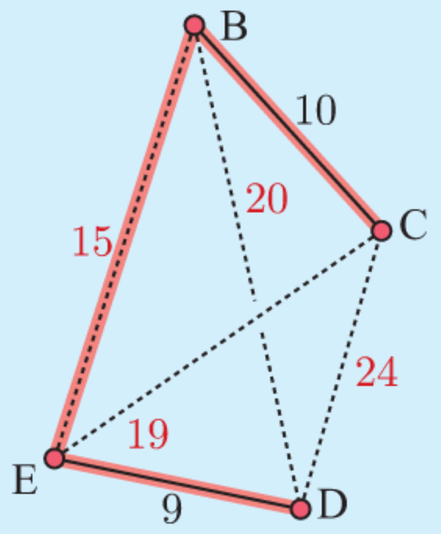
c Use the deleted vertex algorithm with vertex A deleted to find a lower bound for the TSP.

a The shortest distance from B to E is $7 + 8 = 15$ units.
The shortest distance from B to D is $7 + 13 = 20$ units.
The shortest distance from C to E is $11 + 8 = 19$ units.
The shortest distance from C to D is $11 + 13 = 24$ units.



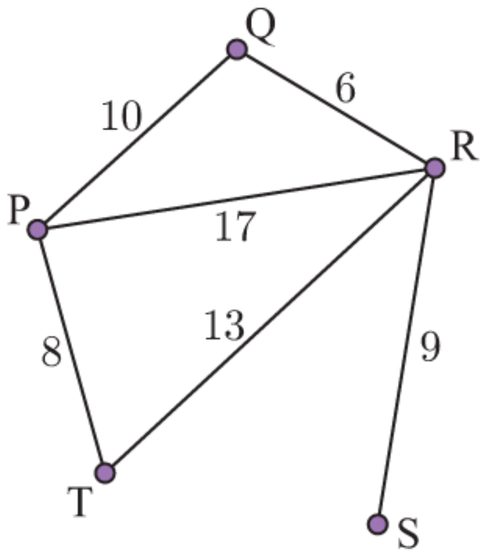
b Starting from vertex A, the nearest neighbour algorithm on the transformed graph gives the Hamiltonian cycle $A \to B \to C \to E \to D \to A$.
So, an upper bound for the TSP is $7 + 10 + 19 + 9 + 13 = 58$.

c Deleting vertex A and the edges connected to it, we apply Kruskal's algorithm to find a minimum spanning tree. The minimum spanning tree has weight 34.
The two shortest deleted edges have weights 7 and 8.
So, a lower bound for the TSP is $34 + 7 + 8 = 49$.



EXERCISE 15K.2

1 Consider the graph alongside.



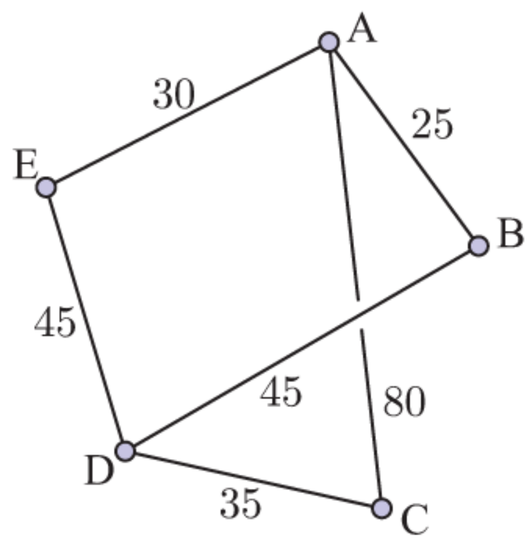
a Complete the graph with edges showing the shortest distance between vertices.

b Use the nearest neighbour algorithm starting at vertex P to find an upper bound for the TSP.

c Use the deleted vertex algorithm with vertex P deleted to find a lower bound for the TSP.

d Solve the practical TSP for the original graph by inspection. Verify that your solution is consistent with the bounds you calculated.

2 This graph shows the railway lines connecting towns A, B, C, D, and E. The weights are the cost in pounds of travelling along each line.



a Complete the graph with edges showing the cheapest route between towns.

b A salesman lives in town B. He must visit every other town before returning to B.

i Use the nearest neighbour algorithm starting at B to find an upper bound for the TSP.

ii Use the deleted vertex algorithm with D deleted to find a lower bound for the TSP.

iii State the total cost of the cheapest route for the salesman, and find a route he should take. Apart from B, which town(s) must he visit more than once?

3 This table shows the distances, in kilometres, of roads between cities. The symbol – indicates there is no direct road between the cities.

	A	B	C	D	E	F
A	0	–	250	–	150	100
B	–	0	130	–	180	–
C	250	130	0	160	140	–
D	–	–	160	0	120	–
E	150	180	140	120	0	80
F	100	–	–	–	80	0

a How many cities are directly connected to A?

b Draw a graph to show the information in the table.

c Construct a table which shows the least distances between the cities.

d Trudi is a sales representative who lives in D. She must visit each of the other cities before returning to D.

i Use the nearest neighbour algorithm starting at D to find an upper bound for the TSP.

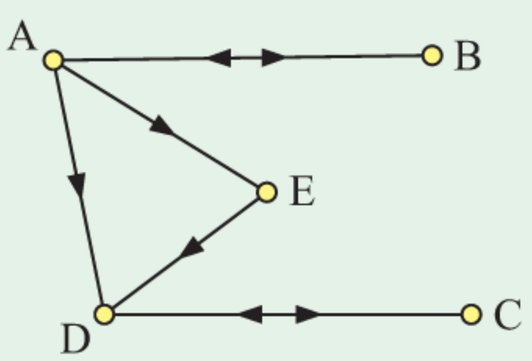
ii Use the deleted vertex algorithm with D deleted to find a lower bound for the TSP.

iii Find a route for Trudi which improves on the upper bound obtained in **i**. Which city must Trudi visit twice in this route?

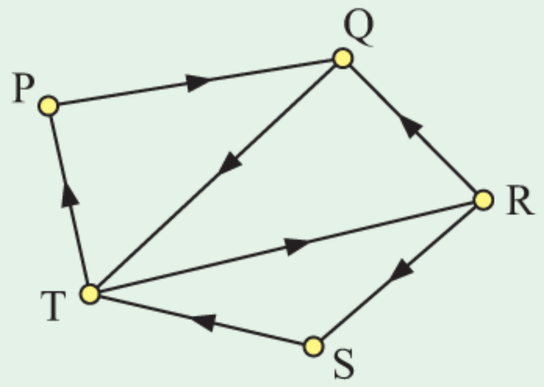
REVIEW SET 15A

1 Determine whether each directed graph is strongly connected:

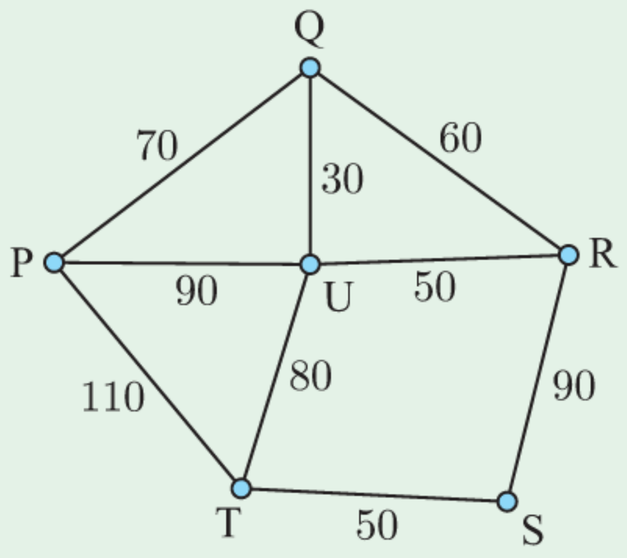
a



b



2 This graph shows the times, in minutes, to travel directly between towns.



a Is the graph:

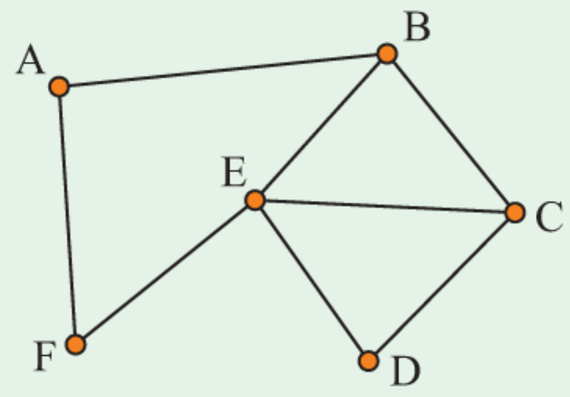
i simple **ii** connected

iii complete?

b Construct a weighted adjacency table to represent this graph.

c Find the quickest route to travel from P to S.

3 Consider the graph alongside.



a Is $B \to A \to F \to E \to B \to C$ a trail?

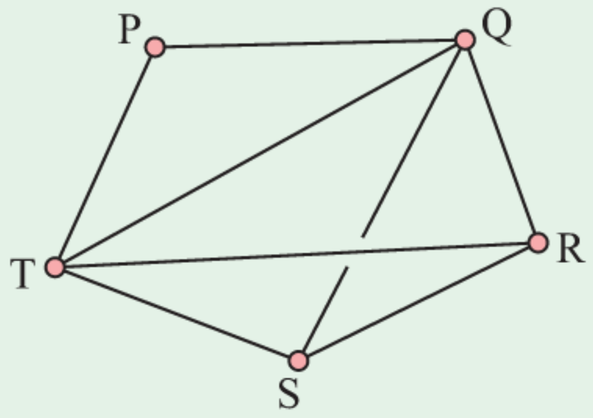
b Is $D \to E \to B \to A \to F$ a path?

c Is $F \to E \to C \to B \to E \to F$ a circuit?

d Is $B \to E \to F \to A \to B$ a cycle?

Give reasons for your answers.

4



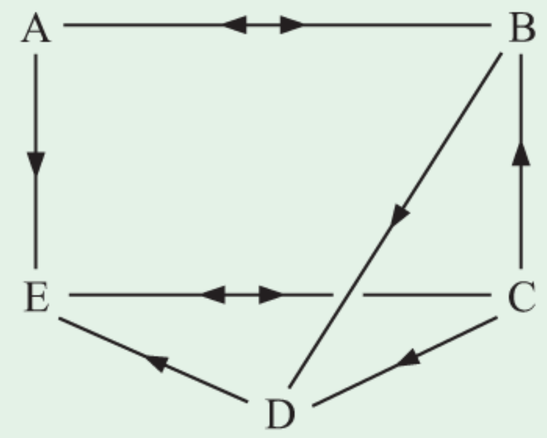
a Find the adjacency matrix $\mathbf{A}$ for this graph.

b Calculate $\mathbf{A}^2$.

c Hence determine the number of 2-step routes from T to Q.

d List the routes in **c**.

5 A new conveyer belt system is set up in a busy warehouse. Its graph is shown alongside.



a Find the adjacency matrix $\mathbf{A}$ for this graph.

b **i** Calculate $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$.

ii Is it possible to move items between any two areas of the warehouse in at most three steps? Explain your answer.

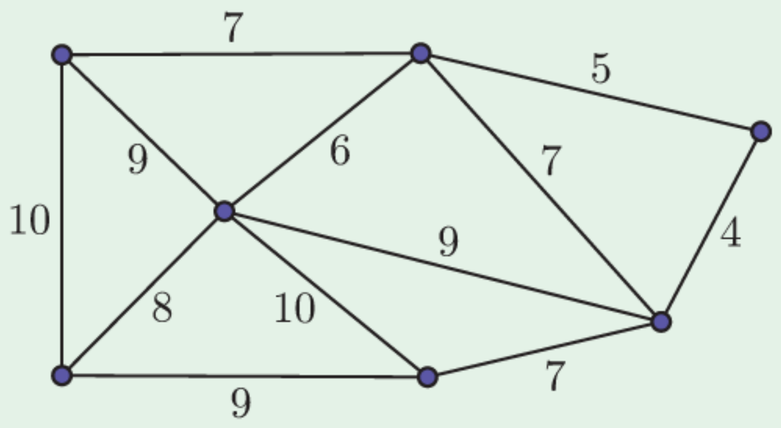
c The belt between areas C and B is shut down due to an electrical fault.

i Adjust the adjacency matrix $\mathbf{A}$, and recalculate $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$.

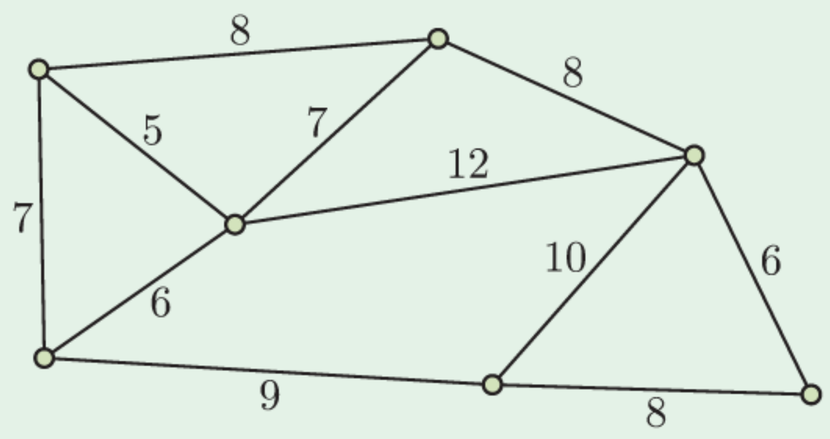
ii Comment on the effect of this breakdown.

6 Use Prim's algorithm to find the minimum spanning tree for each graph:

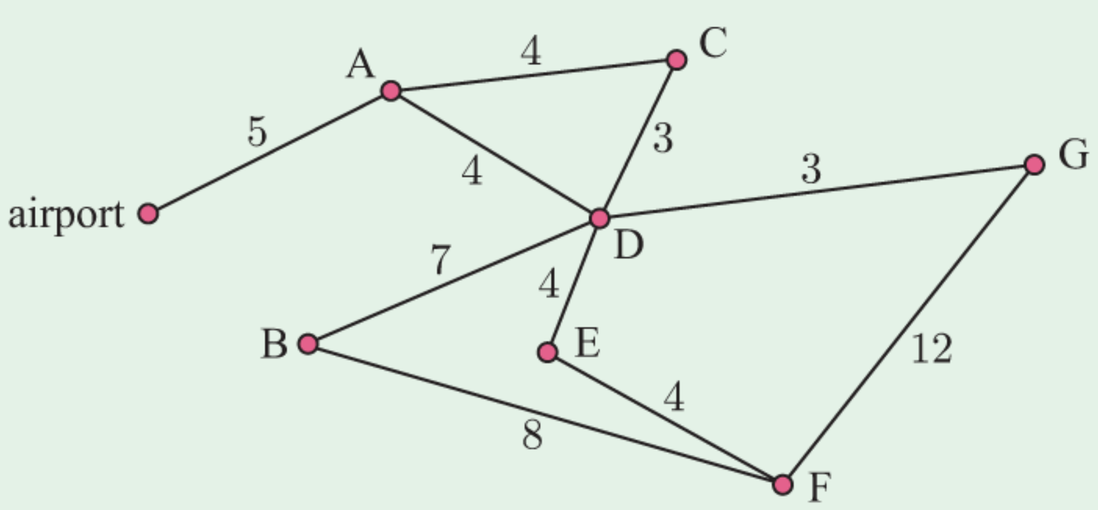
a



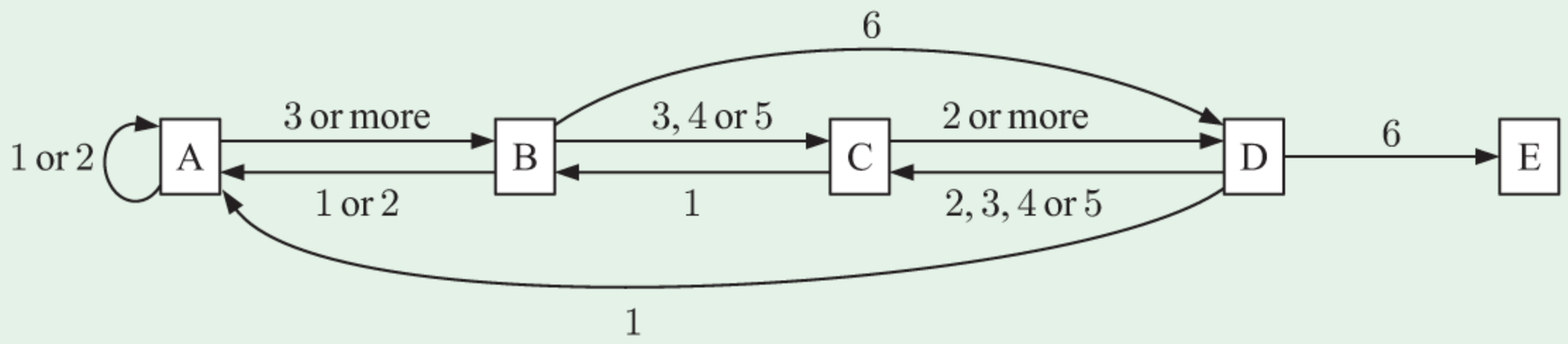
b



7 Fastrail plans to link seven nearby towns to the local airport. The numbers on the graph show distances in kilometres. Show the most efficient way to connect the towns with the airport.



8 In a children's board game, players move between squares based on the roll of a six-sided die. The players start at square A, and their aim is to reach square E.



a Construct a transition matrix for this graph.

b Find the probability that after 3 moves, a player:

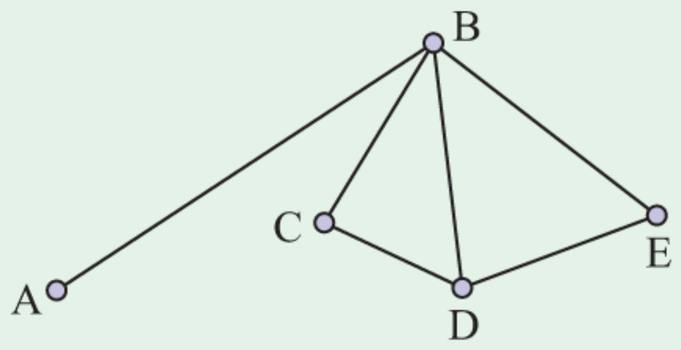
i is still at square A **ii** has reached square E.

Describe the possible routes in each case.

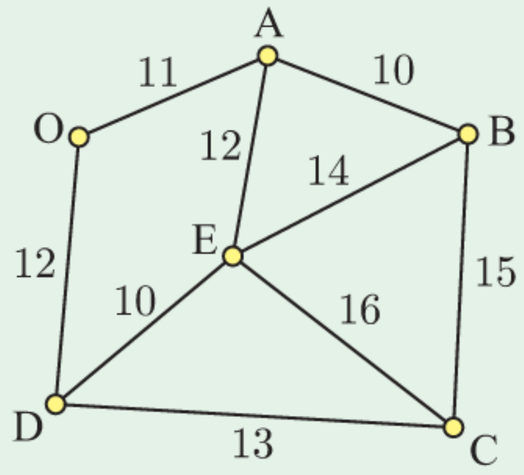
9 **a** Explain why the graph alongside is semi-Eulerian.

b Find an Eulerian trail.

c What additional edge could be added to the graph to make it Eulerian?



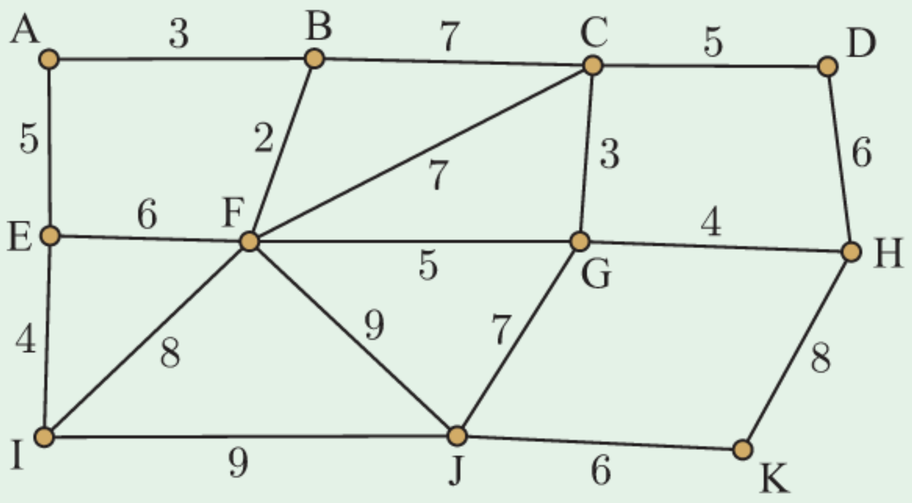
10 Solve the Chinese Postman Problem for the graph shown. Assume the postman starts and finishes at O.



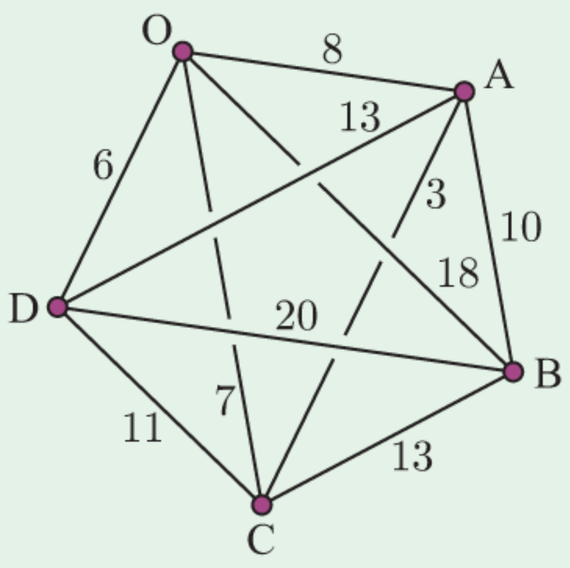
11 Jim is planning his "Trick-or-Treating" route this Halloween. This graph shows Jim's neighbourhood and the time needed to travel down each street, in minutes.

From his house at A, Jim would like to traverse each street in his neighbourhood before returning home.

Can Jim do this in less than 2 hours? If so, suggest a route he should take.



12



Consider the graph alongside.

a Use the nearest neighbour algorithm starting at O to find a Hamiltonian cycle for the graph. Hence find an upper bound for the TSP.

b Apply the deleted vertex algorithm with O deleted to find a lower bound for the TSP.

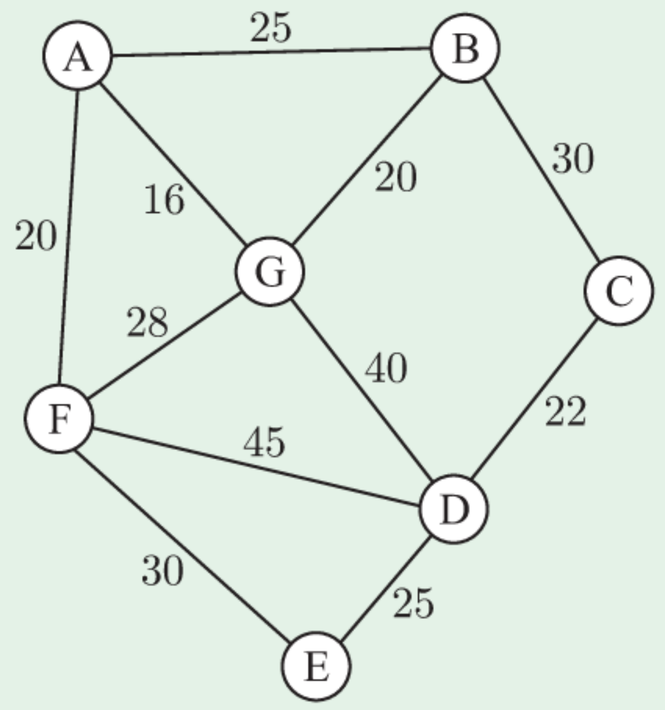
13 The ferry system operating between a group of holiday islands is shown alongside. The numbers show the travel times in minutes.

John is currently on island A. He wants to visit each island once, before returning to A.

a What term is used to describe such a route?

b Given that John first travels to island F, find the route that he can take.

c Find the total travel time for this route.



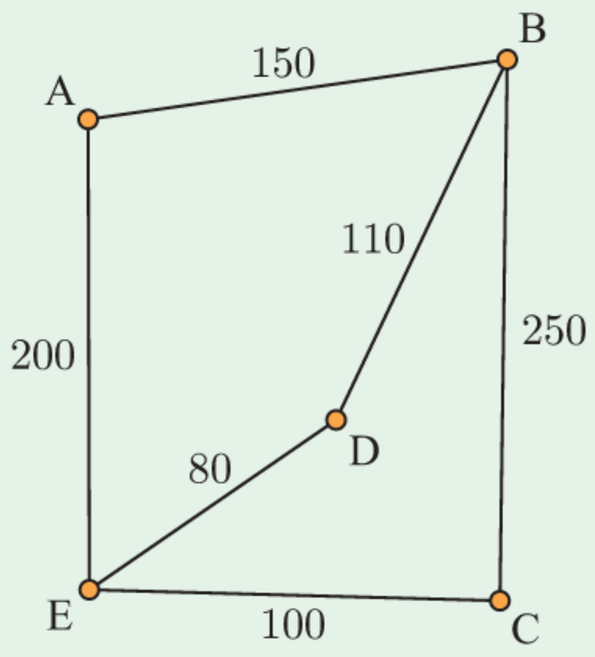
14 This graph shows the roads between towns, and the length of each road in kilometres.

A delivery driver lives in town D, and must visit every other town before returning to D.

a Complete the graph with edges showing the least distance between each pair of towns.

b Use the nearest neighbour algorithm starting at D to find an upper bound for the TSP.

c The delivery van can travel 600 km on one tank of fuel. By applying the deleted vertex algorithm with vertex A deleted, show that the driver must refuel during the trip.



REVIEW SET 15B

1 The graph alongside shows the compatible blood types for kidney transplants.

a Lily has type A blood.

i What blood types can she donate a kidney to?

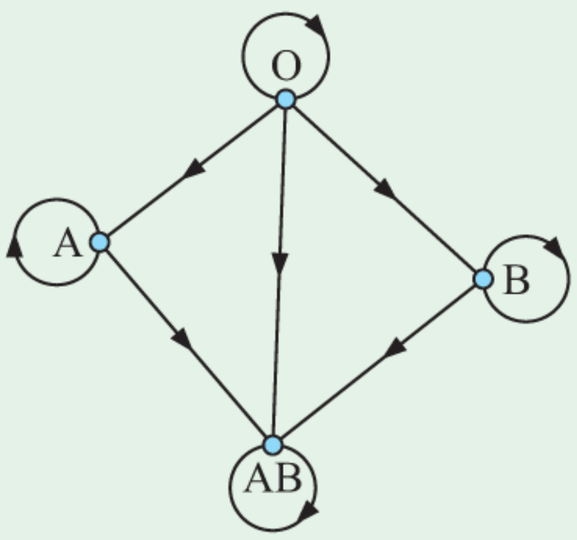
ii What blood types can she receive a kidney from?

b Find the out degree of vertex B. Interpret your answer.

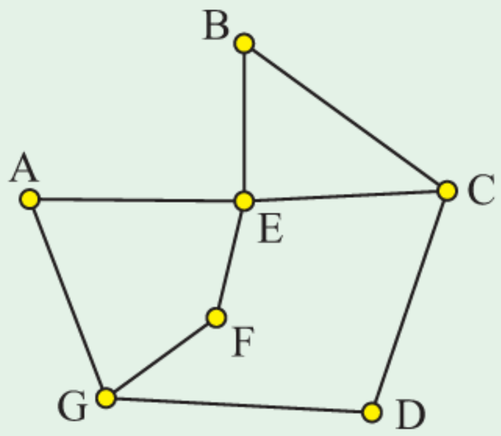
c What blood type is described as:

i the "universal donor"

ii the "universal recipient"?



2



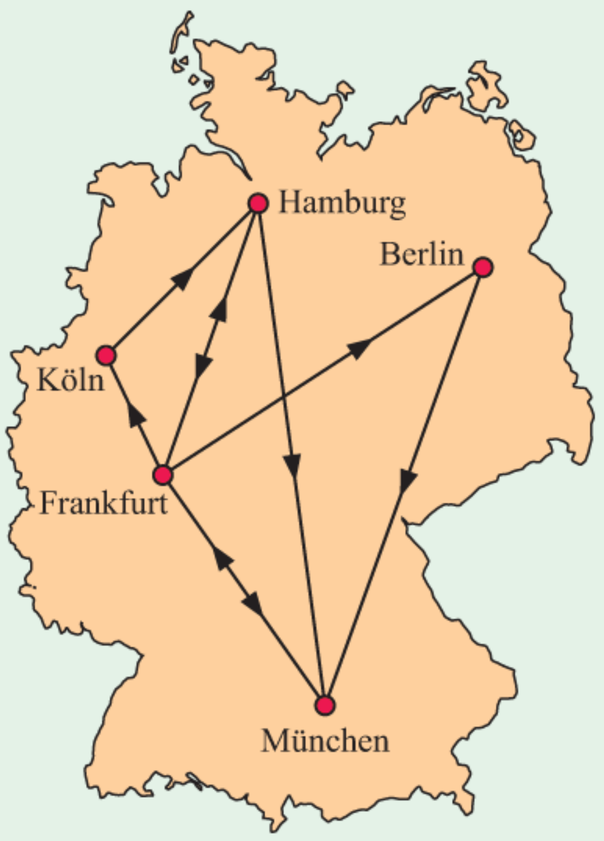
For the given graph, find:

a a trail from E to G that is not a path

b a cycle of length 6.

3 Find the fewest number of vertices required to construct a simple connected graph with at least 500 edges.

4 A German bus company provides the routes shown on the map.



a How many different routes are offered?

b Find the adjacency matrix **A**.

c Use matrix methods to determine:

i how many journeys between two cities cannot be made using at most one stopover

ii the maximum number of trips needed to travel between any two cities.

d The bus company is considering a service upgrade. They can either make the route between Frankfurt and Köln "two-way", or create a Köln to Berlin link. Investigate the effects that these proposed changes would have on your answers to **c**, and advise which would be the better decision.

5 A, B, C, and D are towns connected by canals. Some canals are only wide enough for boats to travel in one direction, while other canals are wide enough for boats to travel in both directions. The following matrix shows the number of ways of travelling from one town to another by canal.



$$\mathbf{A} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{pmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

a Is it possible to travel from town C to town D without going through another town? Explain your answer.

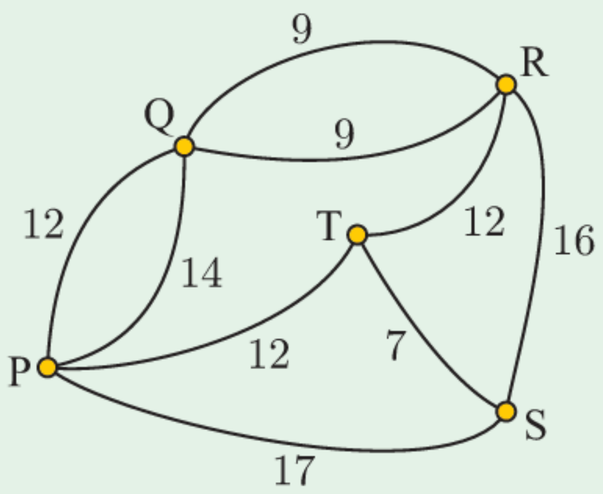
b Calculate $\mathbf{A} + \mathbf{A}^2$.

c Comment on inefficiencies in the canal network.

d **i** In how many ways is it possible to travel from town B and return to town B in exactly 4 trips?

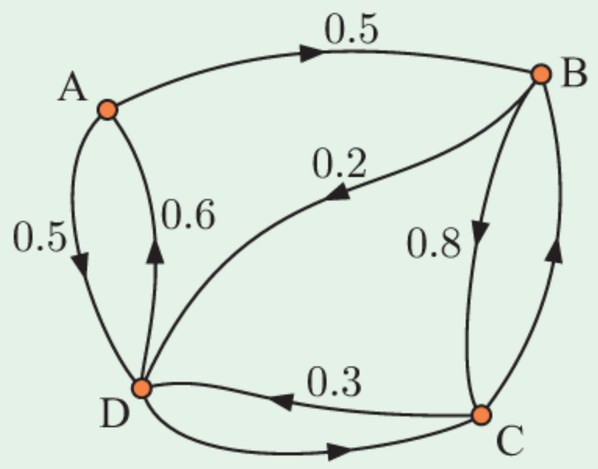
ii Is it possible to do this and visit the other 3 towns exactly once?

6



Find a minimum spanning tree for the graph shown using Kruskal's algorithm.

7 This graph shows the probabilities with which a museum security guard moves between exhibits A, B, C, and D.



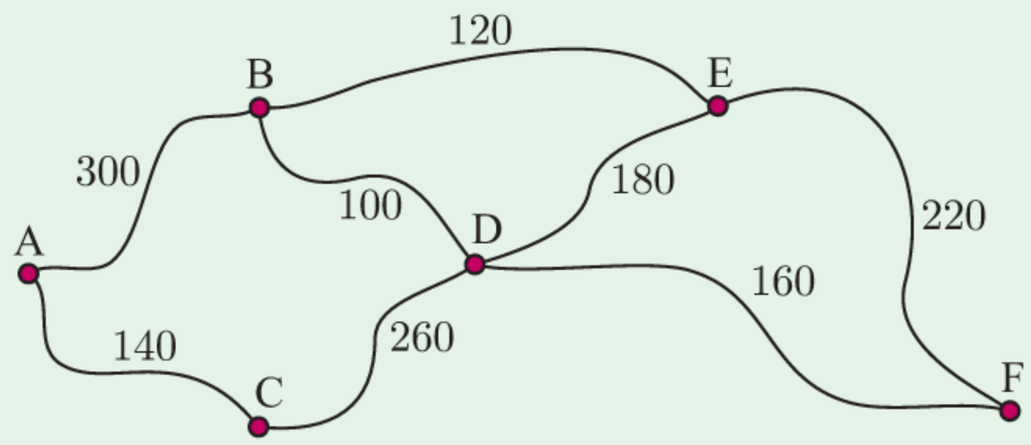
a Is the graph strongly connected?

b Copy and complete the graph.

c Construct a transition matrix for the graph.

d Given that the security guard is currently at B, find the probability that she will return to B after visiting exactly two other exhibits.

8 A telecommunications company wishes to lay cables to connect the six towns shown. The numbers indicate costs for each possible connection, in thousands of dollars.



Suggest how the towns should be connected, to minimise the total cost.

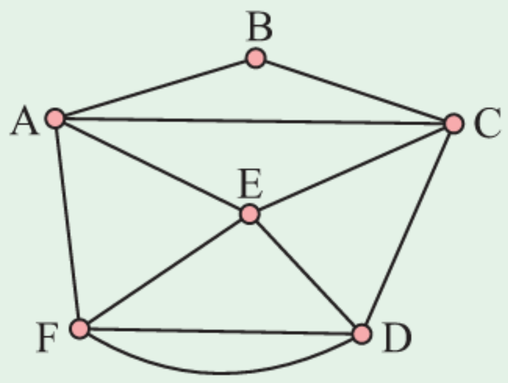
9 An airline is setting up a network of flights between towns A, B, C, D, and E. The numbers shown in the table are flight distances in kilometres.

	A	B	C	D	E
A	–	640	450	320	350
B	640	–	230	410	360
C	450	230	–	220	300
D	320	410	220	–	420
E	350	360	300	420	–

a Apply Prim's algorithm to the table to find the edges in the minimum spanning tree for this flight network. Find the total length of the minimum spanning tree.

b Discuss whether a minimum spanning tree is the most appropriate flight network. What other factors should be considered?

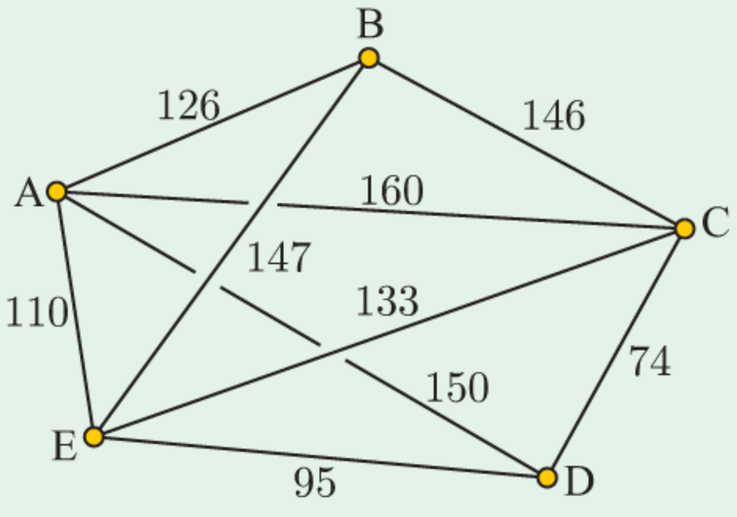
10 Consider the graph alongside.



a Explain why the graph is Eulerian.

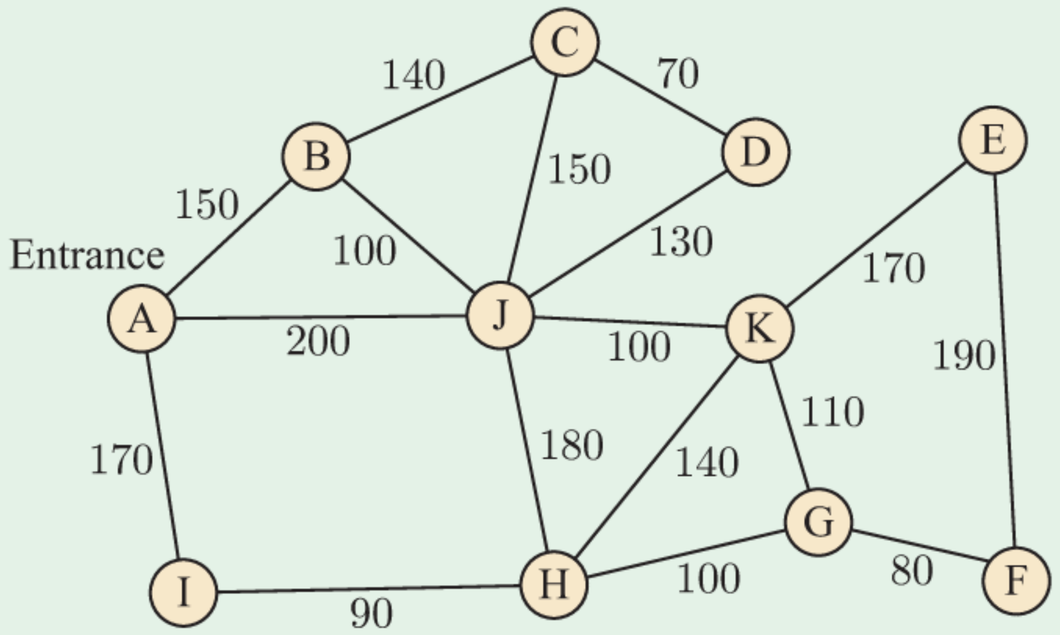
b Find an Eulerian circuit which starts and ends at B.

11 Solve the Chinese Postman Problem for this weighted graph, starting and finishing at A.



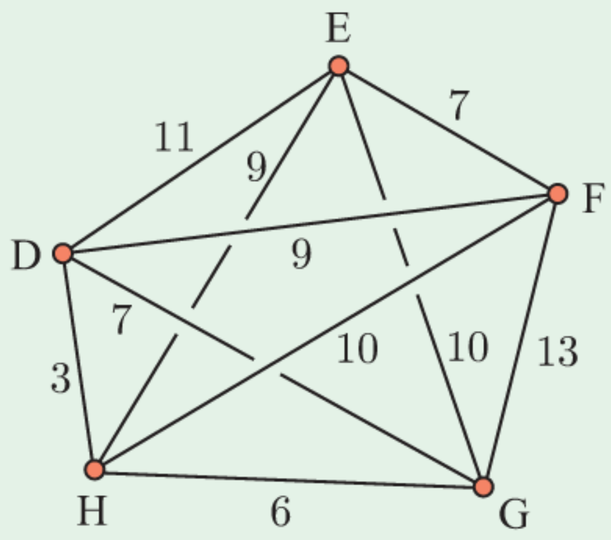
12 The landmarks at a theme park are shown alongside. The numbers on the edges indicate distances in metres.

Yuto is currently at the entrance at A. He wants to walk along each path, and return to A. Find the shortest route Yuto can take, and state its length.



13 Answer the **Opening Problem** on page **376**.

14 From the depot D, a postal van is to collect mail from mailboxes E, F, G, and H, then return to the depot. This graph shows the time in minutes between mailboxes.



a Use the nearest neighbour algorithm to find a Hamiltonian cycle starting at:

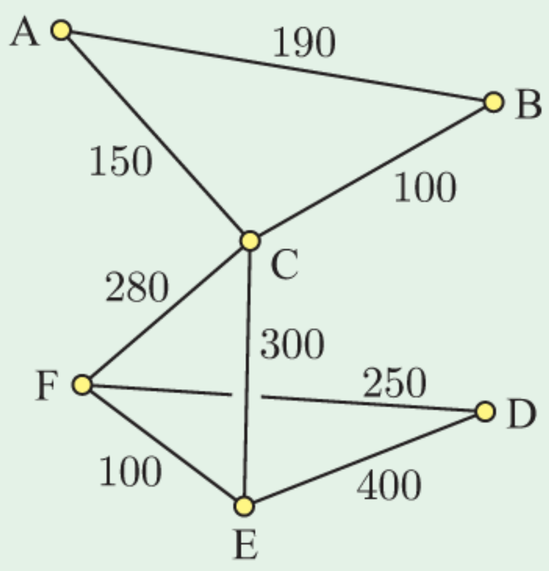
i H **ii** D.

Hence find the best upper bound for the TSP.

b Find a lower bound for the TSP by applying the deleted vertex algorithm with vertex H deleted.

c Suggest a route for the postal van, and determine how long the route will take.

15 The graph below shows the direct flights between cities, and the cost in dollars of each flight. Next to the graph is a table of lowest costs between the cities.



	A	B	C	D	E	F
A	0	190	150			430
B	190	0	100			
C	150	100	0	530	300	280
D			530	0		250
E			300		0	100
F	430		280	250	100	0

a Copy and complete the table.

b Benita works for a bank, and lives in D. She must visit branches in every other city, then return to D.

i By inspecting the graph, which city *must* she visit twice?

ii Use the nearest neighbour algorithm starting at D to find an upper bound for the TSP.

iii Use the deleted vertex algorithm with A deleted to find a lower bound for the TSP.

iv By inspection, find the cheapest route Benita can take, and state its cost. Which cities does Benita visit twice?

v A new flight is introduced between C and D, which costs $400. By adjusting the table of lowest costs and applying the nearest neighbour algorithm starting at D, show that there is now a cheaper route for Benita.

Chapter 16

Voronoi diagrams

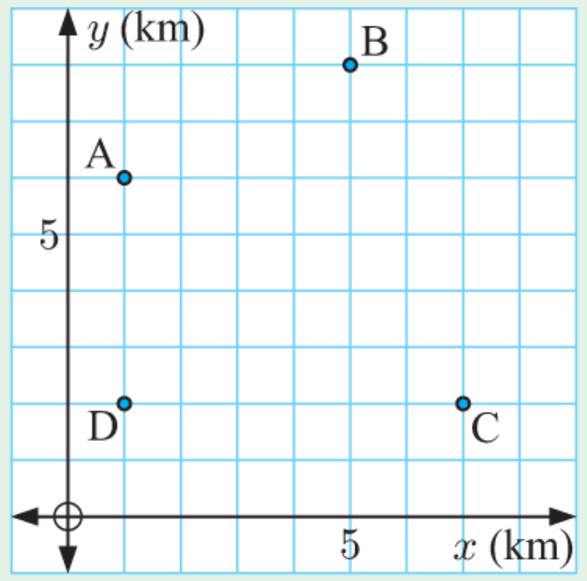
Contents:

A Voronoi diagrams
B Constructing Voronoi diagrams
C Adding a site to a Voronoi diagram
D Nearest neighbour interpolation
E The Largest Empty Circle problem



OPENING PROBLEM

This map shows the locations of the four fire stations in a city.



When a fire starts in the city, it is important that the nearest fire station is alerted.

Things to think about:

a Which fire station is closest to:
i $(6, 3)$ **ii** $(4, 6)$?

b The Station Master at fire station D wants to know the region of the city that is closest to his station. What does this region look like?

c How could we improve the map to make it easier to determine the closest fire station to any given location?

d A new fire station is to be built within the city grid. To maximise the efficiency of the fire stations, the new station should be built as far as possible from existing stations. Where should the new station be built?

To solve problems like the **Opening Problem** we can draw a **Voronoi diagram**. Voronoi diagrams have wide-ranging applications in science, engineering, city planning, health, and meteorology.

HISTORICAL NOTE

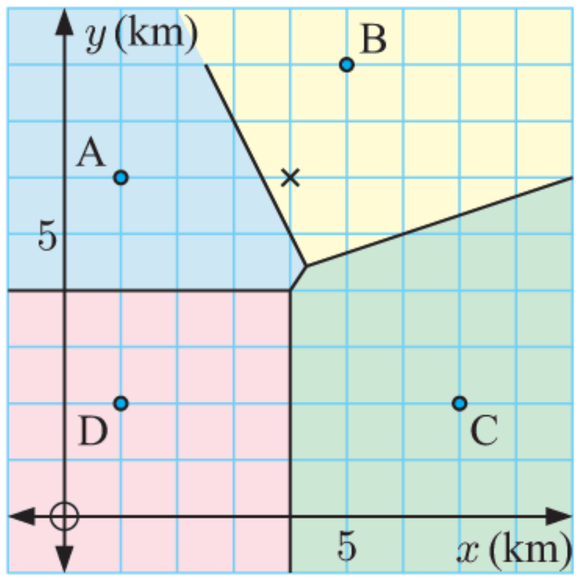
Georgy Feodosevich Voronoy (1868 - 1908) was a mathematician from Pyriatyn (formerly the Russian Empire, now Ukraine). He studied at Saint Petersburg University, and is credited with inventing Voronoi diagrams.



Georgy Voronoy

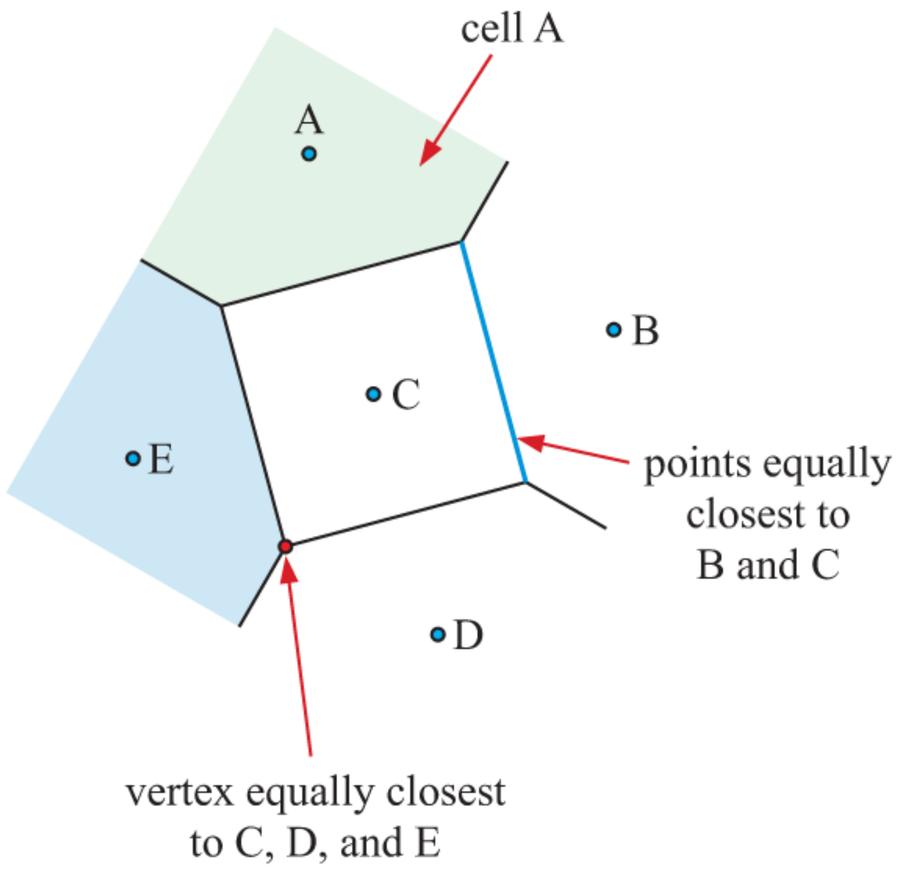
A VORONOI DIAGRAMS

Consider the map of fire stations in the **Opening Problem**. For each station, there is a region which contains all the points that are closer to that station than to any other station. The **Voronoi diagram** alongside illustrates these regions. Notice that each region contains one fire station. This is the closest fire station to every point in the region. For example, $(4, 6)$ lies in the region containing station B, so station B is the closest station to $(4, 6)$.



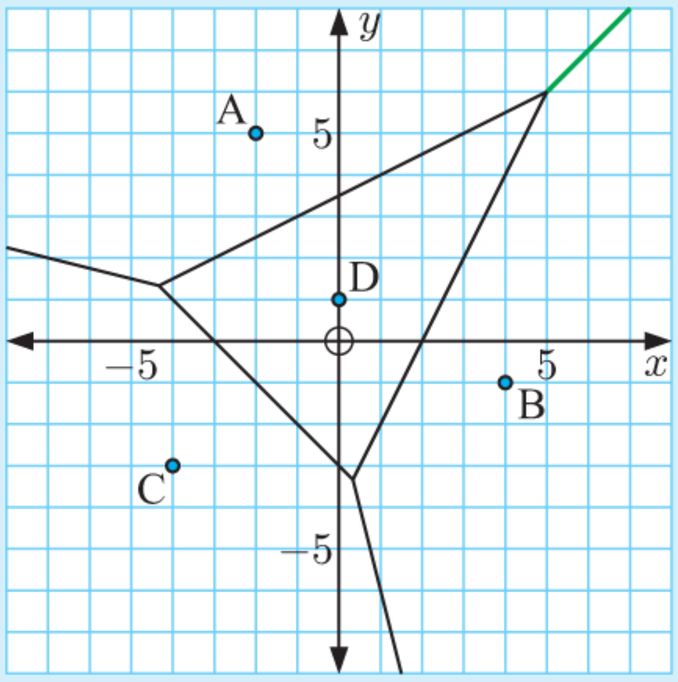
In a Voronoi diagram:

- Important locations are called **sites**. These are the fire stations in the example on the previous page.
- Each site is surrounded by a region or **cell** which contains the points which are closer to that site than to any other site. The cells are labelled according to the site which they contain. For example, the cell which contains site A is called cell A.
- The lines which separate the cells are called **edges**. Each point on an edge is equally closest to the two sites whose cells are adjacent to that edge. For example, any point which lies on the blue edge is equally closest to sites B and C.
- The points at which the edges meet are called **vertices**. Each vertex is equally closest to the sites whose cells meet at that vertex. For example, the red vertex is equally closest to sites C, D, and E.



Example 1 **Self Tutor**

Consider this Voronoi diagram for the sites A, B, C, and D.



a How many cells does the diagram contain?

b How many vertices does the diagram contain?

c Identify the site(s) closest to:

i $(2, -3)$ ii $(1, 5)$ iii $(-1, -2)$

d What can we say about points which lie on the green edge?

a The diagram contains 4 cells, one for each site.

b The diagram contains 3 vertices.

c i $(2, -3)$ lies in cell B, so it is closest to site B.

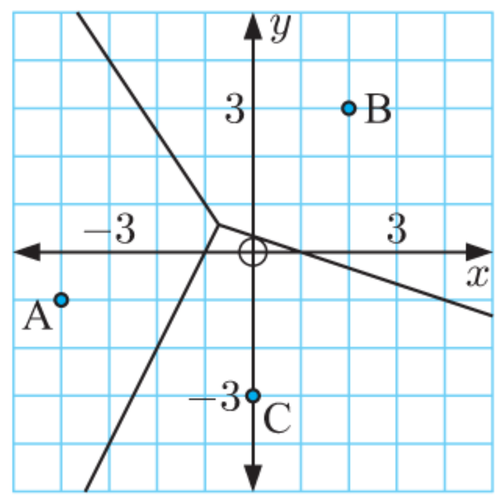
ii $(1, 5)$ lies in cell A, so it is closest to site A.

iii $(-1, -2)$ lies on the edge adjacent to cells C and D, so it is equally closest to sites C and D.

d The green edge is adjacent to cells A and B, so points which lie on this edge are equally closest to sites A and B.

EXERCISE 16A

1 Consider this Voronoi diagram for the sites A, B, and C.



a State the number of:

i cells **ii** edges **iii** vertices.

b The point P has coordinates $(-1, 2)$.

i Which site is closest to P? Explain your answer.

ii Verify your answer by calculating the distance between P and each site.

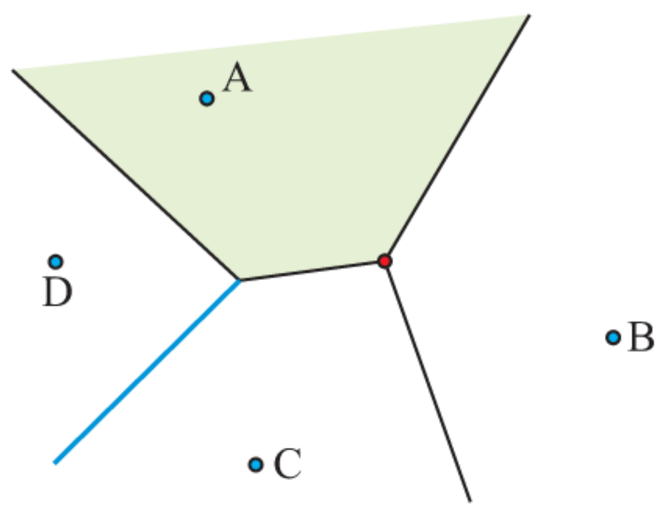
c The point Q has coordinates $(-1, 0)$ and lies on an edge of the Voronoi diagram.

i Identify the site(s) closest to Q. Explain your answer.

ii Verify your answer by calculating the distance between Q and each site.

2 This Voronoi diagram has been constructed for the sites A, B, C, and D.

What can you say about:

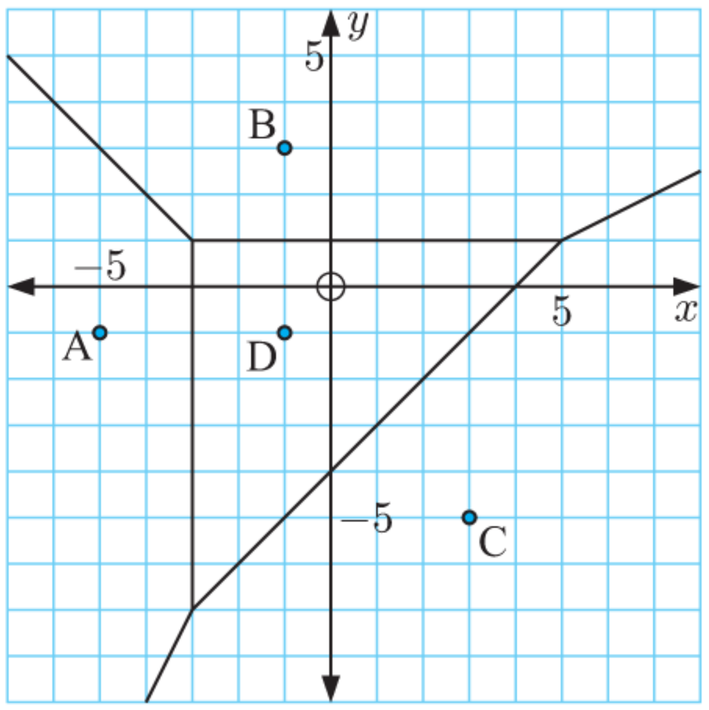


a the points which lie in the green cell

b the points which lie on the blue edge

c the red vertex?

3 Consider this Voronoi diagram for the sites A, B, C, and D.



a Identify the site which is closest to:

i $(2, 3)$ **ii** $(-1, -4)$

iii $(6, 0)$ **iv** $(-4, -3)$

b Verify that:

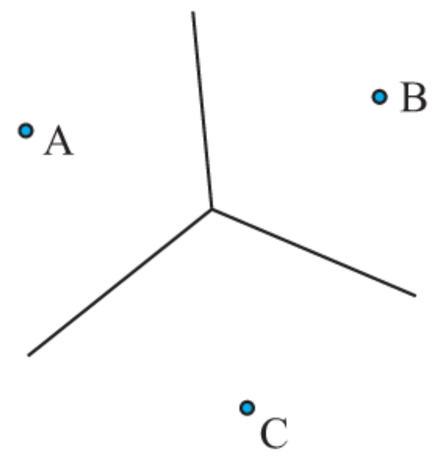
i $(-3, 0)$ is equidistant from A and D

ii $(-3, 2)$ is also equidistant from A and D.

c Explain why $(-3, 0)$ lies on an edge of the Voronoi diagram, but $(-3, 2)$ does not.

d Find the area of cell D.

4 In the Voronoi diagram alongside, suppose point P_A lies in cell A, and P_B lies in cell B.



Discuss whether the following statements are necessarily true:

a P_A is closer to A than to any other site.

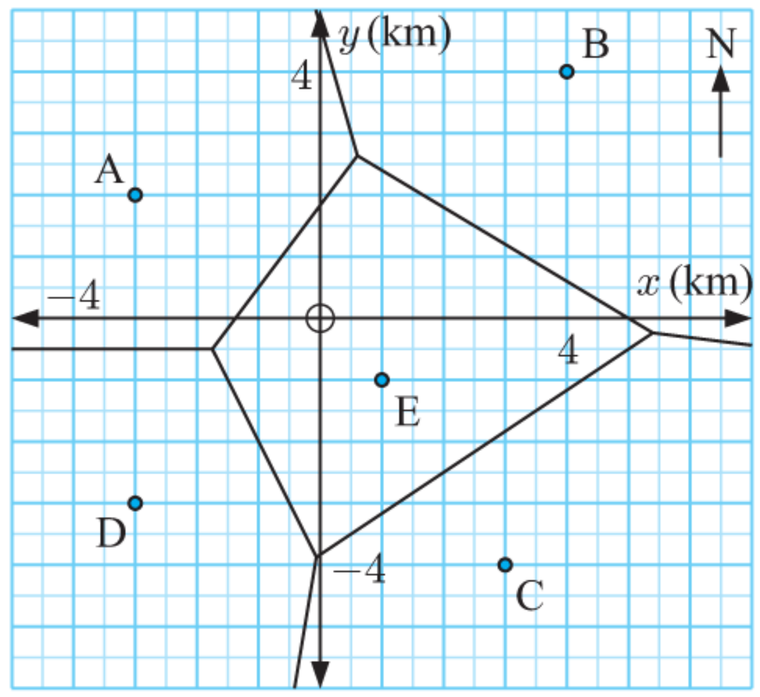
b P_B is closer to B than to C.

c A is closer to P_A than to P_B.

d B is closer to P_B than to C.

5 Let X be a site on a Voronoi diagram, and P be a point within the interior of cell X. Suppose a circle is drawn with centre P, passing through X. Explain why this circle cannot contain any other sites.

6 This Voronoi diagram shows the post offices in a small city. Parcels which cannot be delivered to a location are held at the nearest post office for collection.



a Determine the holding post office for a parcel sent to:

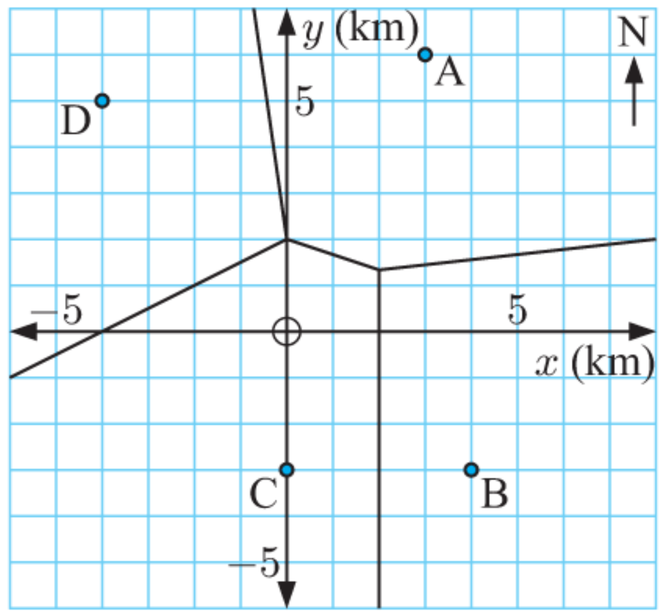
i $(0, 0)$ **ii** $(4, 1)$

iii $(-2, 0)$ **iv** $(3, -2)$

b A company leases two buildings in the city. One building is 2.5 km north of the other. One building is equally closest to post offices D and E, and the other is equally closest to A and E.

Determine the locations of the buildings.

7 This Voronoi diagram shows the public schools in a city. The city is divided into catchment zones, so that students are sent to their nearest public school.



a Identify the nearest public school for a student who lives at:

i $(-3, -2)$ **ii** $(6, 2)$

iii $(-2, 4)$ **iv** $(5, 1)$

b Bailey's family lives at $(-1, 4)$.

i Which school's catchment zone does Bailey live in?

ii Suppose Bailey's family moves to a new home 1 km east of their old home. Show that this will move Bailey into a new catchment zone.

iii Verify by direct calculation that Bailey's new school is now closer to Bailey than his old school.

c Lizzy's house is equally closest to schools A, C, and D.

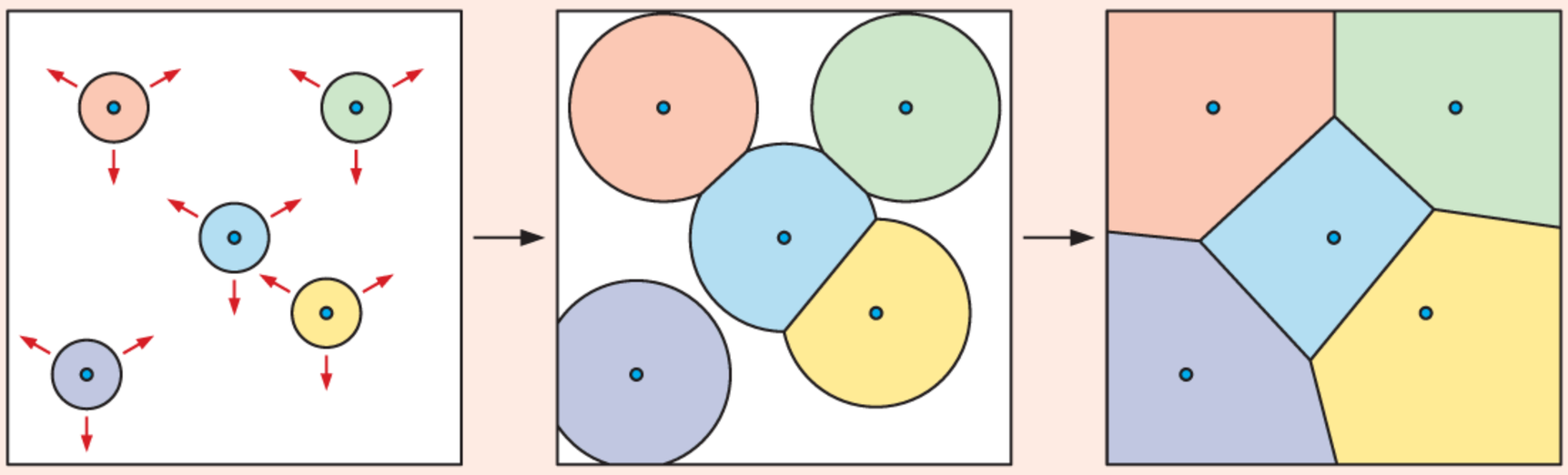
i Where does Lizzy live?

ii How far does Lizzy live from each of these schools?

8 Let V be a vertex of a Voronoi diagram, and X be the site in a cell adjacent to that vertex. Suppose a circle is drawn with centre V, passing through X. Explain why this circle must pass through at least two other sites.

RESEARCH VORONOI DIAGRAMS IN NATURE

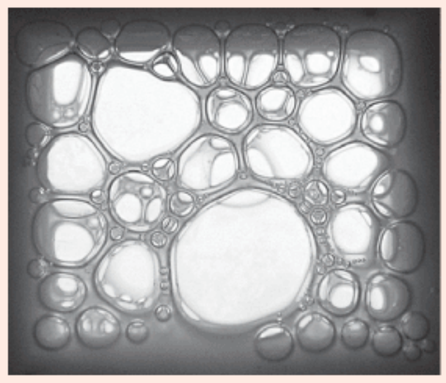
Consider a set of sites. Suppose we create circles centred at each site, which expand until they meet another circle. The regions formed by this process are the cells of the Voronoi diagram for the sites.



ANIMATION

For this reason, we often observe Voronoi diagrams in the world around us. For example, when soap bubbles form, each cell expands until it meets another membrane.

Voronoi diagrams can also be found in the veins of leaves, and the markings of giraffes.





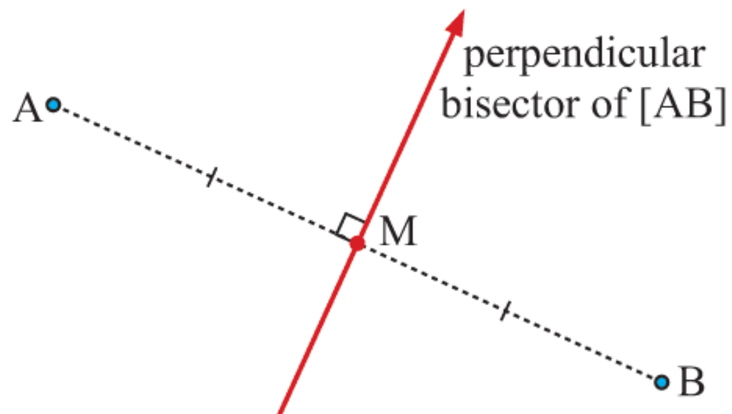


Research other occurrences of Voronoi diagrams in nature. For each occurrence, try to explain why this pattern may have arisen.

B CONSTRUCTING VORONOI DIAGRAMS

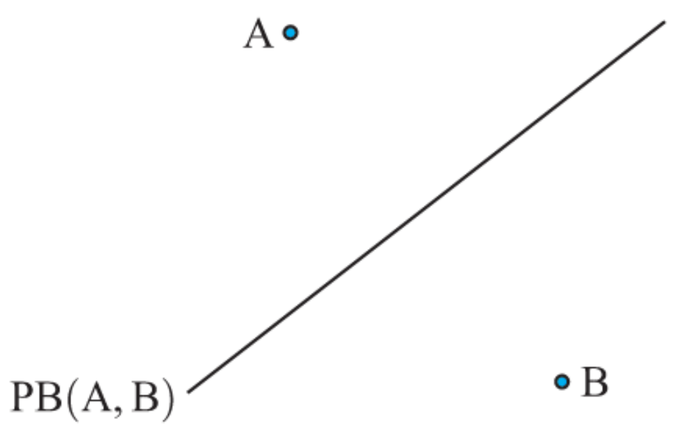
In **Chapter 1** of the **Core Topics HL** book, we saw that:

- The **perpendicular bisector** of a line segment [AB] is the line perpendicular to [AB] which passes through its midpoint.
- All points on the perpendicular bisector are equidistant from A and B.
- The perpendicular bisector divides the plane into two regions. On one side of the line are points that are closer to A than to B, and on the other side are points that are closer to B than to A.



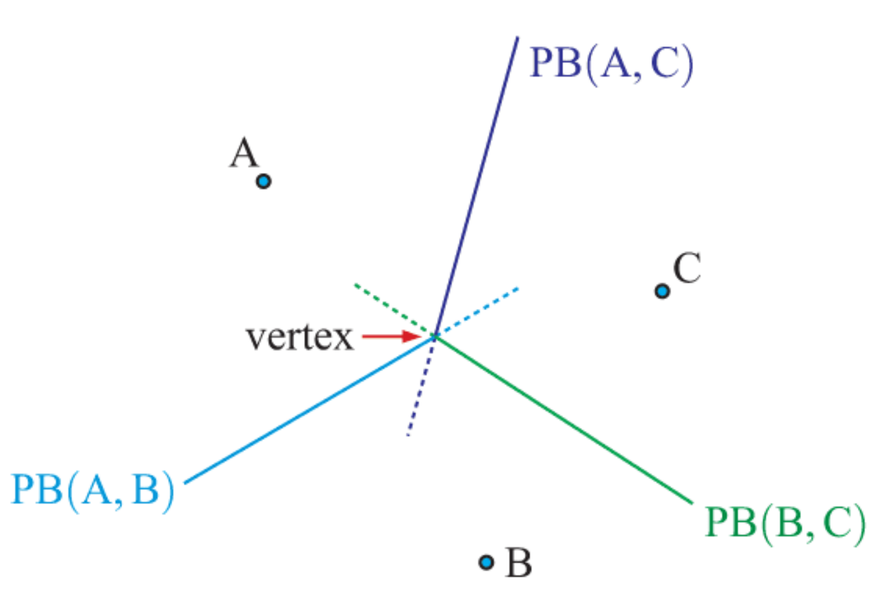
This means that perpendicular bisectors are extremely useful for constructing Voronoi diagrams.

For a plane with two sites A and B, the Voronoi diagram simply consists of the perpendicular bisector of [AB], which we will call PB(A, B).



For a plane with three sites A, B, and C, we draw the perpendicular bisectors of [AB], [AC], and [BC]. Provided A, B, and C are not collinear, these lines will meet at a single point equidistant from A, B, and C. This point is a vertex of the Voronoi diagram.

However, notice that only a section of each perpendicular bisector is included as a Voronoi edge. For example, the points on the dotted blue section of PB(A, B) are *not* part of the Voronoi edge. This is because, although they are equidistant from sites A and B, their *closest* site is site C.



Example 2 **Self Tutor**

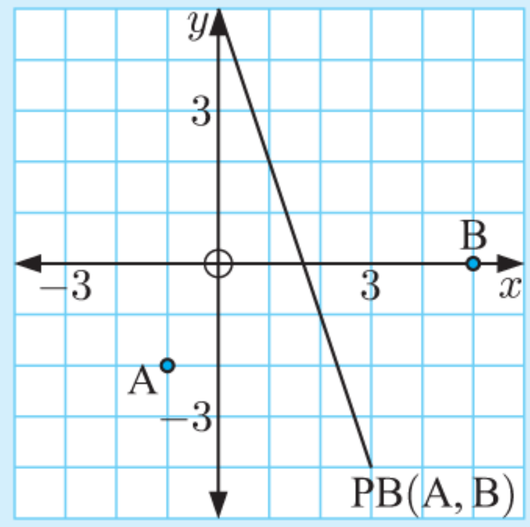
Construct a Voronoi diagram for the sites:

a A$(-1, -2)$ and B$(5, 0)$ **b** A$(-5, 2)$, B$(3, -4)$, and C$(-3, -2)$

a The midpoint of [AB] is $\left(\frac{-1+5}{2}, \frac{-2+0}{2}\right)$ or $(2, -1)$.

The gradient of [AB] is $\frac{0--2}{5--1} = \frac{2}{6} = \frac{1}{3}$.

So, PB(A, B) has gradient -3 and passes through $(2, -1)$.



b The midpoint of [AB] is $\left(\frac{-5+3}{2}, \frac{2+-4}{2}\right)$ or $(-1, -1)$.

The gradient of [AB] is $\frac{-4-2}{3--5} = \frac{-6}{8} = -\frac{3}{4}$.

So, PB(A, B) has gradient $\frac{4}{3}$ and passes through $(-1, -1)$.

The midpoint of [AC] is $\left(\frac{-5+-3}{2}, \frac{2+-2}{2}\right)$ or $(-4, 0)$.

The gradient of [AC] is $\frac{-2-2}{-3--5} = \frac{-4}{2} = -2$.

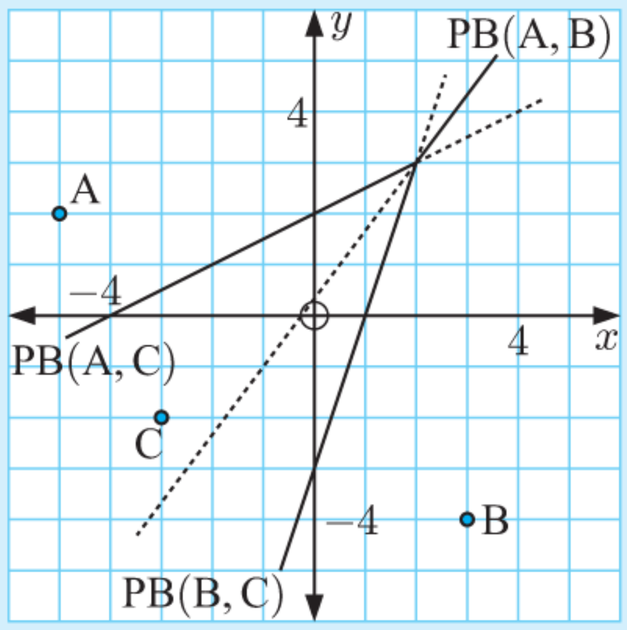
So, PB(A, C) has gradient $\frac{1}{2}$ and passes through $(-4, 0)$.

The midpoint of [BC] is $\left(\frac{3+-3}{2}, \frac{-4+-2}{2}\right)$ or $(0, -3)$.

The gradient of [BC] is $\frac{-2--4}{-3-3} = \frac{2}{-6} = -\frac{1}{3}$.

So, PB(B, C) has gradient 3 and passes through $(0, -3)$.

To draw the Voronoi diagram we plot the three sites on a set of axes. We can draw the perpendicular bisectors as dashed lines, then make solid only the parts which form the Voronoi edges.



If you have trouble determining which part of the perpendicular bisector to include as an edge, remember that all the edges surrounding a cell are perpendicular bisectors involving the corresponding site. For example, in part **b** of the Example above, cell C is formed by PB(A, C) and PB(B, C).

EXERCISE 16B

1 Construct a Voronoi diagram for the sites:

a A$(2, 6)$ and B$(6, 4)$ **b** A$(-3, -4)$ and B$(1, 6)$.

2 **a** Construct a Voronoi diagram for the sites A$(-2, 5)$ and B$(4, 3)$.

b Find the equation of the Voronoi edge.

c Verify that the point $(-2, -5)$:

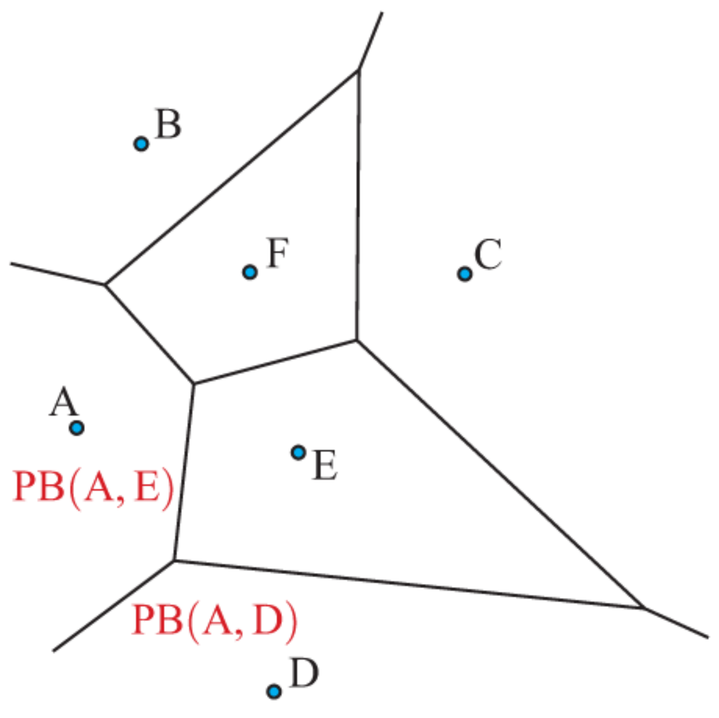
i lies on the Voronoi edge **ii** is equidistant from A and B.

d Use your Voronoi diagram to identify the site closest to:

i $(0, 0)$ **ii** $(3, 6)$ **iii** $(-4, -5)$

3 Copy this Voronoi diagram, and label each of the remaining edges.

PRINTABLE DIAGRAMS



4 Construct a Voronoi diagram for the sites:

a A$(4, 7)$, B$(8, 3)$, and C$(0, -5)$ **b** A$(-1, 4)$, B$(5, 2)$, and C$(-5, -4)$.

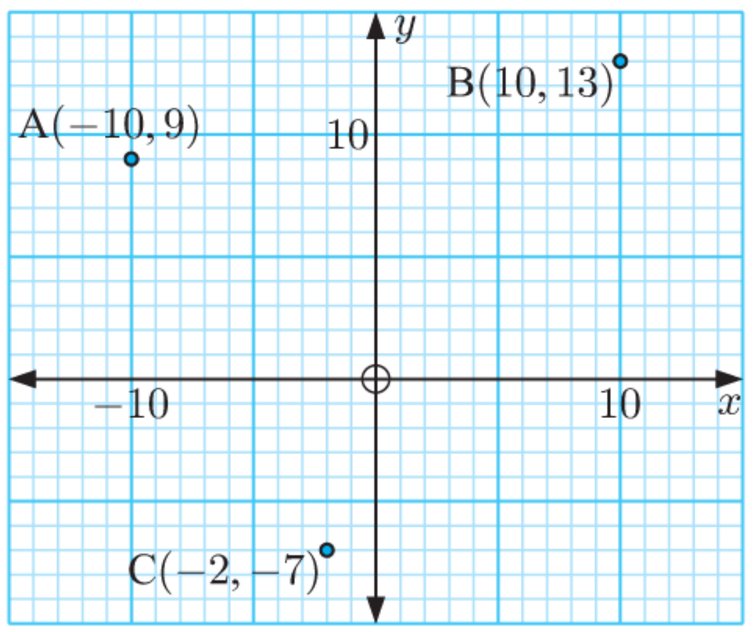
5 **a** Construct a Voronoi diagram for the sites shown.

b Find the equation of each edge of your Voronoi diagram.

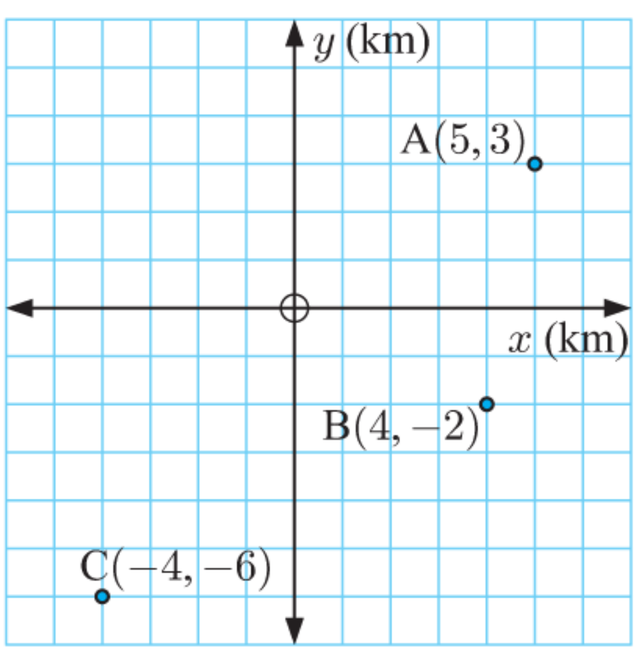
c Find the coordinates of the vertex of your Voronoi diagram. Verify that this point is equidistant from A, B, and C.

d Use your Voronoi diagram to identify the site closest to:

i $(-2, 8)$ **ii** $(5, 5)$ **iii** $(2, -3)$



6



This map shows the locations of the Pablo's Pizza stores in a city. When customers ring Pablo's Pizza, they are automatically transferred to their nearest store.

a Construct a Voronoi diagram for these sites.

b Which store is closest to a customer who rings from:

i $(3, 1)$ **ii** $(-2, 2)$ **iii** $(-5, -1)$?

c Amanda is equidistant from all three stores.

i Where does Amanda live?

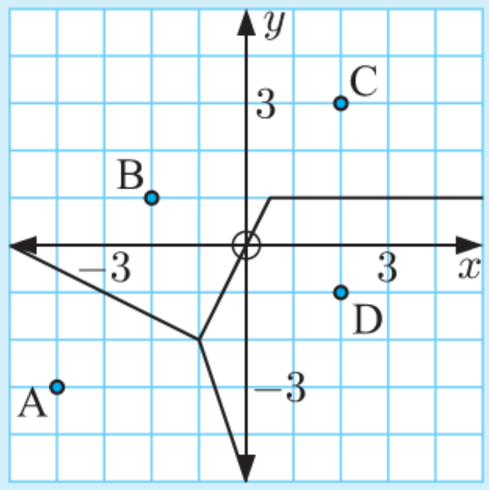
ii How far is Amanda from each store?

Example 3

Self Tutor

This Voronoi diagram is missing an edge.

Complete the diagram, and find the equation of the missing edge.



Sites B$(-2, 1)$ and C$(2, 3)$ are currently in the same cell, so the missing edge must be the perpendicular bisector of [BC].

The midpoint of [BC] is $\left(\frac{-2+2}{2}, \frac{1+3}{2}\right)$

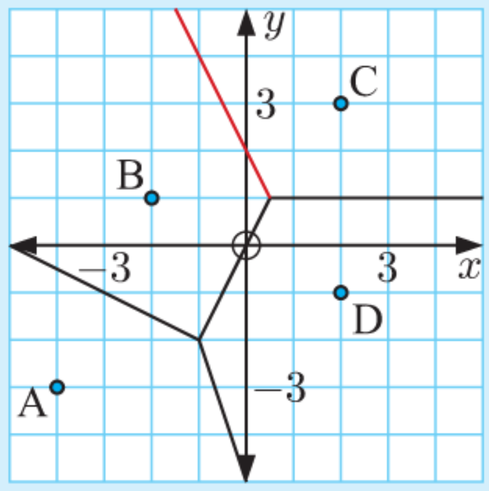
or $(0, 2)$.

The gradient of [BC] is $\frac{3-1}{2--2} = \frac{2}{4} = \frac{1}{2}$.

So, PB(B, C) has gradient -2.

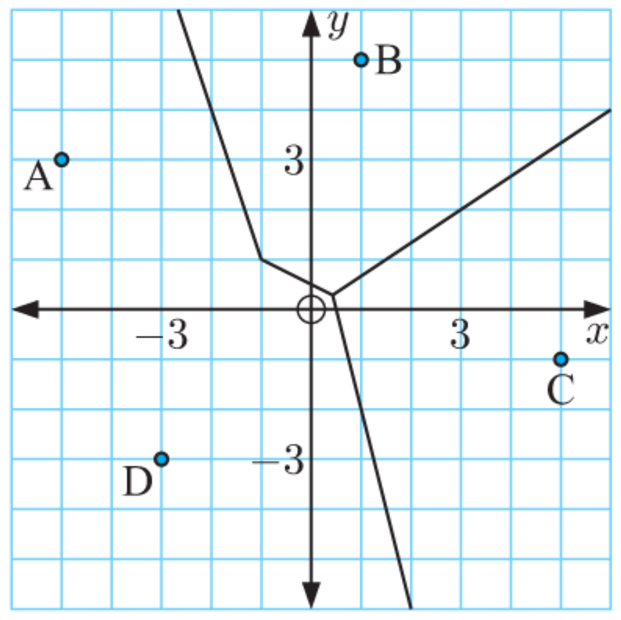
$\therefore$ its equation is $2x + y = 2(0) + 2$

or $2x + y = 2$



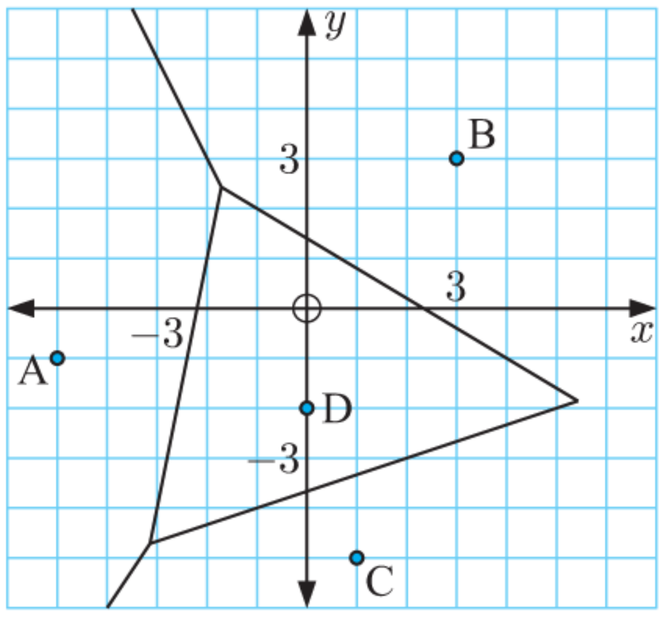
7 **a** Explain why this Voronoi diagram must have an edge missing.

b Copy and complete the diagram, and find the equation of the missing edge.

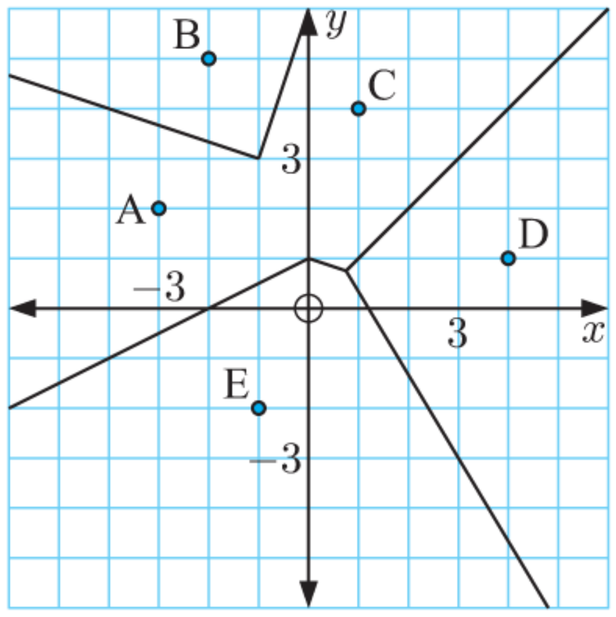


8 These Voronoi diagrams are missing an edge. Complete each diagram, and find the equation of each missing edge. Give your equations in the form $ax + by + d = 0$.

a



b



9 **a** How many edges are missing from this Voronoi diagram?

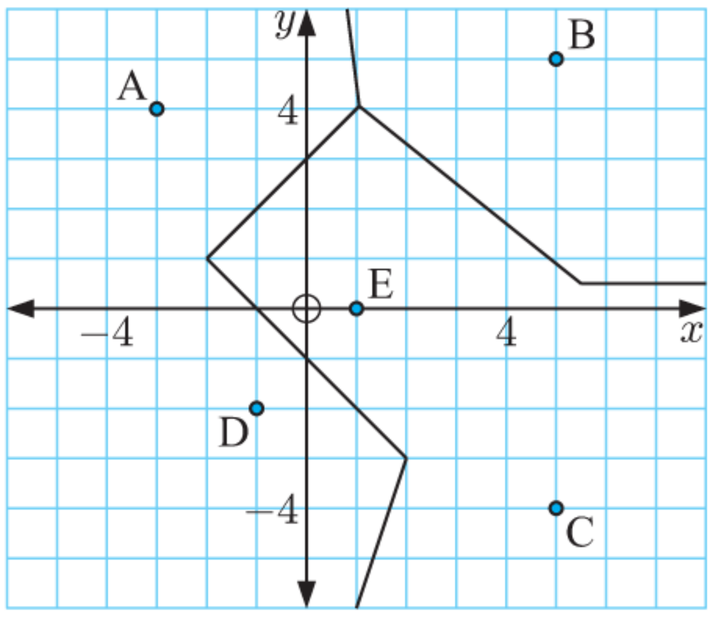
b Complete the diagram, and find the equation of each missing edge.

c Identify the site which is closest to:

i $(-4, 1)$ **ii** $(2, 3)$

d Identify the closest sites to:

i $(4, -1)$ **ii** $(-2, 1)$



10 This map shows the camping sites in a national park.

a Find the equation of the blue edge.

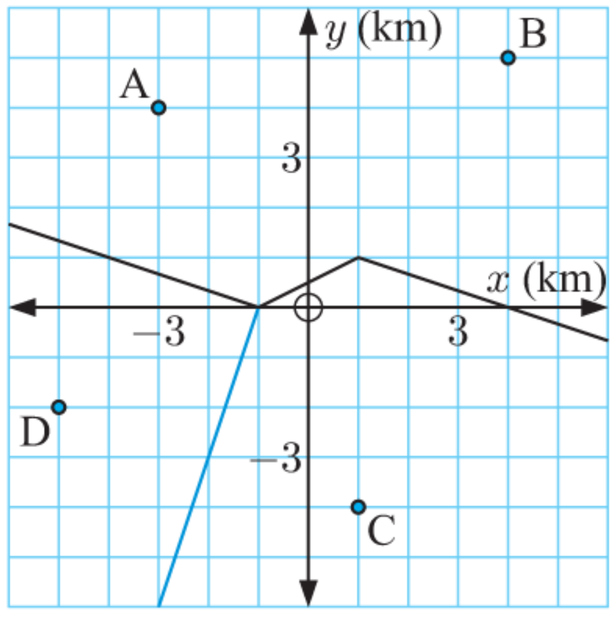
b Complete the diagram, and find the equation of the missing edge.

c Julie is hiking at $(1, 3)$.

i Which camp site is she nearest to?

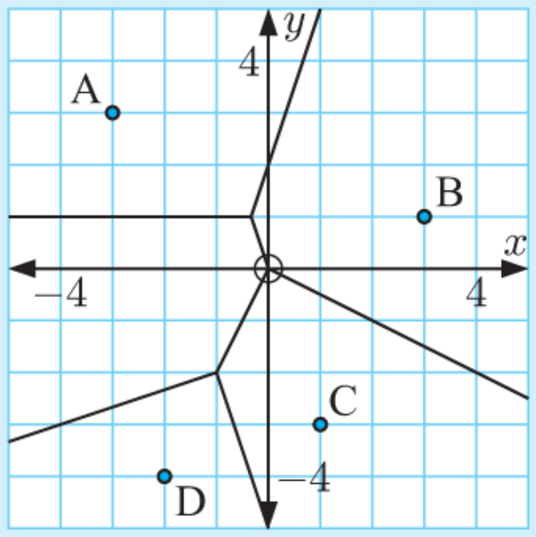
ii How far is she from this camp site?

d Simon is equally closest to camp sites C and D. If he hiked 1 km north, his closest camp site would be A. Mark Simon's possible positions on your diagram.



Example 4 Self Tutor

Find the coordinates of the missing site X in this Voronoi diagram.

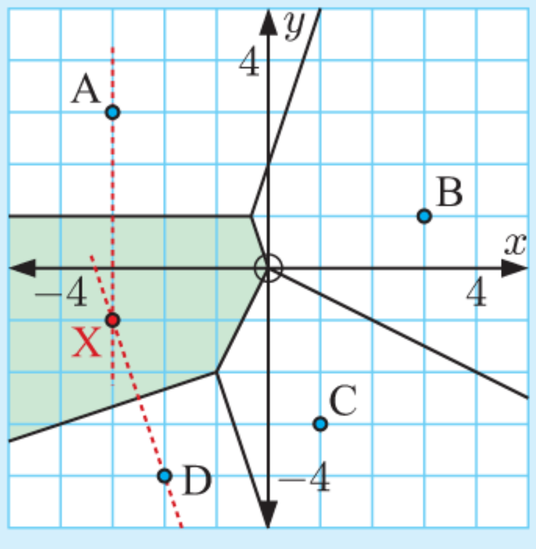


The missing site X must lie in the shaded cell, as this cell currently has no site.

Now PB(A, X) is horizontal, and PB(D, X) has gradient $\frac{1}{3}$.

$\therefore$ [AX] is vertical and [DX] has gradient -3.

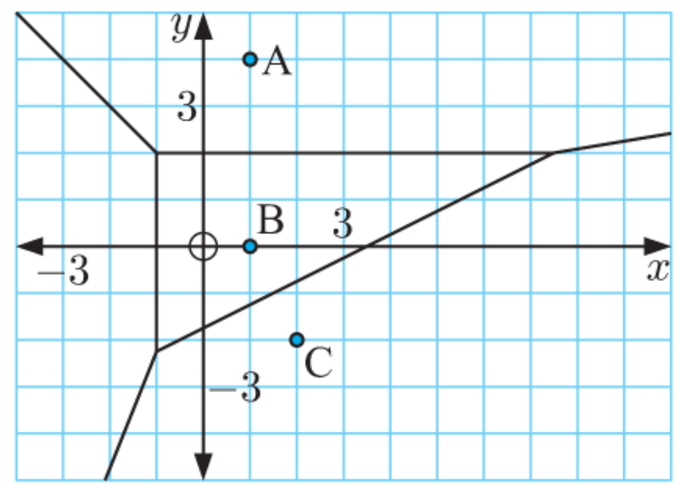
If we draw lines (AX) and (DX) through A and D respectively, their intersection must be point X. We observe that X has coordinates $(-3, 1)$.



11 **a** Explain why the Voronoi diagram alongside must have a site missing.

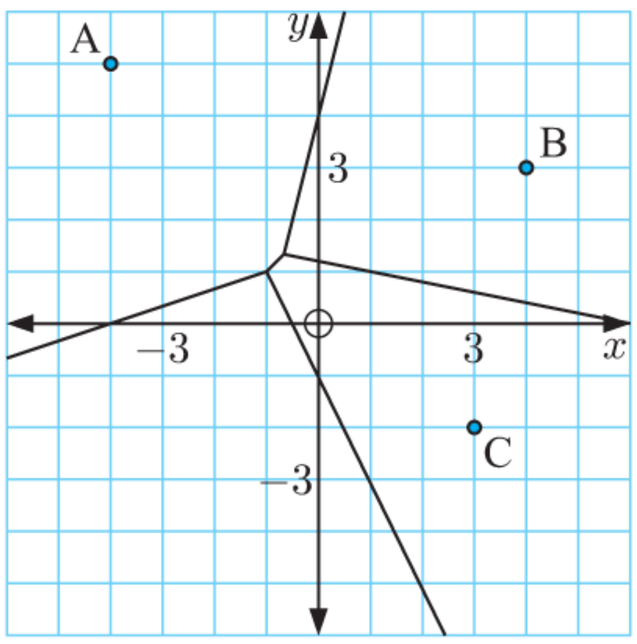
b Explain why the missing site must lie on the x-axis.

c Find the coordinates of the missing site.

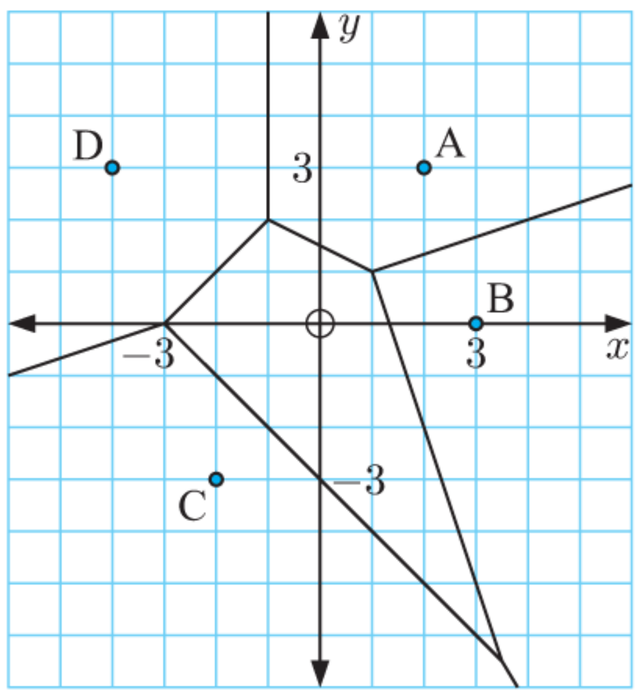


12 Find the coordinates of the missing site X in each Voronoi diagram:

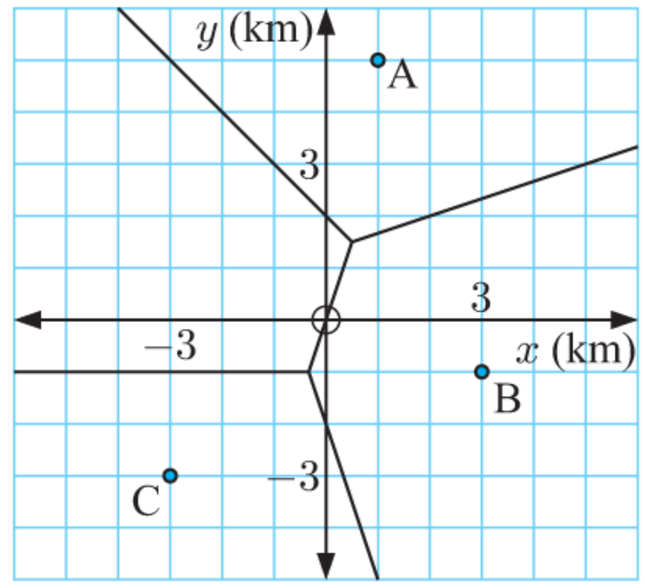
a



b



13 This Voronoi diagram shows the vet clinics in a particular district.



a Copy the diagram, and shade the cell with the missing vet clinic.

b By considering distances from $(0, -1)$, explain why the missing vet clinic is not at $(-2, 1)$.

c Find the coordinates of the missing vet clinic D.

d Find the vet clinic which is closest to:

i $(-1, 4)$ **ii** $(1, -5)$

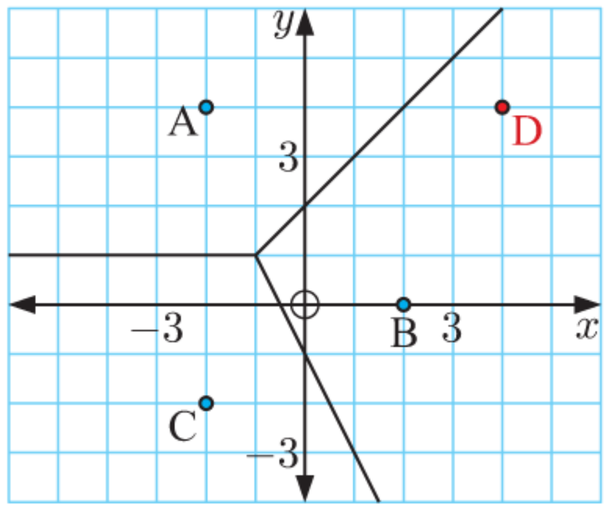
e Larissa lives equally closest to vet clinics A and D. Find the shortest possible distance that Larissa's house could be from these vet clinics.

C ADDING A SITE TO A VORONOI DIAGRAM

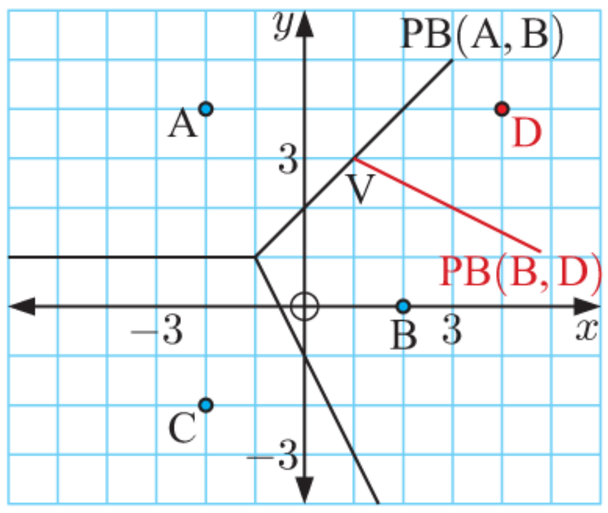
This Voronoi diagram shows the hospitals A, B, and C in a city.

A new hospital is being built at site D marked in red. When it is completed, the Voronoi diagram must be updated to include a cell corresponding to site D.

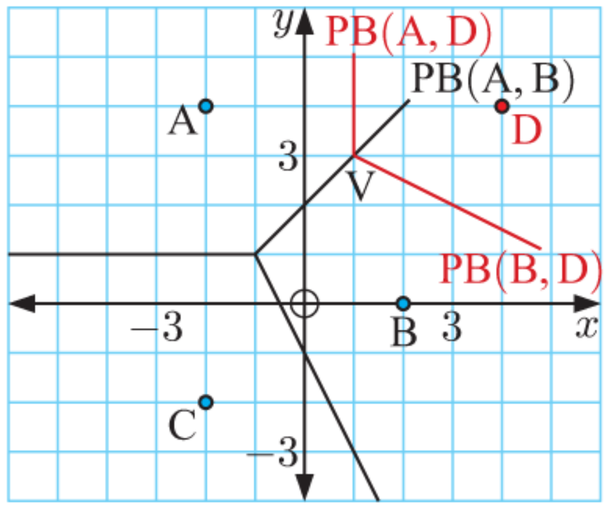
The new site currently lies in cell B, so some areas which were previously closest to Hospital B will now be closest to Hospital D.



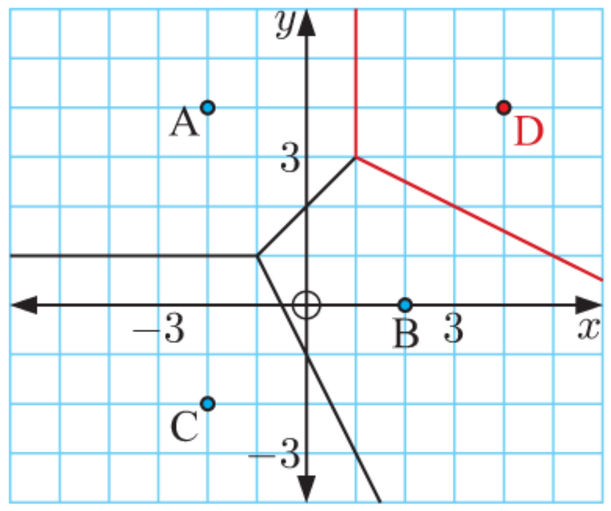
To establish the border between the new cell B and cell D, we draw the perpendicular bisector of [BD]. This line meets the edge PB(A, B) at point V.



V is equidistant from A, B, and D, so is a vertex of the new Voronoi diagram. We therefore need to add PB(A, D) to the diagram, starting at V as shown. This line does not meet any other existing edges, so no more new vertices are created. This tells us that the construction of cell D is complete.



Finally, we remove the part of the existing edge PB(A, B) which now lies within cell D. Notice that the cells for Hospital A and Hospital B were affected by the introduction of Hospital D. Hospital C is relatively further away from Hospital D, so cell C was not affected.



In general, to add the cell for a new site X to an existing Voronoi diagram with sites P_1, P_2, P_3, ..., P_n, we follow these steps:

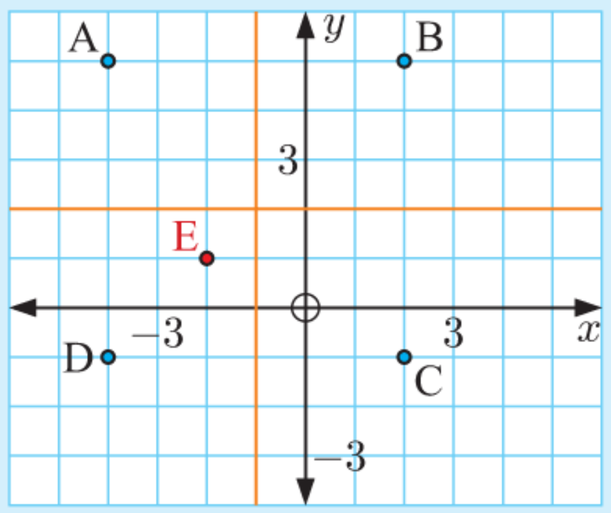
Step 1: Identify the site P_i whose cell contains the new site X. Construct $PB(P_i, X)$ within this cell. At any point where this line meets an existing edge, create a new vertex.

Step 2: For each site P_j whose cell is adjacent to a new vertex, construct $PB(P_j, X)$ within that cell through the vertex. Continue to create new vertices as in *Step 1*. Repeat this process until no more new vertices are created. At this time cell X is complete.

Step 3: Remove any segments of edges from the original Voronoi diagram which now lie within cell X.

Example 5 **Self Tutor**

Redraw this Voronoi diagram with an additional site at E(−1, 1).

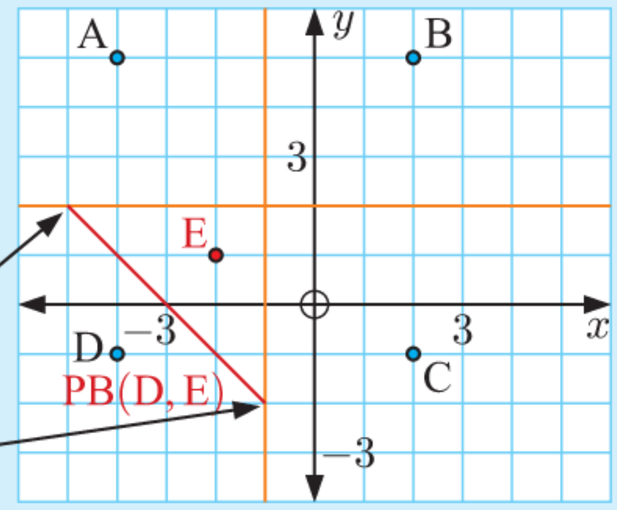


Step 1: E lies in the original cell D, so we construct PB(D, E) within this cell.

We create new vertices at $(-5, 2)$ and $(-1, -2)$.

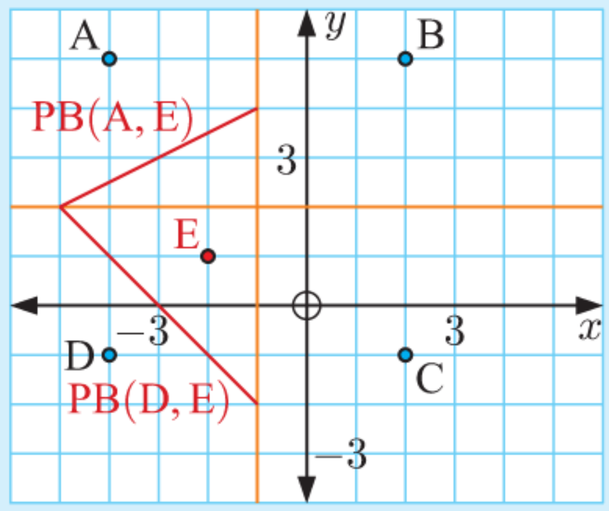


In this case PB(D, E) creates *two* new vertices.



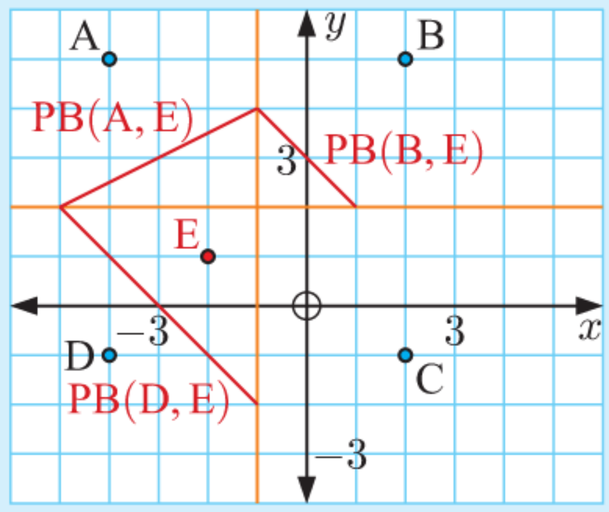
Step 2: Cell A is adjacent to the vertex $(-5, 2)$, so we construct PB(A, E) from $(-5, 2)$ through cell A.

This creates a new vertex at $(-1, 4)$.



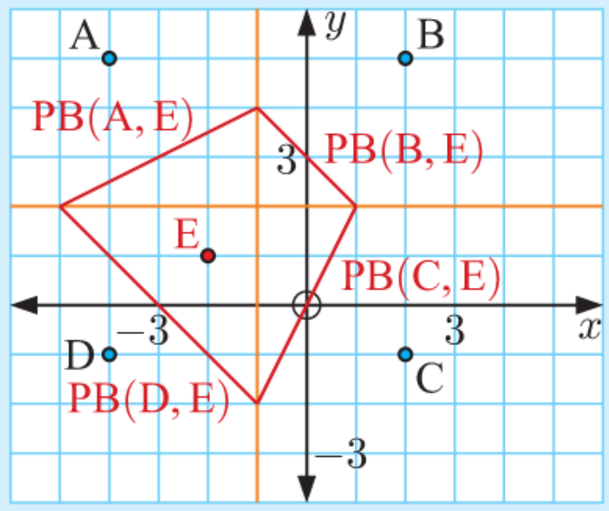
Cell B is adjacent to the vertex $(-1, 4)$, so we construct PB(B, E) from $(-1, 4)$ through cell B.

This creates a new vertex at $(1, 2)$.

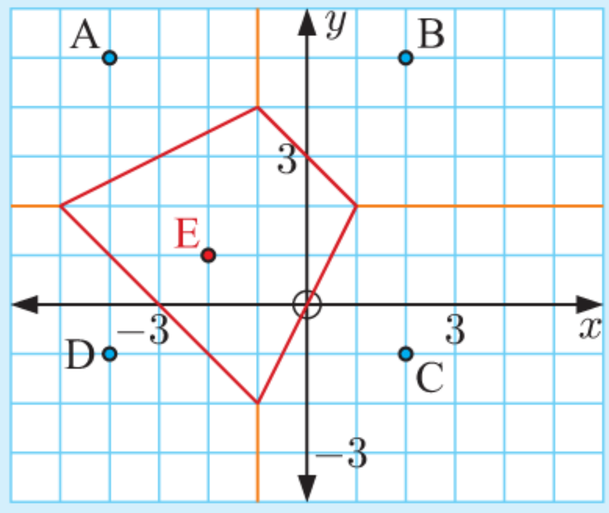


Cell C is adjacent to the vertex $(1, 2)$, so we construct PB(C, E) from $(1, 2)$ through cell C.

This connects us back to the new vertex $(-1, -2)$.

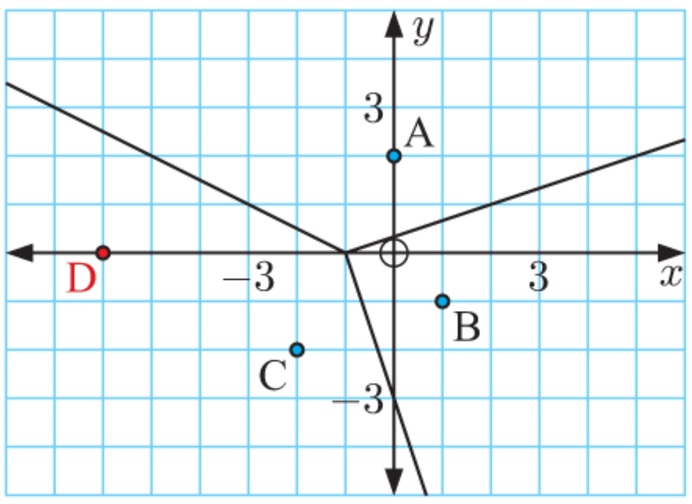


Step 3: We remove the segments of edges from the original Voronoi diagram which now lie within cell E.



EXERCISE 16C

1 Site D is to be added to the Voronoi diagram shown.



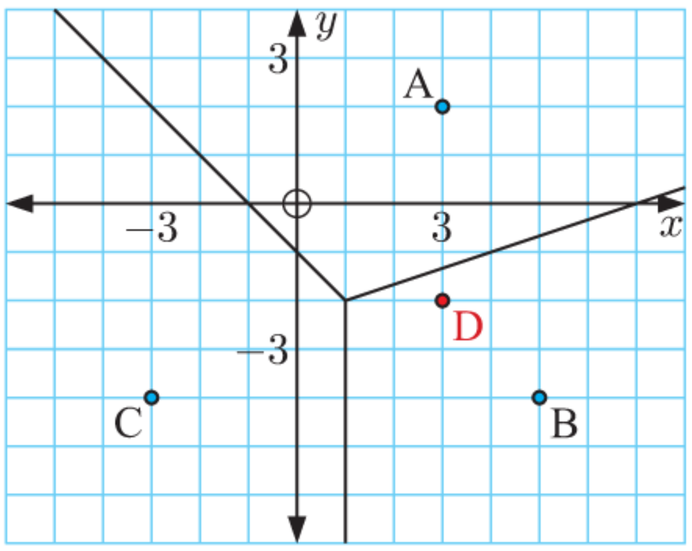
a In which existing cell does site D lie?

b Which of the existing cells do you think will be affected by the introduction of site D? Explain your answer.

c Redraw the Voronoi diagram with site D added.



2 Site D is to be added to the Voronoi diagram shown.



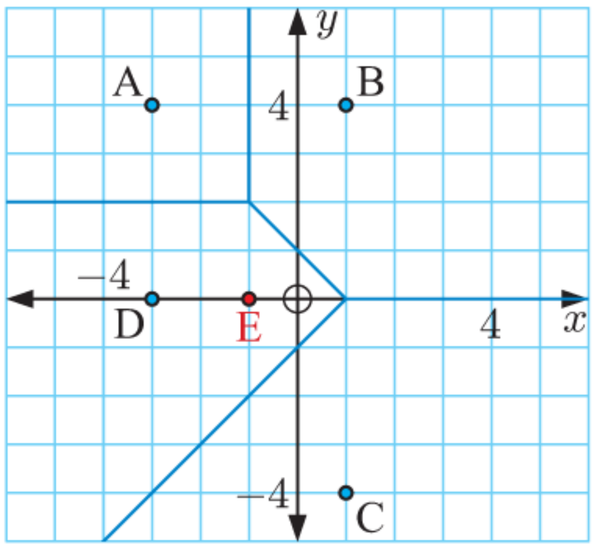
a Explain why you would expect this addition to affect all of the existing cells.

b Redraw the Voronoi diagram with site D added.

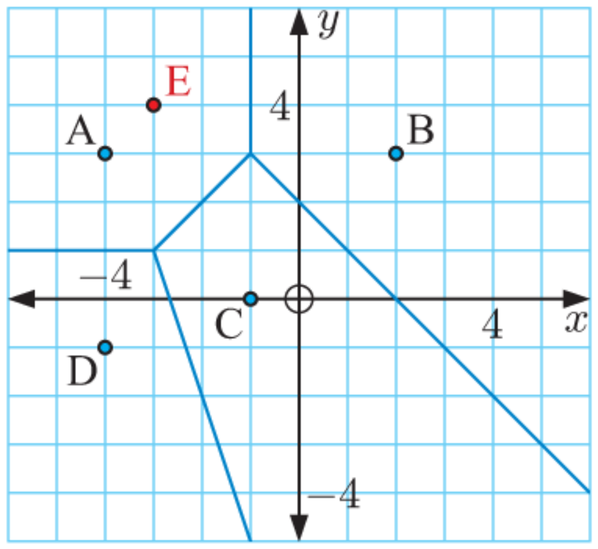
c Find the area of cell D.

3 Redraw the following Voronoi diagrams with the site E added:

a



b

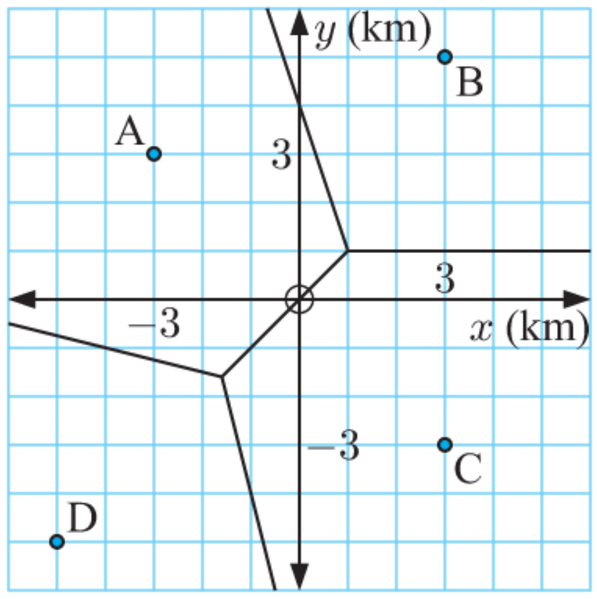


4 For each set of sites, construct a Voronoi diagram for A, B, and C, then add D:

a A$(-3, 1)$, B$(1, 3)$, C$(1, -1)$, D$(0, -2)$

b A$(-3, 3)$, B$(-3, -3)$, C$(1, 1)$, D$(3, -1)$

5 This map shows the automatic teller machines (ATMs) in a suburb.



a Identify the ATM which is closest to:

i $(-2, -1)$ **ii** $(5, 2)$

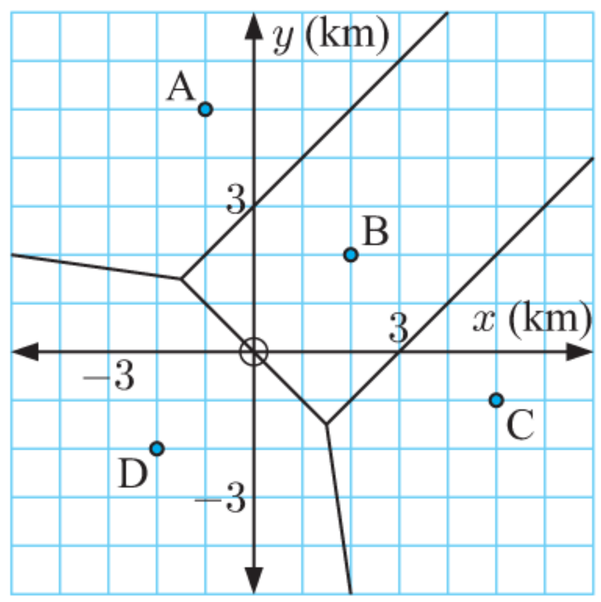
b A new ATM is installed at E$(1, -1)$.

i Redraw the Voronoi diagram to include the new ATM.

ii Are there any residents whose nearest ATM has changed from D to E as a result of the new ATM? Explain your answer.

iii Morris is equally closest to ATMs B, C, and E. Determine Morris' location.

6 This Voronoi diagram shows the location of polling booths in a particular electorate.



a Which polling booth is closest to:

i $(-3, 2)$ **ii** $(1, -4)$?

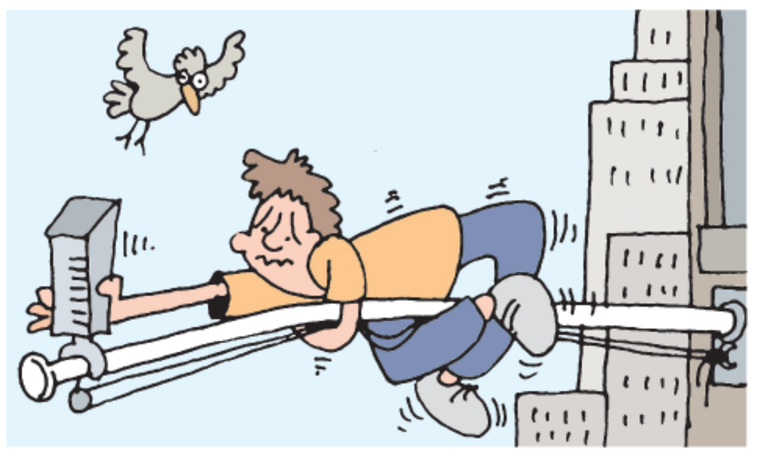
b The polling booth at B was heavily congested at the previous election, so a new polling booth will be set up at $(0, 2)$ for the next election.

i Redraw the Voronoi diagram to include the new polling booth.

ii Which cell will be unaffected by the addition of the new polling booth?

iii Find the area of the cell that is now closest to the new polling booth.

D NEAREST NEIGHBOUR INTERPOLATION



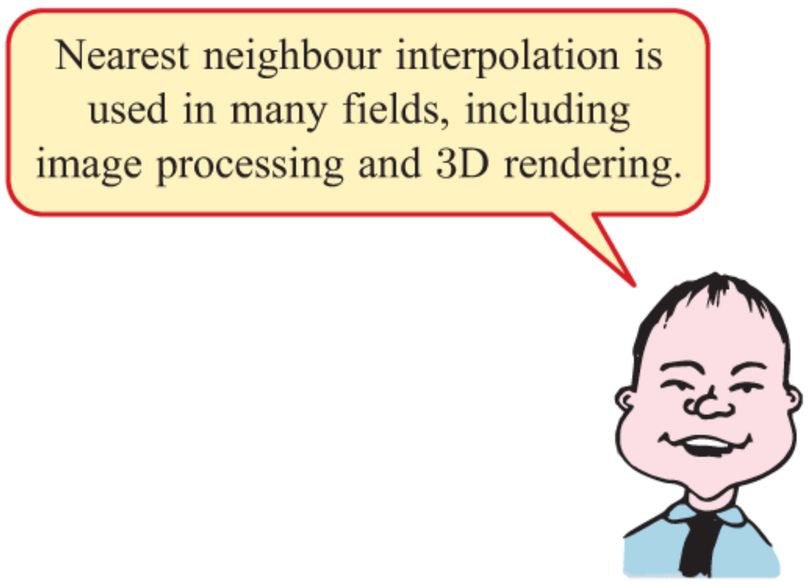
Suppose Ted is interested in the rainfall in his city. He has set up rain gauges at several locations to measure the rainfall. He wants to use these measurements to *estimate* the rainfall in other parts of the city.

On a 2-dimensional grid, the process of using values of a variable at known points to estimate the variable's value at other points is called **interpolation**.

Nearest neighbour interpolation is a simple method of interpolation. To estimate the value of a variable at any point, we use the variable's value at the *nearest* known data point.

By constructing a Voronoi diagram with the known data points as sites, we can quickly identify the nearest known data point to any given point.

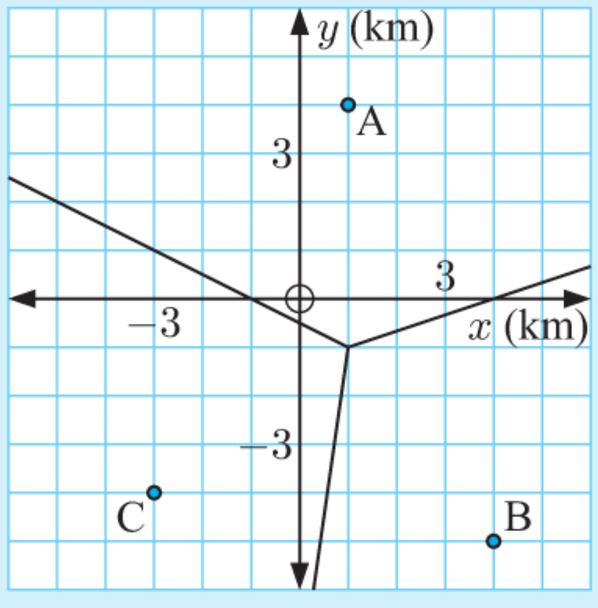
If the given point lies on an edge or at a vertex, we take the average of the closest known data points.



Example 6 **Self Tutor**

Ted measured the rainfall in his city at three locations A, B, and C, marked on the map. The results are shown below.

Location	*Rainfall* (mm)
A	12
B	7
C	15



Use nearest neighbour interpolation to estimate the rainfall at:

a $(-4, 1)$ **b** $(3, -2)$ **c** $(1, -1)$

a $(-4, 1)$ is closest to C, so we estimate 15 mm of rainfall at $(-4, 1)$.

b $(3, -2)$ is closest to B, so we estimate 7 mm of rainfall at $(3, -2)$.

c $(1, -1)$ is the vertex equidistant from A, B, and C, so we estimate $\frac{12+7+15}{3} \approx 11.3$ mm of rainfall at $(1, -1)$.

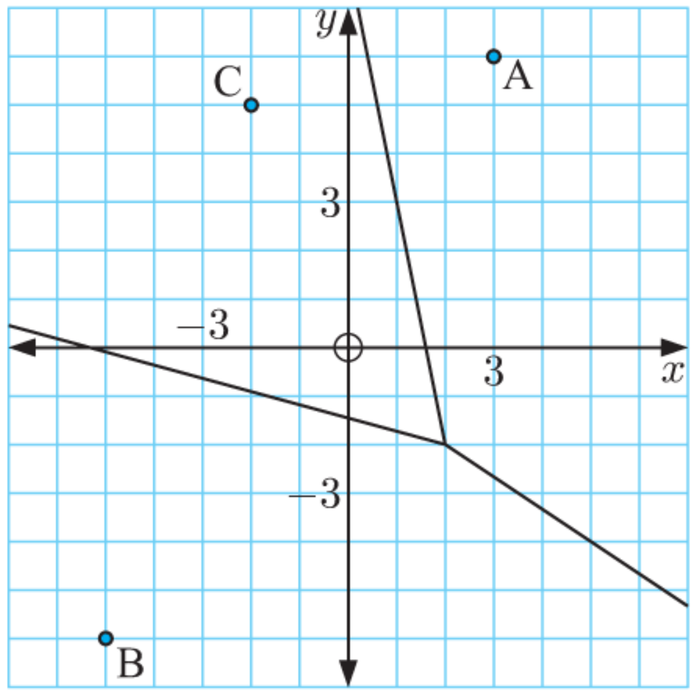
EXERCISE 16D

1 Weather stations measure the temperature in a city at the three locations A, B, and C shown. Their measurements at 3 pm are shown below:

Location	*Temperature* (°C)
A	28.4
B	25.6
C	27.3

Use nearest neighbour interpolation to estimate the 3 pm temperature at:

a $(1, 0)$ **b** $(-3, -1)$ **c** $(5, -2)$



2 John measured the elevation at four locations in a park.

a Construct a Voronoi diagram for these locations.

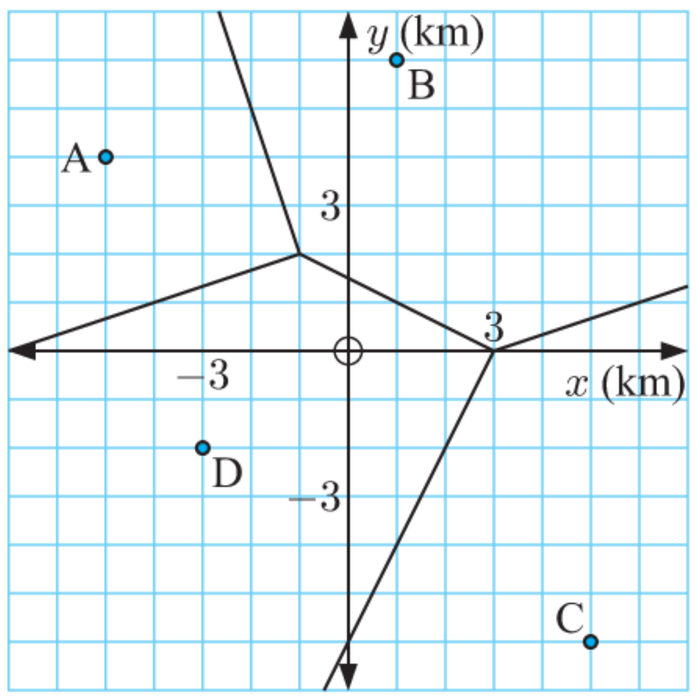
b Estimate the elevation at:

i $(0, 1)$ **ii** $(-4, 2)$ **iii** $(3, -4)$

Location	*Elevation* (m)
A$(-4, 4)$	57
B$(-2, 0)$	48
C$(2, 4)$	55
D$(2, -6)$	36

3 On a snowy day, the snowfall was measured in four locations across a city.

Location	*Snowfall* (inches)
A	7
B	5.5
C	12.2
D	9.3



a Estimate the snowfall received at:

i $(-1, -4)$ **ii** $(2, 1)$ **iii** $(1, -4)$

b A further report indicated that 10.6 inches of snow fell at $E(-1, -6)$.

i Redraw the Voronoi diagram with location E added.

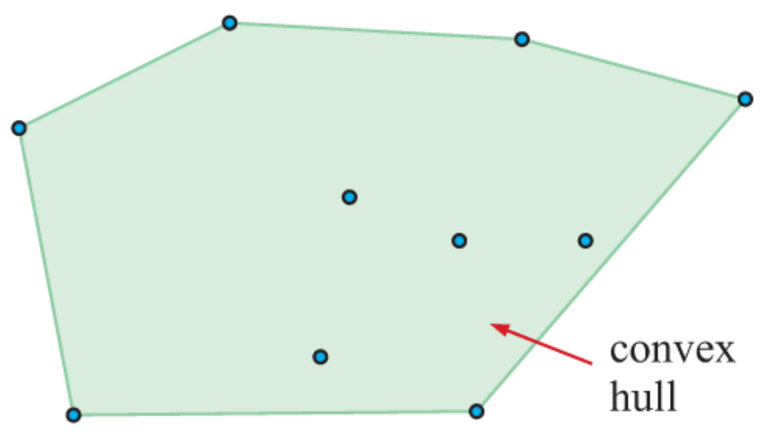
ii Does this change any of your estimates in **a**?

DISCUSSION

What are the advantages and disadvantages of nearest neighbour interpolation? Can you think of a more accurate way to perform the interpolation?

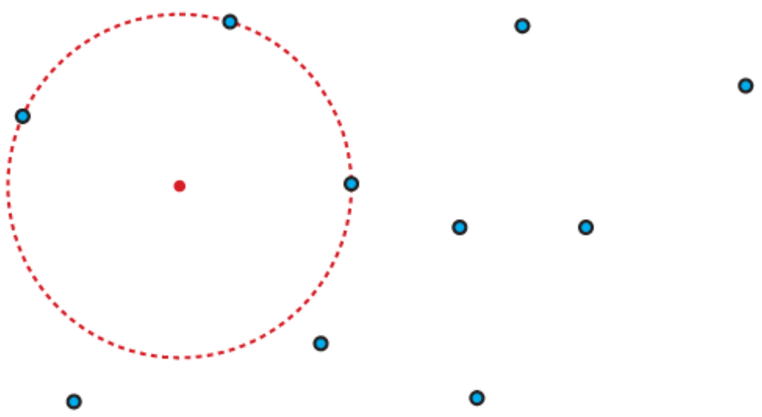
E THE LARGEST EMPTY CIRCLE PROBLEM

Given a set of sites in a plane, the **convex hull** of the sites is the smallest convex polygon which contains all of the sites.



The **Largest Empty Circle problem** is the problem of finding the largest circle, centred within the convex hull, whose interior does not contain any sites.

This problem can be formulated as finding the optimal position for a toxic waste dump, so as to maximise its distance from the nearest town. For this reason, this problem is also known as the **toxic waste dump problem**.

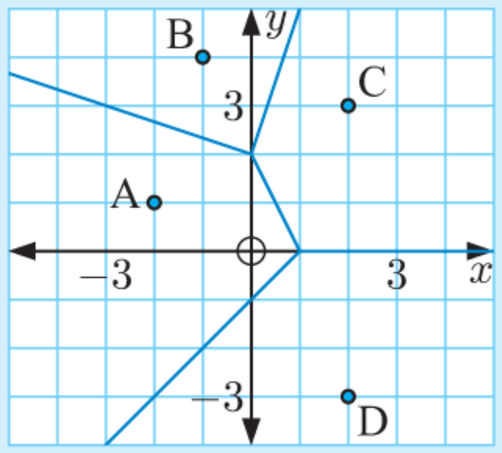


We can solve this problem by drawing the Voronoi diagram for the sites.

In the problems we will consider in this Section, all of the vertices of the Voronoi diagram lie within the convex hull. In this case, the optimal position for the circle's centre will occur at one of these vertices. The vertex with the greatest distance from its nearest site is the optimal position for the circle's centre.

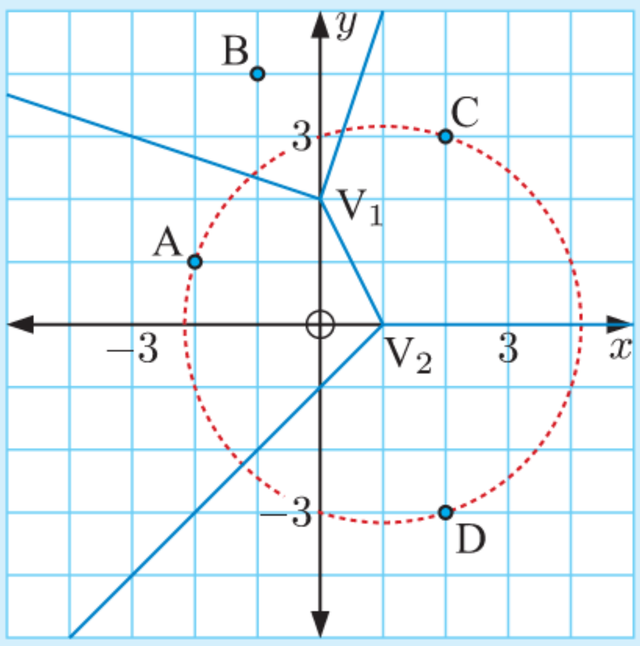
Example 7 Self Tutor

The Voronoi diagram for the sites $A(-2,\ 1)$, $B(-1,\ 4)$, $C(2,\ 3)$, and $D(2,\ -3)$ is shown alongside.



Find the largest empty circle for these sites.

The Voronoi diagram has vertices $V_1(0,\ 2)$ and $V_2(1,\ 0)$.



V_1 is equidistant from A, B, and C.

$$V_1A = \sqrt{(-2-0)^2 + (1-2)^2}$$
$$= \sqrt{(-2)^2 + (-1)^2}$$
$$= \sqrt{5} \text{ units}$$

V_2 is equidistant from A, C, and D.

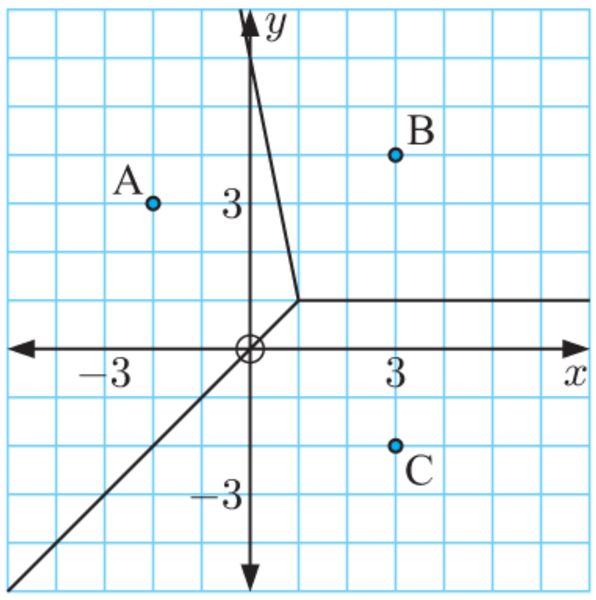
$$V_2A = \sqrt{(-2-1)^2 + (1-0)^2}$$
$$= \sqrt{(-3)^2 + 1^2}$$
$$= \sqrt{10} \text{ units}$$

So, the largest empty circle has centre $V_2(1,\ 0)$ and radius $\sqrt{10}$ units.

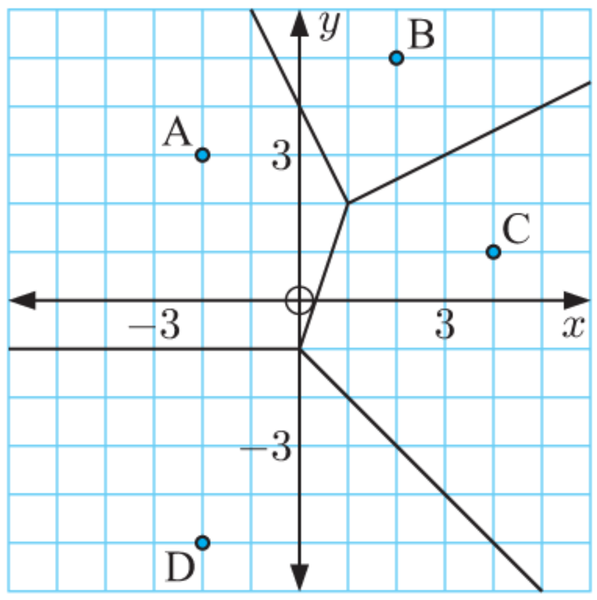
EXERCISE 16E

1 Find the centre and radius of the largest empty circle for these sets of sites:

a



b



2 This Voronoi diagram shows the locations of five towns in a rural area.

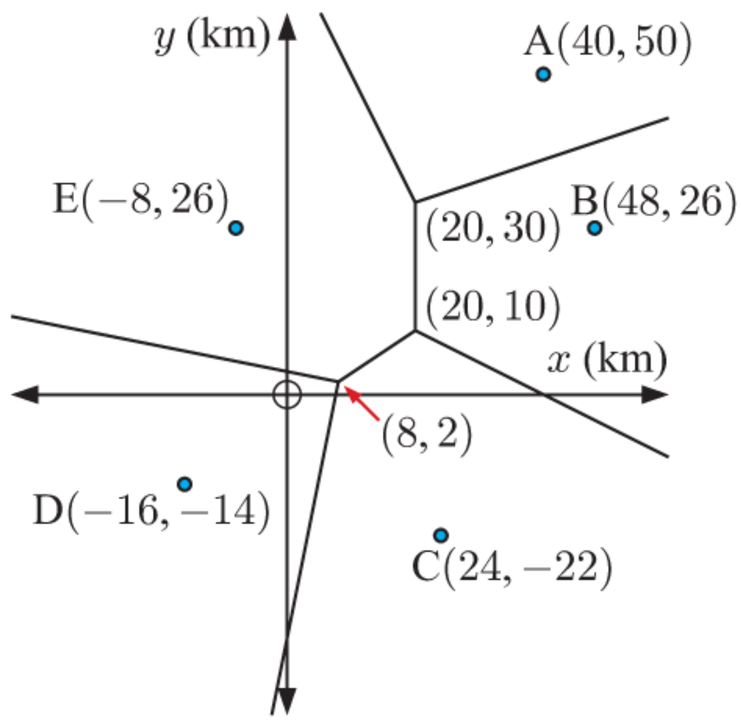
A rubbish dump is to be established somewhere in the region, so that it is as far as possible from the nearest town.

a Where should the rubbish dump be established?

b How far is the dump from the nearest town?

c Which towns are closest to the dump?

d A proposed location for the dump is $(25,\ 15)$. Show by direct calculation that the location you found in **a** is preferable.



3 Explain why it is impossible for the centre of the largest empty circle to lie inside a cell of the Voronoi diagram.

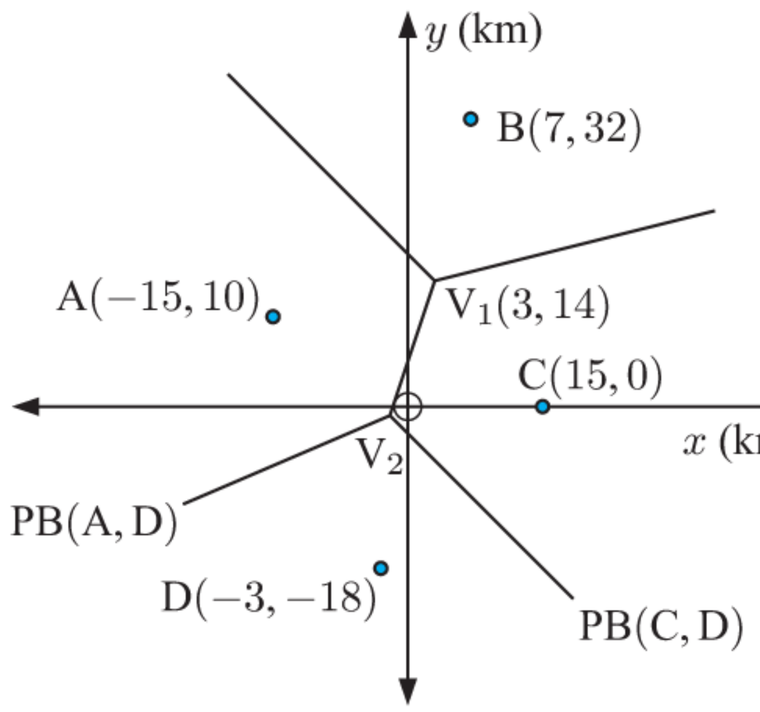
4 Consider this Voronoi diagram for the towns A, B, C, and D.

a Find the equation of:

i PB(A, D) **ii** PB(C, D)

b Hence find the coordinates of V_2.

c Find the optimal position for a toxic waste dump so that it is as far as possible from the nearest town.



5 Consider the sites $A(-5,\ -10)$, $B(11,\ 18)$, and $C(5,\ -12)$.

a Construct the Voronoi diagram for these sites, and find the equation of each edge.

b Find the coordinates of the vertex of the diagram.

c Hence find the centre and radius of the largest empty circle.

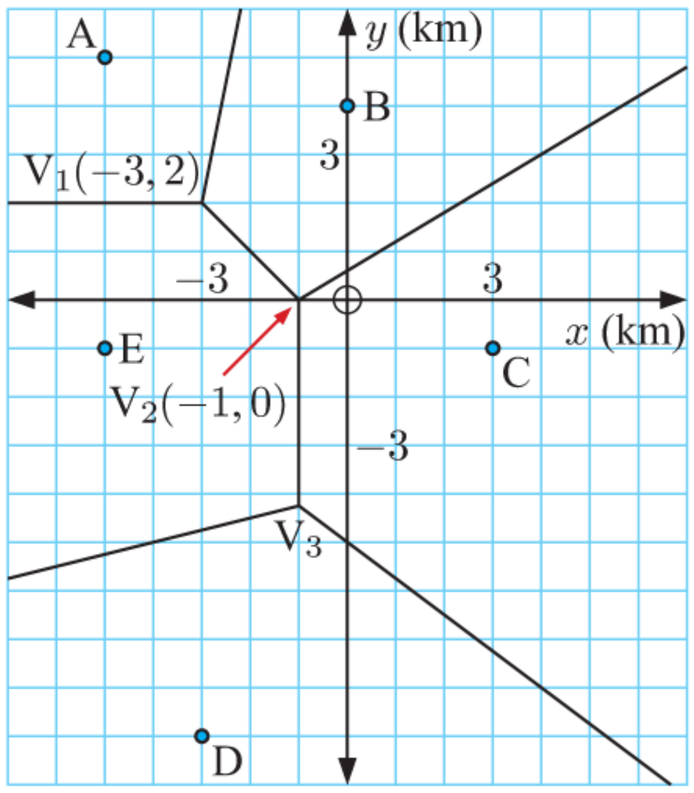
6 Consider the sites A$(-2, 7)$, B$(4, 7)$, C$(6, 3)$, and D$(-2, -5)$.

a Construct a Voronoi diagram for these sites.

b Use your diagram to find the coordinates of the vertices.

c Find the centre and radius of the largest empty circle.

7



This Voronoi diagram shows the police stations in a city.

a Which station is closest to:

i $(3, 4)$ **ii** $(-3, -3)$?

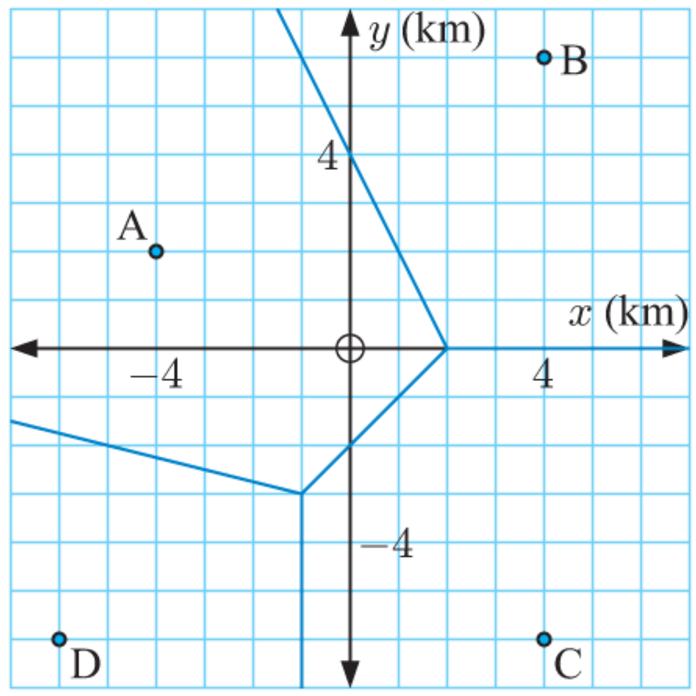
b Find the coordinates of the vertex V_3.

c A new police station is to be built. To maximise the efficiency of the stations, the new station will be built as far as possible from the existing stations.

i Where should the new station be built?

ii Show that this new station will now be the closest station to $(-3, -3)$.

8 Brigette would like to open a burger store. She has drawn a Voronoi diagram showing the existing burger stores in her area so she can locate her own store as far from the existing stores as possible.



a Find the optimal position for Brigette's store, and state how far this location is from her nearest competitor.

b Just before Brigette purchases the site, a new burger store opens at $(4, 2)$.

i Redraw the Voronoi diagram with the new store added.

ii Find the new optimal position for Brigette's store.

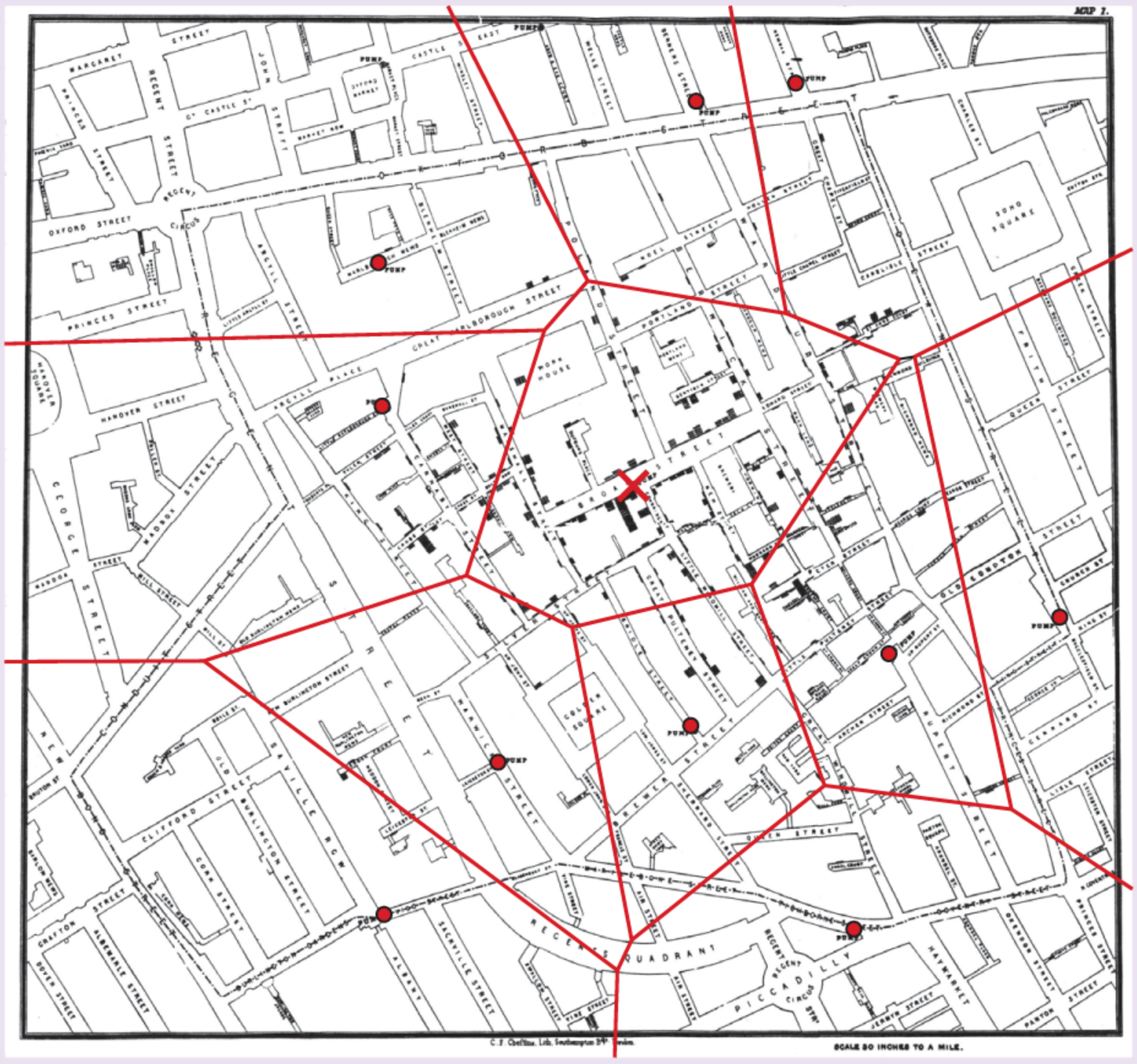
9 Answer the **Opening Problem** on page **424**.

HISTORICAL NOTE — THE BROAD STREET CHOLERA OUTBREAK

In 1854 in the Soho district of London, there was a serious cholera outbreak which killed 616 people. At the time, the cause of cholera was not fully understood. It was thought under the *miasma theory* that diseases such as cholera or the Black Death were caused by some kind of "bad air". The germ theory was not proposed until 1861 by **Louis Pasteur**.

In this outbreak, the English physician **John Snow** looked at the pattern of illness, and decided that the disease was spread by something in the water. At that time, water was drawn directly from the Thames River which was highly contaminated. Snow identified the source of the outbreak as the public water pump on Broad Street, and he persuaded the authorities to disable the pump by removing its handle.

John Snow



The map above shows the locations of the cholera cases. The dark rectangles show where people died of cholera, the red dots show public water pumps, and the large red cross is the Broad Street pump. The Voronoi cells show the closest pump by straight line distance.

At the time of Snow's investigation, he did not have Voronoi diagrams to use. Voronoy had not yet been born! However, the overlaid Voronoi diagram clearly illustrates how the cholera cases were clustered around the Broad Street pump. This is what Snow recognised. He wrote "*it will be observed that the deaths either very much diminished, or ceased altogether, at every point where it becomes decidedly nearer to send to another pump than to the one in Broad Street.*"[1]

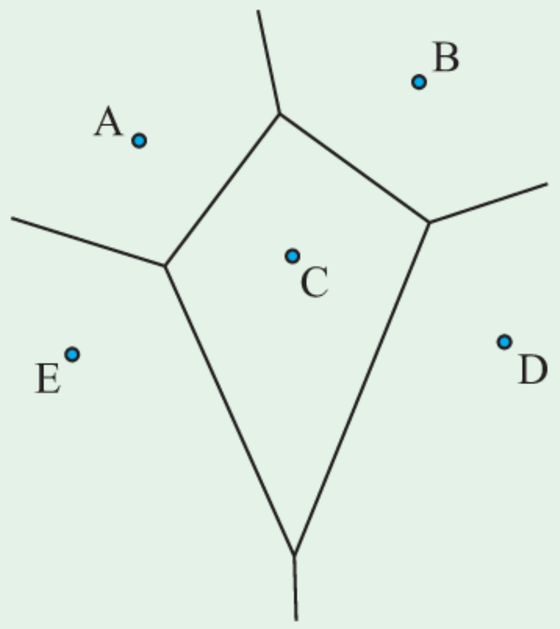
However, not all the cases occurred in the cell around the Broad Street pump. Many cases also occurred in the cell to the left. These people were much closer to the pump in Marlborough Street. However, when Snow investigated where people actually drank from, he found that "*the water of the pump in Marlborough Street, at the end of Carnaby Street, was so impure that many people avoided using it. And I found that the persons who died near this pump in September, had water from the Broad Street pump.*"

It was later discovered that the nappy of a baby who had contracted cholera from another source, had contaminated the water supply near the Broad Street pump.

[1] J. Snow. *On the Mode of Communication of Cholera* p. 47. John Churchill, London, 2nd edition, 1855.

REVIEW SET 16A

1 Copy this Voronoi diagram, and indicate the parts of the diagram that are:

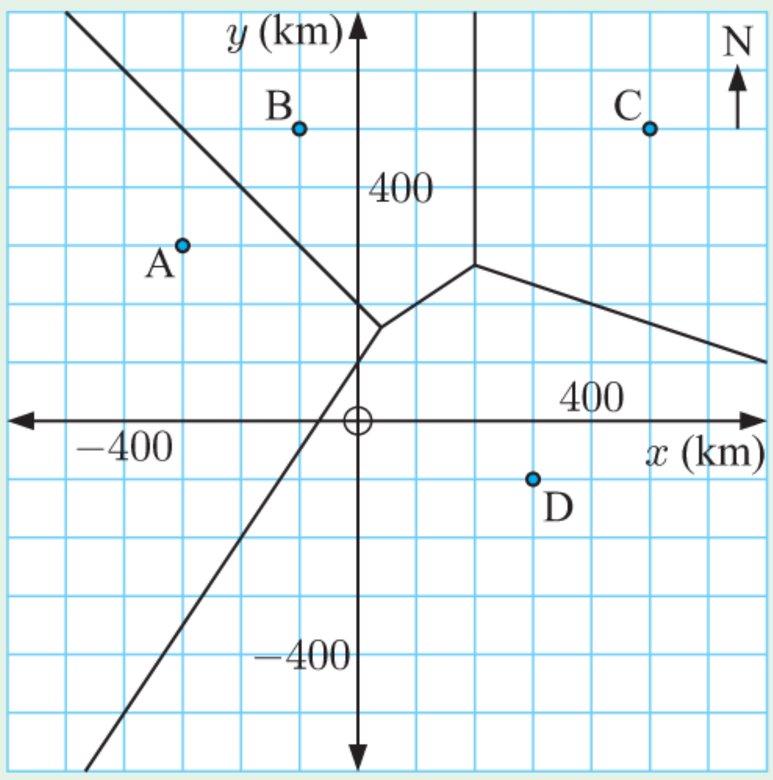


a closest to site B

b equally closest to C and D

c equally closest to A, C, and E.

2



This Voronoi diagram shows the airports in a country. If an aeroplane's passenger falls seriously ill during a flight, the aeroplane must land at the nearest airport.

a Identify the nearest airport to an aeroplane located at:

i $(300, 400)$ **ii** $(-100, 0)$

b An aeroplane is currently at $(100, 200)$.

i Which airports is the aeroplane closest to?

ii How far is the aeroplane from these airports? Give your answer to the nearest kilometre.

iii How far east must the aeroplane travel before it is closest to Airport C?

3 **a** Construct a Voronoi diagram for the sites $A(-3, 2)$ and $B(5, 6)$.

b Find the equation of the edge.

c Verify that the point $(3, 0)$:

i lies on the edge **ii** is equidistant from A and B.

d Use your Voronoi diagram to identify the site closest to:

i $(-1, 7)$ **ii** $(2, -5)$

4 This swimming pool has exits at A, B, and C.

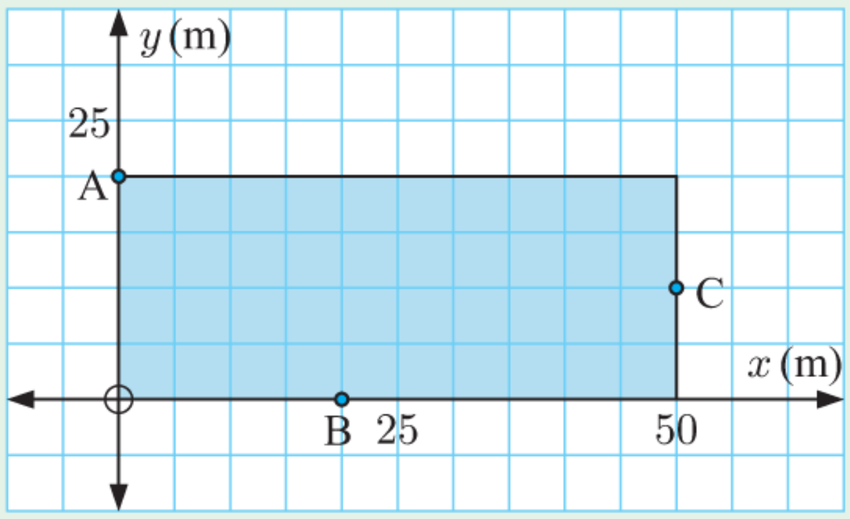
a Draw a Voronoi diagram for A, B, and C.

b Is there any point in the pool which is equidistant from all three exits? Explain your answer.

c Jenny is swimming at $(35, 10)$.

i Which exit is Jenny closest to?

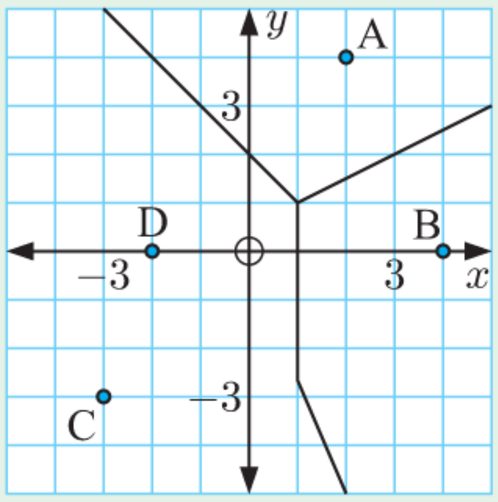
ii How far is Jenny from this exit?



d Find the proportion of the pool that is closest to exit:

i A **ii** B **iii** C.

5



a Explain why this Voronoi diagram must have an edge missing.

b Copy and complete the diagram, and find the equation of the missing edge. Write your equation in the form $ax + by + d = 0$.



6 Consider the Voronoi diagram alongside.

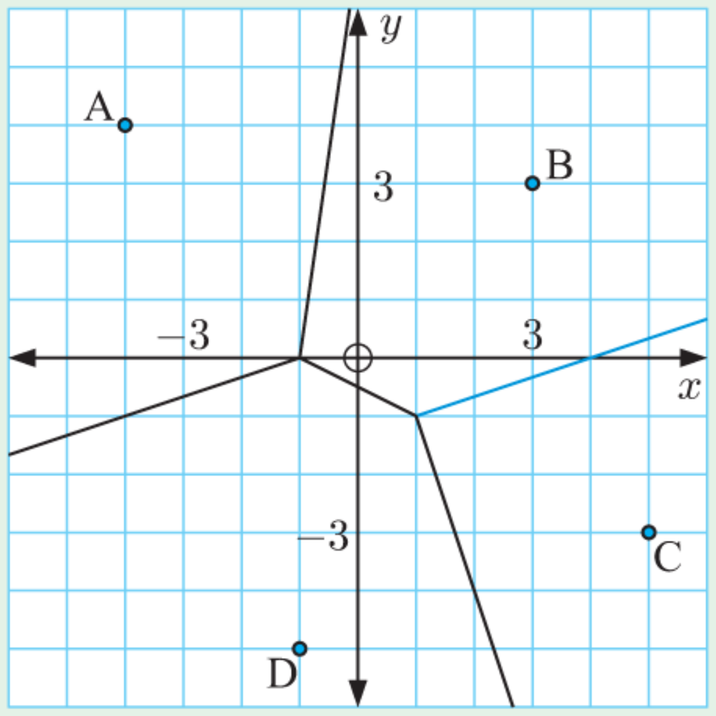
a Find the equation of the blue edge.

b Identify the site which is closest to:

i $(2, -1)$ **ii** $(-5, -2)$

c Redraw the Voronoi diagram with a new site added at $E(-3, -6)$.

d Does the addition of site E affect your answers to **b**?



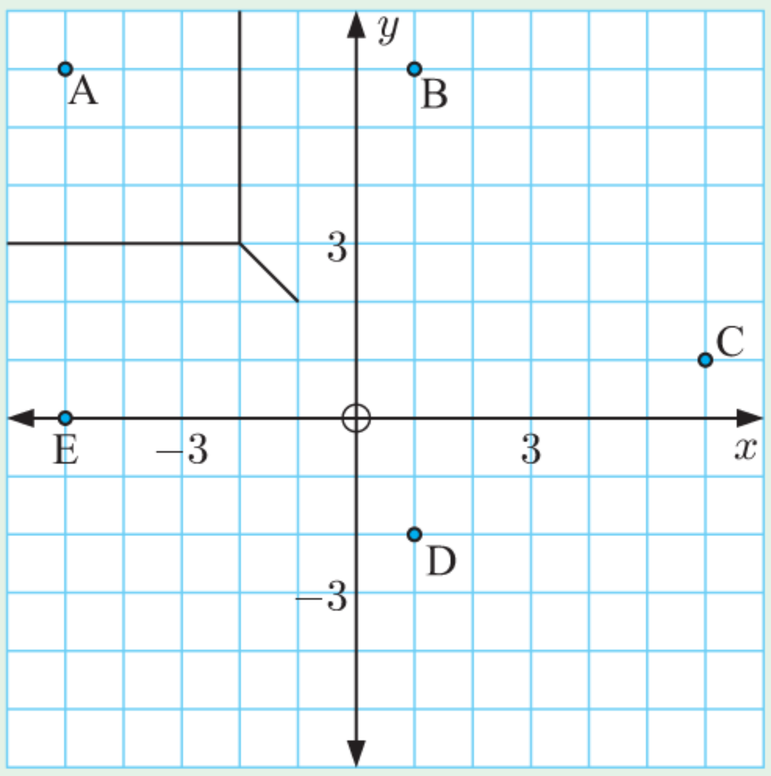
7 The wind speed is measured at three locations on an island.

Location	*Wind speed* (km h^{-1})
A	14
B	11
C	19

Use nearest neighbour interpolation to estimate the wind speed at:

a $(0, 0)$ **b** $(-1, 3)$ **c** $(-4, -2)$

8

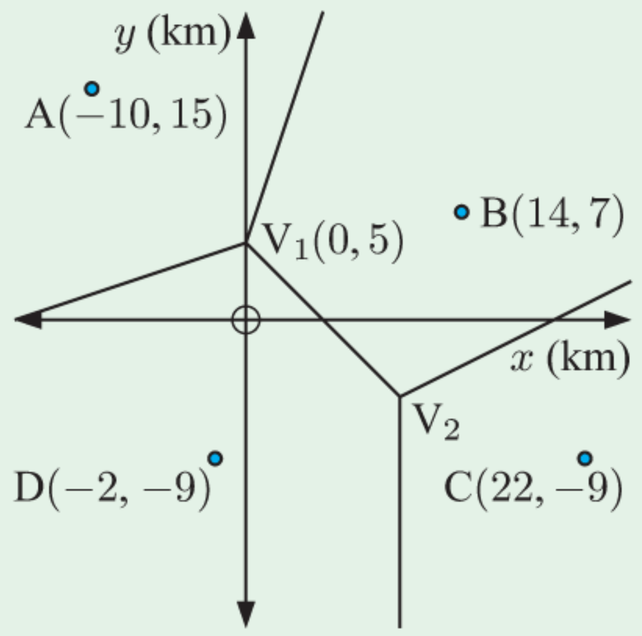


a Copy and complete this Voronoi diagram.

b Find the coordinates of the vertices of the Voronoi diagram.

c Find the centre and radius of the largest empty circle for these sites.

9 Consider this Voronoi diagram for the towns A, B, C, and D.



a Find the equation of:

i PB(B, C) **ii** PB(C, D)

b Hence find the coordinates of V_2.

c An observatory is to be built somewhere in the region, so that it is as far as possible from the town lights.

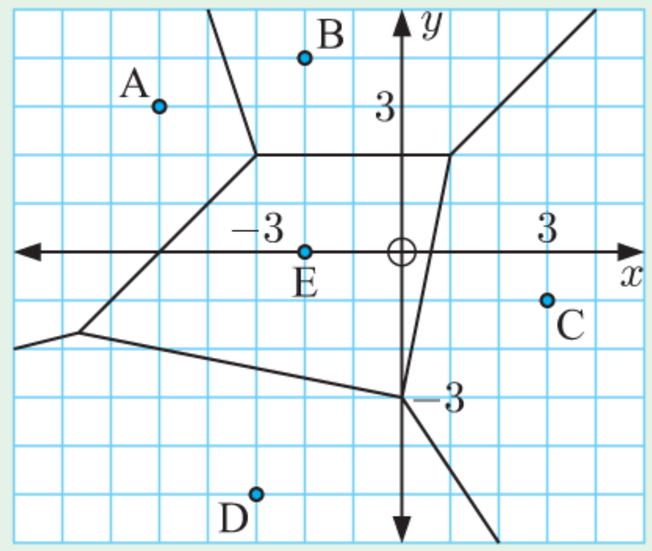
i Where should the observatory be built?

ii How far from the observatory will the nearest town be?

iii Which towns are closest to the observatory?

REVIEW SET 16B

1 Consider this Voronoi diagram for the sites A, B, C, D, and E.



a Identify the site(s) closest to:

i (1, 0) **ii** (−4, −2)

iii (3, −5) **iv** (−3, 2)

b Are there any points that are equally closest to sites B and D? Explain your answer.

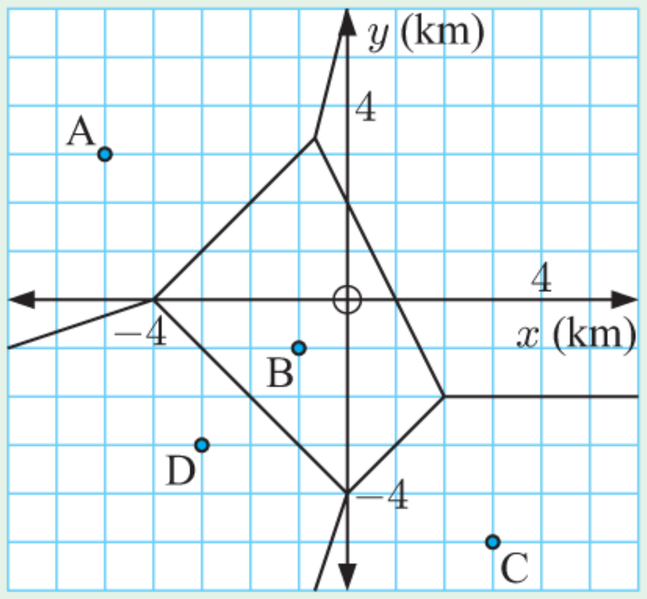
c Which point is equally closest to sites B, C, and E?

2 Let P be a point on an edge of a Voronoi diagram, and X be the site in a cell adjacent to that edge. Suppose a circle is drawn with centre P, passing through X. Explain why this circle must pass through one other site.

3 Construct a Voronoi diagram for the sites:

a A(4, 7) and B(2, −1) **b** A(−5, 0), B(1, 6), and C(7, 4).

4 This Voronoi diagram shows the taxi ranks in a city.



a Explain why the diagram must have a site missing.

b Find the coordinates of the missing site E.

c Elizabeth is at (−3, 2), and needs to catch a taxi.

i Which taxi rank is she closest to?

ii Assuming she can walk at 5 km h^{-1}, how long will she take to walk to this taxi rank? What assumption are you making in your answer?

d Albert is at (−2, −2).

i Which taxi ranks is Albert closest to?

ii What other factors might Albert consider when deciding which taxi rank to use?

5 **a** Construct the Voronoi diagram for the sites shown.

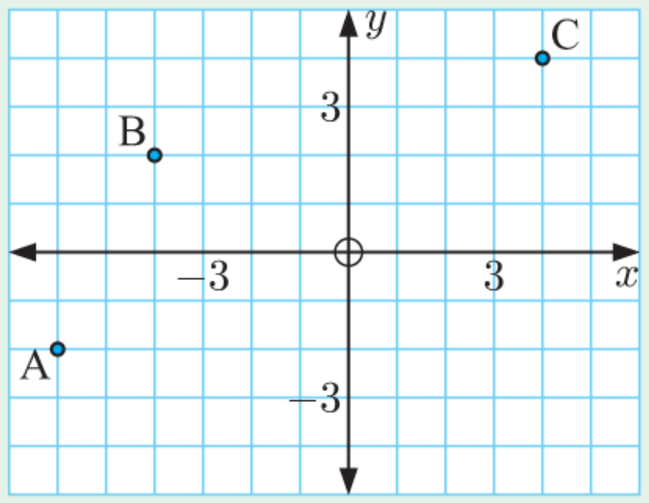
b Use your Voronoi diagram to identify the site(s) closest to:

i $(1, 2)$

ii $(-1, -5)$

iii $(-3, -1)$

PRINTABLE DIAGRAMS



6 This Voronoi diagram shows the locations of rubbish bins in a park.

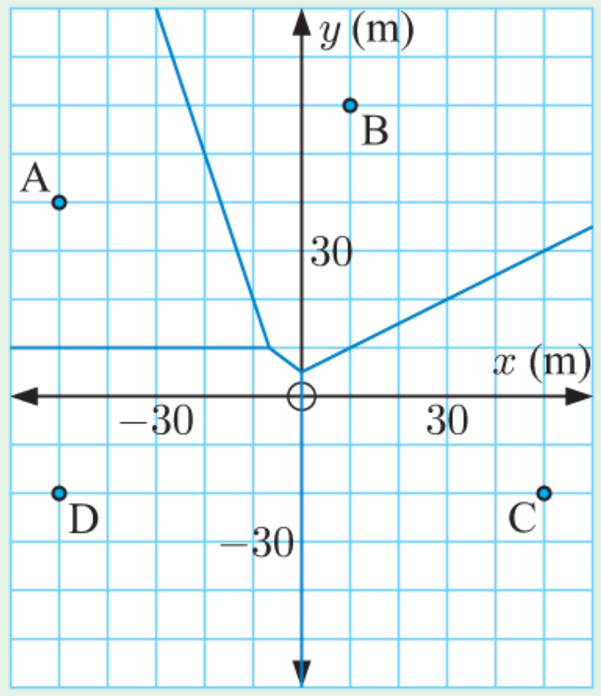
a Boris is at $(20, 20)$. How far is he from the closest bin?

b Suppose a new bin is placed at $E(50, 40)$.

i Redraw the Voronoi diagram to include the new bin.

ii Which cells have been affected by the introduction of the new bin?

iii On your Voronoi diagram, shade the area of the park which was closest to Bin C, but is now closest to Bin E.



7 A diver measured the depth of a lake at four locations.

a Construct a Voronoi diagram for these locations.

b Use nearest neighbour interpolation to estimate the depth of the lake at:

i $(-1, 3)$ **ii** $(4, 0)$

Location	*Depth* (m)
A$(-2, -3)$	7.5
B$(-2, 5)$	9.2
C$(4, 3)$	6.1
D$(6, -1)$	6.9

8 This Voronoi diagram shows the parking lots in a city.

a Identify the nearest parking lot to:

i $(-100, 200)$ **ii** $(0, -400)$

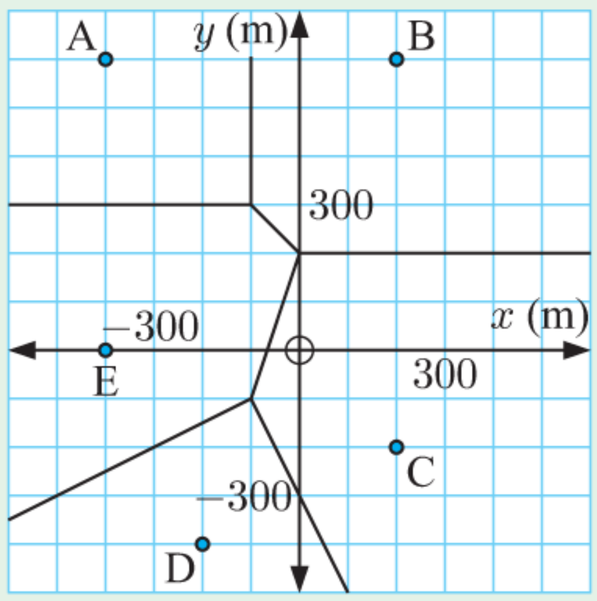
b A new parking lot is to be built in the city. It will be built as far as possible from the existing parking lots.

i Find the optimal position for the new parking lot F.

ii How far is this parking lot from the closest existing parking lots?

iii Redraw the Voronoi diagram with the new parking lot added.

iv Find the area of the region that is closest to the new parking lot.



c Joseph works at $(200, 200)$.

i Show that Joseph is closest to the new parking lot.

ii How much closer is Joseph to a parking lot, now that the new parking lot has been added?

Chapter 17

Introduction to differential calculus

Contents:

A Rates of change
B Instantaneous rates of change
C Limits
D The gradient of a tangent
E The derivative function
F Differentiation from first principles



OPENING PROBLEM

Suppose we add water to a container at a constant rate. The depth of the water increases over time.

Things to think about:

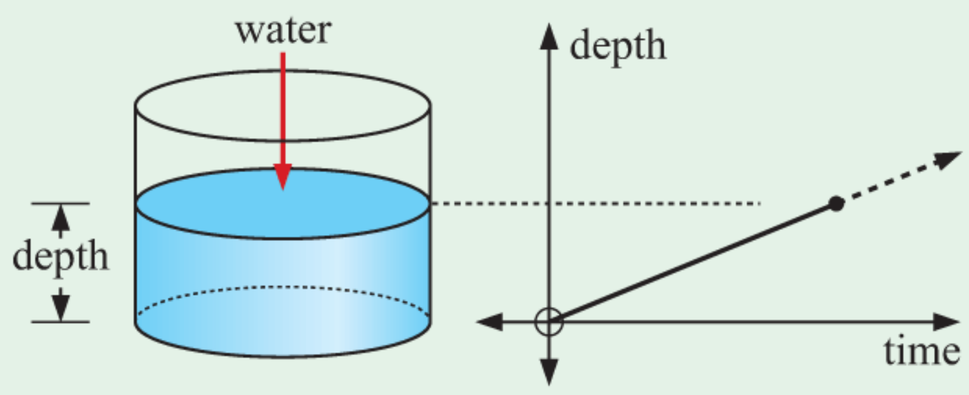
a Think about the graph of the *volume of water added* against *time*.

i Can you explain why this graph is a straight line passing through the origin?

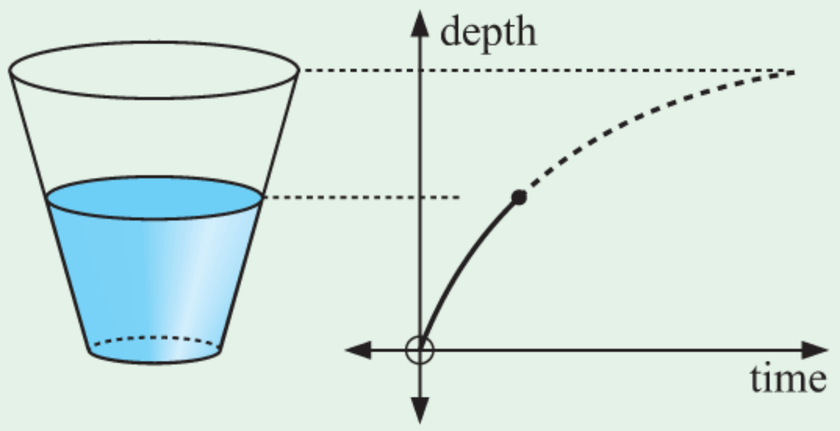
ii What does the *gradient* of the straight line tell us?

b Think about the graph of the *depth of water* against *time*.

i Can you explain why this graph is a straight line for a cylindrical container?

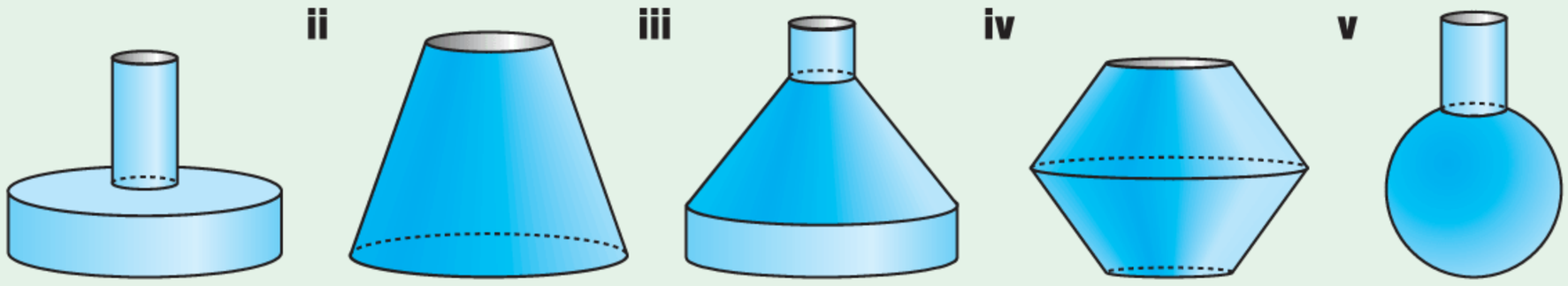


ii Can you explain why this graph is *not* a straight line for the container shown?



c By examining the shape of each container, can you predict the depth-time graph when water is added at a constant rate?

i **ii** **iii** **iv** **v**



Use the water filling demonstration to check your answers.

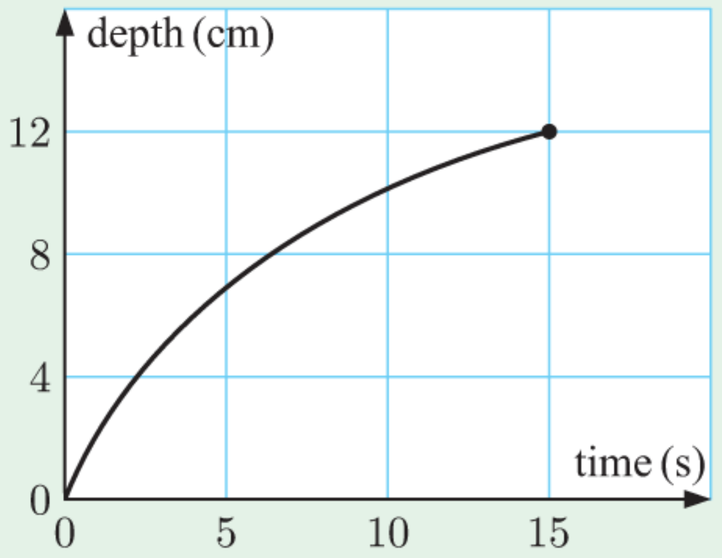
DEMO



d Consider the depth-time graph alongside.

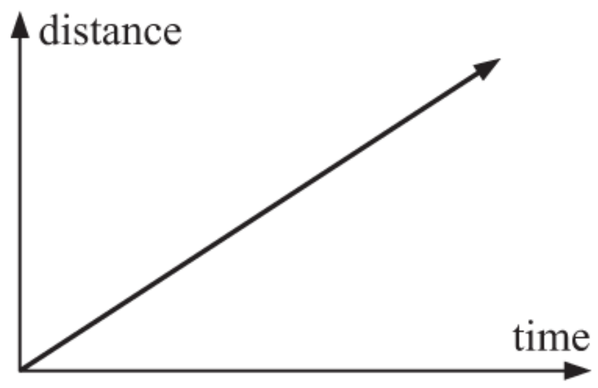
i How can we measure the *average rate* at which the depth increases from $t = 5$ to $t = 10$ seconds?

ii How can we measure the *instantaneous rate* at which the depth is increasing at the instant when $t = 8$ seconds?



If the relationship between two variables is a straight line, the gradient of the line tells us the rate at which one variable changes with respect to the other.

For example, in a travel graph showing distance against time, the gradient of a straight line segment tells us the *speed* of the object.



In the real world, rates such as speed are constantly changing. Their travel graphs are curves rather than straight lines. In order to calculate the instantaneous speed of an object, we need a branch of mathematics called **differential calculus**.

Differential calculus deals with **rates of change**. It has widespread applications in science, engineering, and finance.

A RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by rates. For example:

- Sir Donald Bradman's average batting rate at Test cricket level was 99.94 runs per innings.
- Michael Jordan's average basketball scoring rate was 30.1 points per game.
- Rangi's average typing rate is 63 words per minute with an error rate of 2.3 errors per page.

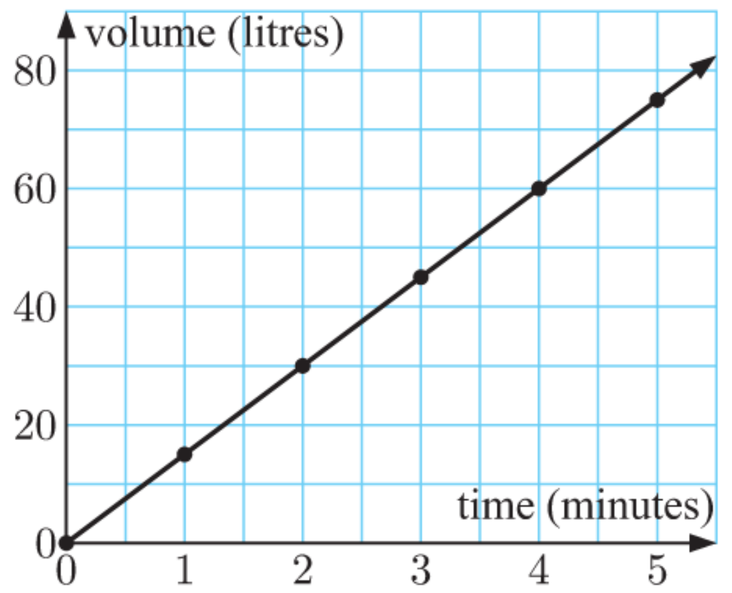
CONSTANT RATES OF CHANGE

Suppose water from a hose is used to fill a swimming pool. The volume of water in the pool is recorded at 1 minute intervals in the table alongside.

Time (minutes)	0	1	2	3	4	5
Volume (litres)	0	15	30	45	60	75

+15 +15 +15 +15 +15

Notice that the volume of water increases by the same amount each time interval. The rate of change in volume is constant, so the graph of *volume* against *time* is a straight line.



The **gradient** of the line gives the rate of change:

$$\text{rate of flow} = \frac{15 - 0}{1 - 0}$$
$$= 15 \text{ litres per minute}$$

EXERCISE 17A.1

1 The table alongside shows the distance travelled by a jogger at 30-second intervals.

Time (seconds)	0	30	60	90	120	150
Distance (metres)	0	90	180	270	360	450

a Is the jogger travelling at a constant speed? Explain your answer.

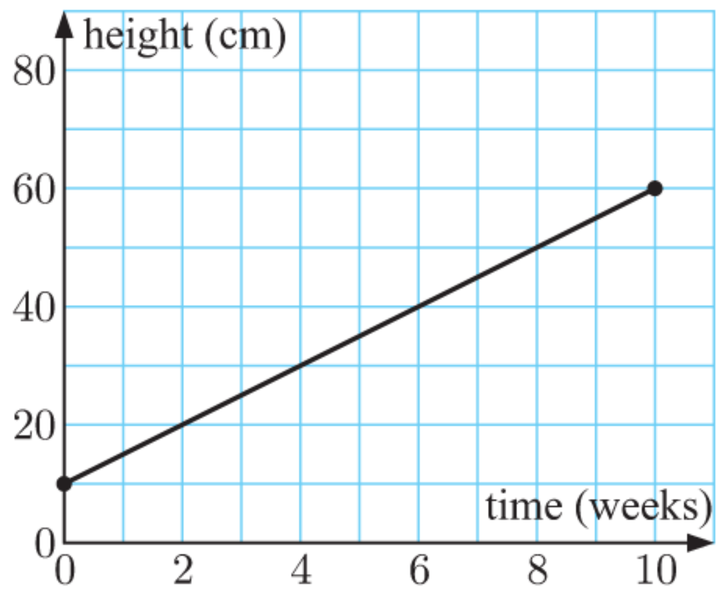
b Draw the graph of distance against time.

c Find the speed of the jogger in metres per second.

2 This graph shows the height of a seedling during its first 10 weeks.

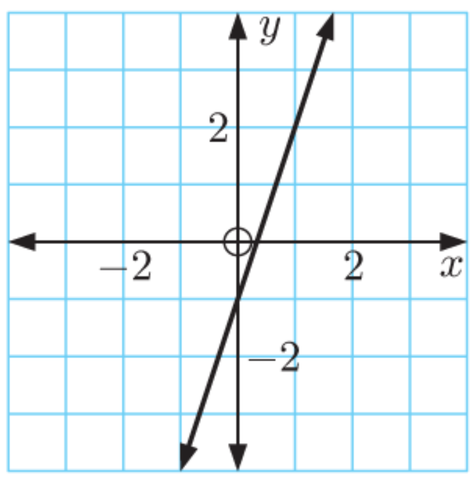
a Is the height changing at a constant rate? Explain your answer.

b Find the rate of change in height.

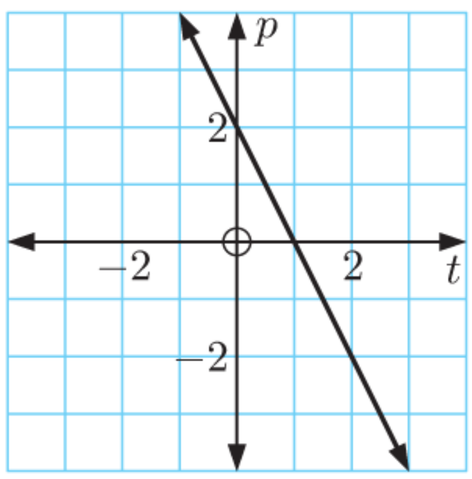


3 Find the rate of change for each function. Do not include units in your answer.

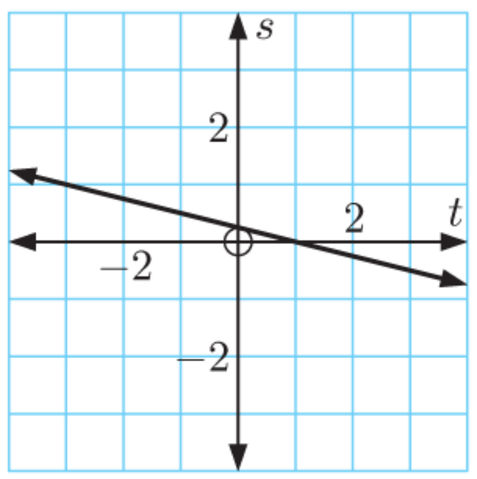
a



b



c

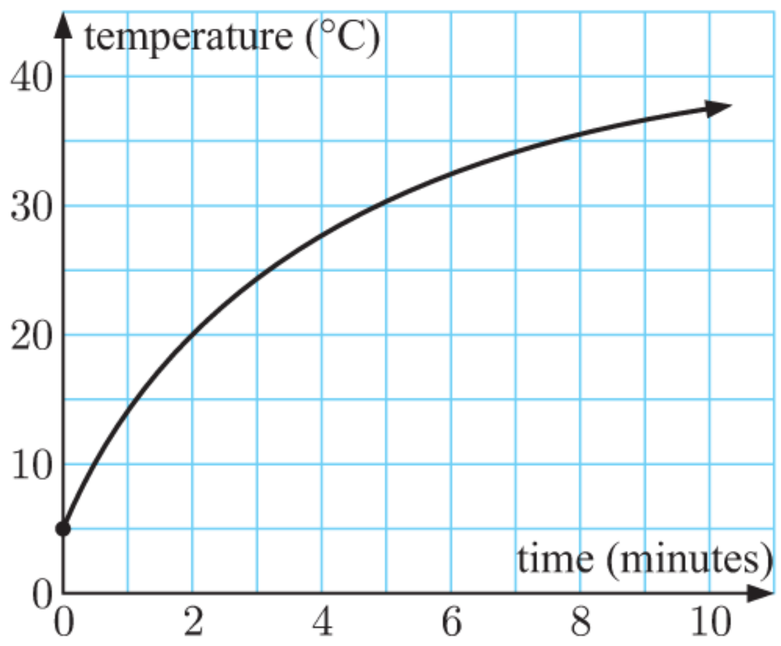


4 Find the rate of change for the function $f(x) = \frac{5}{2}x - 3$.

AVERAGE RATES OF CHANGE

In most real-world situations, rates of change are not constant. They vary over time.

For example, this graph shows the temperature of a glass of water which is left in the sun. The graph is not a straight line, which means the rate of change in temperature is not constant. The temperature increases quickly at first, and then more slowly as time goes by.



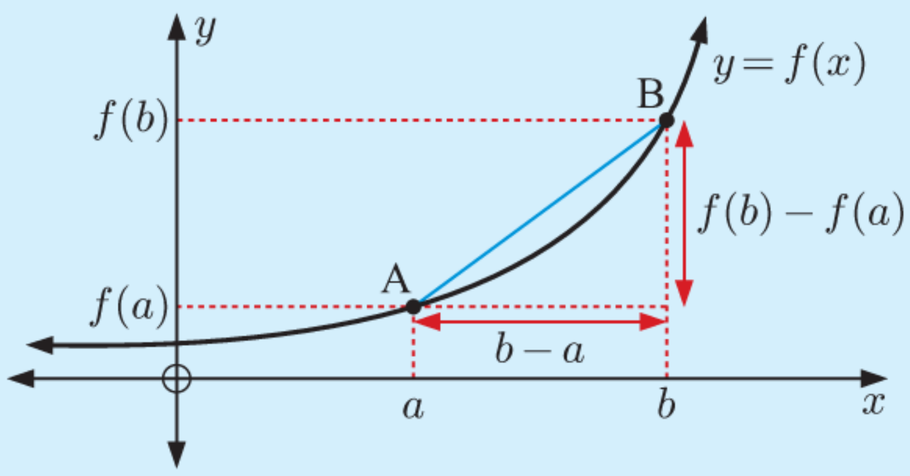
In such cases, we can find an **average rate of change** over a particular time interval.

For example, from time $t = 0$ to $t = 2$ minutes, the temperature increases from 5°C to 20°C. So, the average rate of change is $\frac{20-5}{2-0} = 7.5$°C per minute.

In the context of functions, we say that:

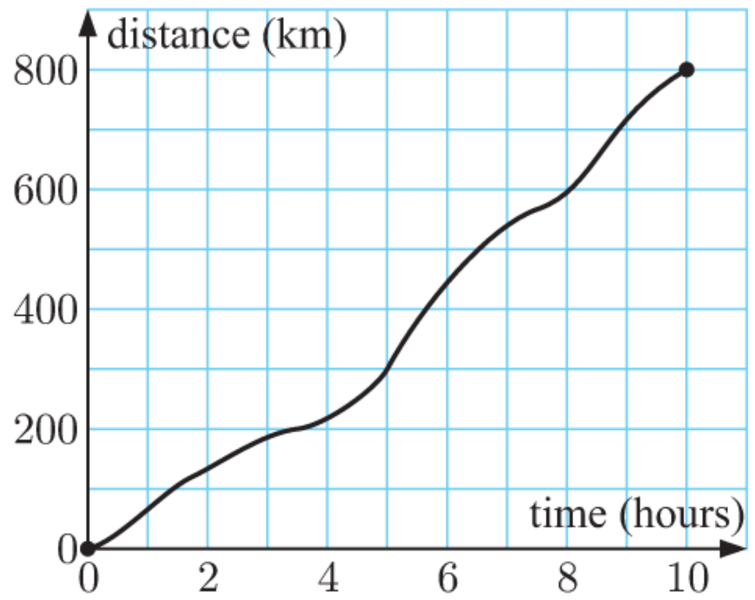
The **average rate of change** in $f(x)$ from $x = a$ to $x = b$ is $\dfrac{f(b) - f(a)}{b - a}$.

This is the **gradient of the chord [AB]**.



EXERCISE 17A.2

1 Aileen is driving from Amsterdam to Zurich. This graph shows the distance travelled against time.

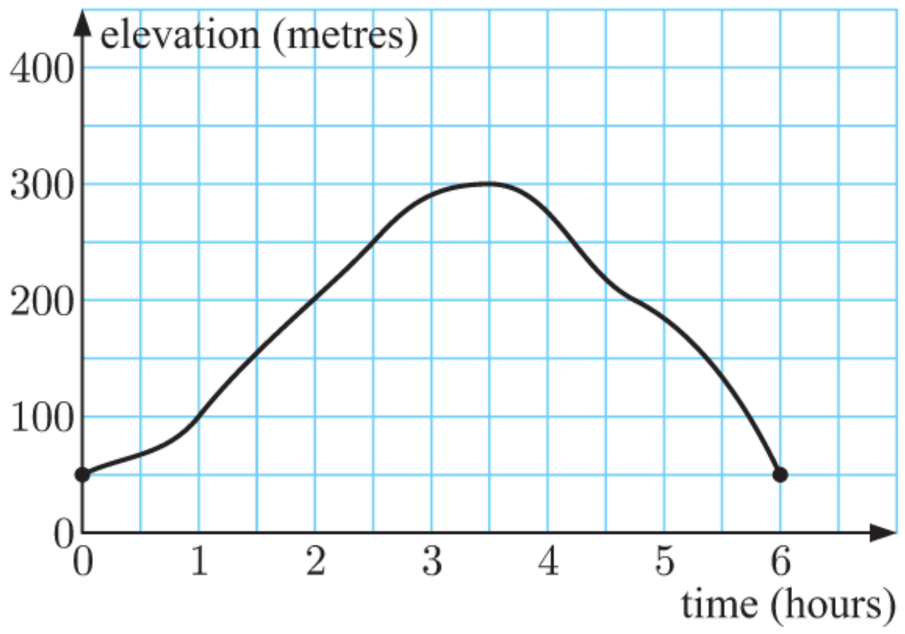


a Did Aileen travel at constant speed? Explain your answer.

b Find Aileen's average speed for:

i the first 5 hours **ii** the final 5 hours.

2 Chris went hiking in the mountains. His elevation above sea level is shown on this graph.



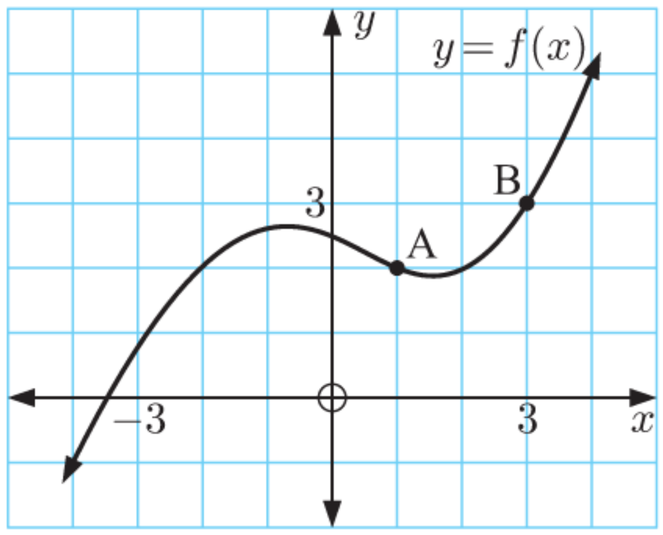
Find Chris' average rate of change in elevation from:

a $t = 1$ hour to $t = 2.5$ hours

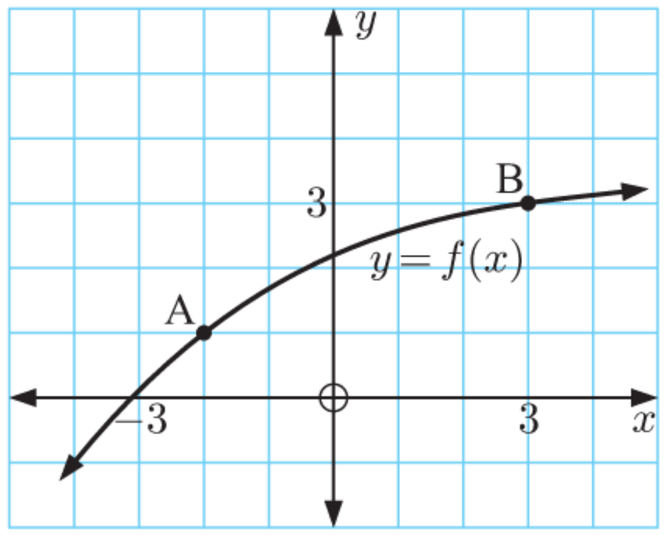
b $t = 3.5$ hours to $t = 6$ hours.

3 For each function, find the average rate of change in $f(x)$ from A to B:

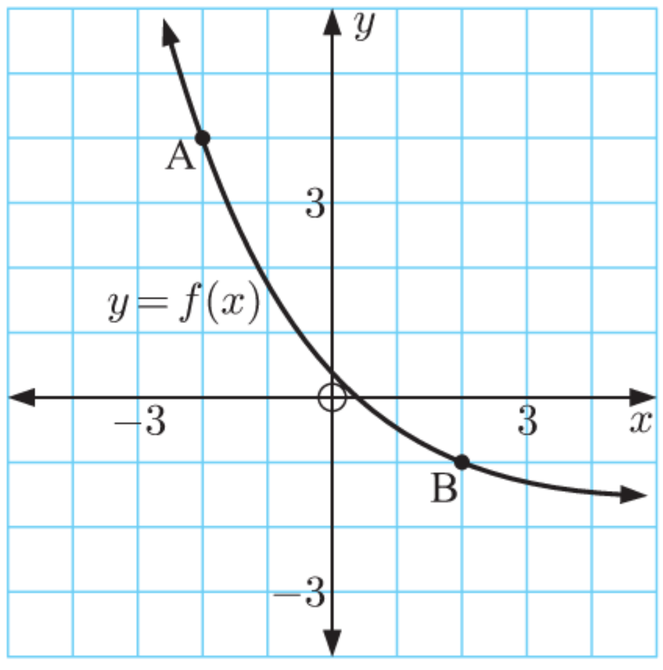
a



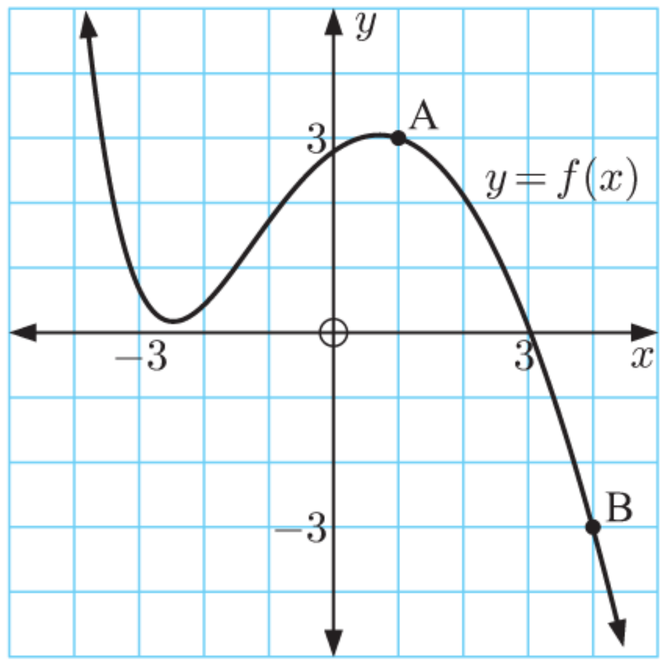
b



c



d

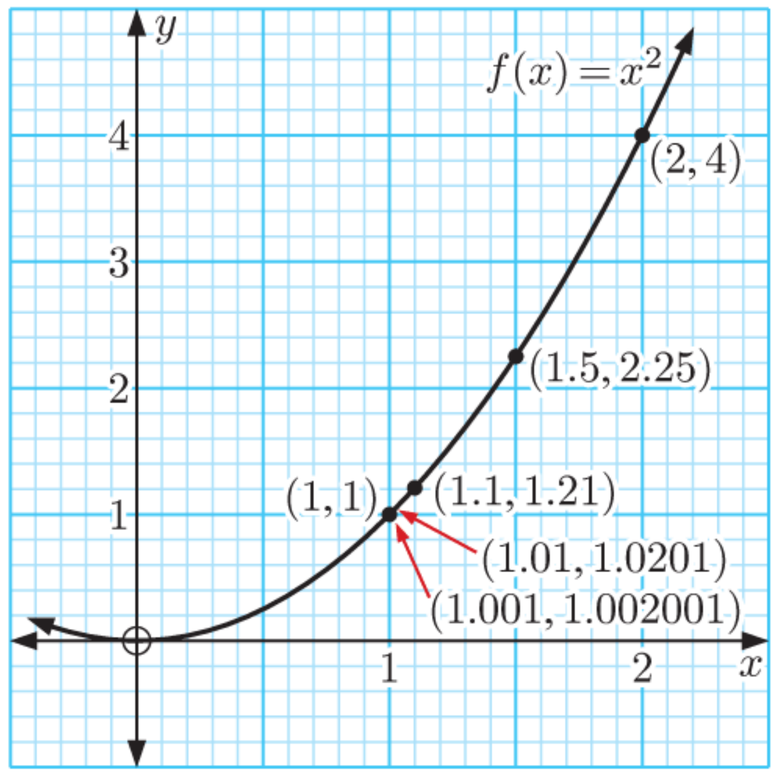


4 Consider the graph of $f(x) = x^2$.

a Find the average rate of change in $f(x)$ from:

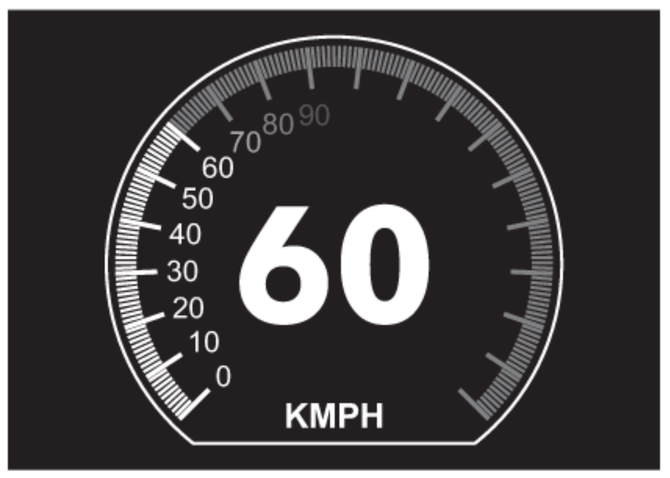
i $x = 1$ to $x = 2$
ii $x = 1$ to $x = 1.5$
iii $x = 1$ to $x = 1.1$
iv $x = 1$ to $x = 1.01$
v $x = 1$ to $x = 1.001$

b Comment on your answers in **a**.



B INSTANTANEOUS RATES OF CHANGE

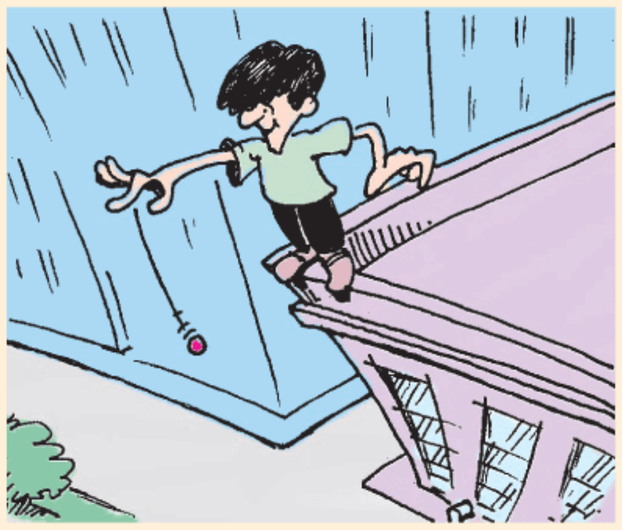
Suppose the speedometer in a car indicates that you are travelling at 60 km per hour. This is not an average speed, but an *instantaneous speed*. It is the speed at which you are travelling at that particular instant.

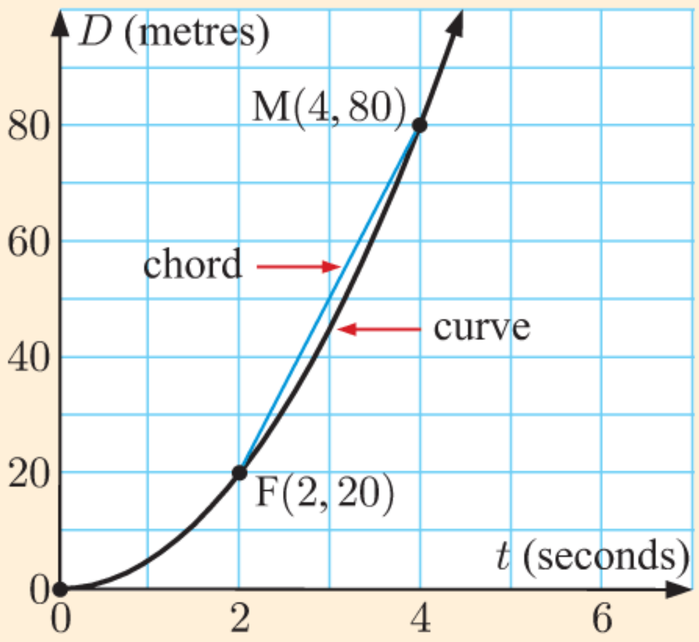


INVESTIGATION 1 INSTANTANEOUS SPEED

A ball bearing is dropped from the top of a tall building. The distance D it has fallen after t seconds is recorded, and the following graph of distance against time is obtained.

Consider a fixed point F on the curve when $t = 2$ seconds. We now choose another point M on the curve, and draw the line segment or **chord** [FM] between the two points. To start with, let M be the point when $t = 4$ seconds.





The *average* speed of the ball bearing over the time interval $2 \leqslant t \leqslant 4$

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80 - 20)\text{ m}}{(4 - 2)\text{ s}}$$

$$= \tfrac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

In this Investigation we will try to measure the *instantaneous* speed of the ball bearing when $t = 2$ seconds.

What to do:

1 Click on the icon to start the demonstration.
The gradient of the chord [FM] is shown. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see that the average speed is 30 m s^{-1}.



2 Click on M and drag it slowly towards F. Copy and complete the table alongside, where M corresponds to the given value of t.

t	gradient of [FM]
4	30
3	
2.5	
2.1	
2.01	

3 Observe what happens as M reaches F. Explain why this is so.

4 What do you suspect is the instantaneous speed of the ball bearing when $t = 2$ seconds? Explain your answer.

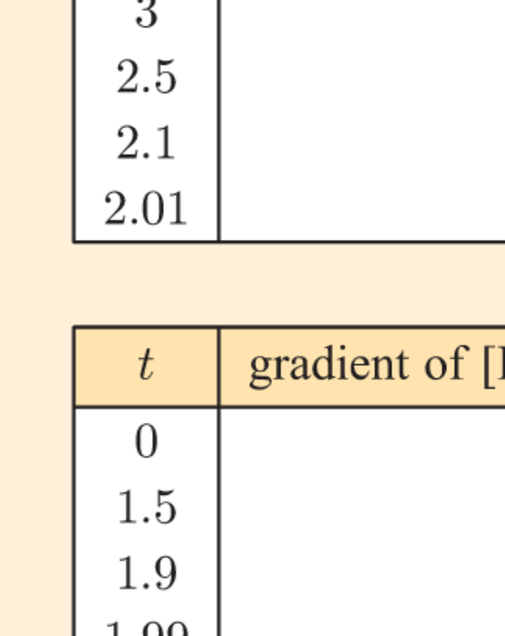
5 Move M to the origin, and then slide it towards F from the left. Copy and complete the table, where M again corresponds to the given value of t.

t	gradient of [FM]
0	
1.5	
1.9	
1.99	

6 Do your results agree with those in **4**?

If A and B are two points on a function, the gradient of the chord [AB] is the average rate of change between these points.

From the **Investigation**, if we let B get closer and closer to A, then the average rate of change from A to B will approach the *instantaneous* rate of change at A.

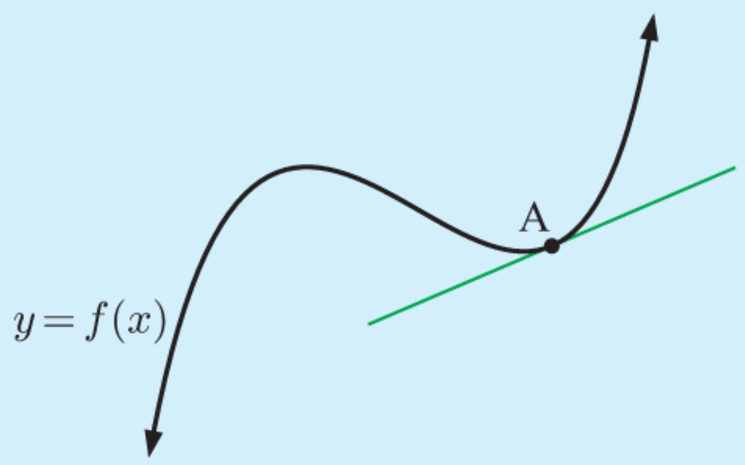


However, as B gets closer to A, the chord [AB] approaches the line which *touches* the curve at A. This line is called the **tangent** to the curve at A.

In particular, as B approaches A, the gradient of [AB] approaches the gradient of the tangent at A.



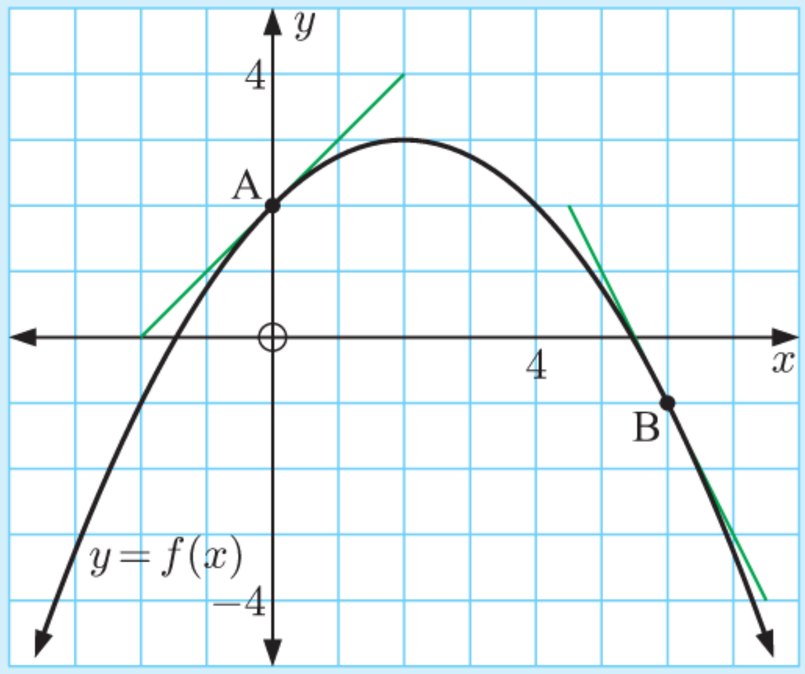
The **instantaneous rate of change** in $f(x)$ at any point A on the curve is the **gradient of the tangent** at A.



Example 1 **Self Tutor**

Use the tangents drawn to find the instantaneous rate of change in $y = f(x)$ at:

a A **b** B



a The tangent at A has gradient 1.
$\therefore$ the instantaneous rate of change at A is 1.

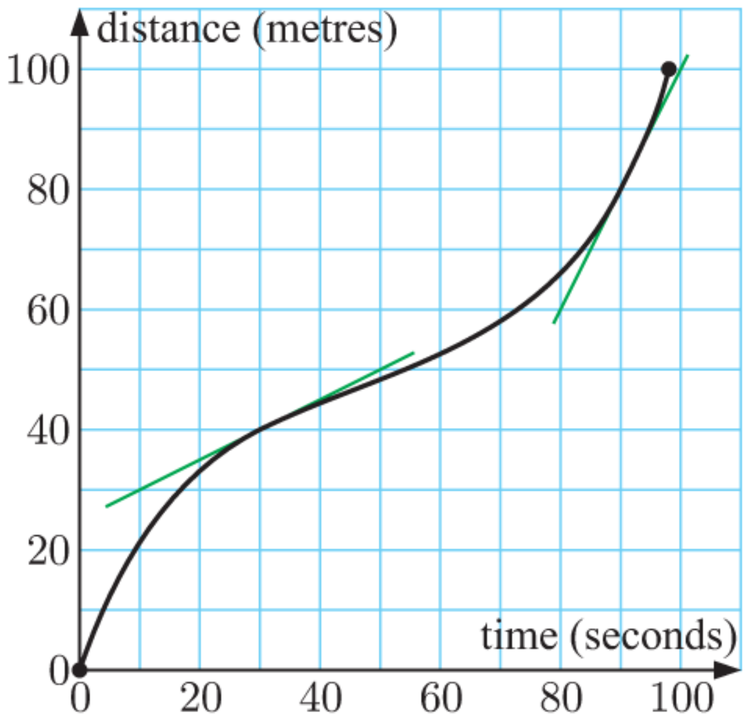
b The tangent at B has gradient -2.
$\therefore$ the instantaneous rate of change at B is -2.

EXERCISE 17B

1 This graph shows the distance travelled by a swimmer in a pool.

Use the tangents drawn to find the swimmer's instantaneous speed after:

a 30 seconds **b** 90 seconds.



2 Water is leaking from a tank. The volume of water left in the tank over time is shown on the graph alongside.

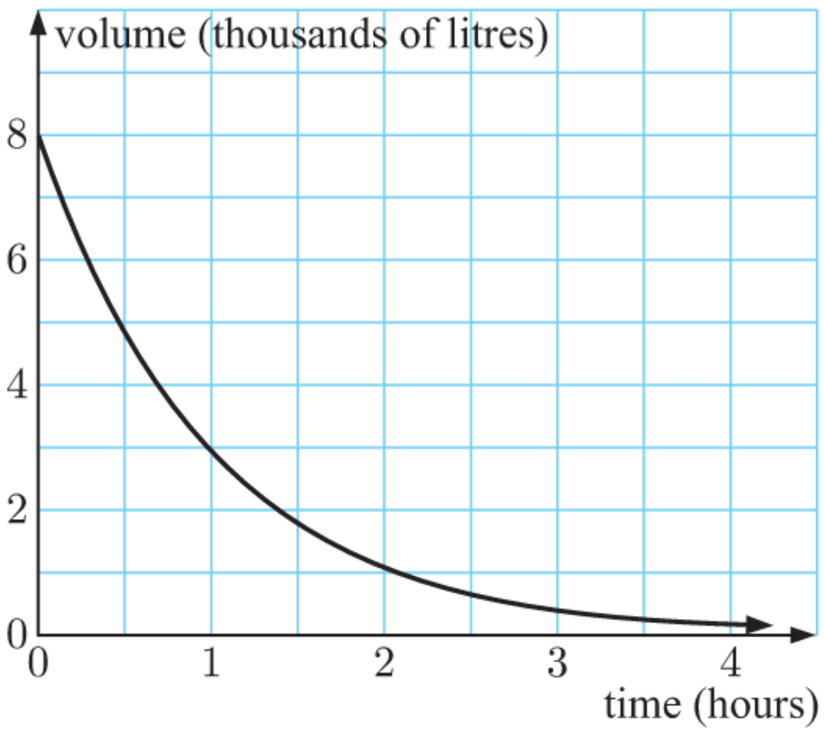
a How much water was in the tank originally?

b How much water was in the tank after 1 hour?

c How quickly was the tank losing water initially?

d How quickly was the tank losing water after 1 hour?

e Describe what happens to the rate at which the water is leaking.



3 **a** Draw an accurate graph of $y = x^2$ on a fine grid.

b Draw, as accurately as possible, the tangent to $y = x^2$ at $x = -1$.

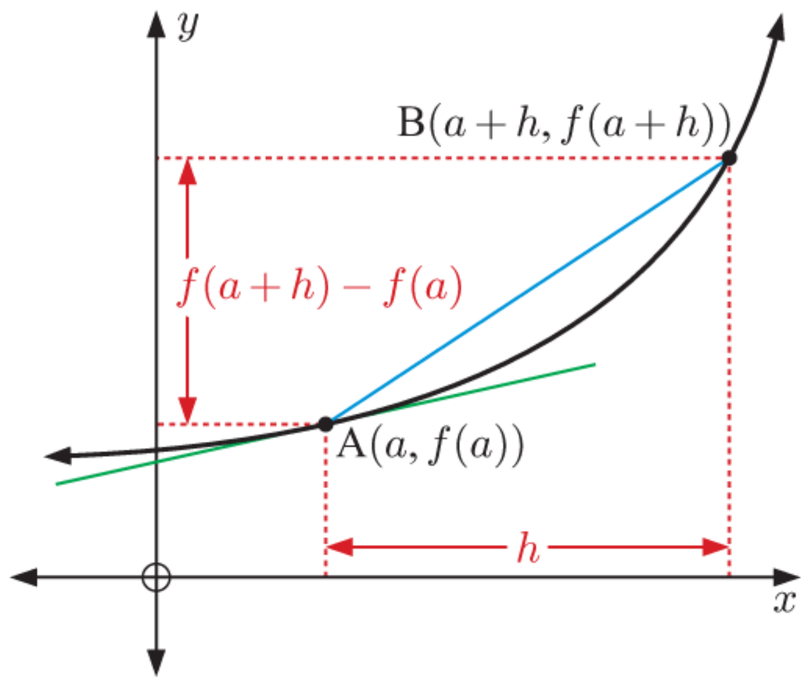
c Hence find the instantaneous rate of change in $y = x^2$ when $x = -1$.

PRINTABLE GRAPH

C LIMITS

Drawing a tangent on a graph and measuring its gradient can be time-consuming and inaccurate. We therefore seek a more efficient and accurate method for finding the gradient of a tangent.

We cannot find the gradient of the tangent at point A by direct calculation, because we only know one point on the tangent. However, if B is another point on the function $y = f(x)$, the gradient of the chord [AB] is $\dfrac{f(a+h) - f(a)}{h}$.



To calculate the gradient of the tangent at A, we let the point B get closer and closer to A. This means that the horizontal step h becomes infinitely small.

To understand what happens when this occurs, we use a mathematical principle called **limits**.

The following definition of a limit is informal but adequate for the purposes of this course:

If $f(x)$ is as close as we like to some real number A for all x sufficiently close to (but not equal to) a, then we say that $f(x)$ has a **limit** of A as x approaches a, and we write $\lim\limits_{x \to a} f(x) = A$.

In this case, $f(x)$ is said to **converge** to A as x approaches a.

Notice that the limit is defined for x close to but *not equal to* a. Whether the function f is defined or not at $x = a$ is not important to the definition of the limit of f as x approaches a. What *is* important is the behaviour of the function as x gets *very close to* a.

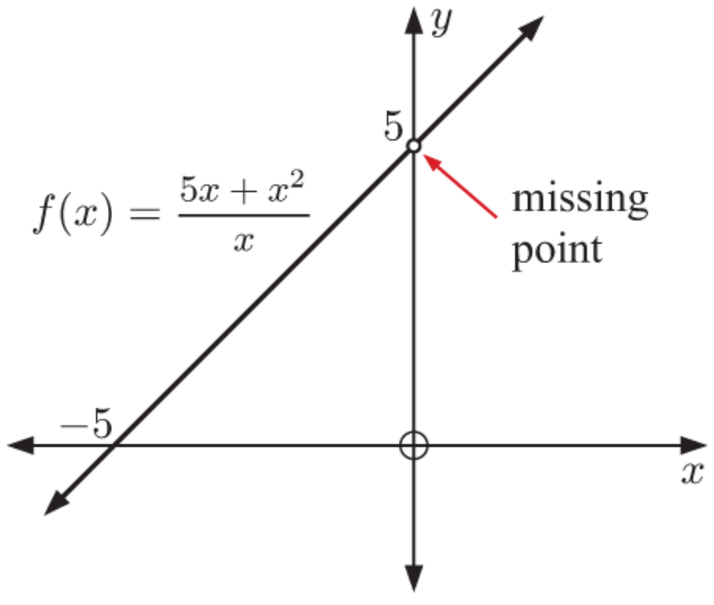
For example, suppose we wish to find the limit of $f(x) = \dfrac{5x + x^2}{x}$ as $x \to 0$.

It is tempting for us to simply substitute $x = 0$ into $f(x)$. However, in doing this, not only do we get the meaningless value of $\frac{0}{0}$, but also we ignore the basic limit definition.

Instead, observe that if $f(x) = \dfrac{5x + x^2}{x} = \dfrac{x(5 + x)}{x}$

$$\text{then} \quad f(x) = \begin{cases} 5 + x & \text{if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$$

The graph of $y = f(x)$ is therefore the straight line $y = x + 5$ with the point $(0, 5)$ missing, called a **point of discontinuity** of the function.



However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does exist. In particular, as $x \to 0$ from either direction, $f(x) \to 5$.

We write $\lim\limits_{x \to 0} \dfrac{5x + x^2}{x} = 5$ which reads:

"the limit as x approaches 0, of $\dfrac{5x + x^2}{x}$, is 5".

In practice we do not need to graph functions each time to determine limits, and most can be found algebraically.

Example 2 Self Tutor

Evaluate:

a $\lim\limits_{x \to 2} x^2$ **b** $\lim\limits_{x \to 0} \dfrac{x^2 + 3x}{x}$ **c** $\lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

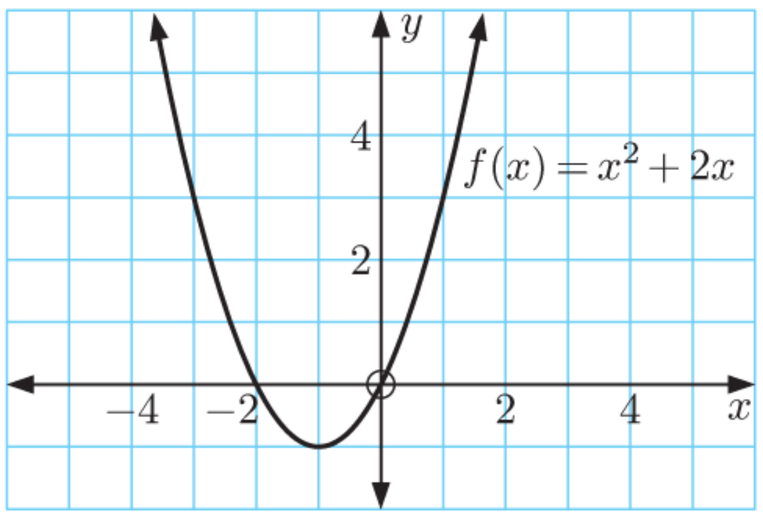
$$\therefore \lim_{x \to 2} x^2 = 4.$$

b
$$\lim_{x \to 0} \frac{x^2 + 3x}{x}$$
$$= \lim_{x \to 0} \frac{x(x+3)}{x}$$
$$= \lim_{x \to 0} (x+3) \quad \{\text{since } x \neq 0\}$$
$$= 3 \quad \{\text{as } x \to 0,\ x+3 \to 3\}$$

c
$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$
$$= \lim_{x \to 3} \frac{(x+3)(x-3)}{x-3}$$
$$= \lim_{x \to 3} (x+3) \quad \{\text{since } x \neq 3\}$$
$$= 6 \quad \{\text{as } x \to 3,\ x+3 \to 6\}$$

EXERCISE 17C.1

1 The graph of $f(x) = x^2 + 2x$ is shown alongside.



a Discuss the behaviour of $f(x)$ as $x \to 1$.

b Hence find $\lim\limits_{x \to 1} (x^2 + 2x)$.

c Find:

i $\lim\limits_{x \to -2} (x^2 + 2x)$ **ii** $\lim\limits_{x \to -1} (x^2 + 2x)$

2 Evaluate:

a $\lim\limits_{x \to 3} (x + 4)$ **b** $\lim\limits_{x \to -1} (5 - 2x)$ **c** $\lim\limits_{x \to 4} (3x - 1)$

d $\lim\limits_{x \to 2} (5x^2 - 3x + 2)$ **e** $\lim\limits_{h \to 0} h^2(1 - h)$ **f** $\lim\limits_{x \to 0} (x^2 + 5)$

3 Evaluate:

a $\lim\limits_{x \to 0} 5$ **b** $\lim\limits_{h \to 2} 7$ **c** $\lim\limits_{x \to 0} c$, c a constant

4 Evaluate:

a $\lim\limits_{x \to 1} \dfrac{x^2 - 3x}{x}$ **b** $\lim\limits_{h \to 2} \dfrac{h^2 + 5h}{h}$ **c** $\lim\limits_{x \to 0} \dfrac{x - 1}{x + 1}$

5 Explain why $\lim\limits_{x \to 0} \dfrac{x}{x} = 1$, even though $\dfrac{0}{0}$ is meaningless.

6 **a** Copy and complete the tables of values alongside.

b Hence predict the value of $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$.

c Prove your result by evaluating $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ directly.

x	$\dfrac{x^2-4}{x-2}$
1.9	
1.99	
1.999	
1.9999	
1.999 99	

x	$\dfrac{x^2-4}{x-2}$
2.1	
2.01	
2.001	
2.0001	
2.000 01	

7 Evaluate:

a $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{x}$ **b** $\lim\limits_{x \to 0} \dfrac{x^2 + 5x}{x}$ **c** $\lim\limits_{x \to 0} \dfrac{2x^2 - x}{x}$

d $\lim\limits_{h \to 0} \dfrac{2h^2 + 6h}{h}$ **e** $\lim\limits_{h \to 0} \dfrac{3h^2 - 4h}{h}$ **f** $\lim\limits_{h \to 0} \dfrac{h^3 - 8h}{h}$

g $\lim\limits_{x \to 1} \dfrac{x^2 - x}{x - 1}$ **h** $\lim\limits_{x \to 2} \dfrac{x^2 - 2x}{x - 2}$ **i** $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

DISCUSSION

1 Do limits always exist?

2 Consider the graph of $f(x) = \dfrac{1}{x}$.

a What happens to the graph as $x \to 0$ from the: **i** left **ii** right?

b Does $f(x) = \dfrac{1}{x}$ have a limit as $x \to 0$?

THEORY OF KNOWLEDGE

The Greek philosopher Zeno of Elea lived in what is now southern Italy, in the 5th century BC. He is most famous for his paradoxes, which were recorded in Aristotle's work *Physics*.

The arrow paradox

"*If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.*"

This argument says that if we choose any particular instant in time, the arrow is motionless. Therefore, how does the arrow actually move?

The dichotomy paradox

"*That which is in locomotion must arrive at the half-way stage before it arrives at the goal.*"

If an object is to move a fixed distance then it must travel half that distance. Before it can travel a half the distance, it must travel a half *that* distance. With this process continuing indefinitely, motion is impossible.

Achilles and the tortoise

"*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point from which the pursued started, and so the slower must always hold a lead.*"

According to this principle, the athlete Achilles will never be able to catch the slow tortoise!

1 A paradox is a logical argument that leads to a contradiction or a situation which defies logic or reason. Can a paradox be the truth?

2 Are Zeno's paradoxes really paradoxes?

3 Are the three paradoxes essentially the same?

4 We know from experience that things *do* move, and that Achilles *would* catch the tortoise. Does that mean that logic has failed?

5 What do Zeno's paradoxes have to do with limits?

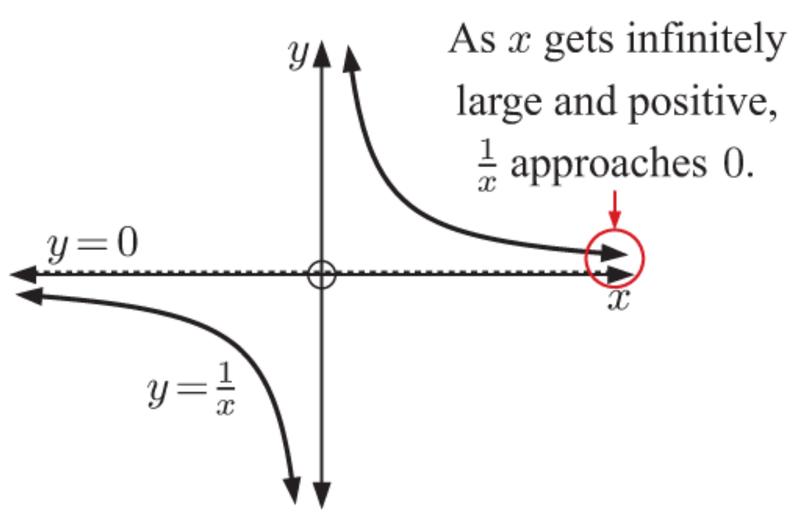
LIMITS AT INFINITY

We can use the idea of limits to discuss the behaviour of functions for extreme values of x.

We write $x \to \infty$ to mean "x tends to plus infinity" and $x \to -\infty$ to mean "x tends to minus infinity".

Notice that as $x \to \infty$, the value of $\frac{1}{x}$ gets very small. In fact, we can make $\frac{1}{x}$ as close to 0 as we like by making x large enough. This means that $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ even though $\frac{1}{x}$ never actually reaches 0.

We observe this on the graph of $y = \frac{1}{x}$ as the function tends towards its asymptote $y = 0$.



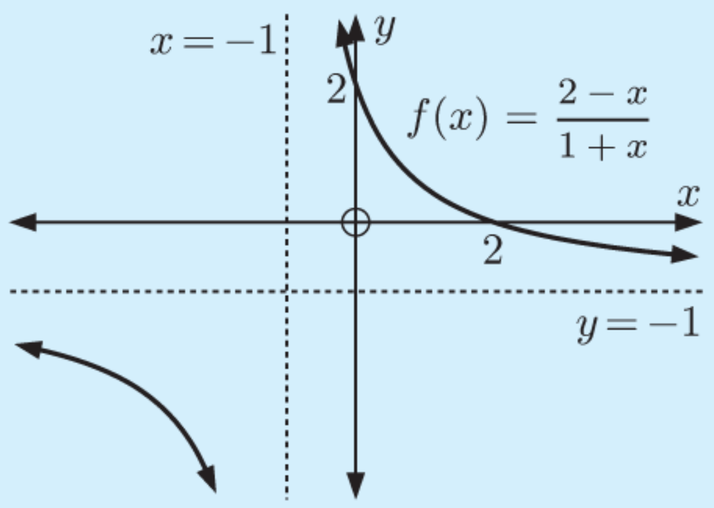
A function has a limit as $x \to \infty$ or as $x \to -\infty$ if the function is either a constant, or if it approaches a constant value. In the latter case, we observe this as a horizontal asymptote.

Example 3 — Self Tutor

a Discuss the behaviour of $f(x) = \dfrac{2-x}{1+x}$ near its asymptotes, and hence deduce their equations.

b If they exist, state the values of $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$.

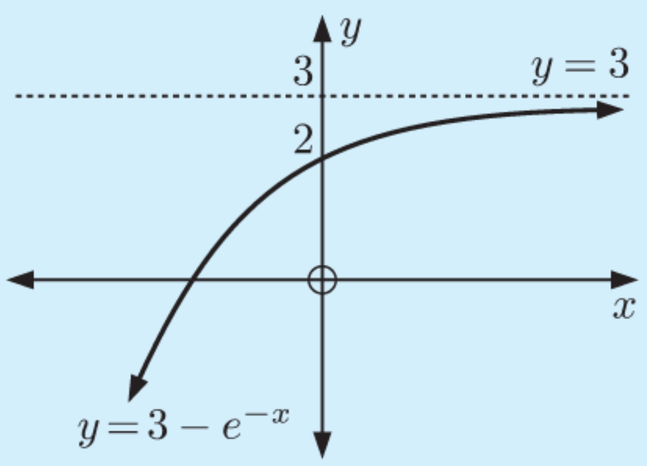
a As $x \to -1^-$, $f(x) \to -\infty$
As $x \to -1^+$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to -1^-$
As $x \to \infty$, $f(x) \to -1^+$
The vertical asymptote is $x = -1$.
The horizontal asymptote is $y = -1$.



b $\lim\limits_{x \to -\infty} f(x) = -1$ and $\lim\limits_{x \to \infty} f(x) = -1$.

Example 4 — Self Tutor

Find, if possible: **a** $\lim\limits_{x \to -\infty} (3 - e^{-x})$ **b** $\lim\limits_{x \to \infty} (3 - e^{-x})$.



a As $x \to -\infty$, $3 - e^{-x} \to -\infty$.
Since $3 - e^{-x}$ does not approach a finite value, $\lim\limits_{x \to -\infty} (3 - e^{-x})$ does not exist.

b As $x \to \infty$, $3 - e^{-x} \to 3^-$
$\therefore \lim\limits_{x \to \infty} (3 - e^{-x}) = 3$

EXERCISE 17C.2

1 For each of the following functions:

i Discuss the behaviour near its asymptotes, and hence deduce their equations.

ii If they exist, state the values of $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{3x-2}{x+3}$ **c** $f(x) = \dfrac{1-2x}{3x+2}$ **d** $f(x) = \dfrac{x}{1-x}$

2 **a** Sketch the graph of $y = e^x - 6$.

b Hence discuss the value and geometric interpretation of:

i $\lim_{x\to-\infty} (e^x - 6)$ **ii** $\lim_{x\to\infty} (e^x - 6)$

3 Find, if possible:

a $\lim_{x\to-\infty} (2e^{-x} - 3)$ **b** $\lim_{x\to\infty} (2e^{-x} - 3)$ **c** $\lim_{x\to-\infty} (30 + 2^{-x})$

d $\lim_{x\to\infty} (30 + 2^{-x})$ **e** $\lim_{x\to-\infty} ((0.4)^x + 10)$ **f** $\lim_{x\to\infty} ((0.4)^x + 10)$

4 **a** Copy and complete this table of values:

x	xe^{-x}
10	
50	
100	
200	

b Hence predict the value of $\lim_{x\to\infty} xe^{-x}$.

c Graph $y = xe^{-x}$ using technology. Does your graph support your prediction in **b**?

INVESTIGATION 2 LIMITS IN NUMBER SEQUENCES

The sequence 0.3, 0.33, 0.333, ... can be defined by the general term $x_n = 0.333....3$ where there are n 3s after the decimal point, $n \in \mathbb{Z}^+$.

What to do:

1 Copy and complete the table alongside:

n	x_n	$3x_n$	$1-3x_n$
1			
2			
3			
4			
5			
10			

2 Consider x_{100} which contains 100 3s. In the number $(1-3x_{100})$, how many 0s are there between the decimal point and the 1?

3 In the limit as n tends to infinity, x_n contains an increasingly large number of 3s. In the number $(1-3x_n)$, how many 0s are there between the decimal point and the 1?

4 Using your answer to **3**, state $\lim_{n\to\infty} (1-3x_n)$.

5 Hence state $\lim_{n\to\infty} x_n$, which is the exact value of $0.\overline{3}$.

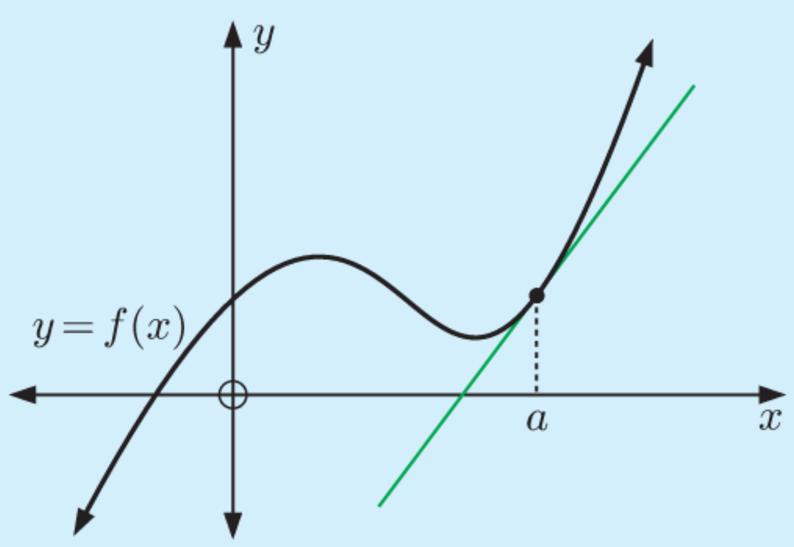
D THE GRADIENT OF A TANGENT

We have seen that for two points $A(a,\ f(a))$ and $B(a+h,\ f(a+h))$ on a function, the gradient of the chord [AB] is $\dfrac{f(a+h)-f(a)}{h}$.

Letting B get infinitely close to A is equivalent to taking the limit as $h \to 0$.

The **gradient of the tangent** to the curve $y = f(x)$ at the point where $x = a$ is

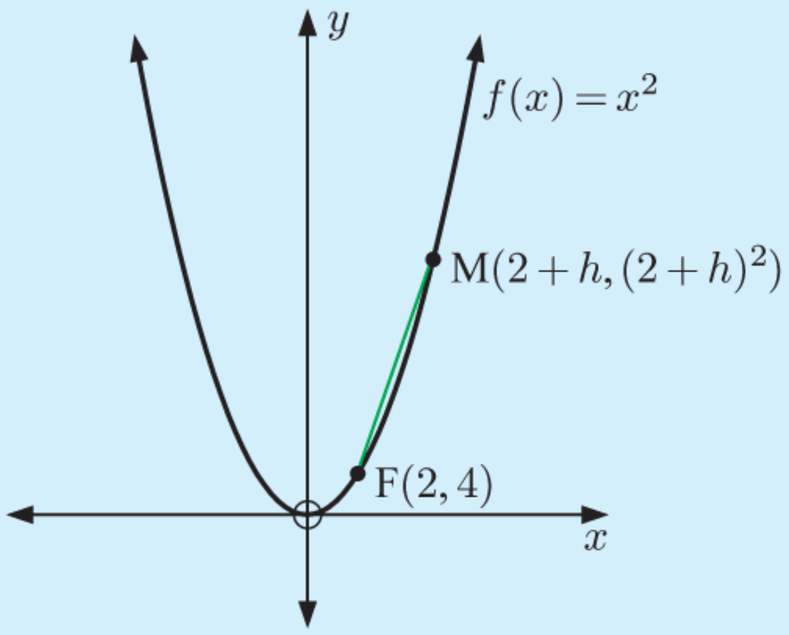
$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$



Example 5 — Self Tutor

Find the gradient of the tangent to $f(x) = x^2$ at the point $(2,\ 4)$.

Let F be the point $(2,\ 4)$. Suppose M has x-coordinate $2+h$ and also lies on the graph, so M is $(2+h,\ (2+h)^2)$.



The gradient of the tangent at F

$$= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{(2+h)^2-4}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{4}+4h+h^2-\cancel{4}}{h}$$

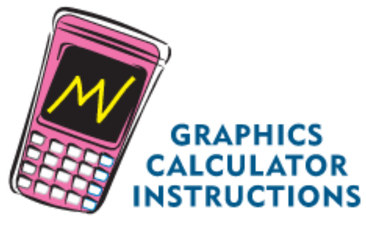
$$= \lim_{h\to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} (4+h) \qquad \{\text{as } h \neq 0\}$$

$$= 4$$

You can use technology to find the gradient of the tangent to a function at a given point, and hence check your answers.





EXERCISE 17D

1 F$(3,\ 9)$ lies on the graph of $f(x) = x^2$. M also lies on the graph, and has x-coordinate $3+h$.

a State the y-coordinate of M.

b Show that the gradient of the line segment [FM] is $6+h$.

c *Hence* find the gradient of [FM] if M has coordinates:

i $(4, 16)$ **ii** $(3.5, 12.25)$ **iii** $(3.1, 9.61)$ **iv** $(3.01, 9.0601)$

d Use limits to find the gradient of the tangent to $f(x) = x^2$ at the point $(3, 9)$.

2 **a** Find the gradient of the tangent to $f(x) = x^2$ at the point where:

i $x = 1$ **ii** $x = 4$

b Use **a** and other results from this Section to complete the table alongside for $f(x) = x^2$.

x-coordinate	*Gradient of tangent to* $f(x) = x^2$
1	
2	
3	
4	

c Predict the gradient of the tangent to $f(x) = x^2$ at the point where $x = a$.

3 Find the gradient of the tangent to:

a $f(x) = 2x^2$ at the point $(1, 2)$ **b** $f(x) = x^2 + x$ at the point $(2, 6)$

c $f(x) = 2x - x^2$ when $x = 1$ **d** $f(x) = x^2 - 3x$ when $x = 0$.

4 **a** Use $(1 + h)^3 = (1 + h)^2(1 + h)$ to expand $(1 + h)^3$ completely.

b F$(1, 1)$ lies on the graph of $y = x^3$. M also lies on the graph, and has x-coordinate $1 + h$.

i Show that the gradient of the chord [FM] is given by $\dfrac{3h + 3h^2 + h^3}{h}$.

ii Find the gradient of the tangent to $f(x) = x^3$ at the point $(1, 1)$.

5 **a** Show that $\dfrac{1}{x+h} - \dfrac{1}{x} = \dfrac{-h}{x(x+h)}$.

b Find the gradient of the tangent to $y = \dfrac{1}{x}$ at the point where:

i $x = 1$ **ii** $x = 3$

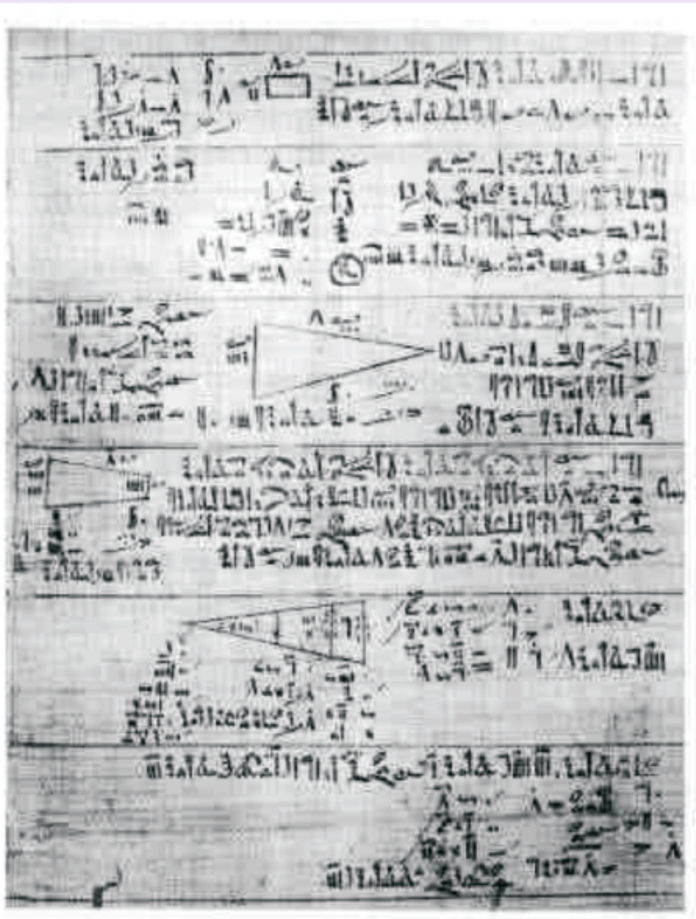
HISTORICAL NOTE

The word "calculus" is a Latin word referring to the small pebbles the ancient Romans used for counting.

The first known description of calculus is found on the **Egyptian Moscow papyrus** from about 1850 BC. Here, it was used to calculate areas and volumes.

Ancient Greek mathematicians such as **Democritus** and **Eudoxus** developed these ideas further by dividing objects into an infinite number of sections. This led to the study of **infinitesimals**, and allowed **Archimedes of Syracuse** to find the tangent to a curve other than a circle.

The methods of Archimedes were the foundation for modern calculus developed almost 2000 years later by mathematicians such as **Johann Bernoulli** and **Isaac Barrow**.

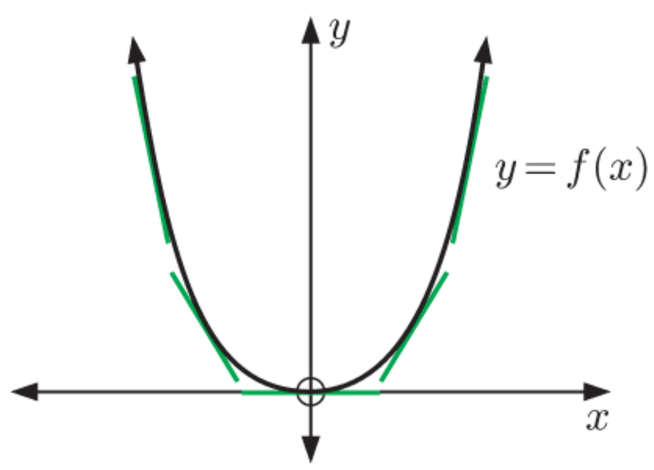


Egyptian Moscow papyrus

E THE DERIVATIVE FUNCTION

For a non-linear function $y = f(x)$, the gradient of the tangent changes as we move along the curve.

We can therefore write a **gradient function** which gives the gradient of the tangent for any given value of x.

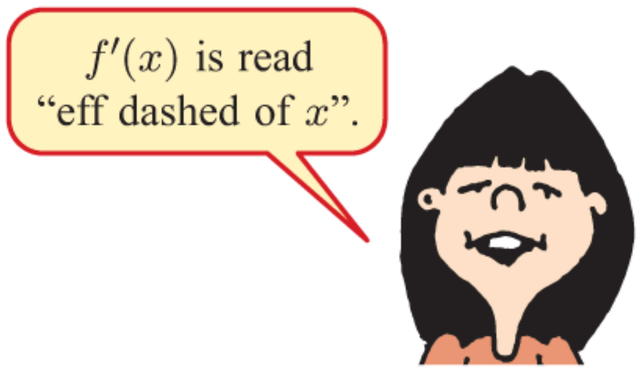


The gradient function of $y = f(x)$ is called its **derivative function** and is labelled $f'(x)$.

For example, in question **2** of the previous Exercise, you should have observed that for $f(x) = x^2$, the gradient of the tangent is always double the x-coordinate.

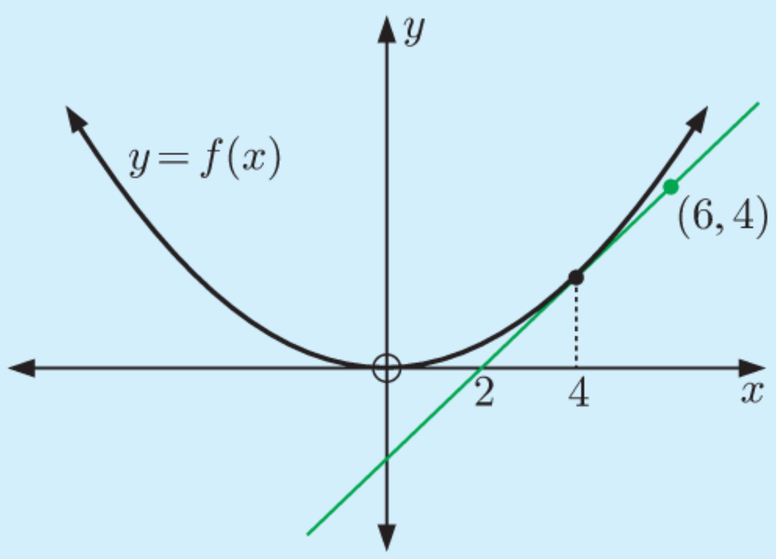
So, for $f(x) = x^2$ we write $f'(x) = 2x$.

Since $f'(3) = 6$, the gradient of the tangent to $f(x) = x^2$ at the point where $x = 3$, is 6.



Example 6 **Self Tutor**

For the given graph, find $f'(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The tangent passes through $(2, 0)$ and $(6, 4)$.

$$\therefore \quad f'(4) = \text{gradient of the tangent}$$
$$= \frac{4 - 0}{6 - 2}$$
$$= 1$$

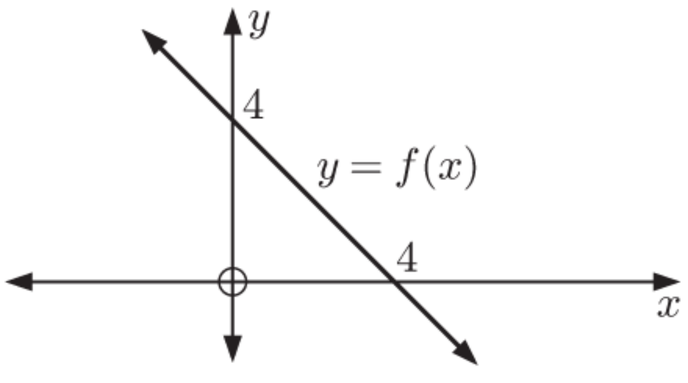
If we are given y written as a function of x, we often write the derivative function as $\frac{dy}{dx}$. This is called “the derivative of y with respect to x”, and is read “dee y by dee x”.

For example, for $y = x^2$ we have $\frac{dy}{dx} = 2x$.

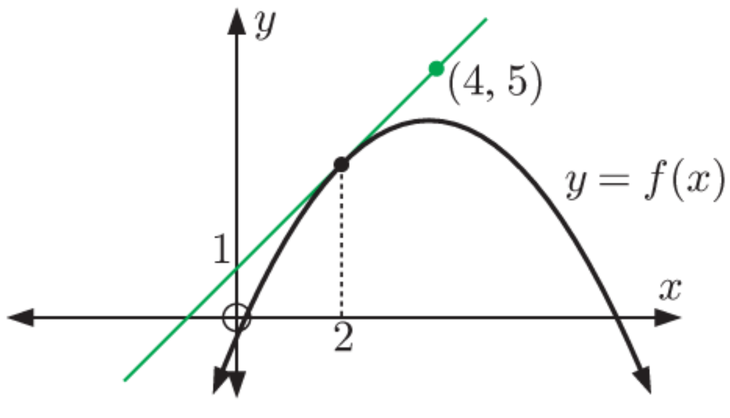
EXERCISE 17E

1 Using the graph below, find:

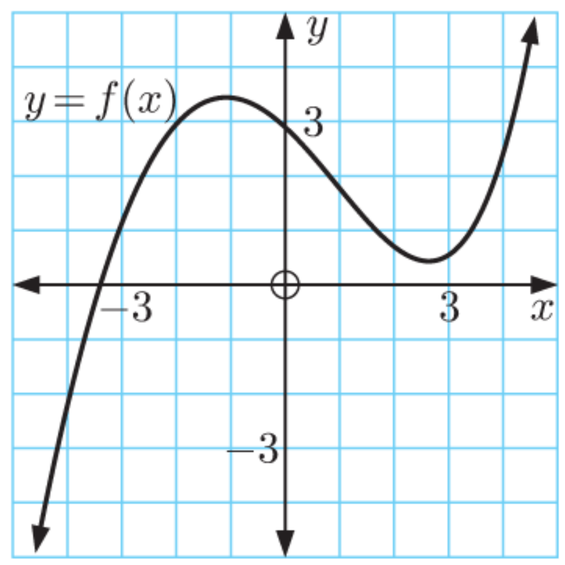
a $f(0)$ **b** $f'(0)$



2 Use the graph below to find $f'(2)$.



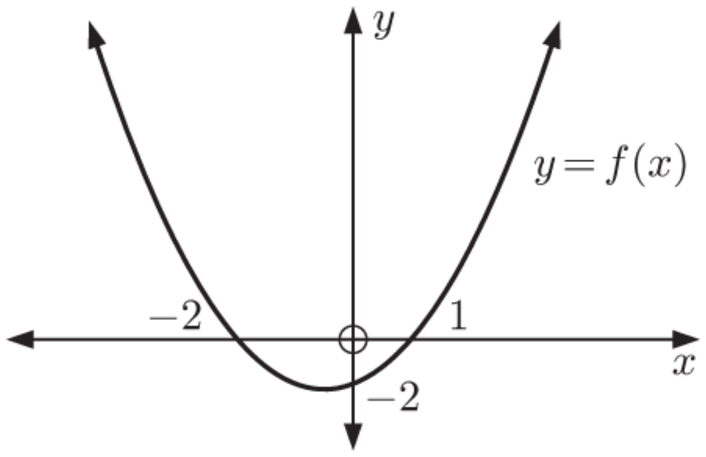
3 For the graph of $y = f(x)$ alongside, decide whether the following are positive or negative:



a $f(3)$ **b** $f'(1)$

c $f(-4)$ **d** $f'(-2)$

4



For the graph of $y = f(x)$ alongside, the derivative function is $f'(x) = 2x + 1$.

a Find and interpret: **i** $f'(-2)$ **ii** $f'(0)$

b Copy the graph, and include the information in **a**.

INVESTIGATION 3 GRADIENT FUNCTIONS

The software, accessible using the icon alongside, can be used to find the gradient of the tangent to a function $f(x)$ at any point. By sliding the point along the graph, we observe the changing gradient of the tangent and hence generate the gradient function $f'(x)$.



What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$.

a For each of these functions, what is the gradient?

b Is the gradient constant for all values of x?

2 Consider the function $f(x) = mx + c$.

a State the gradient of the function.

b Is the gradient constant for all values of x?

c Use the software to graph the following functions and observe the gradient function $f'(x)$. Hence verify your answer to **b**.

i $f(x) = x - 1$ **ii** $f(x) = 3x + 2$ **iii** $f(x) = -2x + 1$

3 **a** Observe the function $f(x) = x^2$ using the software. What *type* of function is the gradient function $f'(x)$?

b Observe the following quadratic functions using the software:

i $f(x) = x^2 + x - 2$ **ii** $f(x) = 2x^2 - 3$

iii $f(x) = -x^2 + 2x - 1$ **iv** $f(x) = -3x^2 - 3x + 6$

c What *type* of function are each of the gradient functions $f'(x)$ in **b**?

4 **a** Observe the function $f(x) = e^x$ using the software.

b What is the gradient function $f'(x)$?

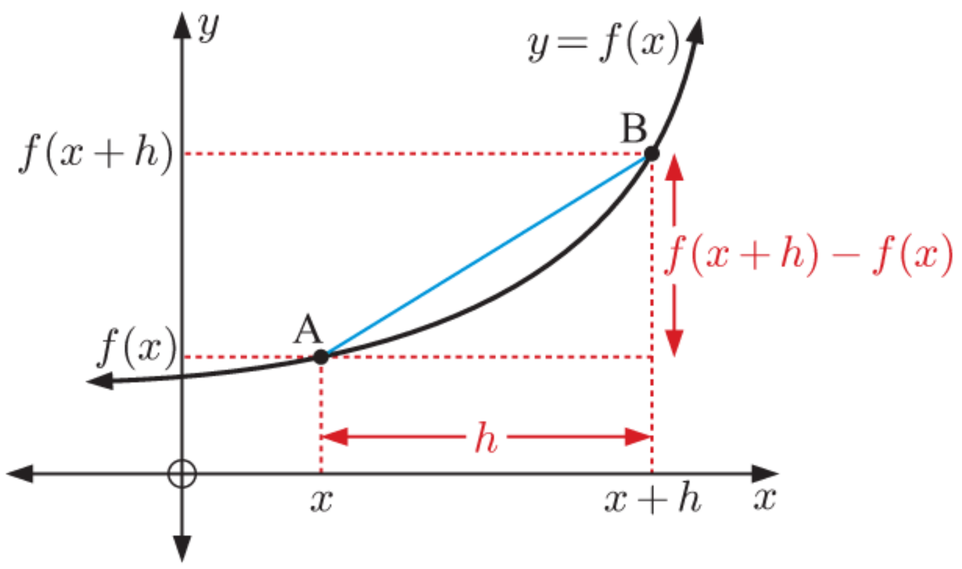
F DIFFERENTIATION FROM FIRST PRINCIPLES

To find the derivative function $f'(x)$ for a function $f(x)$, we use limit theory to find the gradient of the tangent to the curve at a general point $(x,\ f(x))$.

Let $A(x,\ f(x))$ and $B(x+h,\ f(x+h))$ be two points on the curve.

The chord [AB] has gradient $= \dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{f(x+h)-f(x)}{h}$



If we let B approach A, then the gradient of [AB] approaches the gradient of the tangent at A.

So, the gradient of the tangent at the general point $(x,\ f(x))$ is $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

Since there is at most one value of the gradient for each value of x, the formula is actually a function.

The **derivative function** or simply **derivative** of $y=f(x)$ is defined as

$$f'(x) \text{ or } \frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

The domain of $f'(x)$ is the set of values for which this limit exists.

When we evaluate this limit to find a derivative function, we say we are **differentiating from first principles**.

Example 7 **Self Tutor**

Use first principles to find the gradient function $f'(x)$ of $f(x)=x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{x^2}+2hx+h^2-\cancel{x^2}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$
$$= \lim_{h\to 0} (2x+h) \qquad \{\text{as } h\neq 0\}$$
$$= 2x$$

In this Example we prove the result observed in **Exercise 17D**.



EXERCISE 17F

1 For each function $f(x)$, find $f'(x)$ from first principles:

a $f(x) = 1$ **b** $f(x) = x$ **c** $f(x) = 2x - 1$ **d** $f(x) = 3 - x$

2 Find $\frac{dy}{dx}$ from first principles:

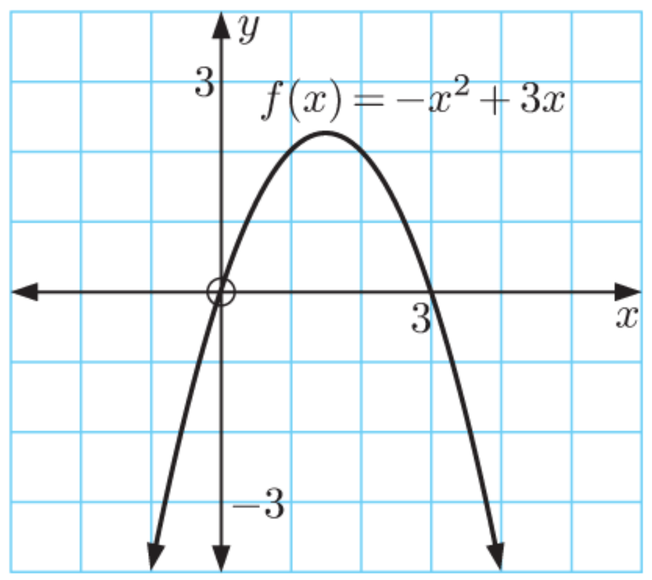
a $y = x^2 + 2$ **b** $y = 3 - x^2$ **c** $y = 2x^2 + x$ **d** $y = -x^2 + 5x - 3$

3 Let $f(x) = 3x^2 - 1$.

a Find $f'(x)$ from first principles.

b Hence find $f'(2)$ and interpret your answer.

4 The graph of $f(x) = -x^2 + 3x$ is shown alongside.



a Use the graph to estimate the gradient of the tangent to the curve at the point where:

i $x = 0$ **ii** $x = 2$.

b Find $f'(x)$ from first principles.

c Find $f'(0)$ and $f'(2)$, and hence check your estimates in **a**.

5 Suppose $f(x) = \frac{1}{2}x^2 - x - 2$.

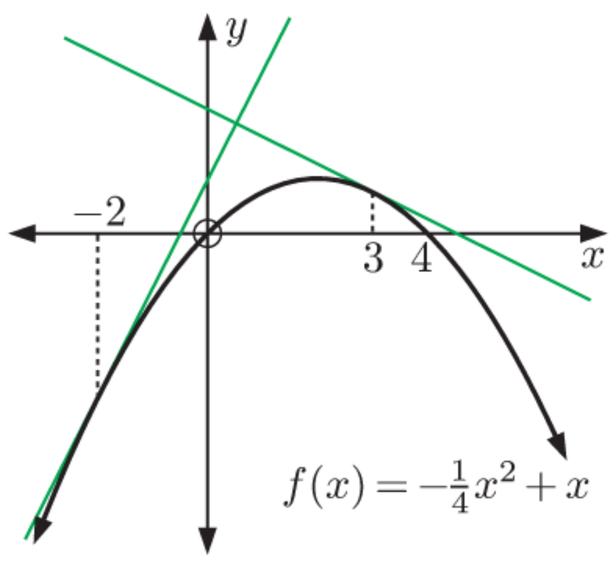
a Find $f'(x)$ from first principles.

b Hence find:

i the gradient of the tangent to $y = f(x)$ at the point where $x = -2$

ii the point at which the tangent to the curve has gradient 4.

6



The graph of $f(x) = -\frac{1}{4}x^2 + x$ is shown alongside.

a Find $f'(x)$.

b Hence show that the illustrated tangents are perpendicular.

7 Consider the function $y = x^3$.

a Use the software or your graphics calculator to find the gradient of the tangent when $x = -3, -2, -1, 0, 1, 2, 3$. Record your results in a table:





GRAPHICS CALCULATOR INSTRUCTIONS

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$							

b Show that $(x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$.

c Hence find $\frac{dy}{dx}$ from first principles. Check your answer using the results you found in **a**.

8 Consider the function $y = \frac{1}{x}$.

a Use technology to complete the following table of derivatives:

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$							

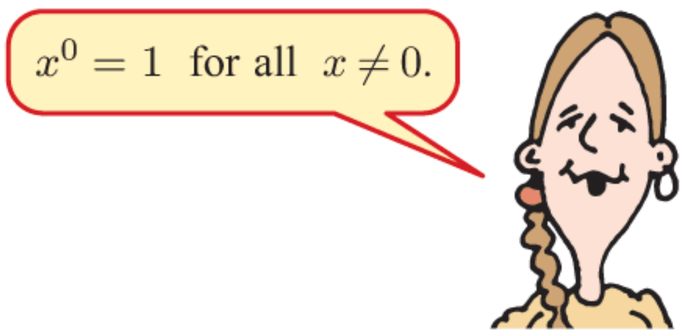
b Use the result from question **5 a** in **Exercise 17D** to find $\frac{dy}{dx}$ from first principles. Check your answer using the results you found in **a**.

9 **a** Use the previous results to copy and complete the table:

$f(x)$	$f'(x)$
x^1	
x^2	$2x$
x^3	
x^{-1}	
x^0	

b Copy and complete:

If $f(x) = x^n$, then $f'(x) =$ .....



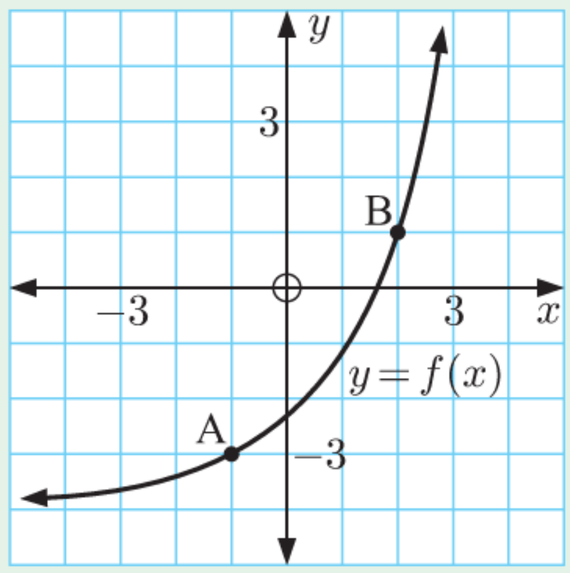
DISCUSSION

1 Does a function always have a derivative function?

2 Are the domains of a function and its derivative always the same?

REVIEW SET 17A

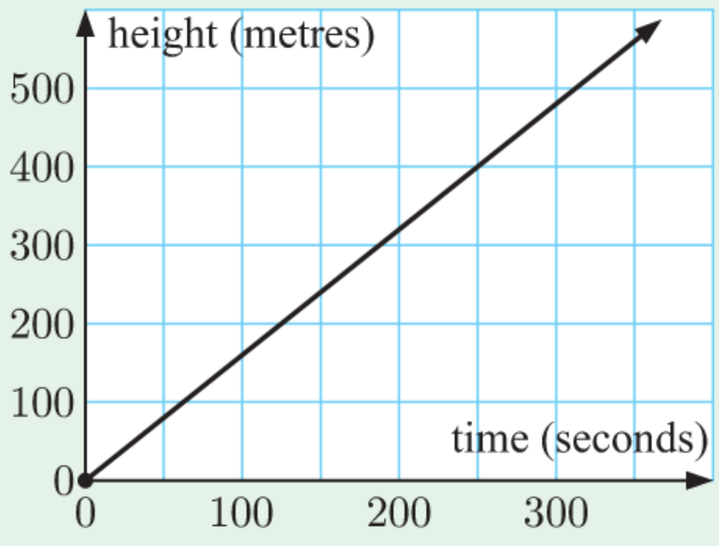
1 Find the average rate of change in $f(x)$ from A to B.



2 Chantelle is riding in a ski-lift. Her height above the base of the mountain is shown on the graph alongside.

a Is the ski-lift increasing in height at a constant rate? Explain your answer.

b Find the rate at which the ski-lift is increasing in height.



3 Evaluate:

a $\lim\limits_{x \to 1} (6x - 7)$ **b** $\lim\limits_{h \to 0} \dfrac{2h^2 - h}{h}$ **c** $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x - 4}$

4 **a** Sketch the graph of $y = \dfrac{2 + x}{x - 4}$.

b Discuss the behaviour of the graph near its asymptotes, and hence deduce their equations.

c State the values of $\lim\limits_{x \to -\infty} \dfrac{2 + x}{x - 4}$ and $\lim\limits_{x \to \infty} \dfrac{2 + x}{x - 4}$.

5 Consider $f(x) = 2x^2$.

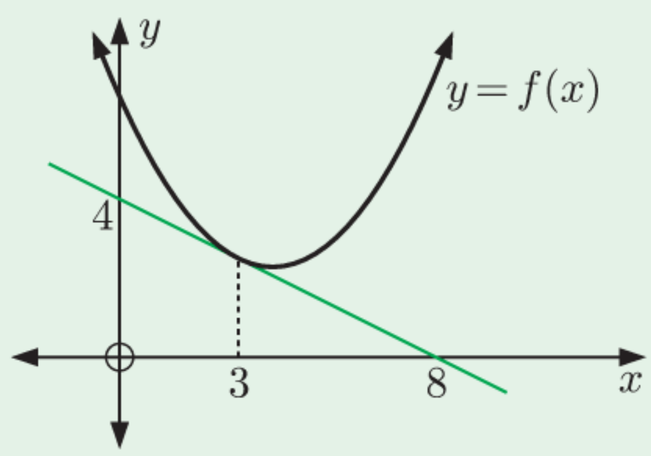
a Show that $\dfrac{f(x + h) - f(x)}{h} = 4x + 2h$ provided $h \neq 0$.

b Hence evaluate $\dfrac{f(3 + h) - f(3)}{h}$ for each value of h in the table.

c Evaluate $\lim\limits_{h \to 0} \dfrac{f(3 + h) - f(3)}{h}$. Give a geometric interpretation of your result.

h	$\dfrac{f(3 + h) - f(3)}{h}$
0.1	
0.01	
0.001	
0.0001	

6 Use the graph alongside to find $f'(3)$.



7 Find, from first principles, the derivative of:

a $f(x) = x^2 + 2x$ **b** $y = 4 - 3x^2$

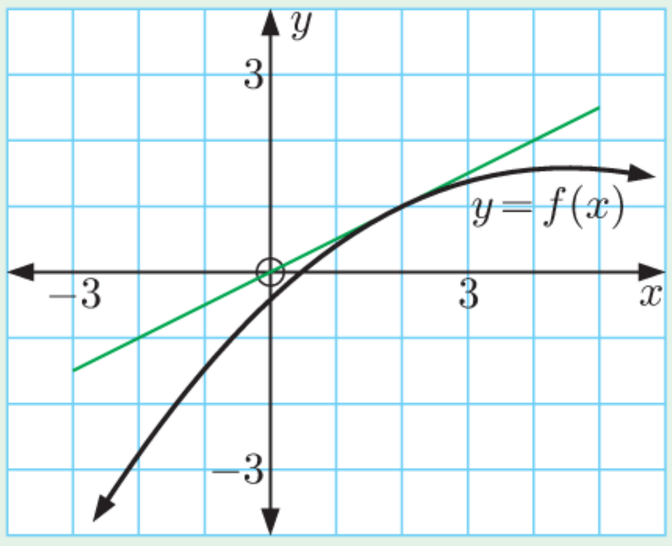
8 **a** Given $y = 2x^2 - 1$, find $\dfrac{dy}{dx}$ from first principles.

b Hence state the gradient of the tangent to $y = 2x^2 - 1$ at the point where $x = 4$.

c For what value of x is the gradient of the tangent to $y = 2x^2 - 1$ equal to -12?

REVIEW SET 17B

1



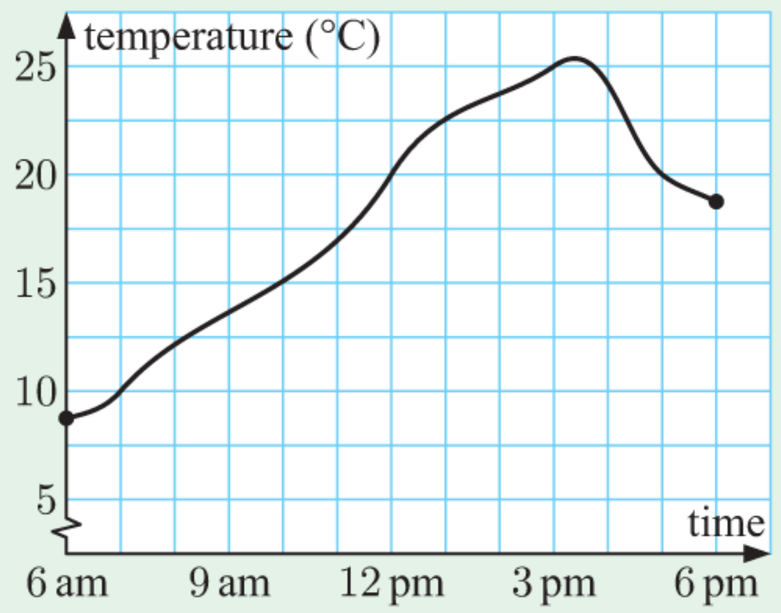
Use the tangent drawn to find the instantaneous rate of change in $f(x)$ at $x = 2$.

2 This graph shows the temperature in Berlin from 6 am to 6 pm on a particular day.

Find the average rate of change in temperature from:

a 7 am to noon

b 3 pm to 5 pm.



3 Evaluate the limits:

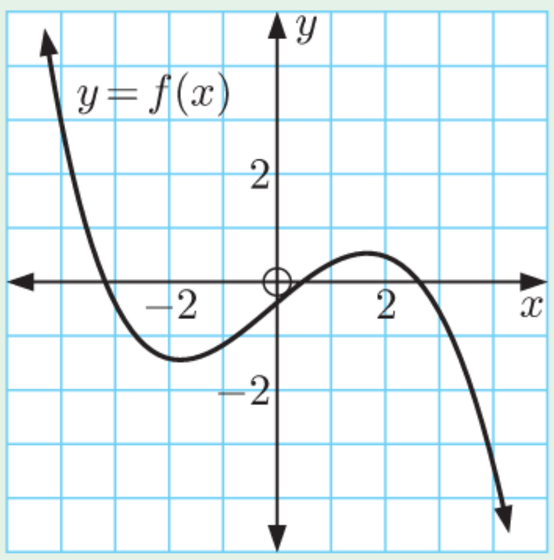
a $\lim\limits_{h \to 0} \dfrac{h^3 - 3h}{h}$ **b** $\lim\limits_{x \to 1} \dfrac{3x^2 - 3x}{x - 1}$ **c** $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{2 - x}$

4 **a** Sketch the graph of $y = e^{x-2} - 3$.

b Hence find, if possible:

i $\lim\limits_{x \to -\infty} (e^{x-2} - 3)$ **ii** $\lim\limits_{x \to \infty} (e^{x-2} - 3)$.

5



Decide whether the following are positive or negative:

a $f(-1)$ **b** $f'(0)$

c $f(2)$ **d** $f'(3)$

6 **a** Draw an accurate graph of $f(x) = x^2 - 2$.

b Draw, as accurately as possible, the tangent to $f(x) = x^2 - 2$ at $x = 2$.

c Hence find the instantaneous rate of change in $f(x) = x^2 - 2$ when $x = 2$.

7 **a** Given $y = x^2 + 5x - 2$, find $\dfrac{dy}{dx}$ from first principles.

b Hence find the point on the graph at which the tangent has gradient -3.

8 In a BASE jumping competition from the Petronas Towers in Kuala Lumpur, the altitude of a professional jumper in the first 3 seconds is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.

a Find the height of the jumper after:

i 1 second **ii** 2 seconds.

b Find $f'(t)$ from first principles.

c Find the speed of the jumper after:

i 1 second **ii** 2 seconds.



Chapter 18

Rules of differentiation

Contents:

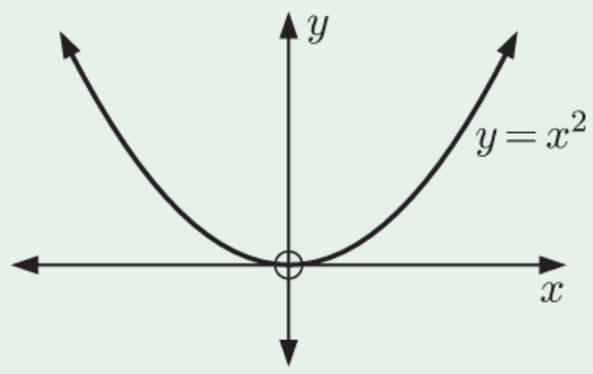
A Simple rules of differentiation
B The chain rule
C The product rule
D The quotient rule
E Derivatives of exponential functions
F Derivatives of logarithmic functions
G Derivatives of trigonometric functions
H Second derivatives



OPENING PROBLEM

Consider the curve $y = x^2$.

In the previous Chapter we found that the gradient function of this curve is $\frac{dy}{dx} = 2x$.



Things to think about:

a At which point on the graph of $y = x^2$ is the tangent to the curve horizontal? How do we observe this from the derivative function $\frac{dy}{dx}$?

b Consider the transformation of $y = x^2$ onto $y = x^2 + 3$.

- **i** What transformation has taken place?
- **ii** For a given value of x, has the gradient of the tangent to the function changed?
- **iii** What is the gradient function of $y = x^2 + 3$?

c Consider the transformation of $y = x^2$ onto $y = (x - 2)^2$.

- **i** What transformation has taken place?
- **ii** How does the gradient function of $y = (x - 2)^2$ relate to the gradient function of $y = x^2$?
- **iii** Can you write down the gradient function of $y = (x - 2)^2$?

DEMO



d Consider the transformation of $y = x^2$ onto $y = 2x^2$.

- **i** What transformation has taken place?
- **ii** How does the gradient function of $y = 2x^2$ relate to the gradient function of $y = x^2$? If necessary, use the software to help you.

In this Chapter we will discover rules which make it easier to find derivative functions.

A SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

Given a function $f(x)$, we obtain $f'(x)$ by **differentiating with respect to** the variable x.

There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use first principles.

For example, in the previous Chapter you should have found that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$.

INVESTIGATION 1 RULES FOR DIFFERENTIATION

In this Investigation we look for other rules which can help us to differentiate functions without needing first principles.

What to do:

1 **a** Differentiate using first principles:

i $y = 3x^2$ **ii** $y = 5x^2$ **iii** $y = -2x^2$

b Copy and complete: "If $f(x) = cx^n$ where c is a constant, then $f'(x) = ...$"

c Check your answer by using the software to consider the derivatives of:



i $y = 4x^4$ **ii** $y = -\dfrac{2}{x}$ **iii** $y = \dfrac{3}{x^2}$

2 **a** Use first principles to find $f'(x)$ for:

i $f(x) = x^2 + 3x$ **ii** $f(x) = x^3 - 2x^2$

b Copy and complete: "If $f(x) = u(x) + v(x)$, then $f'(x) = ...$"

We can summarise the most simple rules of differentiation as follows:

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x) + v(x)$	$u'(x) + v'(x)$	**addition rule**

The last two rules can be proved using the first principles definition of $f'(x)$.

- If $f(x) = c\,u(x)$,
  then $f'(x) = c\,u'(x)$.

  Proof:

$$\begin{aligned} & f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{c\,u(x+h) - c\,u(x)}{h} \\ &= \lim_{h\to 0} c\left[\frac{u(x+h) - u(x)}{h}\right] \\ &= c \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x) + v(x)$,
  then $f'(x) = u'(x) + v'(x)$

  Proof:

$$\begin{aligned} & f'(x) \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{u(x+h) + v(x+h) - [u(x) + v(x)]}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{u(x+h) - u(x) + v(x+h) - v(x)}{h}\right) \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x) \end{aligned}$$

Using the rules we have now developed, we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$\begin{aligned} f'(x) &= 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0 \\ &= 12x^3 + 6x^2 - 10x + 7 \end{aligned}$$

Example 1 — Self Tutor

Find $f'(x)$ for $f(x)$ equal to:

a $5x^3 + 6x^2 - 3x + 2$

b $7x - \frac{4}{x} + \frac{3}{x^3}$

a
$$f(x) = 5x^3 + 6x^2 - 3x + 2$$
$$\therefore\ f'(x) = 5(3x^2) + 6(2x) - 3(1)$$
$$= 15x^2 + 12x - 3$$

b
$$f(x) = 7x - \frac{4}{x} + \frac{3}{x^3}$$
$$= 7x - 4x^{-1} + 3x^{-3}$$
$$\therefore\ f'(x) = 7(1) - 4(-1x^{-2}) + 3(-3x^{-4})$$
$$= 7 + 4x^{-2} - 9x^{-4}$$
$$= 7 + \frac{4}{x^2} - \frac{9}{x^4}$$

Example 2 — Self Tutor

Find the gradient function for:

a $f(x) = 3\sqrt{x} + \frac{2}{x}$

b $g(x) = x^2 - \frac{4}{\sqrt{x}}$

a
$$f(x) = 3\sqrt{x} + \frac{2}{x}$$
$$= 3x^{\frac{1}{2}} + 2x^{-1}$$
$$\therefore\ f'(x) = 3(\tfrac{1}{2}x^{-\frac{1}{2}}) + 2(-1x^{-2})$$
$$= \tfrac{3}{2}x^{-\frac{1}{2}} - 2x^{-2}$$
$$= \frac{3}{2\sqrt{x}} - \frac{2}{x^2}$$

b
$$g(x) = x^2 - \frac{4}{\sqrt{x}}$$
$$= x^2 - 4x^{-\frac{1}{2}}$$
$$\therefore\ g'(x) = 2x - 4(-\tfrac{1}{2}x^{-\frac{3}{2}})$$
$$= 2x + 2x^{-\frac{3}{2}}$$
$$= 2x + \frac{2}{x\sqrt{x}}$$

EXERCISE 18A

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** x^8 **c** x^{11} **d** $6x$

e $2x^3$ **f** $7x^2$ **g** $3x^5$ **h** $5x^6$

i $5x - 2$ **j** $x^2 - 3$ **k** $x^2 + x$ **l** $x^2 + 3x - 5$

m $2x^2 + x - 1$ **n** $3x^2 - 7x + 8$ **o** $4 - 2x^2$ **p** $\frac{1}{2}x^4 - 6x^2$

q $x^3 - 4x^2 + 6x$ **r** $2x^3 + x - 1$ **s** $7 - x - 4x^3$ **t** $\frac{1}{5}x^3 - \frac{7}{2}x^2 - 2$

2 Differentiate with respect to x:

a $\frac{1}{x^2}$ **b** $\frac{1}{x^5}$ **c** $\frac{1}{3x}$ **d** $\frac{4}{x^3}$

e $-\frac{7}{x^4}$ **f** $2x + \frac{3}{x^2}$ **g** $x^2 - \frac{6}{x}$ **h** $\frac{2}{x^2} + \frac{9}{x^4}$

i $3x - \frac{1}{x} + \frac{2}{x^2}$ **j** $5 - \frac{8}{x^2} + \frac{1}{2x^3}$ **k** $\frac{x^3 + 4}{x}$ **l** $\frac{2x - 5}{x^2}$

3 Find the gradient function for $f(x)$ where $f(x)$ is:

a $\sqrt{x}$ **b** $\sqrt[3]{x}$ **c** $2x - \sqrt{x}$ **d** $x\sqrt{x}$

e $\dfrac{1}{\sqrt{x}}$ **f** $-\dfrac{1}{x\sqrt{x}}$ **g** $2x^2 - \dfrac{3}{\sqrt{x}}$ **h** $\dfrac{\sqrt{x}-4}{x}$

i $\dfrac{x+5}{\sqrt{x}}$ **j** $\dfrac{7-x^2}{\sqrt{x}}$ **k** $2x - \dfrac{3}{x^2\sqrt{x}}$ **l** $\dfrac{x^2-x+2}{\sqrt[3]{x}}$

4 Find $\dfrac{dy}{dx}$ for:

a $y = \pi x^2$ **b** $y = 2.5x^3 - 1.4x^2 - 1.3$ **c** $y = 3x^2 - \dfrac{8}{x^2}$

d $y = (x+1)(x-2)$ **e** $y = (2x+1)(3x-2)$ **f** $y = (5-x)^2$

g $(2x-1)^2$ **h** $y = x(x+1)(2x-5)$ **i** $y = \dfrac{(x-3)^2}{\sqrt{x}}$

5 Find:

a $\dfrac{dy}{dt}$ for $y = \frac{1}{2}t^4 - \frac{1}{3}t$ **b** $\dfrac{dy}{dt}$ for $y = 7 - \dfrac{6}{\sqrt{t}}$

c $\dfrac{dT}{dt}$ for $T = \sqrt[3]{t} - \dfrac{2}{t^2}$ **d** $\dfrac{dP}{du}$ for $P = \dfrac{5}{u} - 10u\sqrt{u}$

Example 3 **Self Tutor**

Find the gradient of the tangent to $y = x^2 - \dfrac{4}{x}$ at the point where $x = 2$.

$y = x^2 - \dfrac{4}{x}$ $\quad\therefore\quad \dfrac{dy}{dx} = 2x - 4(-1x^{-2})$

$= x^2 - 4x^{-1}$ $\qquad\qquad = 2x + 4x^{-2}$

$= 2x + \dfrac{4}{x^2}$

When $x = 2$, $\dfrac{dy}{dx} = 4 + 1 = 5$. So, the tangent has gradient 5.

6 Find the gradient of the tangent to:

a $y = x^2$ at $x = 2$ **b** $y = x^3 - 5x + 2$ at the point $(3, 14)$

c $y = \dfrac{8}{x^2}$ at the point $\left(9, \frac{8}{81}\right)$ **d** $y = 2x^2 - 3x + 7$ at $x = -1$

e $y = 3\sqrt{x}$ at the point $(1, 3)$ **f** $y = 2x - \dfrac{5}{x}$ at the point $\left(2, \frac{3}{2}\right)$

g $y = \dfrac{x^2-4}{x^2}$ at the point $\left(4, \frac{3}{4}\right)$ **h** $y = \dfrac{x^3 - 4x - 8}{x^2}$ at $x = -1$

7 The tangent to $f(x) = x^3 + ax + 5$ at the point where $x = 1$, has gradient 10. Find a.

8 The tangent to $f(x) = -3x^2 + ax + b$ at the point $(-3, 8)$ has gradient 9. Find a and b.

9 The tangent to $f(x) = 2x^2 + a + \dfrac{b}{x}$ at the point $(1, 11)$ has gradient -2. Find a and b.

10 Find a and b given $f(x) = ax + \dfrac{b}{x}$, $f(3) = 5$, and $f'(1) = 5$.

Example 4 **Self Tutor**

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$.

$\frac{dy}{dx}$ is the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient of the tangent at any point on the curve can be found. It is also the instantaneous rate of change of y with respect to x.

11 If $y = 4x - \frac{3}{x}$, find $\frac{dy}{dx}$ and interpret its meaning.

12 Consider the function $f(x) = \sqrt{x} - \frac{4}{\sqrt{x}}$.

a State the domain of $f(x)$.

b Find the derivative function $f'(x)$.

c State the domain of $f'(x)$.

d Find $f'(1)$ and interpret your answer.

13 The position of a car moving along a straight road is given by $S = 2t^2 + 4t$ metres where t is the time in seconds.

a Find $\frac{dS}{dt}$ and interpret its meaning.

b Find the value of $\frac{dS}{dt}$ when $t = 3$, and interpret your answer.

14 The cost of producing x toasters each week is given by $C = 1785 + 3x + 0.002x^2$ pounds. Find the value of $\frac{dC}{dx}$ when $x = 1000$, and interpret its meaning.

B THE CHAIN RULE

In **Chapter 15** of the **Core Topics HL** book, we saw that the **composite** of two functions g and f is $(g \circ f)(x)$ or $g(f(x))$.

We can often write complicated functions as the composite of two or more simpler functions.

For example $y = (x^2 + 3x)^4$ could be rewritten as $y = u^4$ where $u = x^2 + 3x$, or as $y = g(f(x))$ where $g(x) = x^4$ and $f(x) = x^2 + 3x$.

Example 5 **Self Tutor**

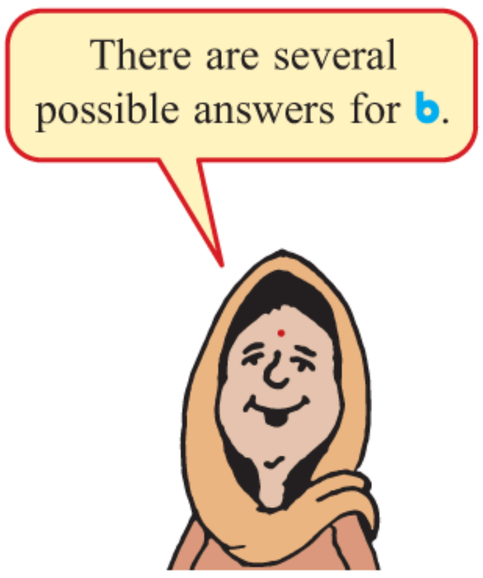
Find:

a $g(f(x))$ if $g(x) = \sqrt{x}$ and $f(x) = 2 - 3x$

b $g(x)$ and $f(x)$ such that $g(f(x)) = \frac{1}{x - x^2}$.

a $g(f(x)) = g(2 - 3x)$
$= \sqrt{2 - 3x}$

b If we let $f(x) = x - x^2$ then
$g(f(x)) = \frac{1}{f(x)}$
$\therefore \ g(x) = \frac{1}{x}$ and $f(x) = x - x^2$



EXERCISE 18B.1

1 Find $g(f(x))$ if:

a $g(x) = x^2$ and $f(x) = 2x + 7$

b $g(x) = 2x + 7$ and $f(x) = x^2$

c $g(x) = \sqrt{x}$ and $f(x) = 3 - 4x$

d $g(x) = 3 - 4x$ and $f(x) = \sqrt{x}$

e $g(x) = \dfrac{2}{x}$ and $f(x) = x^2 + 3$

f $g(x) = x^2 + 3$ and $f(x) = \dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $g(f(x))$ is:

a $(3x + 10)^3$

b $(7 - 2x)^5$

c $\dfrac{1}{2x + 4}$

d $\sqrt{x^2 - 3x}$

e $\dfrac{1}{(5x - 1)^4}$

f $\dfrac{10}{(3x - x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

INVESTIGATION 2 DIFFERENTIATING COMPOSITE FUNCTIONS

In this Investigation we want to learn how to differentiate composite functions.

Based on the rule "if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$", we might suspect that if $y = (2x + 1)^2$ then $\dfrac{dy}{dx} = 2(2x + 1)^1$. But is this so?

What to do:

1 Expand $y = (2x + 1)^2$, and hence find $\dfrac{dy}{dx}$. How does this compare with $2(2x + 1)^1$?

2 Expand $y = (3x + 1)^2$, and hence find $\dfrac{dy}{dx}$. How does this compare with $2(3x + 1)^1$?

3 Expand $y = (ax + 1)^2$ where a is a constant, and hence find $\dfrac{dy}{dx}$. How does this compare with $2(ax + 1)^1$?

4 Suppose $y = u^2$.

a Find $\dfrac{dy}{du}$.

b Now suppose $u = ax + 1$, so $y = (ax + 1)^2$.

i Find $\dfrac{du}{dx}$.

ii Write $\dfrac{dy}{du}$ from **a** in terms of x.

iii Hence find $\dfrac{dy}{du} \times \dfrac{du}{dx}$.

iv Compare your answer to the result in **3**.

c If $y = u^2$ where u is a function of x, what do you suspect $\dfrac{dy}{dx}$ will be equal to?

5 Expand $y = (x^2 + 3x)^2$ and hence find $\dfrac{dy}{dx}$.

Does your answer agree with the rule you suggested in **4 c**?

6 Consider $y = (2x + 1)^3$.

a Expand the brackets and hence find $\dfrac{dy}{dx}$.

b If we let $u = 2x + 1$, then $y = u^3$.

i Find $\frac{du}{dx}$.

ii Find $\frac{dy}{du}$, and write it in terms of x.

iii Hence find $\frac{dy}{du} \times \frac{du}{dx}$.

iv Compare your answer to the result in **a**.

7 Copy and complete: "If y is a function of u, and u is a function of x, then $\frac{dy}{dx} =$ ....."

THE CHAIN RULE

If $y = g(u)$ where $u = f(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

This rule is extremely important and allows us to differentiate complicated functions much faster.

For example, for any function $f(x)$:

If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$.

Example 6 — Self Tutor

Find $\frac{dy}{dx}$ if: **a** $y = (x^2 - 2x)^4$ **b** $y = \dfrac{4}{\sqrt{1-2x}}$

a

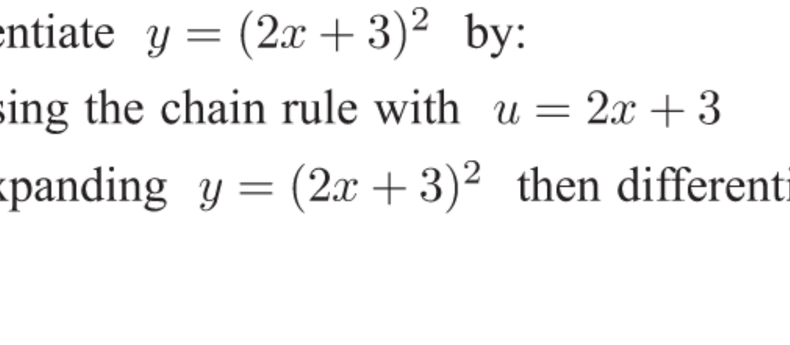
$y = (x^2 - 2x)^4$

$\therefore\ y = u^4$ where $u = x^2 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 4u^3(2x - 2)$

$= 4(x^2 - 2x)^3(2x - 2)$



b

$y = \dfrac{4}{\sqrt{1-2x}}$

$\therefore\ y = 4u^{-\frac{1}{2}}$ where $u = 1 - 2x$

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= 4 \times (-\frac{1}{2}u^{-\frac{3}{2}}) \times (-2)$

$= 4u^{-\frac{3}{2}}$

$= 4(1 - 2x)^{-\frac{3}{2}}$

EXERCISE 18B.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$ **b** $\sqrt{x^2 - 3x}$ **c** $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3 - x^2}$ **e** $\dfrac{4}{(3-x)^3}$ **f** $\dfrac{10}{x^2 - 3}$

2 Differentiate $y = (2x + 3)^2$ by:

a using the chain rule with $u = 2x + 3$

b expanding $y = (2x + 3)^2$ then differentiating term-by-term.

3 Find the derivative function $\frac{dy}{dx}$ for:

a $y = (4x - 5)^2$ **b** $y = \dfrac{1}{5 - 2x}$ **c** $y = \sqrt{3x - x^2}$ **d** $y = (1 - 3x)^4$

e $y = 6(5 - x)^3$ **f** $y = \sqrt[3]{2x^3 - x^2}$ **g** $y = \dfrac{6}{(5x - 4)^2}$ **h** $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

4 Find the gradient of the tangent to:

a $y = \sqrt{1 - x^2}$ at $x = \frac{1}{2}$ **b** $y = (3x + 2)^6$ at $x = -1$

c $y = \dfrac{1}{(2x - 1)^4}$ at $x = 1$ **d** $y = 6 \times \sqrt[3]{1 - 2x}$ at $x = 0$

e $y = \dfrac{4}{x + 2\sqrt{x}}$ at $x = 4$ **f** $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$.

GRAPHING PACKAGE

Check your answers using technology.

5 The gradient function of $f(x) = (2x - b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

6 Suppose $y = \dfrac{a}{\sqrt{1 + bx}}$ where $a, b \in \mathbb{R}$. At the point $(3, 1)$, $\dfrac{dy}{dx} = -\frac{1}{8}$. Find a and b.

7 Suppose $f(x) = 3\left(ax - \dfrac{b}{x}\right)^3$. Given that $f(\frac{3}{2}) = 3$ and $f'(\frac{3}{2}) = 30$, find a and b.

C THE PRODUCT RULE

We have seen the addition rule which allows us to differentiate term-by-term:

$$\text{If } f(x) = u(x) + v(x) \text{ then } f'(x) = u'(x) + v'(x).$$

If we now consider the case $f(x) = u(x)\,v(x)$, we might wonder if $f'(x) = u'(x)\,v'(x)$.

In other words, does the derivative of a product of two functions equal the product of their derivatives?

INVESTIGATION 3 THE PRODUCT RULE

Suppose $u(x)$ and $v(x)$ are two functions of x, and that $f(x) = u(x)\,v(x)$ is the product of these functions. In this Investigation we attempt to find a rule for determining $f'(x)$.

What to do:

1 Suppose $u(x) = x$ and $v(x) = x$, so $f(x) = x^2$.

a Find $f'(x)$ by direct differentiation. **b** Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

2 Suppose $u(x) = x$ and $v(x) = \sqrt{x}$, so $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$.

a Find $f'(x)$ by direct differentiation. **b** Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

3 Copy and complete the following table, finding $f'(x)$ by direct differentiation.

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)\,v(x) + u(x)\,v'(x)$
x^2		x	x			
$x^{\frac{3}{2}}$		x	$\sqrt{x}$			
$x(x+1)$		x	$x+1$			
$(x-1)(2-x^2)$		$x-1$	$2-x^2$			

4 Copy and complete: "If $f(x) = u(x)\,v(x)$ then $f'(x) = \ldots\ldots$"

THE PRODUCT RULE

If $f(x) = u(x)\,v(x)$ then $f'(x) = u'(x)\,v(x) + u(x)\,v'(x)$.

Alternatively, if $y = uv$ where u and v are functions of x, then

$$\frac{dy}{dx} = u'v + uv' = \frac{du}{dx}\,v + u\,\frac{dv}{dx}.$$

Example 7 — Self Tutor

If $y = x^2(x^2-2x)^4$, find $\dfrac{dy}{dx}$.

$y = x^2(x^2-2x)^4$ is the product of

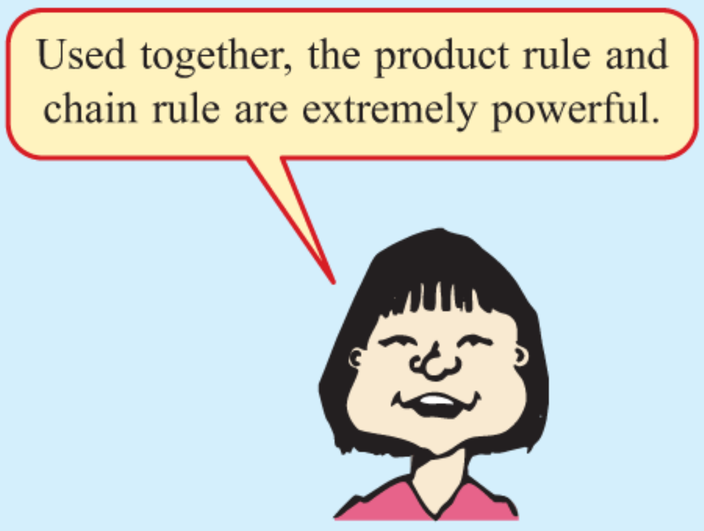
$u = x^2$ and $v = (x^2-2x)^4$

$\therefore\ u' = 2x$ and $v' = 4(x^2-2x)^3(2x-2)$ {chain rule}

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= 2x(x^2-2x)^4 + x^2 \times 4(x^2-2x)^3(2x-2)$

$= 2x(x^2-2x)^4 + 4x^2(x^2-2x)^3(2x-2)$



Example 8 — Self Tutor

Find $\dfrac{dy}{dx}$ given $y = \sqrt{x}(2x+1)^3$.

$y = \sqrt{x}(2x+1)^3$ is the product of $u = x^{\frac{1}{2}}$ and $v = (2x+1)^3$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 3(2x+1)^2 \times 2$ {chain rule}

$= 6(2x+1)^2$

Now $\dfrac{dy}{dx} = u'v + uv'$ {product rule}

$= \frac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + x^{\frac{1}{2}} \times 6(2x+1)^2$

$= \frac{1}{2}x^{-\frac{1}{2}}(2x+1)^3 + 6x^{\frac{1}{2}}(2x+1)^2$

EXERCISE 18C

1 Use the product rule to differentiate:

a $f(x) = x(x-1)$ **b** $f(x) = 2x(x+1)$ **c** $f(x) = x^2\sqrt{x+1}$

d $f(x) = (x+3)(x-1)$ **e** $f(x) = x\sqrt{x^2-1}$ **f** $f(x) = x(x+1)^2$

2 Find $\dfrac{dy}{dx}$ using the product rule:

a $y = x^2(2x-1)$ **b** $y = 4x(2x+1)^3$ **c** $y = x^2\sqrt{3-x}$

d $y = \sqrt{x}(x-3)^2$ **e** $y = 5x^2(3x^2-1)^2$ **f** $y = \sqrt{x}(x-x^2)^3$

3 Find the gradient of the tangent to:

a $y = x^4(1-2x)^2$ at $x = -1$ **b** $y = \sqrt{x}(x^2-x+1)^2$ at $x = 4$

c $y = x\sqrt{1-2x}$ at $x = -4$ **d** $y = x^3\sqrt{5-x^2}$ at $x = 1$.

4 Consider $y = \sqrt{x}(3-x)^2$.

a Show that $\dfrac{dy}{dx} = \dfrac{(3-x)(3-5x)}{2\sqrt{x}}$.

b Find the x-coordinates of all points on $y = \sqrt{x}(3-x)^2$ where the tangent is horizontal.

c State the domain of $\dfrac{dy}{dx}$. Discuss how it differs from the domain of the original function.

5 Suppose $y = -2x^2(x+4)$. For what values of x does $\dfrac{dy}{dx} = 10$?

6 Suppose $y = (x+3)(x-2)^2$. For what values of x does $\dfrac{dy}{dx} = -7$?

7 Find the value of x for which the tangent to $f(x) = ax\sqrt{1-x}$, $a \neq 0$ has gradient:

a 0 **b** a.

8 Find the values of a such that $f(x) = x^2\sqrt{x^2+a}$ and $f'(-2) = -\frac{34}{3}$.

D THE QUOTIENT RULE

Expressions like $\dfrac{x^2+1}{2x-5}$, $\dfrac{\sqrt{x}}{1-3x}$, and $\dfrac{x^3}{(x-x^2)^4}$ are **quotients** because they represent the division of one expression by another.

Quotient functions have the form $Q(x) = \dfrac{u(x)}{v(x)}$.

$$\text{Notice that} \quad u(x) = Q(x)\,v(x)$$

$$\therefore \quad u'(x) = Q'(x)\,v(x) + Q(x)\,v'(x) \qquad \{\text{product rule}\}$$

$$\therefore \quad u'(x) - Q(x)\,v'(x) = Q'(x)\,v(x)$$

$$\therefore \quad Q'(x)\,v(x) = u'(x) - \frac{u(x)}{v(x)}\,v'(x)$$

$$\therefore \quad Q'(x)\,v(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)}$$

$$\therefore \quad Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2} \qquad \text{when this exists.}$$

THE QUOTIENT RULE

If $Q(x) = \dfrac{u(x)}{v(x)}$ then $Q'(x) = \dfrac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$.

Alternatively, if $y = \dfrac{u}{v}$ where u and v are functions of x, then

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{\dfrac{du}{dx}\,v - u\,\dfrac{dv}{dx}}{v^2}$$

Example 9 — Self Tutor

Use the quotient rule to find $\dfrac{dy}{dx}$ given $y = \dfrac{1+3x}{x^2+1}$.

$y = \dfrac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore\ u' = 3$ and $v' = 2x$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}$$

$$= \frac{3x^2 + 3 - 2x - 6x^2}{(x^2+1)^2}$$

$$= \frac{3 - 2x - 3x^2}{(x^2+1)^2}$$

Example 10 — Self Tutor

Find $\dfrac{dy}{dx}$ given $y = \dfrac{\sqrt{x}}{(1-2x)^2}$.

$y = \dfrac{\sqrt{x}}{(1-2x)^2}$ is a quotient with

$u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore\ u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times (-2)$ {chain rule}

$= -4(1-2x)$

Now $\dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}$$

$$= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}$$



If you only need the gradient of a tangent at a given point, you will not need to simplify $\dfrac{dy}{dx}$.

In such cases, substitute the value for x into the derivative function immediately.

$$= \frac{(1-2x)\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{4\,3}}$$

$$= \frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}$$

$$= \frac{6x+1}{2\sqrt{x}(1-2x)^3}$$

EXERCISE 18D

1 Use the quotient rule to find $\frac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{2-x}$ **b** $y = \dfrac{x^2}{2x+1}$ **c** $y = \dfrac{x}{x^2-3}$

d $y = \dfrac{\sqrt{x}}{1-2x}$ **e** $y = \dfrac{x^2-3}{3x-x^2}$ **f** $y = \dfrac{x}{\sqrt{1-3x}}$

2 Find:

a $\dfrac{d}{dx}\left(\dfrac{x+1}{3-x}\right)$ **b** $\dfrac{d}{dx}\left(\dfrac{3x}{x^2-1}\right)$

c $\dfrac{d}{dx}\left(\dfrac{x^3}{2x-1}\right)$ **d** $\dfrac{d}{dx}\left(\dfrac{4x}{\sqrt{x-5}}\right)$

e $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{3-x^2}\right)$ **f** $\dfrac{d}{dx}\left(-\dfrac{x^2}{\sqrt{x^2+3}}\right)$

$\frac{d}{dx}(....)$ reads "the derivative of (....) with respect to x".



3 Find the gradient of the tangent to:

a $y = \dfrac{x}{1-2x}$ at $x = 1$ **b** $y = \dfrac{x^3}{x^2+1}$ at $x = -1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ at $x = 4$ **d** $y = \dfrac{x^2}{\sqrt{x^2+5}}$ at $x = -2$.

4 Suppose $f(x) = \dfrac{x}{\sqrt{x-1}}$. Find $f'(x)$ using the quotient rule.

Check your answer by writing $f(x) = \dfrac{x-1}{\sqrt{x-1}} + \dfrac{1}{\sqrt{x-1}}$ and then differentiating.

5 **a** If $y = \dfrac{2\sqrt{x}}{1-x}$, show that $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}(1-x)^2}$.

b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

6 **a** If $y = \dfrac{x^2+6}{2x+1}$, show that $\dfrac{dy}{dx} = \dfrac{2x^2+2x-12}{(2x+1)^2}$.

b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

7 **a** If $y = \dfrac{x^2-3x+1}{x+2}$, show that $\dfrac{dy}{dx} = \dfrac{x^2+4x-7}{(x+2)^2}$.

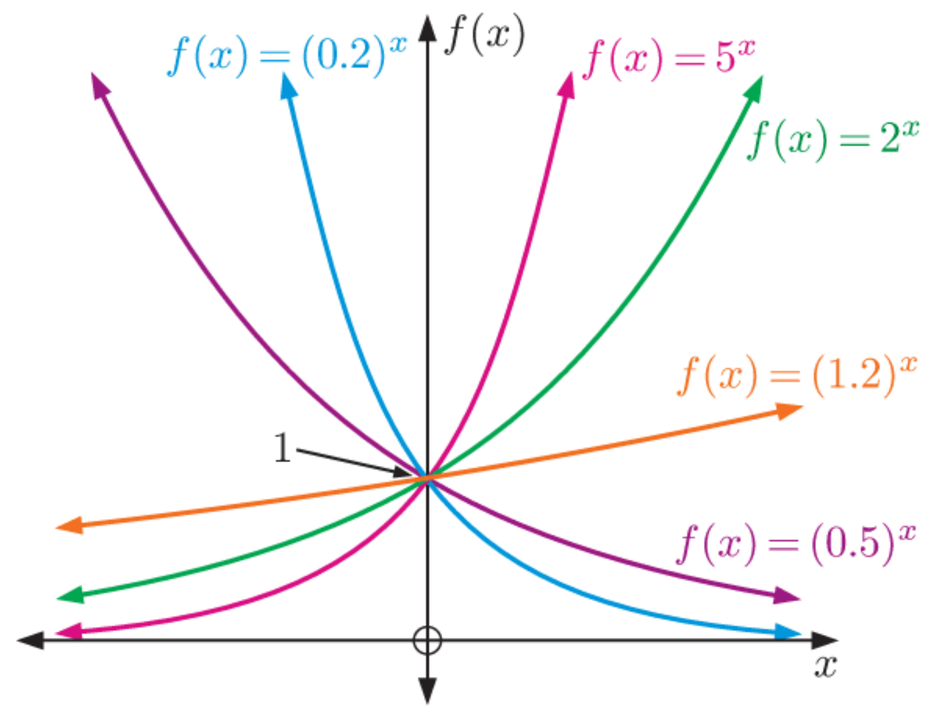
b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

E DERIVATIVES OF EXPONENTIAL FUNCTIONS

We have seen previously that the simplest **exponential functions** have the form $f(x) = b^x$ where b is any positive constant, $b \neq 1$.

The graphs of all members of the exponential family $f(x) = b^x$:

- pass through the point $(0, 1)$
- are asymptotic to the x-axis at one end
- lie above the x-axis for all x.



INVESTIGATION 4 THE DERIVATIVE OF b^x

The purpose of this Investigation is to observe the nature of the derivatives of $f(x) = b^x$ for various values of b.

What to do:

1 Use the software provided to help fill in the table for $y = 2^x$:

CALCULUS DEMO



x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0			
0.5			
1			
1.5			
2			

2 Repeat **1** for the following functions:

a $y = 3^x$ **b** $y = 5^x$ **c** $y = (0.5)^x$

3 Use your observations from **1** and **2** to write a statement about the derivative of the general exponential $y = b^x$ for $b > 0$, $b \neq 1$.

From the **Investigation** you should have found that:

If $f(x) = b^x$ then $f'(x) = f'(0) \times b^x$.

Proof:

If $f(x) = b^x$ then $f'(x) = \lim_{h \to 0} \dfrac{b^{x+h} - b^x}{h}$ {definition of the derivative}

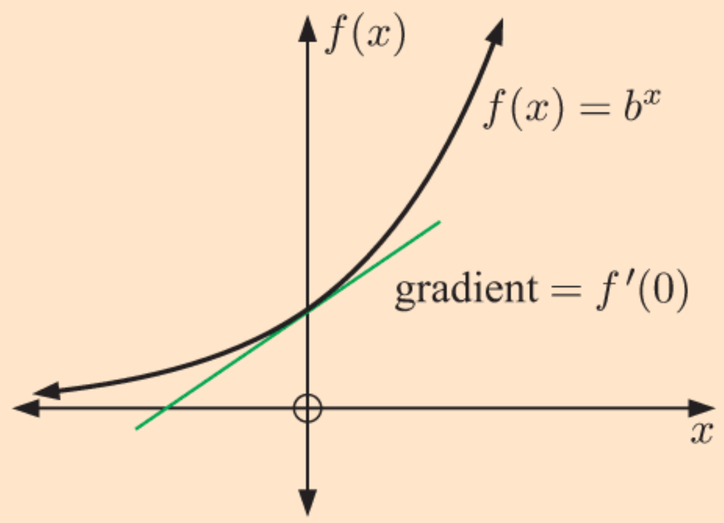
$= \lim_{h \to 0} \dfrac{b^x(b^h - 1)}{h}$

$= b^x \times \left(\lim_{h \to 0} \dfrac{b^h - 1}{h} \right)$ {as b^x is independent of h}

But $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0} \dfrac{b^h - 1}{h}$

$\therefore \quad f'(x) = b^x \times f'(0)$



Given this result, if we can find a value of b such that $f'(0) = 1$, then we will have found *a function which is its own derivative*!

INVESTIGATION 5 SOLVING FOR b IF $f(x) = b^x$ AND $f'(x) = b^x$

Click on the icon to graph $f(x) = b^x$ and its derivative function $y = f'(x)$.

Experiment with different values of b until the graphs of $f(x) = b^x$ and $y = f'(x)$ appear the same.

Estimate the corresponding value of b to 3 decimal places.



You should have discovered that $f(x) = f'(x) = b^x$ when $b \approx 2.718$.

To find b more accurately, we return to the requirement that $f'(0) = \lim\limits_{h \to 0} \dfrac{b^h - 1}{h} = 1$.

Now if $\dfrac{b^h - 1}{h} = 1$ then $b = (1+h)^{\frac{1}{h}}$,

so taking the limit on both sides gives $b = \lim\limits_{h \to 0} (1+h)^{\frac{1}{h}}$.

Letting $h = \dfrac{1}{n}$, we notice that $h \to 0$ as $n \to \infty$.

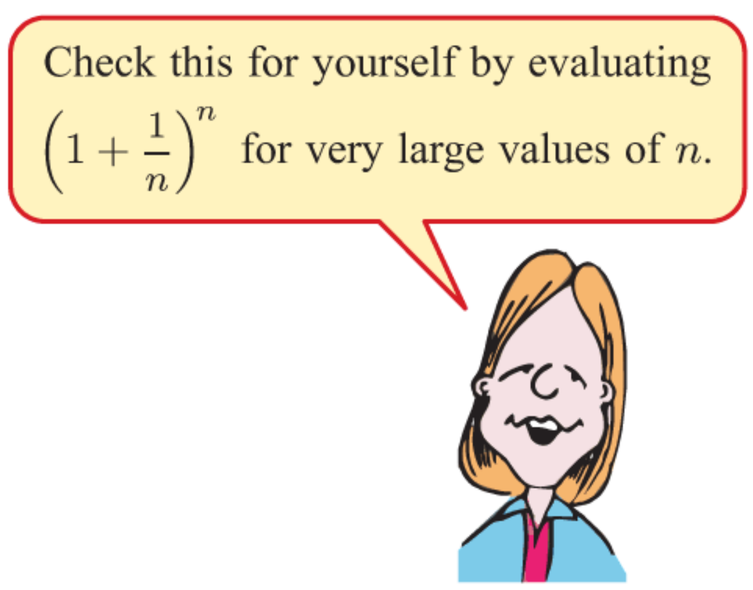
$\therefore \quad b = \lim\limits_{n \to \infty} \left(1 + \dfrac{1}{n}\right)^n$ if this limit exists.

We have in fact already seen this limit previously in an **Investigation** on continuously compounding interest.

We found that as $n \to \infty$, $\left(1 + \dfrac{1}{n}\right)^n \to 2.718\,281\,828\,459\ldots$.

and this irrational number is the natural exponential e.

We therefore conclude: If $f(x) = e^x$ then $f'(x) = e^x$.



THE DERIVATIVE OF $e^{f(x)}$

Suppose $y = e^{f(x)} = e^u$ where $u = f(x)$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$= e^u \dfrac{du}{dx}$

$= e^{f(x)} \times f'(x)$

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)} \times f'(x)$

Example 11 — Self Tutor

Find the gradient function for y equal to:

a $2e^x + e^{-3x}$ **b** x^2e^{-x} **c** $\dfrac{e^{2x}}{x}$

a If $y = 2e^x + e^{-3x}$ then $\dfrac{dy}{dx} = 2e^x + e^{-3x}(-3)$ {addition rule}
$= 2e^x - 3e^{-3x}$

b If $y = x^2e^{-x}$ then $\dfrac{dy}{dx} = 2xe^{-x} + x^2e^{-x}(-1)$ {product rule}
$= 2xe^{-x} - x^2e^{-x}$

c If $y = \dfrac{e^{2x}}{x}$ then $\dfrac{dy}{dx} = \dfrac{e^{2x}(2)x - e^{2x}(1)}{x^2}$ {quotient rule}
$= \dfrac{e^{2x}(2x-1)}{x^2}$

EXERCISE 18E

1 Find the gradient function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** e^{-2x} **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1 + e^{2x})$ **l** $20(1 - e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $20xe^{-0.5x}$ **h** $\dfrac{e^x + 2}{e^{-x} + 1}$

Example 12 — Self Tutor

Find the gradient function for y equal to:

a $(e^x - 1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x} + 1}}$

a $y = (e^x - 1)^3$
$= u^3$ where $u = e^x - 1$
$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}
$= 3u^2\dfrac{du}{dx}$
$= 3(e^x - 1)^2 \times e^x$
$= 3e^x(e^x - 1)^2$

b $y = (2e^{-x} + 1)^{-\frac{1}{2}}$
$= u^{-\frac{1}{2}}$ where $u = 2e^{-x} + 1$
$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}
$= -\frac{1}{2}u^{-\frac{3}{2}}\dfrac{du}{dx}$
$= -\frac{1}{2}(2e^{-x} + 1)^{-\frac{3}{2}} \times 2e^{-x}(-1)$
$= e^{-x}(2e^{-x} + 1)^{-\frac{3}{2}}$

3 Find the gradient function for y equal to:

a $(2+e^x)^4$ **b** $\sqrt{e^x-1}$ **c** $(e^x+e^{-x})^{\frac{3}{2}}$ **d** $\dfrac{1}{\sqrt{e^{2x}+2}}$

4 Find the gradient of the tangent to:

a $y=(e^x+2)^4$ at $x=0$ **b** $y=\dfrac{1}{2-e^{-x}}$ at $x=0$

c $y=\sqrt{e^{2x}+10}$ at $x=\ln 3$ **d** $y=\dfrac{2-x}{e^{3x}}$ at $x=1$.

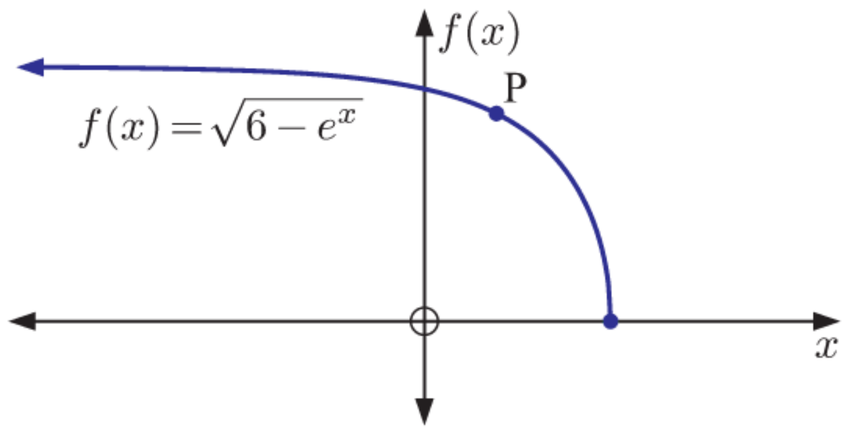
5 The graph of $f(x)=\sqrt{6-e^x}$ is shown alongside.

a State the domain of the function.

b The point P has y-coordinate 2. Find exactly:

i the coordinates of P

ii the gradient of the tangent at P.



6 Given $f(x)=e^{kx}+x$ and $f'(0)=-8$, find k.

7 **a** By substituting $e^{\ln 2}$ for 2 in $y=2^x$, find $\dfrac{dy}{dx}$.

b Show that if $y=b^x$ where $b>0$, $b\neq 1$, then $\dfrac{dy}{dx}=b^x\times \ln b$.

c Find $\dfrac{dy}{dx}$ for: **i** $y=5^x$ **ii** $y=8\times 10^x$

8 The tangent to $f(x)=x^2e^{-x}$ at point P is horizontal. Find the possible coordinates of P.

9 Suppose $S(x)=\frac{1}{2}(e^x-e^{-x})$ and $C(x)=\frac{1}{2}(e^x+e^{-x})$.

a Show that $[C(x)]^2-[S(x)]^2=1$.

b Show that $\dfrac{d}{dx}[S(x)]=C(x)$.

c Find $\dfrac{d}{dx}[C(x)]$ in terms of $S(x)$.

d If $T(x)=\dfrac{S(x)}{C(x)}$, find $\dfrac{d}{dx}[T(x)]$ in terms of $C(x)$.

F DERIVATIVES OF LOGARITHMIC FUNCTIONS

INVESTIGATION 6 THE DERIVATIVE OF $\ln x$

What to do:

1 Click on the icon to see the graph of $y=\ln x$. Observe the gradient function being drawn as the point moves from left to right along the graph.

CALCULUS DEMO

2 Predict a formula for the gradient function of $y=\ln x$.

3 Find the gradient of the tangent to $y=\ln x$ for $x=0.25$, 0.5, 1, 2, 3, 4, and 5. Do your results confirm your prediction in **2**?

From the **Investigation** you should have observed:

PROOF



If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$, $x > 0$.

THE DERIVATIVE OF $\ln f(x)$

Suppose $y = \ln f(x)$

$\therefore\ y = \ln u$ where $u = f(x)$.

Now $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$\therefore\ \dfrac{dy}{dx} = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{f'(x)}{f(x)}$

Function	*Derivative*
$\ln x$	$\dfrac{1}{x}$, $x > 0$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$

Example 13 Self Tutor

Find the gradient function of:

a $y = \ln(1 - 3x)$ **b** $y = x^3 \ln x$

a

$y = \ln(1 - 3x)$

$\therefore\ \dfrac{dy}{dx} = \dfrac{-3}{1 - 3x}$

$= \dfrac{3}{3x - 1}$

b

$y = x^3 \ln x$

$\therefore\ \dfrac{dy}{dx} = 3x^2 \ln x + x^3\left(\dfrac{1}{x}\right)$ {product rule}

$= 3x^2 \ln x + x^2$

$= x^2(3 \ln x + 1)$

The laws of logarithms can help us to differentiate some logarithmic functions more easily.

For $a > 0$, $b > 0$, $n \in \mathbb{R}$:

$$\ln(ab) = \ln a + \ln b$$

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(a^n) = n \ln a$$

EXERCISE 18F

1 Find the gradient function of:

a $y = \ln(7x)$ **b** $y = \ln(2x + 1)$ **c** $y = \ln(x - x^2)$ **d** $y = 3 - 2\ln x$

e $y = x^2 \ln x$ **f** $y = \dfrac{\ln x}{2x}$ **g** $y = e^x \ln x$ **h** $y = (\ln x)^2$

i $y = \sqrt{\ln x}$ **j** $y = e^{-x} \ln x$ **k** $y = \sqrt{x}\ln(2x)$ **l** $y = -4\ln(1 - x)$

m $y = \dfrac{2\sqrt{x}}{\ln x}$ **n** $y = x\ln(x^2 + 1)$ **o** $y = \dfrac{\ln x}{x^2}$

2 Given $f(x) = \ln(kx)$ where $k \neq 0$, find $f'(x)$.
Use the laws of logarithms to explain why this derivative does not depend on the value of k.

3 Find $\dfrac{dy}{dx}$ for:

a $y = x \ln 5$ **b** $y = \ln(x^3)$ **c** $y = \ln(x^4 + x)$ **d** $y = \ln(10 - 5x)$

e $y = [\ln(2x + 1)]^3$ **f** $y = \dfrac{\ln(4x)}{x}$ **g** $y = \ln\left(\dfrac{1}{x}\right)$ **h** $y = \dfrac{1}{\ln x}$

Example 14 **Self Tutor**

Differentiate with respect to x:

a $y = \ln(xe^{-x})$ **b** $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a
$$\begin{aligned} y &= \ln(xe^{-x}) \\ &= \ln x + \ln e^{-x} && \{\ln(ab) = \ln a + \ln b\} \\ &= \ln x - x && \{\ln e^a = a\} \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} - 1$$

b
$$\begin{aligned} y &= \ln\left[\frac{x^2}{(x+2)(x-3)}\right] \\ &= \ln x^2 - \ln[(x+2)(x-3)] && \left\{\ln\left(\frac{a}{b}\right) = \ln a - \ln b\right\} \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \end{aligned}$$

$$\therefore \frac{dy}{dx} = \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3}$$

A derivative function will only be valid on *at most* the domain of the original function.



4 Use the laws of logarithms to help differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$ **b** $y = \ln\left(\dfrac{1}{2x+3}\right)$ **c** $y = \ln(e^x\sqrt{x})$

d $y = \ln(x\sqrt{2-x})$ **e** $y = \ln\left(\dfrac{x+3}{x-1}\right)$ **f** $y = \ln\left(\dfrac{x^2}{3-x}\right)$

5 Differentiate with respect to x:

a $f(x) = \ln((3x-4)^3)$ **b** $f(x) = \ln(x(x^2+1))$ **c** $f(x) = \ln\left(\dfrac{x^2+2x}{x-5}\right)$

6 Find the gradient of the tangent to $y = x\ln x$ at the point where $x = e$.

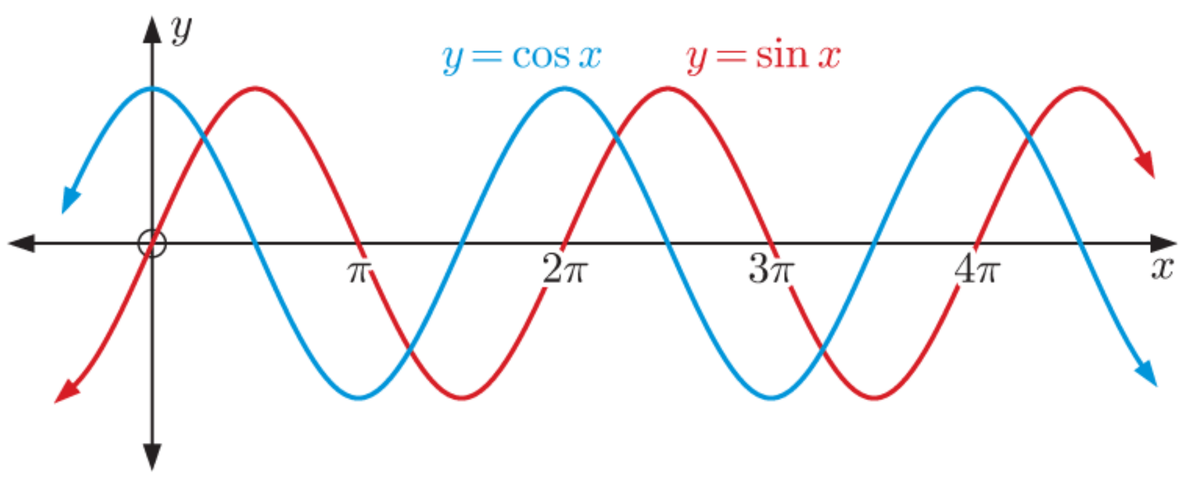
7 Suppose $f(x) = a\ln(bx^2)$ where $f(e) = 3$ and $f'(1) = 6$. Find the constants a and b.

8 Suppose $f(x) = ax\ln(bx)$ where $f(1) = 12$ and $f'(1) = 16$. Find the constants a and b.

9 Find the point(s) at which the tangent to $y = \ln(15 - x^2)$ has gradient 1.

G DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

We have seen that the sine and cosine curves naturally arise from circular motion.



INVESTIGATION 7 DERIVATIVES OF $\sin x$ AND $\cos x$

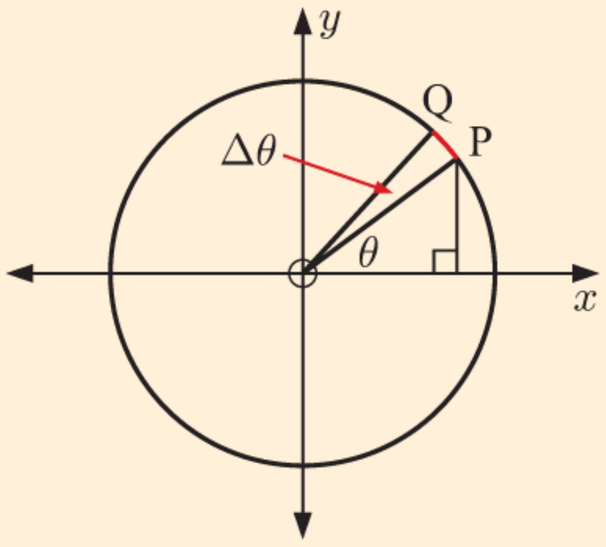
What to do:

1 Click on the icon to observe the graph of $y = \sin x$. A tangent with x-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Predict the derivative of the function $y = \sin x$.

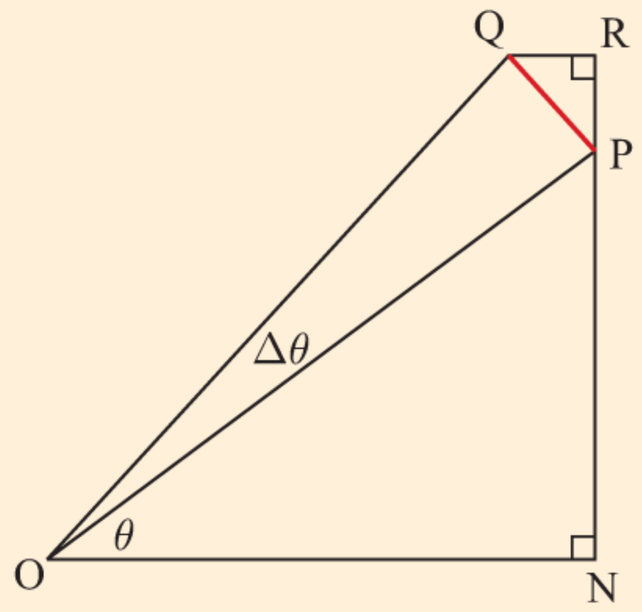


2 Repeat the process in **1** for the graph of $y = \cos x$. Hence predict the derivative of the function $y = \cos x$.

3



Suppose P and Q are points on the unit circle corresponding to angles θ and $\theta + \Delta\theta$ respectively from the positive x-axis.



a Explain why $PR = \sin(\theta + \Delta\theta) - \sin\theta$.

b If P and Q are close together then $\Delta\theta$ is very small. Explain why as Q approaches P:

- **i** the arc PQ resembles line segment [PQ]
- **ii** the length of the line segment $PQ \approx \Delta\theta$
- **iii** $Q\widehat{P}O$ approaches a right angle
- **iv** $Q\widehat{P}R \approx \theta$.

c Use right angled triangle trigonometry in $\triangle QRP$ to show that $\cos\theta \approx \dfrac{\sin(\theta + \Delta\theta) - \sin\theta}{\Delta\theta}$.

Hence explain why in the limit as $\Delta\theta \to 0$, $\cos\theta = \dfrac{d}{d\theta}(\sin\theta)$.

From the **Investigation** you should have deduced that:

For x in radians:

If $f(x) = \sin x$ then $f'(x) = \cos x$.

If $f(x) = \cos x$ then $f'(x) = -\sin x$.

THE DERIVATIVE OF $\tan x$

Consider $y = \tan x = \dfrac{\sin x}{\cos x}$

We let $u = \sin x$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$

$\therefore \dfrac{dy}{dx} = \dfrac{u'v - uv'}{v^2}$ {quotient rule}

$= \dfrac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2}$

$= \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$ {since $\sin^2 x + \cos^2 x = 1$}



Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$, AND $\tan[f(x)]$

Suppose $y = \sin[f(x)]$

$\therefore\ y = \sin u$ where $u = f(x)$.

But $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$\therefore\ \dfrac{dy}{dx} = \cos u \times f'(x)$

$= \cos[f(x)] \times f'(x)$

We can perform the same procedure for $\cos[f(x)]$ and $\tan[f(x)]$, giving the following results:

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\dfrac{1}{\cos^2[f(x)]} \times f'(x)$

Example 15 — Self Tutor

Differentiate with respect to x:

a $x \sin x$

$y = x \sin x$

$\therefore\ \dfrac{dy}{dx} = (1)\sin x + (x)\cos x$ {product rule}

$= \sin x + x\cos x$

b $4\tan^2 3x$

$y = 4\tan^2 3x$

$= 4u^2$ where $u = \tan 3x$

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$\therefore\ \dfrac{dy}{dx} = 8u \times \dfrac{du}{dx}$

$= 8\tan 3x \times \dfrac{3}{\cos^2 3x}$

$= \dfrac{24\sin 3x}{\cos^3 3x}$

EXERCISE 18G

1 Find $\dfrac{dy}{dx}$ for:

a $y = \sin 2x$ **b** $y = \sin x + \cos x$ **c** $y = \cos 3x - \sin x$

d $y = \sin(x+1)$ **e** $y = \cos(3-2x)$ **f** $y = \tan 5x$

g $y = \sin\frac{x}{2} - 3\cos x$ **h** $y = 4\sin x - \cos 2x$ **i** $y = \frac{1}{2}\cos 6x - 5\sin 4x$

2 Differentiate with respect to x:

a $x^2 + \cos x$ **b** $\tan x - 3\sin x$ **c** $e^x \cos x$

d $e^{-x}\sin x$ **e** $\ln(\sin x)$ **f** $e^{2x}\tan x$

g $3\tan \pi x$ **h** $\cos\frac{x}{2}$ **i** $3\tan 2x$

j $x\cos x$ **k** $\dfrac{\sin x}{x}$ **l** $x\tan x$

m $e^{\cos\sqrt{x}}$ **n** $\dfrac{2\sqrt{x}}{\tan x}$ **o** $\dfrac{\cos x + \sin 2x}{x^3}$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos\sqrt{x}$ **c** $\sqrt{\cos x}$ **d** $\sin(e^x)$

e $\sin^2 x$ **f** $\cos^3 x$ **g** $5\cos^2 2x$ **h** $\cos x \sin 2x$

i $\cos(\cos x)$ **j** $\cos^3 4x$ **k** $\dfrac{2}{\sin^2 4x}$ **l** $\frac{1}{3}\tan^2 2x$

4 Find the gradient of the tangent to:

a $f(x) = \sin^3 x$ at the point where $x = \frac{2\pi}{3}$

b $f(x) = \tan(2x - \frac{\pi}{6})$ at the point where $x = \frac{\pi}{6}$

c $f(x) = \cos x \sin x$ at the point where $x = \frac{\pi}{4}$.

5 Consider the function $f(x) = 2\cos^2 x + 2\sin^2 x + 1$.

a Find $f'(x)$. **b** Explain your answer to **a**.

6 **a** Show that $\dfrac{\cos\theta + i\sin\theta}{e^{i\theta}}$ is a constant by showing that its derivative is zero.

b Find the value of this constant by substituting $\theta = 0$.

c Hence establish Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$.

H SECOND DERIVATIVES

Given a function $f(x)$, the derivative $f'(x)$ is known as the **first derivative**.

The **second derivative** of $f(x)$ is the derivative of $f'(x)$, or **the derivative of the first derivative**.

We use $f''(x)$, y'', or $\dfrac{d^2y}{dx^2}$ to represent the second derivative.

$f''(x)$ reads "f *double dashed* x".

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ reads "*dee two y by dee x squared*".



Example 16 Self Tutor

Find $f''(x)$ given:

a $f(x) = x^3 - \dfrac{3}{x}$ **b** $f(x) = x\cos x + \dfrac{1}{x}$

a
$$f(x) = x^3 - 3x^{-1}$$
$$\therefore\ f'(x) = 3x^2 + 3x^{-2}$$
$$\therefore\ f''(x) = 6x - 6x^{-3}$$
$$= 6x - \frac{6}{x^3}$$

b
$$f(x) = x\cos x + x^{-1}$$
$$\therefore\ f'(x) = (1)\cos x + (x)(-\sin x) - x^{-2}$$
$$= \cos x - x\sin x - x^{-2}$$
$$\therefore\ f''(x) = -\sin x - (1)\sin x - (x)\cos x + 2x^{-3}$$
$$= -2\sin x - x\cos x + \frac{2}{x^3}$$

EXERCISE 18H

1 Find $f''(x)$ given that:

a $f(x) = 3x^2 - 6x + 2$ **b** $f(x) = \dfrac{2}{\sqrt{x}} - 1$ **c** $f(x) = 2x^3 - 3x^2 - x + 5$

d $f(x) = \dfrac{2 - 3x}{x^2}$ **e** $f(x) = (1 - 2x)^2$ **f** $f(x) = \dfrac{x + 2}{2x - 1}$

2 Find $\dfrac{d^2y}{dx^2}$ given that:

a $y = x - x^3$ **b** $y = x^2 - \dfrac{5}{x^2}$ **c** $y = 2 - \dfrac{3}{\sqrt{x}}$

d $y = \dfrac{4 - x}{x}$ **e** $y = (x^2 - 3x)^2$ **f** $y = x^2 - x + \dfrac{1}{1 - x}$

g $y = e^{3x} + 2x$ **h** $y = \dfrac{1 - e^{-x}}{x}$ **i** $y = \dfrac{3 - x}{xe^x}$

3 Given $f(x) = x^3 - 2x + 5$, find:

a $f(2)$ **b** $f'(2)$ **c** $f''(2)$

4 Find the value(s) of x such that $f''(x) = 0$, given:

a $f(x) = 2x^3 - 6x^2 + 5x + 1$ **b** $f(x) = x^4 - 10x^3 + 36x^2 - 72x + 108$

5 Consider the function $f(x) = 2x^3 - x$.

Copy and complete the table alongside by indicating whether $f(x)$, $f'(x)$, and $f''(x)$ are positive $(+)$, negative $(-)$, or zero (0) at the given values of x.

x	-1	0	1
$f(x)$	$-$		
$f'(x)$			
$f''(x)$			

6 Given $f(x) = x^2 - \dfrac{1}{x}$, find:

a $f(1)$ **b** $f'(1)$ **c** $f''(1)$

7 Given $f(x) = 3e^x - 2x$, find:

a $f(1)$ **b** $f'(1)$ **c** $f''(1)$

8 Find $\dfrac{d^2y}{dx^2}$ given that:

a $y = 3\tan x$ **b** $y = x\sin x$ **c** $y = \sin^2 3x$

d $y = \dfrac{\cos^2 x - x}{x^2}$ **e** $y = e^{-x}\sin x$ **f** $y = \tan(x^2)$

9 Suppose $y = Ae^{kx}$ where A and k are constants. Show that:

a $\dfrac{dy}{dx} = ky$ **b** $\dfrac{d^2y}{dx^2} = k^2y$

10 Suppose $f(x) = 2\sin^3 x - 3\sin x$.

a Show that $f'(x) = -3\cos x\cos 2x$. **b** Find $f''(x)$.

11 Given $f(x) = \frac{2}{3}\sin 3x$, find $f''(\frac{2\pi}{9})$.

12 Find $\dfrac{d^2y}{dx^2}$ given:

a $y = -\ln x$ **b** $y = x\ln x$ **c** $y = (\ln x)^2$

13 If $y = 2e^{3x} + 5e^{4x}$, show that $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$.

14 If $y = \sin(2x+3)$, show that $\dfrac{d^2y}{dx^2} + 4y = 0$.

15 If $y = 2\sin x + 3\cos x$, show that $\dfrac{d^2y}{dx^2} + y = 0$.

REVIEW SET 18A

1 Find $f'(x)$ given that $f(x)$ is:

a $5x^3$ **b** $x^6 - 5x$ **c** $7x^2 - \dfrac{3}{x}$

d $3x - \dfrac{4}{x^2}$ **e** $2x\sqrt{x}$ **f** $4\sqrt{x} - \dfrac{1}{\sqrt{x}}$

2 Find $\dfrac{dy}{dx}$ for:

a $y = 3x^2 - x^4$ **b** $y = \dfrac{x^3 - x}{x^2}$ **c** $y = x^2\sqrt{x-2}$

3 **a** Find $f'(x)$ given $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

b At what point on the curve $y = f(x)$ does the tangent have gradient 1?

4 Find $\dfrac{dy}{dx}$ if:

a $y = e^{x^3+2}$ **b** $y = \ln\left(\dfrac{x+3}{x^2}\right)$ **c** $y = x^3e^{2x}$

5 Find the gradient of the tangent to:

a $f(x) = -x^2 + 4x - 2$ at $(-3, -23)$ **b** $y = (2-3x)^5$ at $x = 1$.

6 Differentiate with respect to x:

a $5x - 3x^{-1}$ **b** $(3x^2 + \sqrt{x})^4$ **c** $(x^2+1)(1-x^2)^3$

7 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

8 Differentiate with respect to x:

a $\sin 5x \ln x$ **b** $\sin x \cos 2x$ **c** $e^{-2x}\tan x$

9 Find the gradient of the tangent to $y = \sin^2 x$ at the point where $x = \frac{\pi}{3}$.

10 Consider the function $f(x) = \dfrac{x^2 - 4x - 1}{e^x}$.

a Find $f'(x)$.

b Find the gradient of the tangent to $y = f(x)$ at $x = 1$.

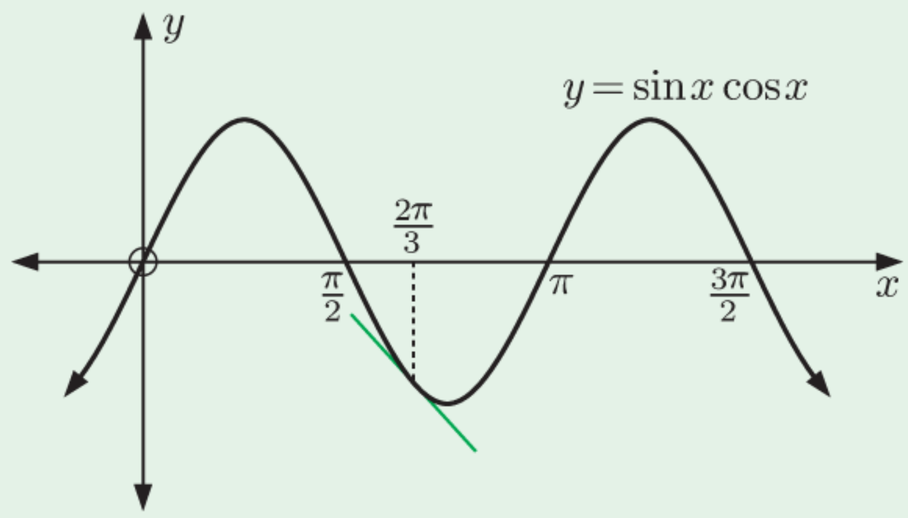
c For what values of x is the tangent to $y = f(x)$ horizontal?

11 Given $y = 3e^x - e^{-x}$, show that $\dfrac{d^2y}{dx^2} = y$.

12 Find the derivative with respect to x of:

a $f(x) = (x^2 + 3)^4$ **b** $g(x) = \dfrac{\sqrt{x+5}}{x^2}$ **c** $h(x) = \dfrac{e^{4x}}{1-2x}$

13 The graph of $y = \sin x \cos x$ is shown alongside.



a Find $\dfrac{dy}{dx}$.

b Show that $y = \frac{1}{2}\sin 2x$ has the same derivative.

c Find the gradient of the illustrated tangent.

14 For $f(x) = 2\sin x + \cos 2x$, find:

a $f(\frac{\pi}{2})$ **b** $f'(\frac{\pi}{2})$ **c** $f''(\frac{\pi}{2})$

15 Find the gradient of the tangent to $y = 3\tan\frac{x}{2}$ at $x = \frac{\pi}{3}$.

16 Find $\dfrac{d^2y}{dx^2}$ for:

a $y = \frac{1}{8}x^4 + \frac{1}{6}x^3 - \frac{1}{4}x^2$ **b** $y = xe^{-x}$

17 For the function $f(x) = \sqrt{x}\cos 4x$:

a Find $f'(x)$ and $f''(x)$.

b Hence find: **i** $f'(\frac{\pi}{16})$ **ii** $f''(\frac{\pi}{8})$

REVIEW SET 18B

1 Find $f'(x)$ given that $f(x)$ is:

a $3x^2 - 7x + 4$ **b** $(x+5)^2$ **c** $2\sqrt{x} - \dfrac{3}{x}$ **d** $6x^2\sqrt{x}$

2 Find $\dfrac{dy}{dx}$ for:

a $y = 2x^3 - 6x^2 + 7x - 4$ **b** $y = \dfrac{3}{x} - \dfrac{5}{x^3}$ **c** $y = \dfrac{15}{\sqrt[3]{x}}$

3 If $f(x) = 7 + x - 3x^2$, find:

a $f(3)$ **b** $f'(3)$ **c** $f''(3)$

4 Differentiate with respect to x:

a $y = x^3\sqrt{1-x^2}$ **b** $y = \dfrac{x^2 - 3x}{\sqrt{x+1}}$ **c** $y = x^3 - x + \dfrac{1}{\sqrt{x}}$

5 **a** Find $\dfrac{dy}{dx}$ for $y = xe^x$.

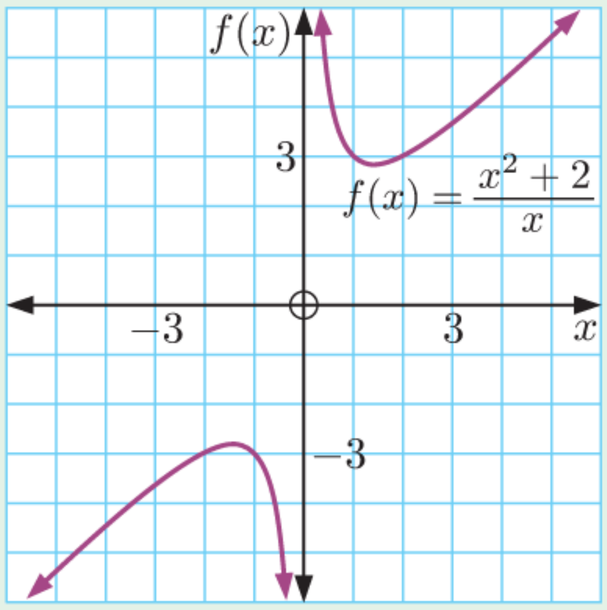
b Find all points on the curve $y = xe^x$ where the gradient of the tangent is $2e$.

6 Differentiate with respect to x:

a $f(x) = \ln(e^x + 3)$ **b** $f(x) = \ln\left[\dfrac{(x+2)^3}{x}\right]$ **c** $f(x) = \ln(\tan x)$

7 Suppose $y = \left(x - \dfrac{1}{x}\right)^4$. Find $\dfrac{dy}{dx}$ at the point where $x = 1$.

8 The graph of $f(x) = \dfrac{x^2+2}{x}$ is shown alongside.



PRINTABLE GRAPH

a Find $f'(x)$.

b Hence find the gradient of the tangent at:

i $x = 1$ **ii** $x = -2$

c Copy the graph, and include the information from **b**.

9 Find $\dfrac{dy}{dx}$ if:

a $y = \ln(x^3 - 3x)$ **b** $y = \dfrac{e^x}{x^2}$ **c** $y = e^{2x}\sin x$

10 Suppose $f(x) = 2x^4 - 4x^3 - 9x^2 + 4x + 7$.

a Find $f''(x)$. **b** Find x such that $f''(x) = 0$.

11 Differentiate with respect to x:

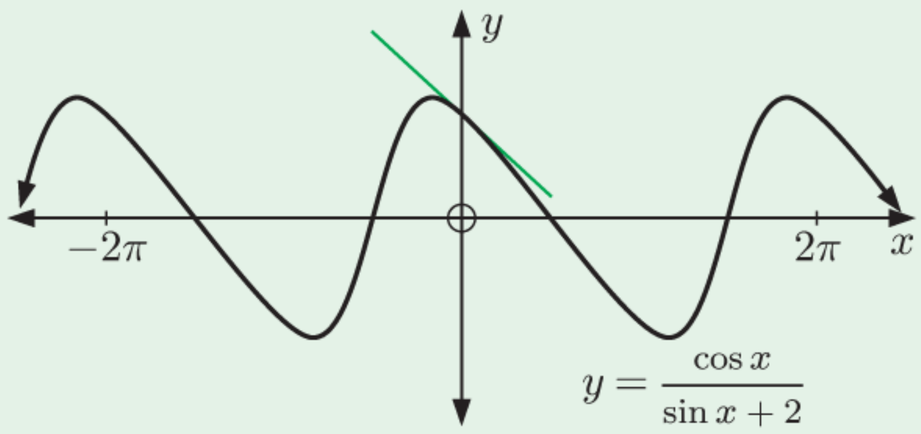
a $10x - \sin 10x$ **b** $\ln\left(\dfrac{1}{\cos x}\right)$ **c** $\sin 5x \ln(2x)$

12 Find the gradient of the tangent to:

a $y = \dfrac{x^3}{x+1}$ at $x = 2$ **b** $y = \tan 4x$ at $x = \frac{\pi}{3}$.

13 Suppose $f(x) = a\ln(bx)$ where $f(e) = 12$ and $f'(2) = 2$. Find the constants a and b.

14 The graph of $y = \dfrac{\cos x}{\sin x + 2}$ is shown alongside.



a Find the gradient of the illustrated tangent.

b Show that it is impossible to draw a tangent to the graph with gradient $-\frac{1}{2}$.

15 **a** If $y = \dfrac{e^x}{\sqrt{x}}$, show that $\dfrac{dy}{dx} = \dfrac{e^x(2x-1)}{2x\sqrt{x}}$.

b For what values of x is $\dfrac{dy}{dx}$: **i** zero **ii** undefined?

16 Suppose $y = 3\sin 2x + 2\cos 2x$. Show that $4y + \dfrac{d^2y}{dx^2} = 0$.

17 Let $f(x) = \dfrac{6x}{3+x^2}$. Find the value(s) of x such that:

a $f(x) = -\frac{1}{2}$ **b** $f'(x) = 0$ **c** $f''(x) = 0$

Chapter 19

Properties of curves

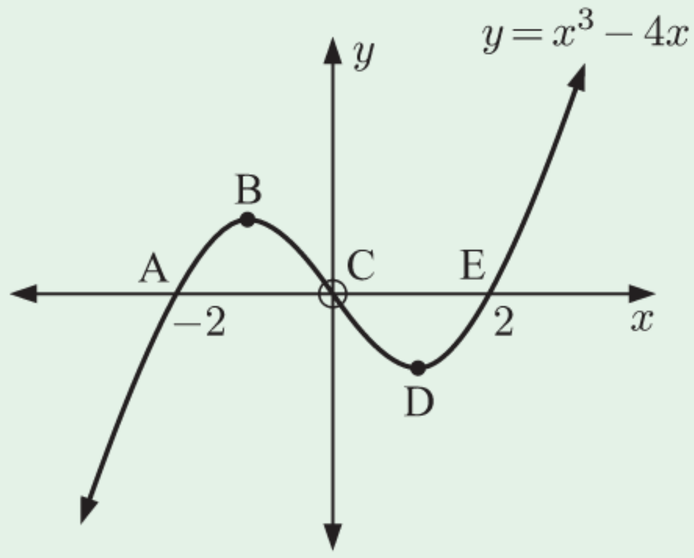
Contents:

A Tangents
B Normals
C Increasing and decreasing
D Stationary points
E Shape
F Inflection points



OPENING PROBLEM

The curve $y = x^3 - 4x$ is shown alongside.



Things to think about:

a What features of this curve could you describe to someone?

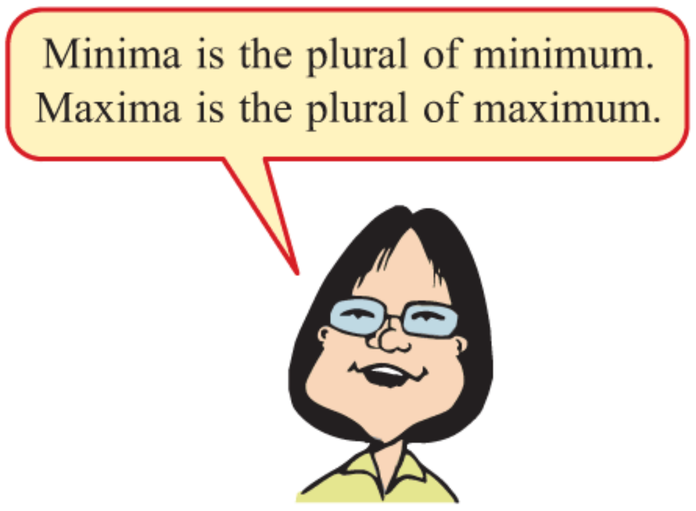
b How can we use calculus to help identify important features?

c How would you describe the *shape* of a curve? What rules would you use to identify a curve's shape?

In the previous Chapter we saw how to differentiate many types of functions.

In this Chapter we will use derivatives to find:

- tangents and normals to curves
- turning points, which are local minima and maxima
- inflection points where the curve changes shape.

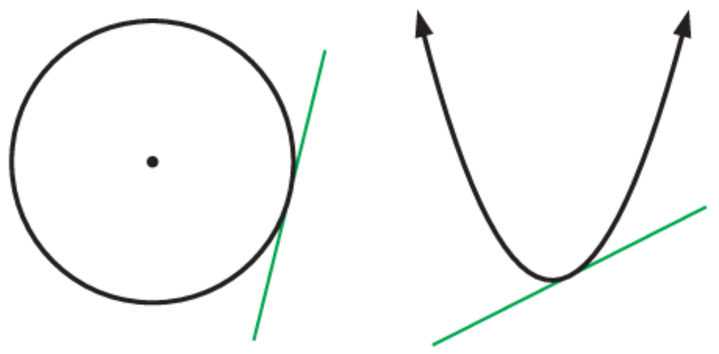


A TANGENTS

The **tangent** to a curve at point A is the best approximating straight line to the curve at A.

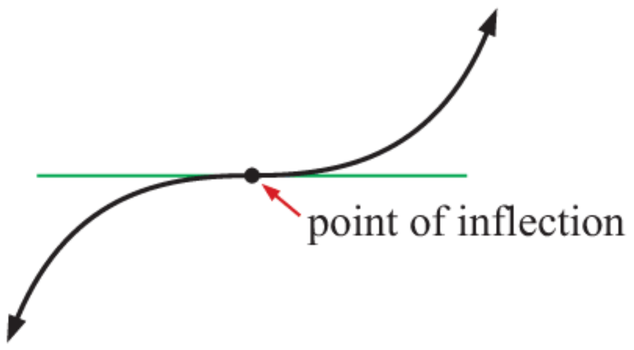
In cases we have seen already, the tangent *touches* the curve.

For example, consider the tangents to the circle and parabola shown.



However, we note that for some functions:

- The tangent may intersect the curve again somewhere else.
- It is possible for the tangent to pass through the curve at the point of tangency. If this happens, we call it a **point of inflection**.



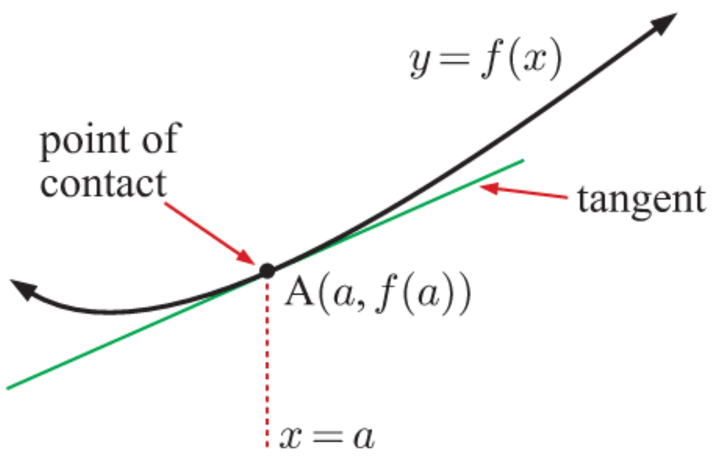
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a)$.

The equation of the tangent is

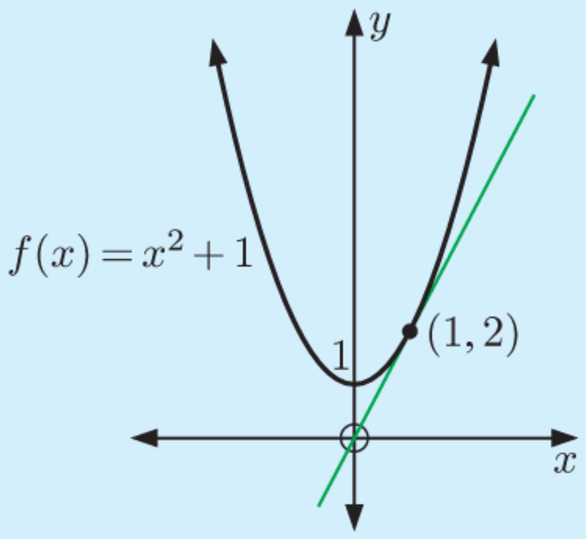
$$y - f(a) = f'(a)(x - a)$$

or $$\boldsymbol{y = f'(a)(x - a) + f(a).}$$



Example 1 **Self Tutor**

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1^2 + 1 = 2$, the point of contact is $(1, 2)$.

Now $f'(x) = 2x$, so at $x = 1$ the tangent has gradient $f'(1) = 2$.

$\therefore$ the tangent has equation $y = 2(x - 1) + 2$

which is $y = 2x$.



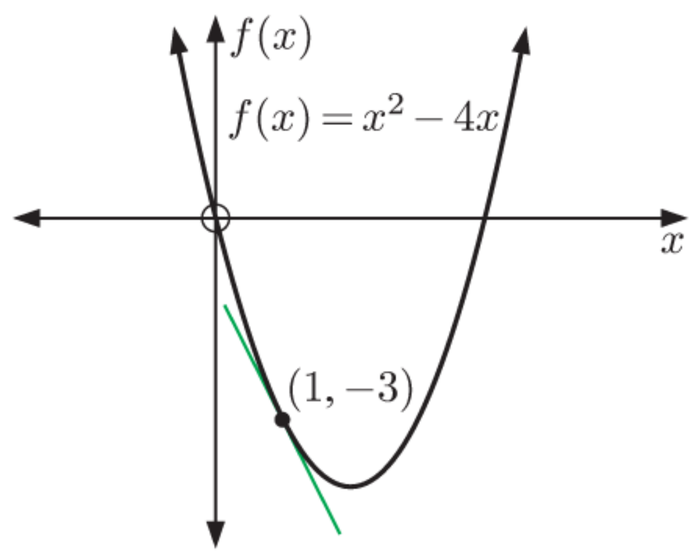
GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 19A

1 The graph of $f(x) = x^2 - 4x$ is shown alongside.

a Find $f'(x)$.

b Hence find the equation of the illustrated tangent.



2 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.



GRAPHING PACKAGE

Check your answers using technology.

Example 2 **Self Tutor**

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\dfrac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0,

so $3x^2 - 12 = 0$

$\therefore\ 3(x^2 - 4) = 0$

$\therefore\ 3(x + 2)(x - 2) = 0$

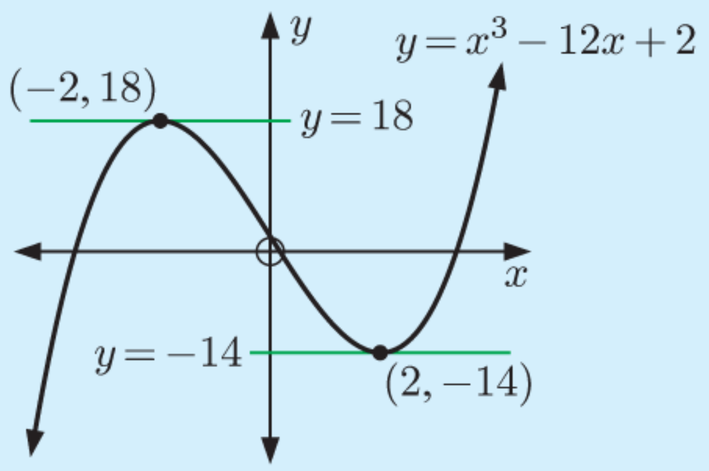
$\therefore\ x = -2$ or 2

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.



3 Find the equations of any horizontal tangents to:

a $y = 2x^3 + 3x^2 - 12x + 1$ **b** $y = -x^3 + 3x^2 + 9x - 4$ **c** $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

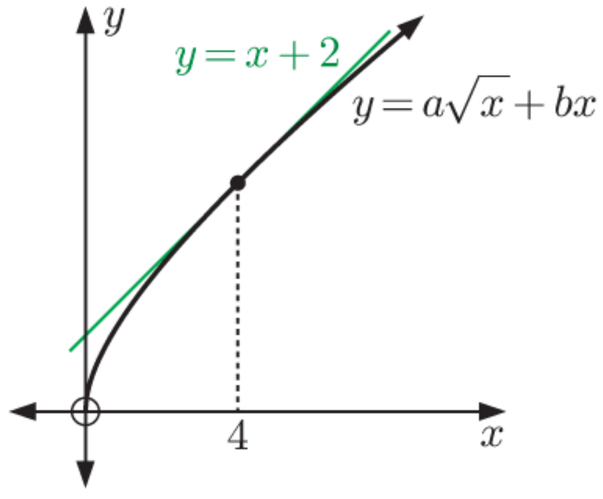
4 The tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

a Find k. **b** Hence find the equation of this tangent.

5 Find another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

6 Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

7 Find the values of a and b.



8 Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

9 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$.

a Find $f'(x)$.

b Find the values of x at which the tangent to the curve is horizontal.

c Show that the tangents at these points are the same line.

10 Consider the curve $y = a\sqrt{1 - bx}$ where a and b are constants. The tangent to this curve at the point where $x = -1$ is $3x + y = 5$. Find the values of a and b.

Example 3 **Self Tutor**

Show that the equation of the tangent to $y = \ln x$ at the point where $y = -1$ is $y = ex - 2$.

When $y = -1$, $\ln x = -1$

$\therefore\ x = e^{-1} = \frac{1}{e}$

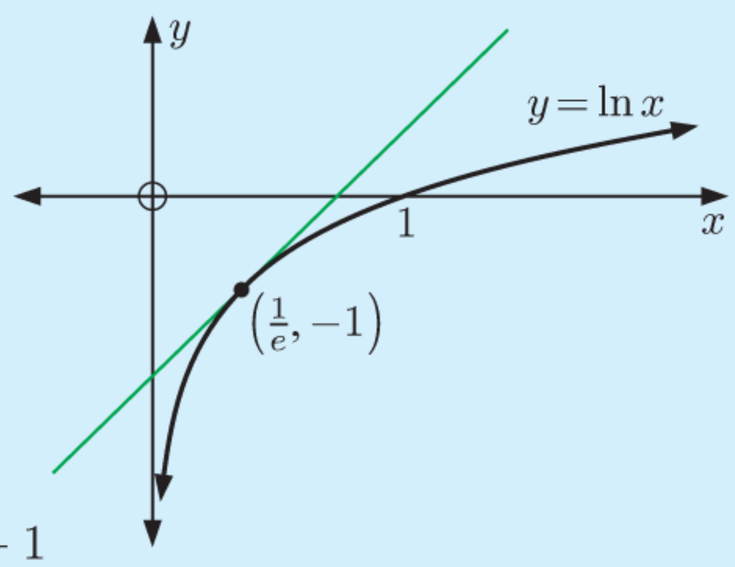
$\therefore$ the point of contact is $\left(\frac{1}{e}, -1\right)$.

Now $f(x) = \ln x$ has derivative $f'(x) = \dfrac{1}{x}$

$\therefore$ the tangent at $\left(\frac{1}{e}, -1\right)$ has gradient $\dfrac{1}{\frac{1}{e}} = e$

$\therefore$ the tangent has equation $y = e\left(x - \frac{1}{e}\right) - 1$

which is $y = ex - 2$



11 Find the equation of the tangent to:

a $f(x) = e^{-x}$ at the point where $x = 2$ **b** $y = \ln(2 - x)$ at the point where $x = -1$

c $y = (x + 2)e^x$ at the point where $x = 1$ **d** $y = \ln\sqrt{x}$ at the point where $y = -1$

e $y = e^{3x-5}$ at the point where $y = e$.

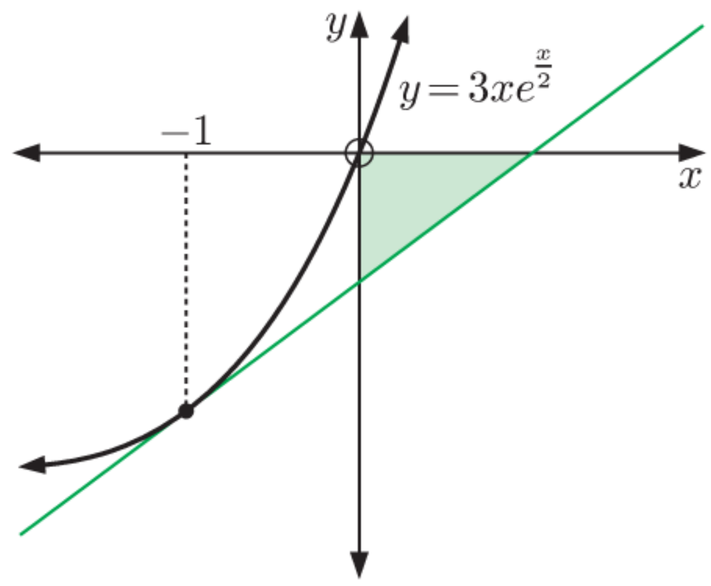
12 Consider $f(x) = \ln(x(x-2))$.

a State the domain of $f(x)$. **b** Find $f'(x)$.

c Find the equation of the tangent to $y = f(x)$ at the point where $x = 3$.

13 Find the axes intercepts of the tangent to $y = x^2e^x$ at $x = 1$.

14 Find the exact area of the shaded triangle.



Example 4 **Self Tutor**

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

When $x = \frac{\pi}{4}$, $y = \tan\frac{\pi}{4} = 1$

$\therefore$ the point of contact is $(\frac{\pi}{4}, 1)$.

Now $f(x) = \tan x$ has derivative $f'(x) = \dfrac{1}{\cos^2 x}$

$\therefore$ the tangent at $(\frac{\pi}{4}, 1)$ has gradient $\dfrac{1}{\cos^2(\frac{\pi}{4})} = \dfrac{1}{\frac{1}{2}} = 2$

$\therefore$ the tangent has equation $y = 2(x - \frac{\pi}{4}) + 1$

which is $y = 2x + (1 - \frac{\pi}{2})$

15 Find the equation of the tangent to:

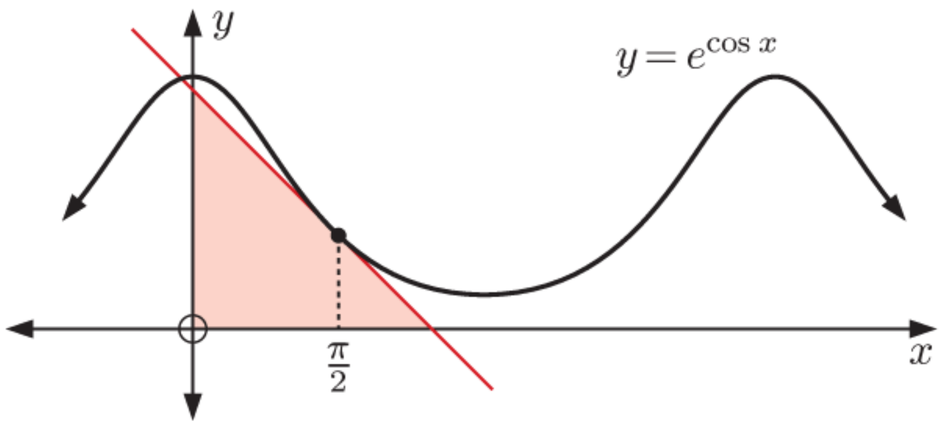
a $y = \sin x$ at the origin **b** $y = \tan x$ at the origin

c $y = \cos x$ at $x = \frac{\pi}{6}$ **d** $y = \dfrac{1}{\sin 2x}$ at $x = \frac{\pi}{4}$

e $y = \cos 2x + 3\sin x$ at $x = \frac{\pi}{2}$.

16 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

17 The graph of $y = e^{\cos x}$ is shown alongside. Find the area of the shaded triangle.



Example 5 **Self Tutor**

Find where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.

Let $f(x) = x^3 + x + 2$

$\therefore\ f'(x) = 3x^2 + 1$ and $\therefore\ f'(1) = 3 + 1 = 4$

$\therefore$ the equation of the tangent at $(1, 4)$ is $4x - y = 4(1) - 4$

or $y = 4x$.

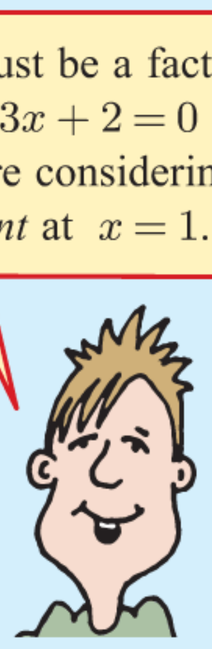
The curve meets the tangent again when $x^3 + x + 2 = 4x$

$\therefore\ x^3 - 3x + 2 = 0$

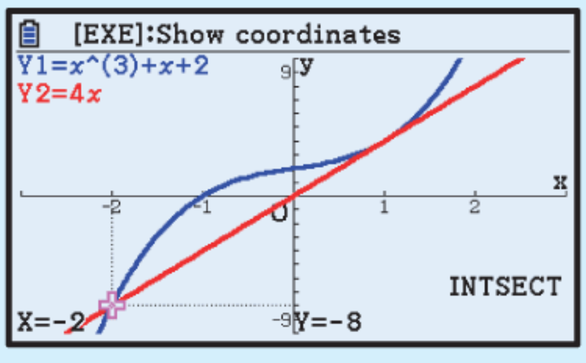
$\therefore\ (x - 1)^2(x + 2) = 0$

When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

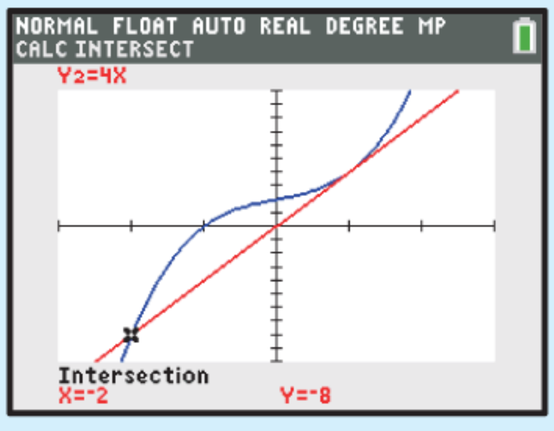
$\therefore$ the tangent meets the curve again at $(-2, -8)$.



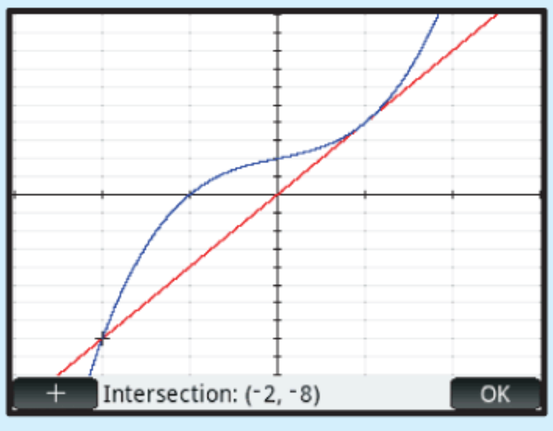
Casio fx-CG50



TI-84 Plus CE



HP Prime



18 Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

19 Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

20 Find where the tangent to the curve $y = \frac{1}{x} - \frac{1}{x^2}$ at the point where $x = 1$, meets the curve again.

Example 6 **Self Tutor**

Find the equations of the tangents to $y = x^2$ from the external point $(2, 3)$.

Let (a, a^2) be a general point on $f(x) = x^2$.

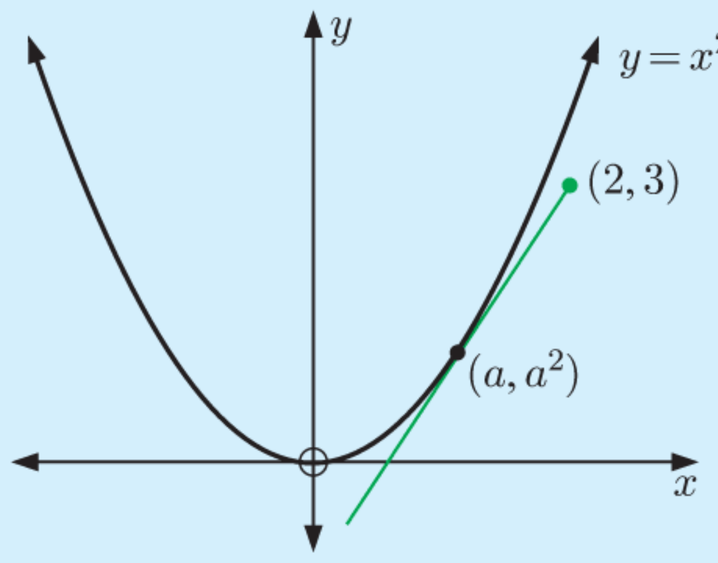
Now $f'(x) = 2x$, so $f'(a) = 2a$

$\therefore$ the equation of the tangent at (a, a^2) is

$y = 2a(x - a) + a^2$ which is $y = 2ax - a^2$.

Thus the tangents which pass through $(2, 3)$ satisfy

$$3 = 2a(2) - a^2$$
$$\therefore\ a^2 - 4a + 3 = 0$$
$$\therefore\ (a - 1)(a - 3) = 0$$
$$\therefore\ a = 1 \text{ or } 3$$



$\therefore$ two tangents pass through the external point $(2, 3)$.

If $a = 1$, the tangent has equation $y = 2x - 1$ with point of contact $(1, 1)$.

If $a = 3$, the tangent has equation $y = 6x - 9$ with point of contact $(3, 9)$.

21 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$.

b Hence find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

22 **a** Find the equation of the tangent to $y = x^2 + 4x$ at the point where $x = a$.

b Hence find the equations of the tangents to $y = x^2 + 4x$ which pass through the external point $(1, -4)$. State the coordinates of the points of contact.

23 Find the equations of the tangents to $y = x^2 - 3x + 1$ which pass through $(1, -10)$.

24 **a** Find the equation of the tangent to $y = e^x$ at the point where $x = a$.

b Hence find the equation of the tangent to $y = e^x$ which passes through the origin.

25 Consider the function $y = 2x^2$.

a Find the equations of the tangents to the function from the external point $(1, -6)$.

b Find the points of contact for the tangents.

c Show that no tangents to the function pass through the point $(1, 4)$.

d Draw a graph of $y = 2x^2$ showing the information above.

26 Consider $f(x) = \dfrac{8}{x^2}$.

a Sketch the graph of the function.

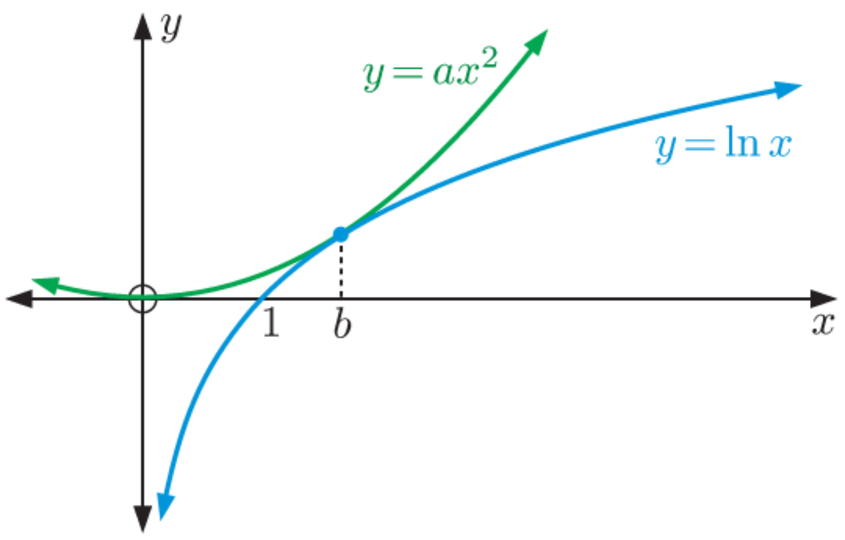
b Find the equation of the tangent at the point where $x = a$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d Find the area of triangle OAB and discuss the area of the triangle as $a \to \infty$.

27 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

28 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.





a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \dfrac{1}{b}$.

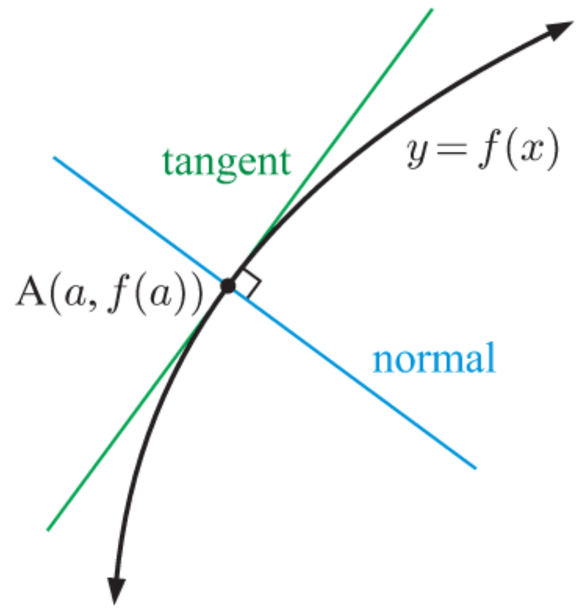
b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.

B NORMALS

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.



The gradients of perpendicular lines are negative reciprocals of each other, so:

The gradient of the normal to the curve at $x = a$ is $-\dfrac{1}{f'(a)}$.

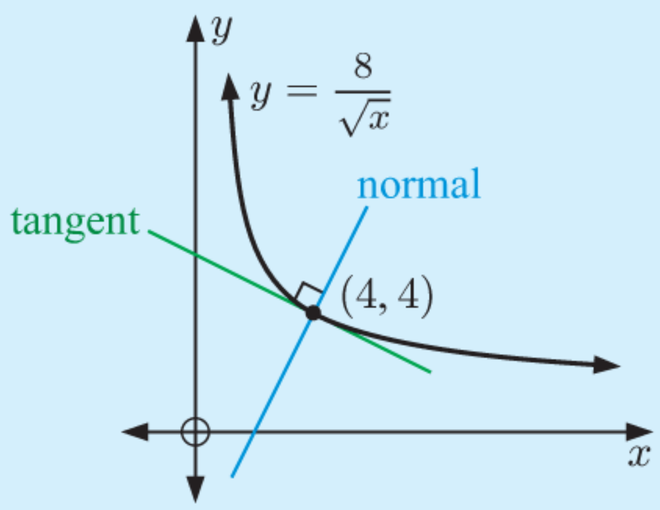
The equation of the normal to the curve at $x = a$ is

$$y = -\frac{1}{f'(a)}(x - a) + f(a).$$

Example 7 **Self Tutor**

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \dfrac{8}{\sqrt{4}} = \dfrac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $\dfrac{dy}{dx} = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $\frac{2}{1}$.

$\therefore$ the equation of the normal is

$$2x - 1y = 2(4) - 1(4)$$

or $2x - y = 4$

EXERCISE 19B

1 Find the equation of the normal to:

a $y = x^2$ at the point $(4, 16)$ **b** $y = x^3 - 5x + 2$ at $x = -2$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$ **d** $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$

e $f(x) = \dfrac{x}{1 - 3x}$ at $(-1, -\frac{1}{4})$ **f** $f(x) = \dfrac{x^2}{1 - x}$ at $(2, -4)$.



Use technology to check your answers.

2 Suppose $f(x) = x^2 - \dfrac{8}{x}$.

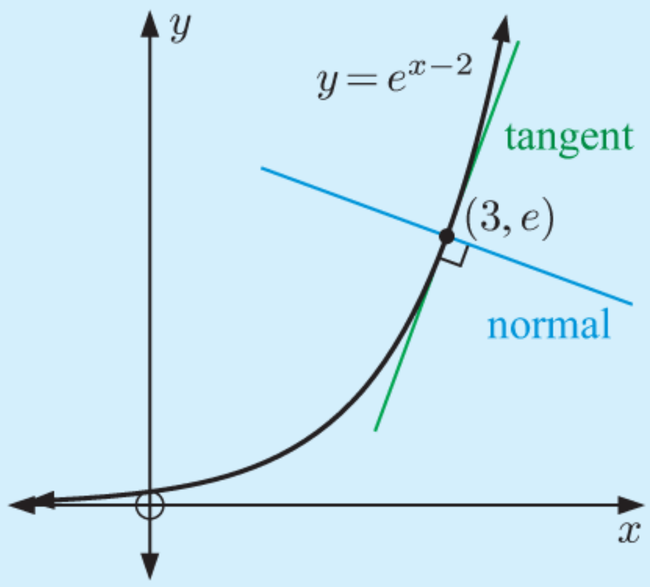
a Find the equation of the tangent to $y = f(x)$ at $x = -2$.

b Find the equation of the normal to $y = f(x)$ at $x = 3$.

Example 8 **Self Tutor**

Find the equation of the normal to $y = e^{x-2}$ at the point where $x = 3$.

When $x = 3$, $y = e^{3-2} = e$. So, the point of contact is $(3, e)$.



Now as $y = e^{x-2}$, $\dfrac{dy}{dx} = e^{x-2}$

$\therefore$ when $x = 3$, $\dfrac{dy}{dx} = e^{3-2} = e$

$\therefore$ the normal at $(3, e)$ has gradient $-\dfrac{1}{e}$.

$\therefore$ the equation of the normal is

$$y = -\frac{1}{e}(x - 3) + e$$

$\therefore$ $x + ey = 3 + e^2$

3 Find the equation of the normal to:

a $y = e^{-x}$ at $x = 0$ **b** $y = \ln x$ at $x = e$

c $y = e^{2x-1}$ at $x = 1$ **d** $y = \sin x$ at $x = \frac{\pi}{3}$

e $y = \tan 3x$ at $x = \frac{\pi}{4}$ **f** $y = \cos(2x - \frac{\pi}{3})$ at $x = \frac{\pi}{2}$.

4 Consider the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

5 Find the points where the normal to $y = x^3 - 2x^2 + 1$ at $x = 1$, meets the curve again.

6 Find the equation of the normal to $f(x) = \cos x$ which passes through the origin.

7 Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.
Hint: There is no normal at the point where $x = 0$, as this is the end point of the function.

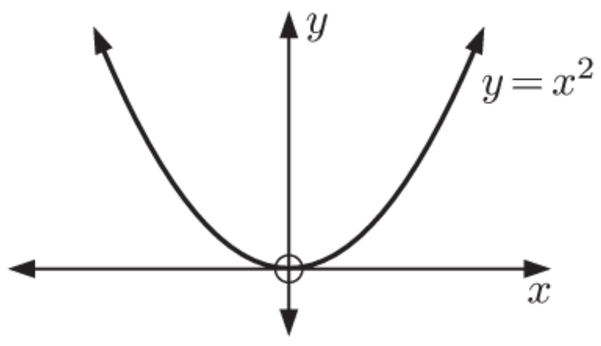
C INCREASING AND DECREASING

When we draw a graph of a function, we may notice that the function is **increasing** or **decreasing** over particular intervals.

Suppose S is an interval in the domain of $f(x)$, so $f(x)$ is defined for all x in S.

- $f(x)$ is **increasing** on $S \Leftrightarrow f(a) \leqslant f(b)$ for all $a, b \in S$ such that $a < b$.
- $f(x)$ is **decreasing** on $S \Leftrightarrow f(a) \geqslant f(b)$ for all $a, b \in S$ such that $a < b$.

For example, $y = x^2$ is decreasing for $x \leqslant 0$ and increasing for $x \geqslant 0$.



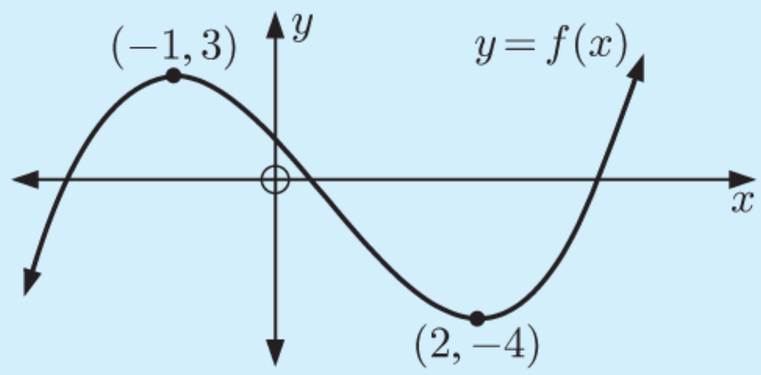
Important: In the example $y = x^2$, people often get confused about the point $x = 0$. They wonder how the curve can be both increasing and decreasing at this point. The answer is that the notion of increasing and decreasing is associated with *intervals*, not particular values for x. $y = x^2$ is decreasing *on the interval* $x \leqslant 0$ and increasing *on the interval* $x \geqslant 0$.

Example 9 **Self Tutor**

Find intervals where $f(x)$ is:

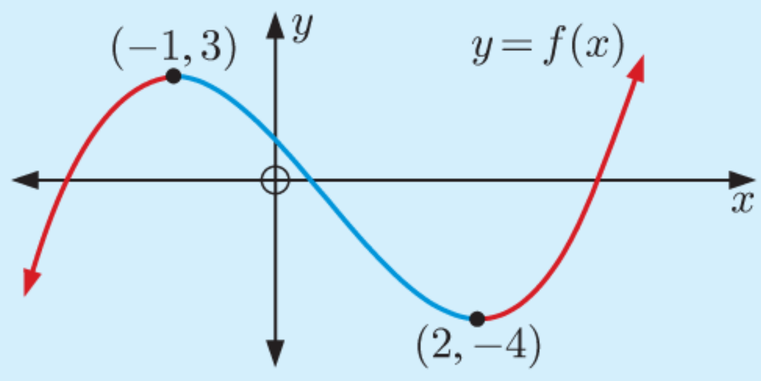
a increasing

b decreasing.



a $f(x)$ is increasing for $x \leqslant -1$ and for $x \geqslant 2$.

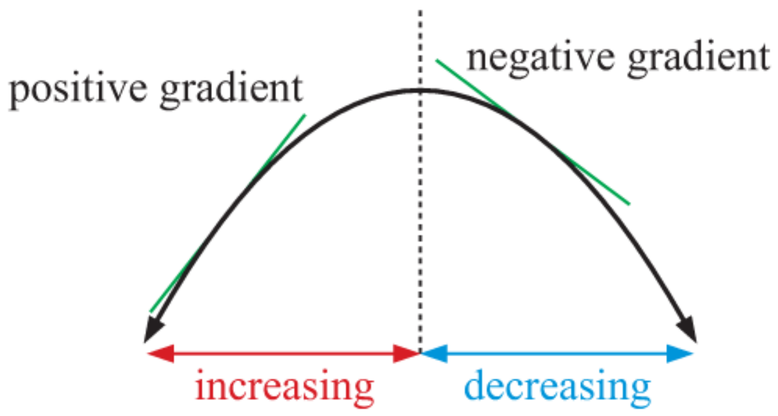
b $f(x)$ is decreasing for $-1 \leqslant x \leqslant 2$.



We can determine intervals where a curve $y = f(x)$ is increasing or decreasing by considering a **sign diagram** of the derivative function $f'(x)$.

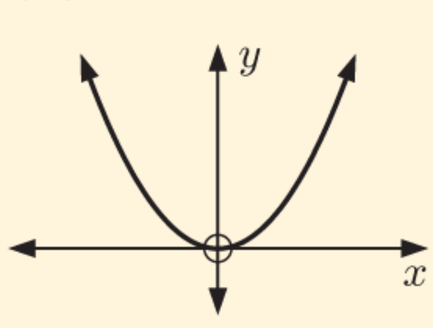
For most functions that we deal with in this course:

- $f(x)$ is **increasing** on $S \Leftrightarrow f'(x) \geqslant 0$ for all x in S
- $f(x)$ is **decreasing** on $S \Leftrightarrow f'(x) \leqslant 0$ for all x in S.



Sign diagrams for the derivative are extremely useful for determining intervals where a function is increasing or decreasing. Consider the following examples:

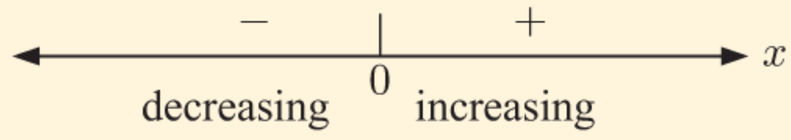
$f(x) = x^2$



DEMO

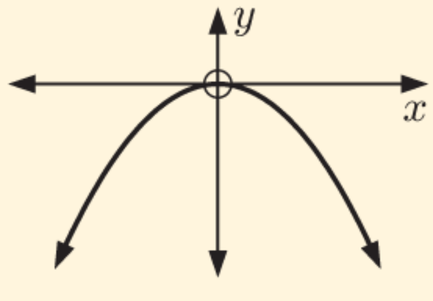


$f'(x) = 2x$ which has sign diagram



$\therefore \quad f(x) = x^2$ is $\begin{cases} \text{decreasing for } x \leqslant 0 \\ \text{increasing for } x \geqslant 0. \end{cases}$

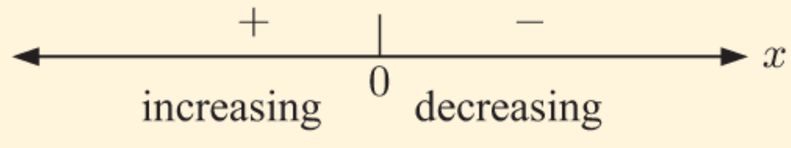
$f(x) = -x^2$



DEMO

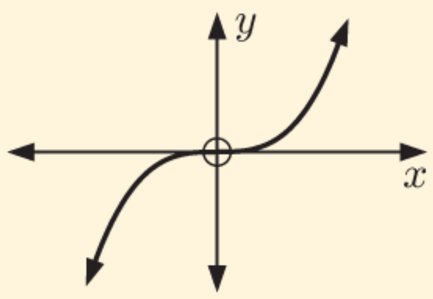


$f'(x) = -2x$ which has sign diagram



$\therefore \quad f(x) = -x^2$ is $\begin{cases} \text{increasing for } x \leqslant 0 \\ \text{decreasing for } x \geqslant 0. \end{cases}$

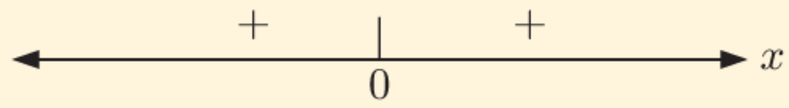
$f(x) = x^3$



DEMO



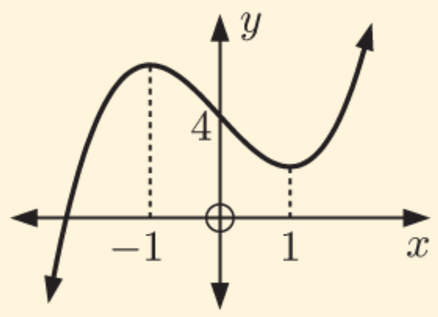
$f'(x) = 3x^2$ which has sign diagram



increasing for all x
(never negative)

$\therefore$ $f(x) = x^3$ is increasing for all $x \in \mathbb{R}$.

$f(x) = x^3 - 3x + 4$

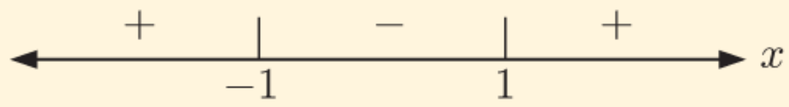


DEMO



$f'(x) = 3x^2 - 3$
$= 3(x^2 - 1)$
$= 3(x + 1)(x - 1)$

which has sign diagram



increasing decreasing increasing

$\therefore$ $f(x) = x^3 - 3x + 4$ is $\begin{cases} \text{increasing for } x \leqslant -1 \text{ and for } x \geqslant 1 \\ \text{decreasing for } -1 \leqslant x \leqslant 1. \end{cases}$

Example 10

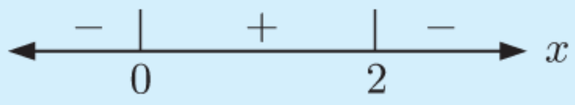
Self Tutor

Find the intervals where the following functions are increasing or decreasing:

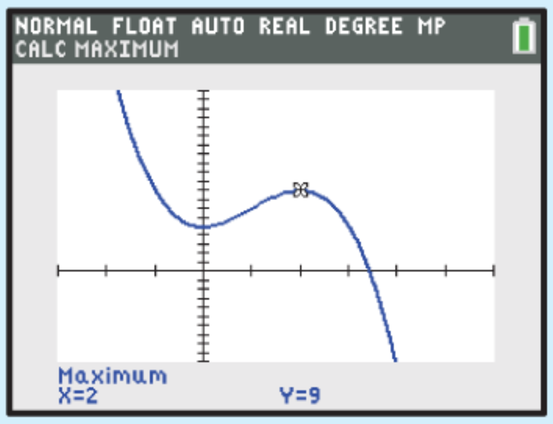
a $f(x) = -x^3 + 3x^2 + 5$ **b** $f(x) = 3x^4 - 8x^3 + 2$

a $f(x) = -x^3 + 3x^2 + 5$
$\therefore$ $f'(x) = -3x^2 + 6x$
$= -3x(x - 2)$

which has sign diagram:



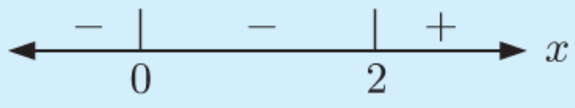
TI-84 Plus CE



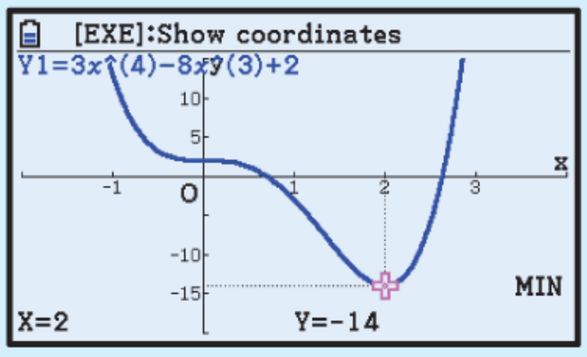
So, $f(x)$ is decreasing for $x \leqslant 0$ and for $x \geqslant 2$, and increasing for $0 \leqslant x \leqslant 2$.

b $f(x) = 3x^4 - 8x^3 + 2$
$\therefore$ $f'(x) = 12x^3 - 24x^2$
$= 12x^2(x - 2)$

which has sign diagram:



Casio fx-CG50



So, $f(x)$ is decreasing for $x \leqslant 2$, and increasing for $x \geqslant 2$.

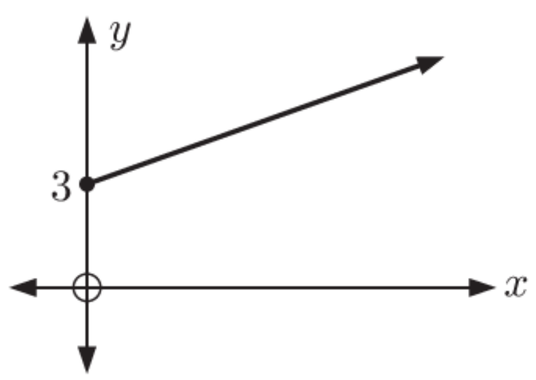
Remember that $f(x)$ must be defined for all x on an interval before we can classify the function as increasing or decreasing on that interval. We need to take care with vertical asymptotes and other values for x where the function is not defined.

EXERCISE 19C

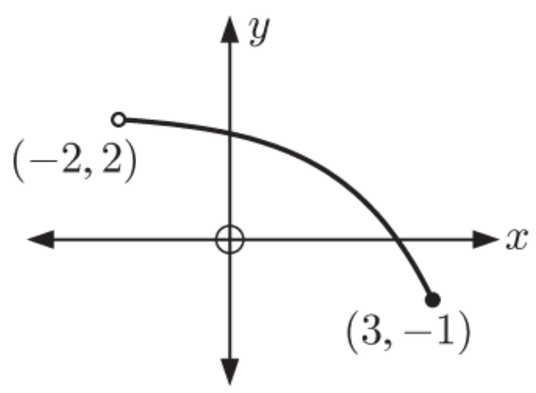
1 Write down the intervals where the graphs are:

i increasing **ii** decreasing.

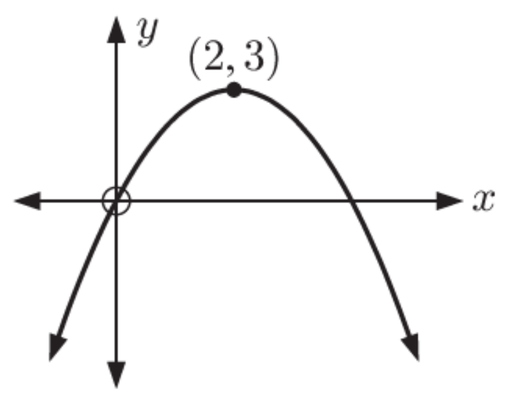
a



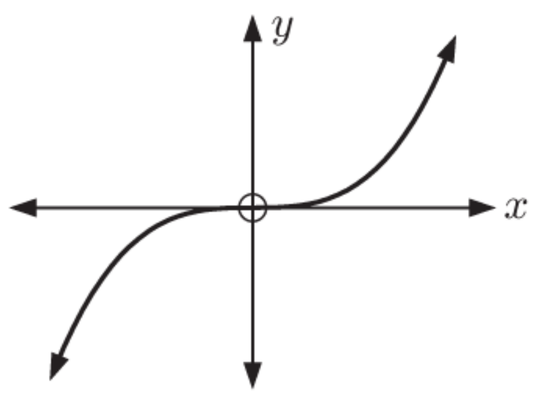
b



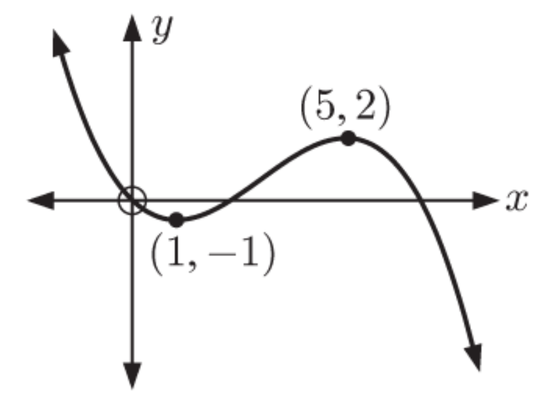
c



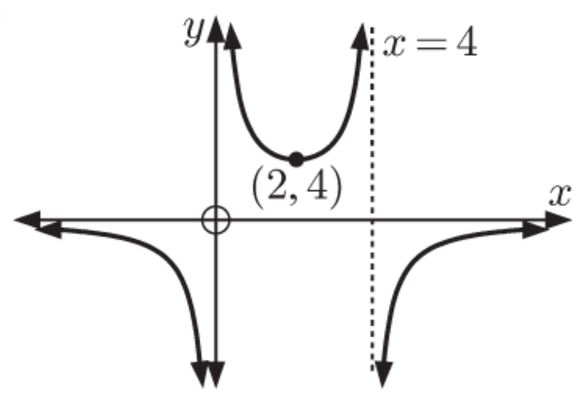
d



e



f

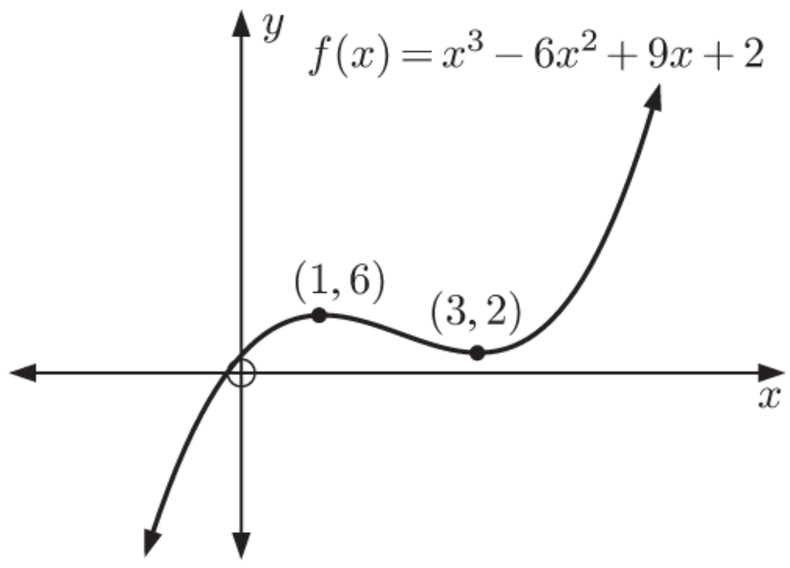


2 The graph of $f(x) = x^3 - 6x^2 + 9x + 2$ is shown alongside.

a Use the graph to write down the intervals where the function is:

i increasing **ii** decreasing.

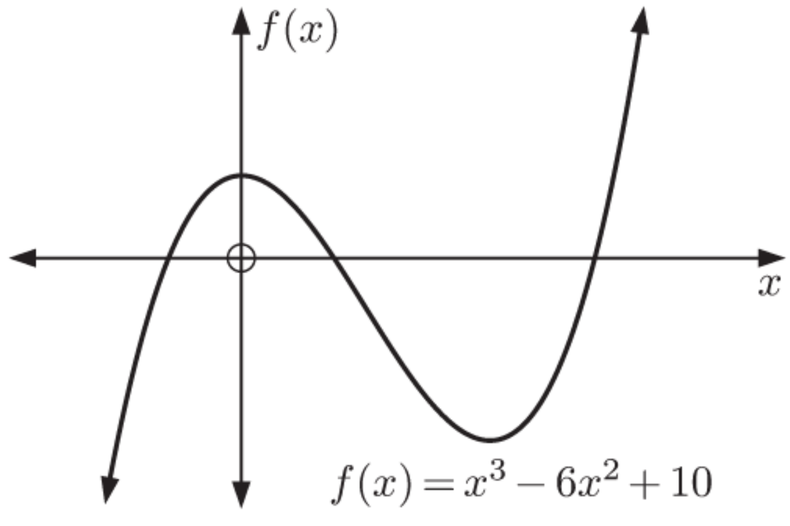
b Check your answer by finding $f'(x)$ and constructing its sign diagram.



3 The graph of $f(x) = x^3 - 6x^2 + 10$ is shown alongside.

a Find $f'(x)$, and draw its sign diagram.

b Find the intervals where $f(x)$ is increasing or decreasing.



4 Find the intervals where $f(x)$ is increasing or decreasing:

a $f(x) = x^2$ **b** $f(x) = -x^3$

c $f(x) = 2x^2 + 3x - 4$ **d** $f(x) = \dfrac{1}{x}$

e $f(x) = -\dfrac{1}{x^2}$ **f** $f(x) = \dfrac{2}{\sqrt{x}}$

g $f(x) = x^3 - 6x^2$ **h** $f(x) = -2x^3 + 4x$

i $f(x) = -4x^3 + 15x^2 + 18x + 3$ **j** $f(x) = x^3 - 3x^2 + 5x + 2$

k $f(x) = 3x^4 - 16x^3 + 24x^2 - 2$ **l** $f(x) = x^3 - 6x^2 + 3x - 1$

Check your answers using technology.

Example 11 **Self Tutor**

Consider $f(x) = \dfrac{2x-3}{x^2+2x-3}$.

a Show that $f'(x) = \dfrac{-2x(x-3)}{(x-1)^2(x+3)^2}$, and draw its sign diagram.

b Hence find the intervals where $y = f(x)$ is increasing or decreasing.

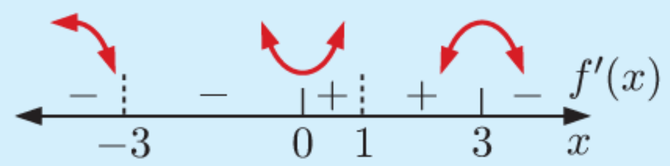
a $f(x) = \dfrac{2x-3}{x^2+2x-3}$

$\therefore\ f'(x) = \dfrac{2(x^2+2x-3)-(2x-3)(2x+2)}{(x^2+2x-3)^2}$ {quotient rule}

$= \dfrac{2x^2+4x-6-(4x^2-2x-6)}{((x-1)(x+3))^2}$

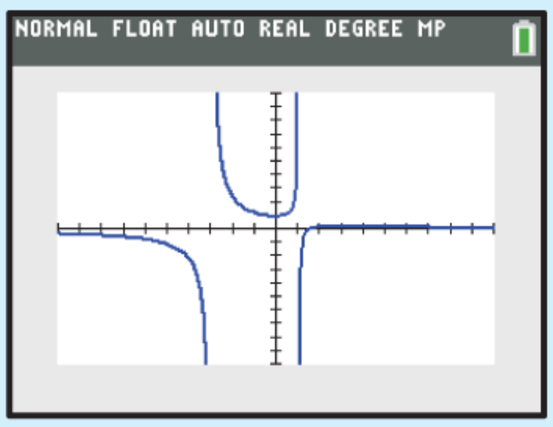
$= \dfrac{-2x^2+6x}{(x-1)^2(x+3)^2}$

$= \dfrac{-2x(x-3)}{(x-1)^2(x+3)^2}$ which has sign diagram:



b $f(x)$ is increasing for $0 \leqslant x < 1$
and for $1 < x \leqslant 3$.

$f(x)$ is decreasing for $x < -3$
and for $-3 < x \leqslant 0$
and for $x \geqslant 3$.



5 Let $f(x) = x + \dfrac{9}{x}$.

a Show that $f'(x) = \dfrac{(x+3)(x-3)}{x^2}$ and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is increasing or decreasing.

6 Consider $f(x) = \dfrac{4x}{x^2+1}$.

a Show that $f'(x) = \dfrac{-4(x+1)(x-1)}{(x^2+1)^2}$ and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is increasing or decreasing.

7 Consider $f(x) = \dfrac{4x}{(x-1)^2}$.

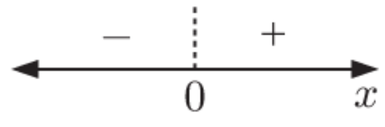
a Show that $f'(x) = \dfrac{-4(x+1)}{(x-1)^3}$ and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is increasing or decreasing.

8 Consider $f(x) = \dfrac{-x^2+4x-7}{x-1}$.

a Show that $f'(x) = \dfrac{-(x+1)(x-3)}{(x-1)^2}$ and draw its sign diagram.

b Hence find intervals where $y = f(x)$ is increasing or decreasing.

9 To find the intervals where $f(x) = \ln x$ is increasing or decreasing, Kenneth states that $f'(x) = \dfrac{1}{x}$, which has sign diagram 

. He therefore concludes that $f(x) = \ln x$ is increasing for $x > 0$, and decreasing for $x < 0$. Explain the mistake Kenneth has made.

10 Find the intervals where $f(x)$ is increasing or decreasing:

a $f(x) = e^x$ **b** $f(x) = \ln(x + 2)$ **c** $f(x) = 3 + e^{-x}$

d $f(x) = xe^x$ **e** $f(x) = x - 2\sqrt{x}$ **f** $f(x) = x^3 \ln x$

g $f(x) = \dfrac{x^3}{x^2 - 1}$ **h** $f(x) = e^{-x^2}$ **i** $f(x) = (3x^2 + 1)^4$

j $f(x) = x^2 + \dfrac{4}{x - 1}$ **k** $f(x) = \ln(x^2 + 4)$ **l** $f(x) = \dfrac{e^{-x}}{x}$

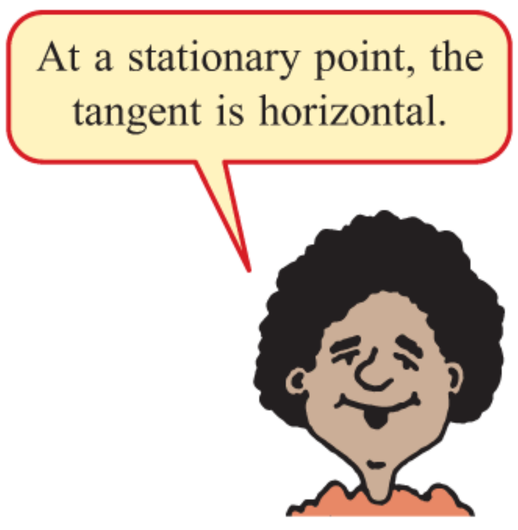
11 Discuss intervals where the following functions are increasing or decreasing:

a $y = \cos x$ **b** $y = \sin x$ **c** $y = \tan x$ **d** $y = x - 2\sin x$

D STATIONARY POINTS

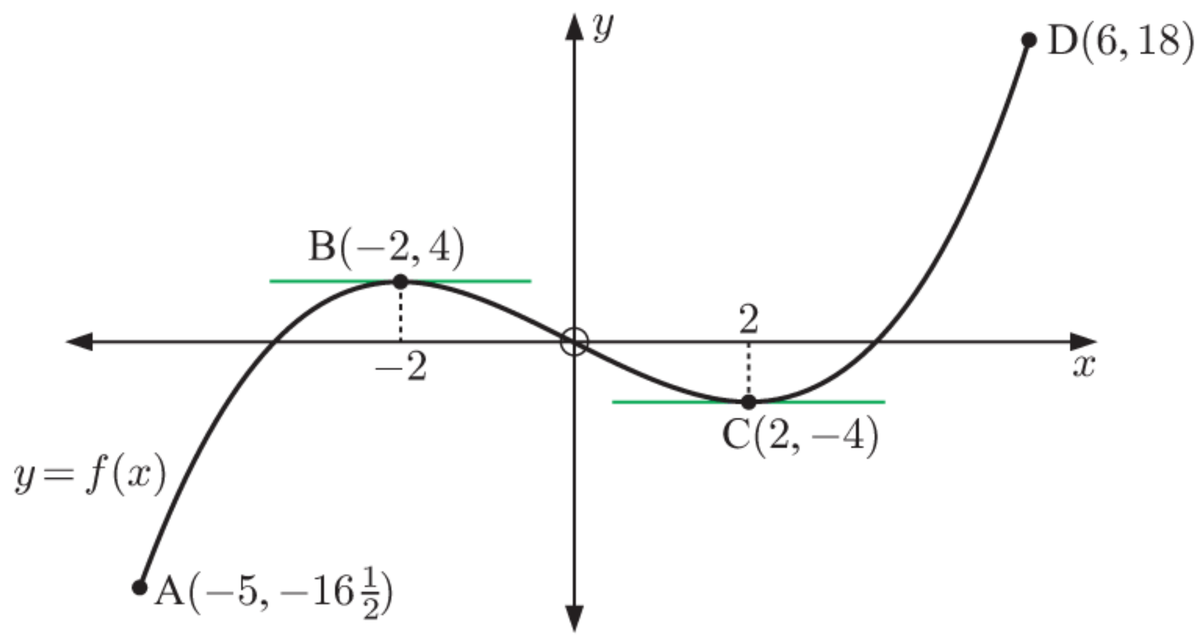
A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a **local maximum** or **local minimum**, or else a **stationary inflection**.



TURNING POINTS (MAXIMA AND MINIMA)

The graph shown has the restricted domain $-5 \leqslant x \leqslant 6$.



A is a **global minimum** as it has the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

C is a **local minimum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

D is a **global maximum** as it is the maximum value of y on the entire domain.

For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y = x^2$ the point $(0, 0)$ is a local minimum and is also the global minimum.

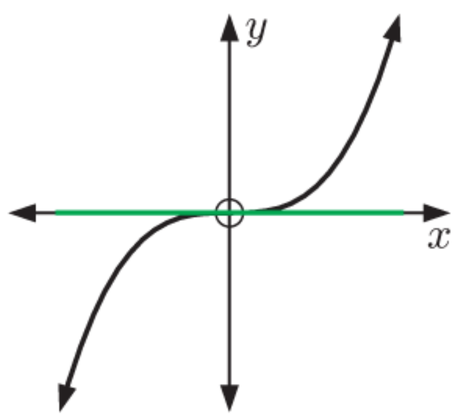
STATIONARY POINTS OF INFLECTION

It is not always true that whenever we find a value of x where $f'(x) = 0$, we have a local maximum or minimum.

For example, $f(x) = x^3$ has $f'(x) = 3x^2$, so $f'(x) = 0$ when $x = 0$.

The tangent to the curve crosses over the curve at $\mathrm{O}(0, 0)$. This tangent is horizontal, but $\mathrm{O}(0, 0)$ is neither a local maximum nor a local minimum.

Rather, this point is called a **stationary inflection** (or **inflexion**) as the curve changes its curvature or shape.



SIGN DIAGRAMS

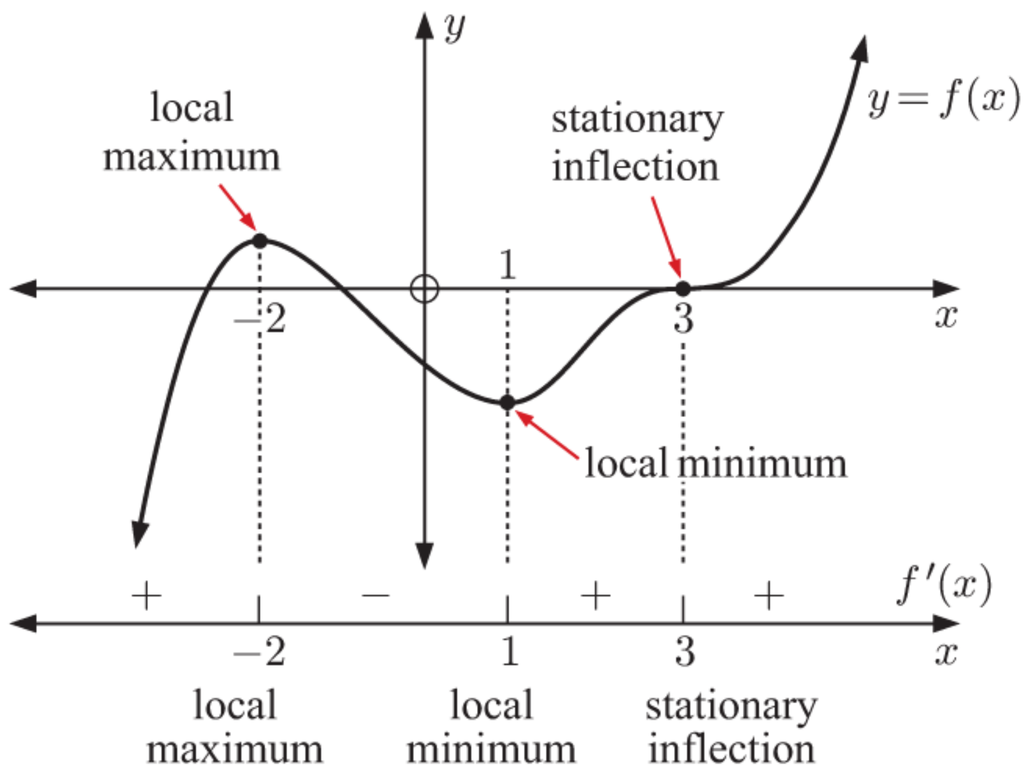
In calculus we commonly use sign diagrams of the derivative function $f'(x)$ to determine the nature of a stationary point.

Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

The signs on the sign diagram of $f'(x)$ indicate whether the gradient of $y = f(x)$ is positive or negative on that interval.





We observe the following properties:

Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum	+ a − $f'(x)$ x	$x = a$
local minimum	− a + $f'(x)$ x	$x = a$
stationary inflection	+ a + $f'(x)$ x ; − a − $f'(x)$ x	$x = a$; $x = a$

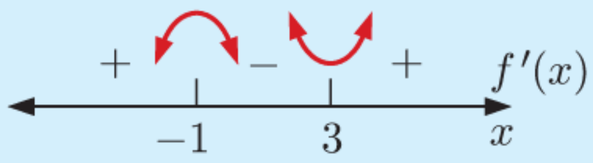
To find the stationary points of a function $f(x)$, we find the values of x for which $f'(x) = 0$. The sign diagram of $f'(x)$ tells us whether each stationary point is a local maximum, local minimum, or stationary inflection.

Example 12 **Self Tutor**

Find and classify all stationary points of $f(x) = x^3 - 3x^2 - 9x + 5$.

$$f(x) = x^3 - 3x^2 - 9x + 5$$
$$\therefore\ f'(x) = 3x^2 - 6x - 9$$
$$= 3(x^2 - 2x - 3)$$
$$= 3(x - 3)(x + 1)$$

$\therefore\ f'(x)$ has sign diagram:

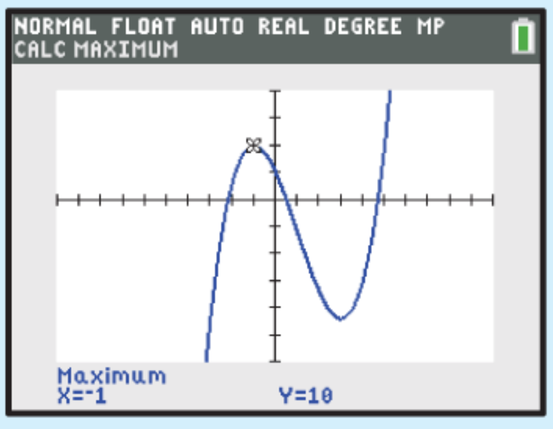


So, we have a local maximum at $x = -1$ and a local minimum at $x = 3$.

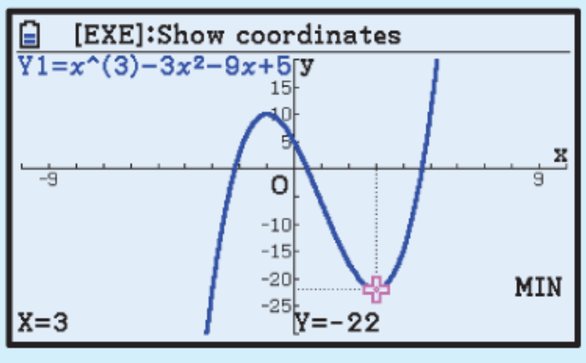
$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5$
$= 10$

$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5$
$= -22$

TI-84 Plus CE



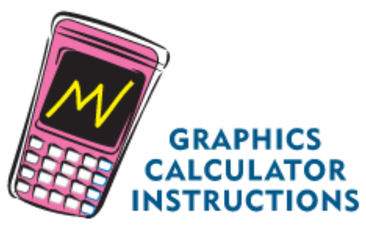
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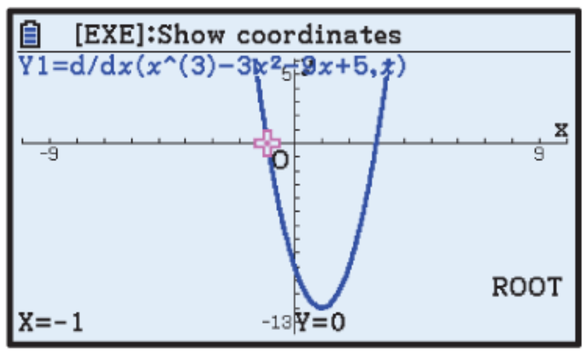


There is a local maximum at $(-1,\ 10)$.

There is a local minimum at $(3,\ -22)$.

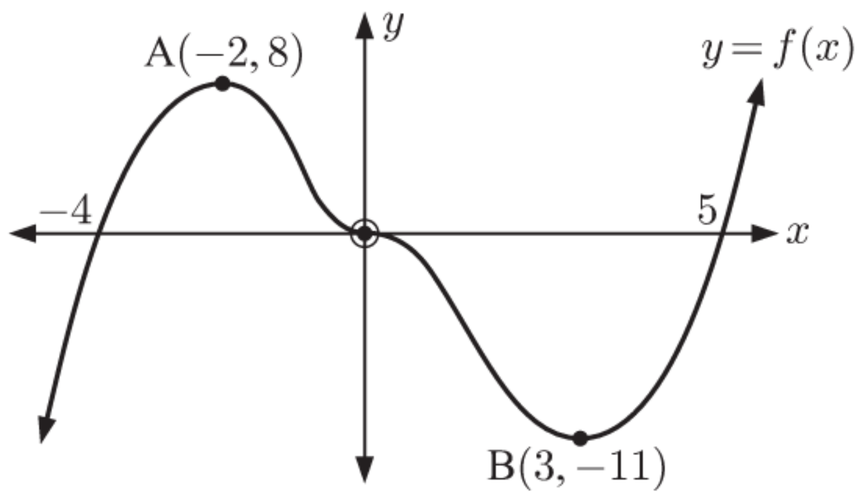
To help construct the sign diagram for $f'(x)$, you can use technology to draw the graph of $f'(x)$, and solve $f'(x) = 0$.





EXERCISE 19D

1 The tangents at points A, O, and B are horizontal.



a Classify points A, O, and B.

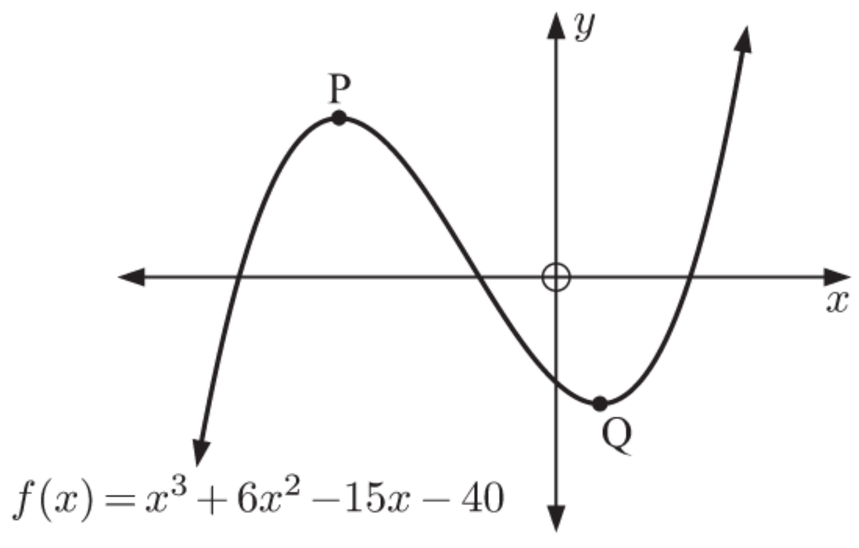
b Draw a sign diagram for the gradient function $f'(x)$ for all x.

c State intervals where $y = f(x)$ is:

i increasing **ii** decreasing.

d Draw a sign diagram for $f(x)$ for all x.

2



The graph of $f(x) = x^3 + 6x^2 - 15x - 40$ is shown alongside. P and Q are stationary points.

a Classify points P and Q.

b Find $f'(x)$.

c Find the coordinates of P and Q.

3 Consider the function $f(x) = \frac{1}{3}x^3 - 9x + 4$.

a Find $f'(x)$, and draw its sign diagram.

b Find intervals where the function is increasing and decreasing.

c Find and classify any stationary points.

d Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

e Sketch the graph of $y = f(x)$, showing the features you have found.

4 Consider the function $g(x) = -2x^3 + 6x^2 + 18x - 7$.

a Find $g'(x)$, and draw its sign diagram.

b Find intervals where the function is increasing and decreasing.

c Find and classify any stationary points.

d Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

e Sketch the graph of $y = g(x)$, showing the features you have found.

5 For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

a $f(x) = x^2 - 2$

b $f(x) = x^3 + 1$

c $f(x) = x^3 - 3x + 2$

d $f(x) = x^4 - 2x^2$

e $f(x) = x^3 - 6x^2 + 12x + 1$

f $f(x) = \sqrt{x} + 2$

g $f(x) = x - \sqrt{x}$

h $f(x) = x^4 - 6x^2 + 8x - 3$

i $f(x) = 1 - x\sqrt{x}$

j $f(x) = x^4 - 2x^2 - 8$

6 **a** For what value of x does the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, have a stationary point?

b Under what conditions is the stationary point a local maximum or a local minimum?

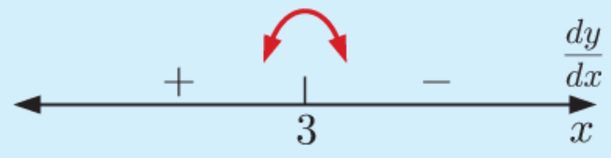
Example 13 **Self Tutor**

Find the exact position and nature of the stationary point of $y = (x-2)e^{-x}$.

$$\frac{dy}{dx} = (1)e^{-x} + (x-2)e^{-x}(-1) \qquad \{\text{product rule}\}$$
$$= e^{-x}(1 - (x-2))$$
$$= \frac{3-x}{e^x} \quad \text{where } e^x \text{ is positive for all } x.$$

So, $\frac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\frac{dy}{dx}$ is



$\therefore$ at $x = 3$ we have a local maximum.

When $x = 3$, $y = (1)e^{-3} = \frac{1}{e^3}$ $\therefore$ the local maximum is at $\left(3, \frac{1}{e^3}\right)$.

7 Find the position and nature of the stationary point(s) of:

a $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \frac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

8 $f(x) = 2x^3 + ax^2 - 24x + 1$ has a local maximum at $x = -4$.

a Find a. **b** Find the coordinates of the local maximum.

9 $f(x) = x^3 + ax + b$ has a stationary point at $(-2, 3)$.

a Find the values of a and b. **b** Find the position and nature of all stationary points.

10 $y = \dfrac{e^{ax}}{bx}$ has a stationary point at $\left(\dfrac{1}{3}, \dfrac{e}{2}\right)$.

a Find the values of a and b. **b** State the nature of the stationary point.

11 Consider $f(x) = x \ln x$.

a For what values of x is $f(x)$ defined?

b Show that the minimum value of $f(x)$ is $-\dfrac{1}{e}$.

12 For each of the following, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

a $f(x) = \sin x$ **b** $f(x) = \cos 2x$

c $f(x) = e^{\sin x}$ **d** $f(x) = \cos x - \sin x$

13 The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

Example 14 Self Tutor

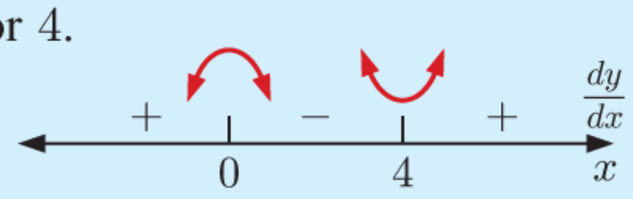
Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

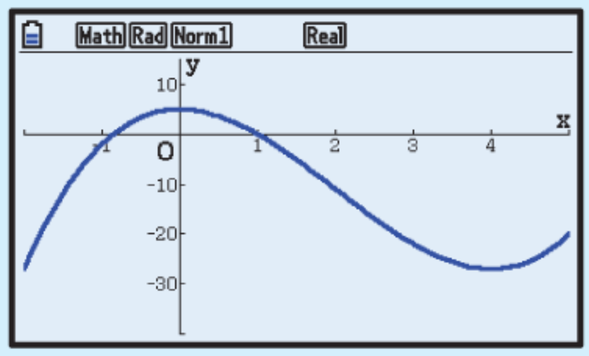
Now $\dfrac{dy}{dx} = 3x^2 - 12x$

$= 3x(x - 4)$

$\therefore \dfrac{dy}{dx} = 0$ when $x = 0$ or 4.

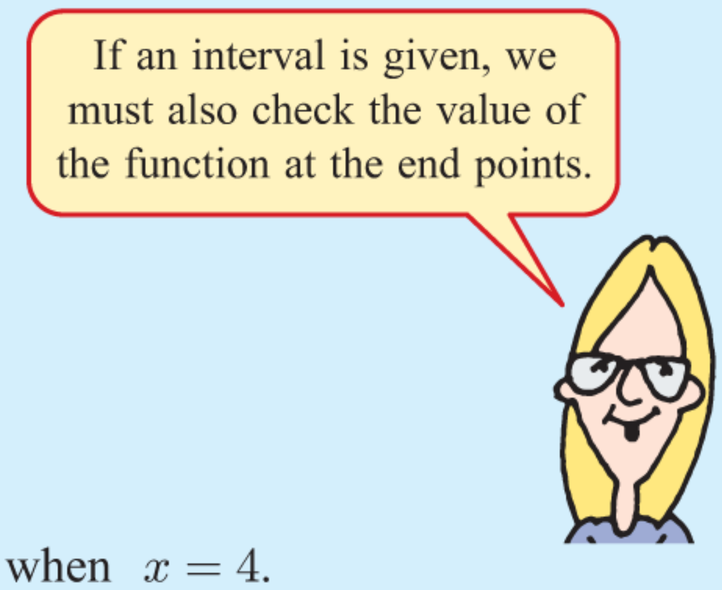
The sign diagram of $\dfrac{dy}{dx}$ is:





$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

Critical value (x)	y
-2 (end point)	-27
0 (local maximum)	5
4 (local minimum)	-27
5 (end point)	-20



The greatest of these values is 5 when $x = 0$.

The least of these values is -27 when $x = -2$ and when $x = 4$.

14 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

c $x^2 + \dfrac{16}{x}$ for $1 \leqslant x \leqslant 4$ **d** $x - 4\sqrt{x}$ for $0 \leqslant x \leqslant 5$.

15 Show that:

a $y = \tan x - 2x$ has a stationary point when $x = -\frac{\pi}{4}$

b $y = 4e^{-x} \sin x$ has a local maximum when $x = \frac{\pi}{4}$.

16 $f(t) = ate^{bt^2}$ has a maximum value of 1 when $t = 2$. Find constants a and b.

17 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$.

18 Consider the function $f(x) = x - \ln x$.

a Show that $y = f(x)$ has a local minimum and that this is the only turning point.

b Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

E SHAPE

We have seen that the first derivative $f'(x)$ gives the gradient of the curve $y = f(x)$ for any value of x.

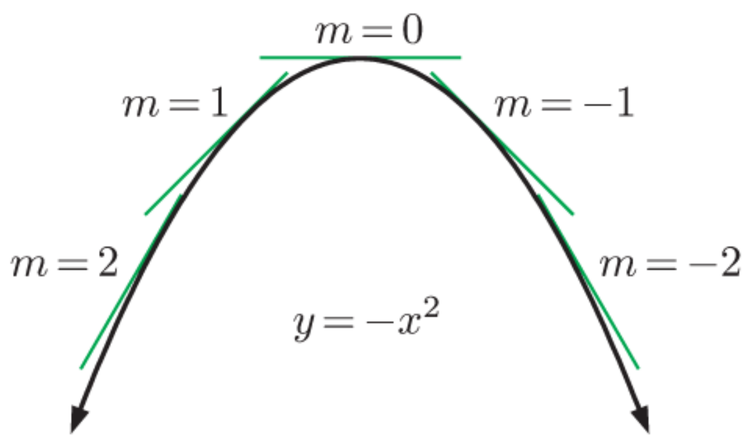
The second derivative $f''(x)$ tells us the rate of change of the gradient $f'(x)$. It therefore gives us information about the **shape** or **curvature** of the curve $y = f(x)$.

When a curve, or part of a curve, has shape:

-  we say it is **concave down**
-  we say it is **concave up**.

For example:

- the curve $f(x) = -x^2$ is concave down

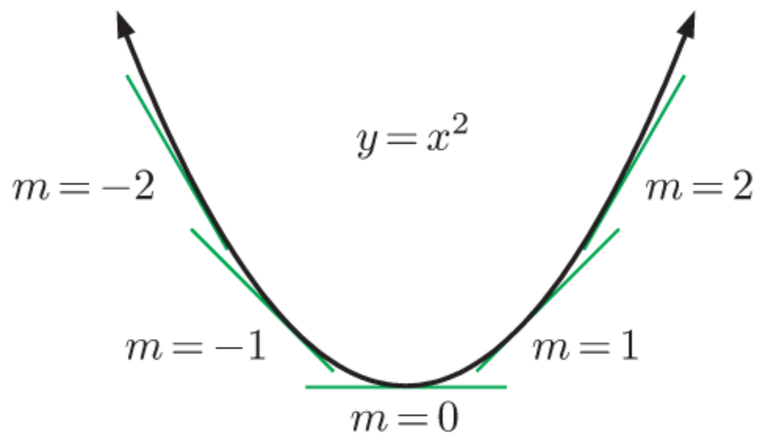


Wherever we are on the curve, as x increases, the gradient of the tangent decreases.

$\therefore$ $f'(x)$ is decreasing

$\therefore$ $f''(x) < 0$.

- the curve $f(x) = x^2$ is concave up.



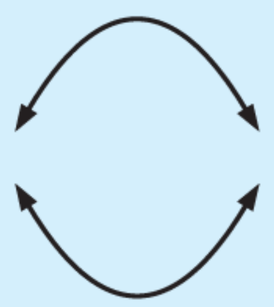
Wherever we are on the curve, as x increases, the gradient of the tangent increases.

$\therefore$ $f'(x)$ is increasing

$\therefore$ $f''(x) > 0$.

We conclude that:

- A curve is **concave down** on an interval $S \Leftrightarrow f''(x) \leqslant 0$ for all $x \in S$.
- A curve is **concave up** on an interval $S \Leftrightarrow f''(x) \geqslant 0$ for all $x \in S$.

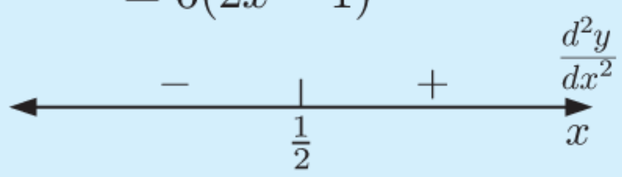


Example 15 **Self Tutor**

Find intervals where the curve is concave up or concave down:

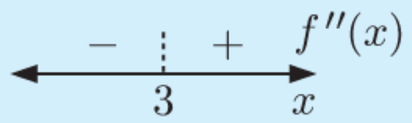
a $y = 2x^3 - 3x^2 + 4x - 6$ **b** $f(x) = \dfrac{x+1}{x-3}$

a
$$y = 2x^3 - 3x^2 + 4x - 6$$
$$\therefore \frac{dy}{dx} = 6x^2 - 6x + 4$$
$$\therefore \frac{d^2y}{dx^2} = 12x - 6$$
$$= 6(2x - 1)$$



The curve is concave up for $x \geqslant \frac{1}{2}$ and concave down for $x \leqslant \frac{1}{2}$.

b
$$f(x) = \frac{x+1}{x-3}$$
$$\therefore f'(x) = \frac{(1)(x-3) - (x+1)(1)}{(x-3)^2}$$
$$= \frac{\cancel{x} - 3 - \cancel{x} - 1}{(x-3)^2}$$
$$= -4(x-3)^{-2}$$
$$\therefore f''(x) = 8(x-3)^{-3}$$

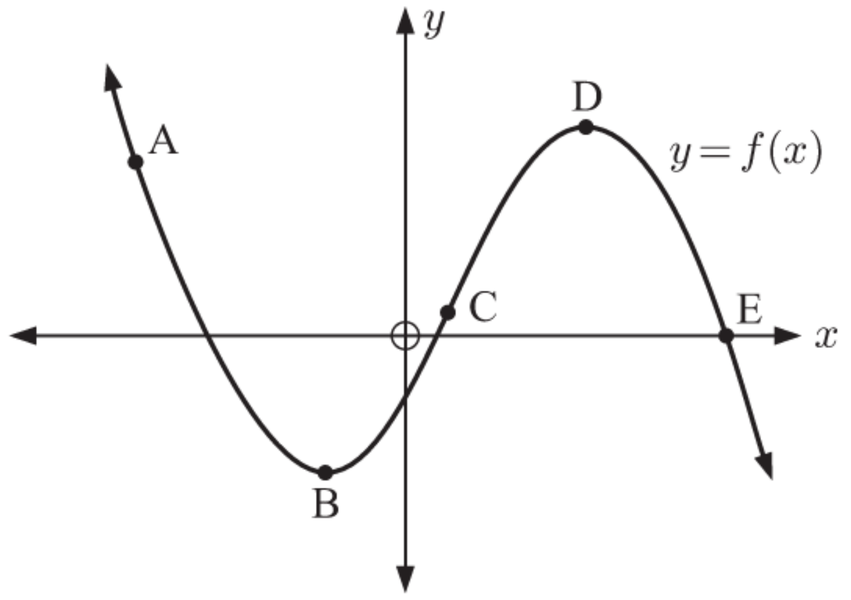


The curve is concave up for $x > 3$ and concave down for $x < 3$.

EXERCISE 19E

1 **a** Complete the table by indicating whether each value is zero, positive, or negative:

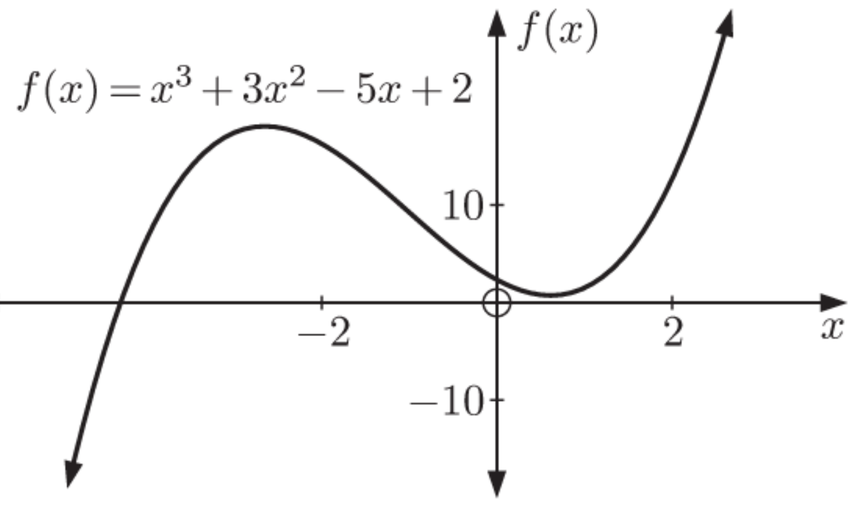
Point	$f(x)$	$f'(x)$	$f''(x)$
A	+		
B			
C			
D		0	
E			



b Describe the turning points of $y = f(x)$.

c At which point does the shape of $y = f(x)$ change?

2 The graph of $f(x) = x^3 + 3x^2 - 5x + 2$ is shown alongside.



a Find $f'(x)$ and $f''(x)$.

b Draw the sign diagram of $f''(x)$.

c State the interval on which the function is:

i concave up **ii** concave down.

3 Determine the shape of each quadratic function:

a $y = 2x^2 - 3x + 4$ **b** $y = -2(x-3)(x+1)$

c $y = -4 - x^2 + 6x$ **d** $y = (5-x)(1-2x)$

4 For each of the following functions, determine the interval(s) on which the function is:

i increasing **ii** decreasing **iii** concave up **iv** concave down.

a $f(x) = x^2 + 1$ **b** $f(x) = -x^3$ **c** $f(x) = e^x$

d $f(x) = \sqrt{x} - 2$ **e** $f(x) = -\dfrac{1}{\sqrt{x}}$ **f** $f(x) = x^4 - 12x^2$

5 Consider the function $f(x) = e^{2x} - 3$.

a Find the x and y-intercepts.

b Show algebraically that the function is increasing for all x.

c Find $f''(x)$, and hence explain why $f(x)$ is concave up for all x.

6 Consider $f(x) = \ln(2x - 1) - 3$.

a Can $f(0)$ be found? State the significance of this result.

b Find the domain of f.

c Find the gradient of the tangent to the curve at $x = 1$.

d Find $f''(x)$, and hence explain why $f(x)$ is concave down for all x in the domain of f.

e Graph the function, showing the features you have found.

7 Consider $f(x) = \ln x$.

a For what values of x is $f(x)$ defined?

b Draw the sign diagrams of $f'(x)$ and $f''(x)$, and give a geometrical interpretation of each.

c Find the equation of the normal to $y = f(x)$ at the point where $y = 1$.

8 Consider the function $f(x) = \dfrac{e^x}{x}$.

a Does the graph of $y = f(x)$ have any x or y-intercepts?

b Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

c Find and classify any stationary points of $y = f(x)$.

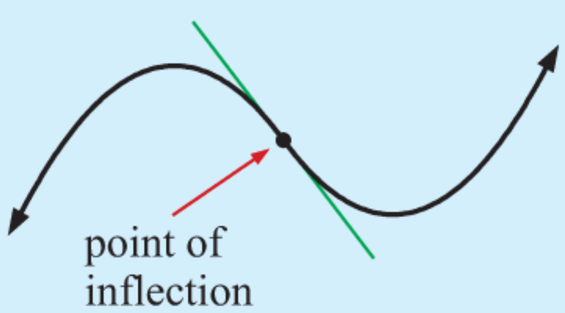
d Find the intervals where $f(x)$ is: **i** concave up **ii** concave down.

e Sketch the graph of $y = f(x)$, showing all important features.

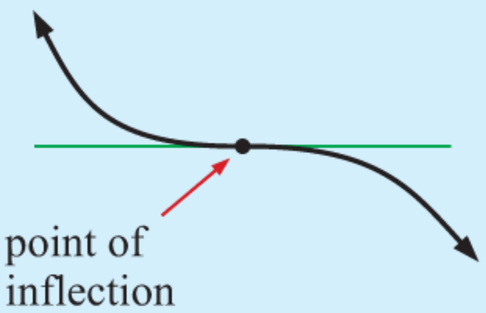
f Find the equation of the tangent to $f(x) = \dfrac{e^x}{x}$ at the point where $x = -1$.

F INFLECTION POINTS

A **point of inflection** is a point at which the tangent to the curve crosses the curve.
At a point of inflection, $f''(x) = 0$.

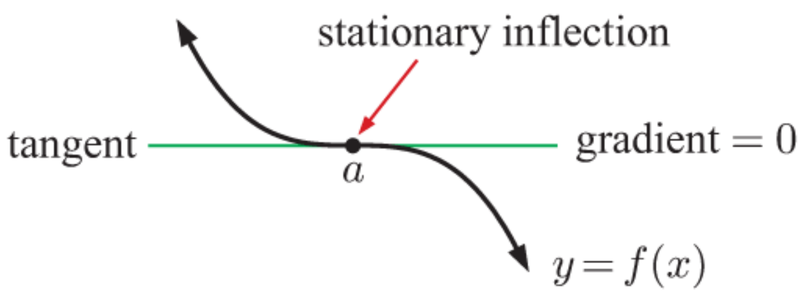


or

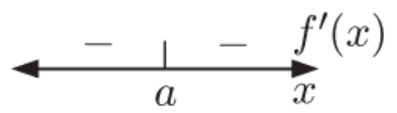




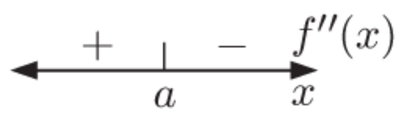
If the tangent at a point of inflection is horizontal, then this is a **stationary inflection point**.



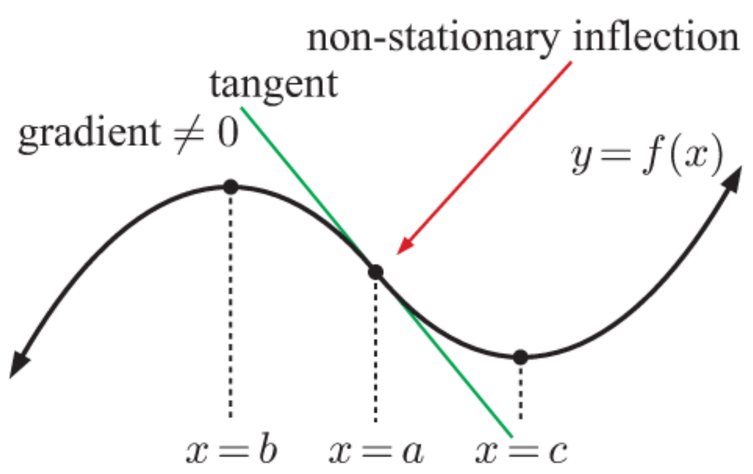
$f'(x)$ has sign diagram



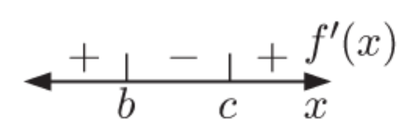
$f''(x)$ has sign diagram



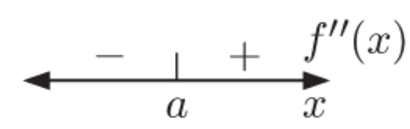
If the tangent at a point of inflection is *not* horizontal, then this is a **non-stationary inflection point**.



$f'(x)$ has sign diagram



$f''(x)$ has sign diagram



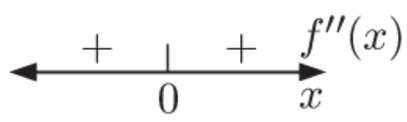
The tangent at the point of inflection, also called the **inflecting tangent**, crosses the curve at that point.

There is a **point of inflection** at $x = a$ if $f''(a) = 0$ **and** the sign of $f''(x)$ changes at $x = a$.
The point of inflection is a:

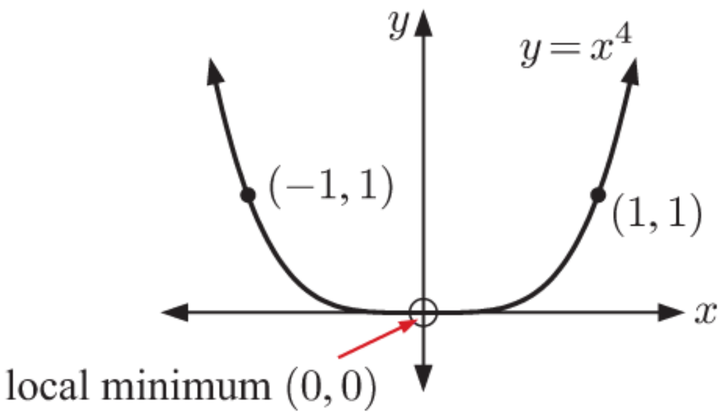
- **stationary inflection** if $f'(a) = 0$
- **non-stationary inflection** if $f'(a) \neq 0$.

Notice that if $f(x) = x^4$ then $f'(x) = 4x^3$
and $f''(x) = 12x^2$.

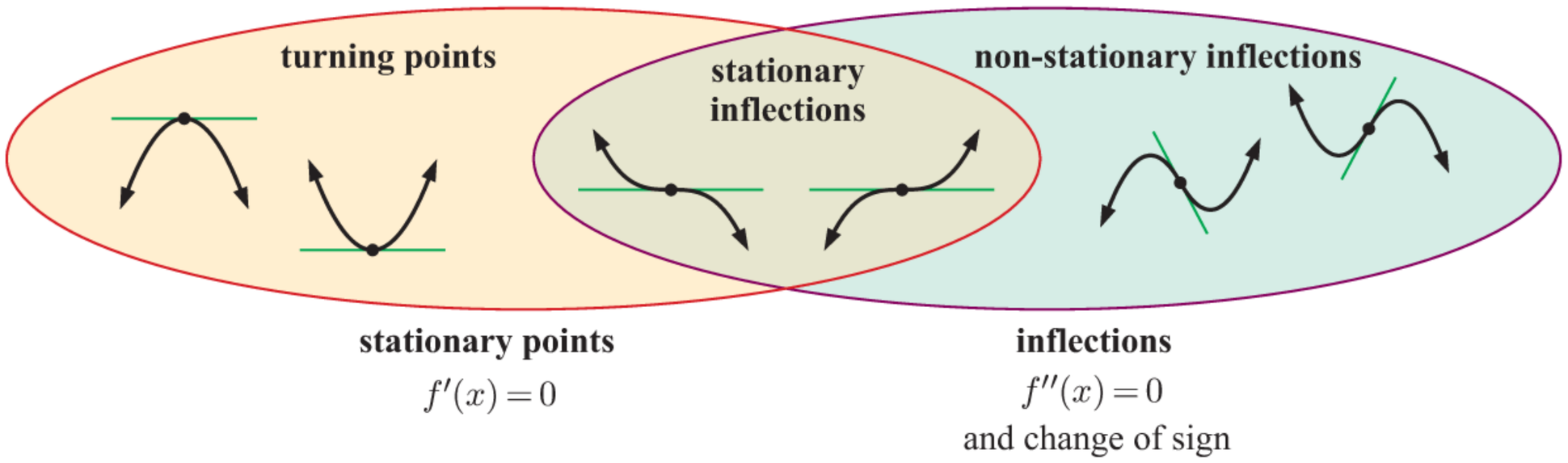
$f''(x)$ has sign diagram



Although $f''(0) = 0$ we do not have a point of inflection at $(0, 0)$ because the sign of $f''(x)$ does not change at $x = 0$.



SUMMARY



Click on the demo icon to examine some common functions for turning points, points of inflection, and intervals where the function is increasing, decreasing, and concave up or down.



Example 16 **Self Tutor**

Consider $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$.

a Find and classify all points where $f'(x) = 0$.

b Find and classify all points of inflection.

c Find intervals where the function is increasing or decreasing.

d Find intervals where the function is concave up or down.

e Sketch the function showing the features you have found.

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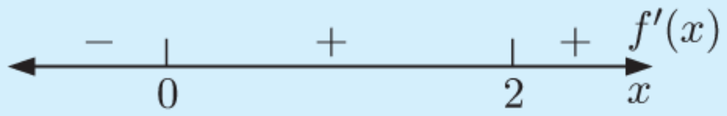
a $f(x) = 3x^4 - 16x^3 + 24x^2 - 9$

$\therefore \quad f'(x) = 12x^3 - 48x^2 + 48x$

$= 12x(x^2 - 4x + 4)$

$= 12x(x-2)^2$

$\therefore$ $f'(x)$ has sign diagram:



Now $f(0) = -9$ and $f(2) = 7$

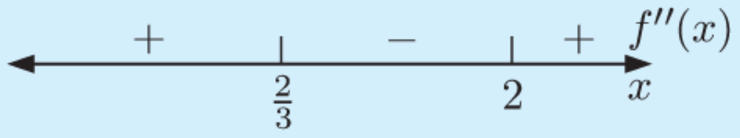
$\therefore$ $(0, -9)$ is a local minimum and $(2, 7)$ is a stationary inflection.

b $f''(x) = 36x^2 - 96x + 48$

$= 12(3x^2 - 8x + 4)$

$= 12(x-2)(3x-2)$

$\therefore$ $f''(x)$ has sign diagram:



Now $f(\frac{2}{3}) \approx -2.48$

$\therefore$ $(2, 7)$ is a stationary inflection and $(\frac{2}{3}, -2.48)$ is a non-stationary inflection.

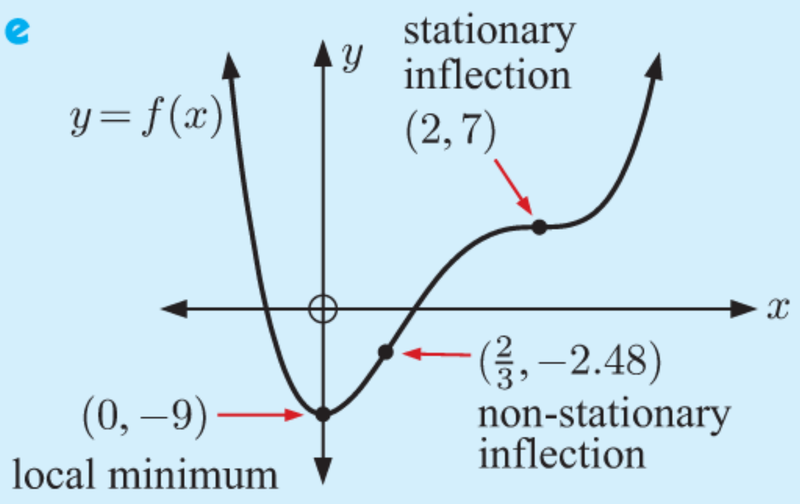
c $f(x)$ is decreasing for $x \leqslant 0$

$f(x)$ is increasing for $x \geqslant 0$.

d $f(x)$ is concave up for $x \leqslant \frac{2}{3}$ and $x \geqslant 2$

$f(x)$ is concave down for $\frac{2}{3} \leqslant x \leqslant 2$.

e



EXERCISE 19F

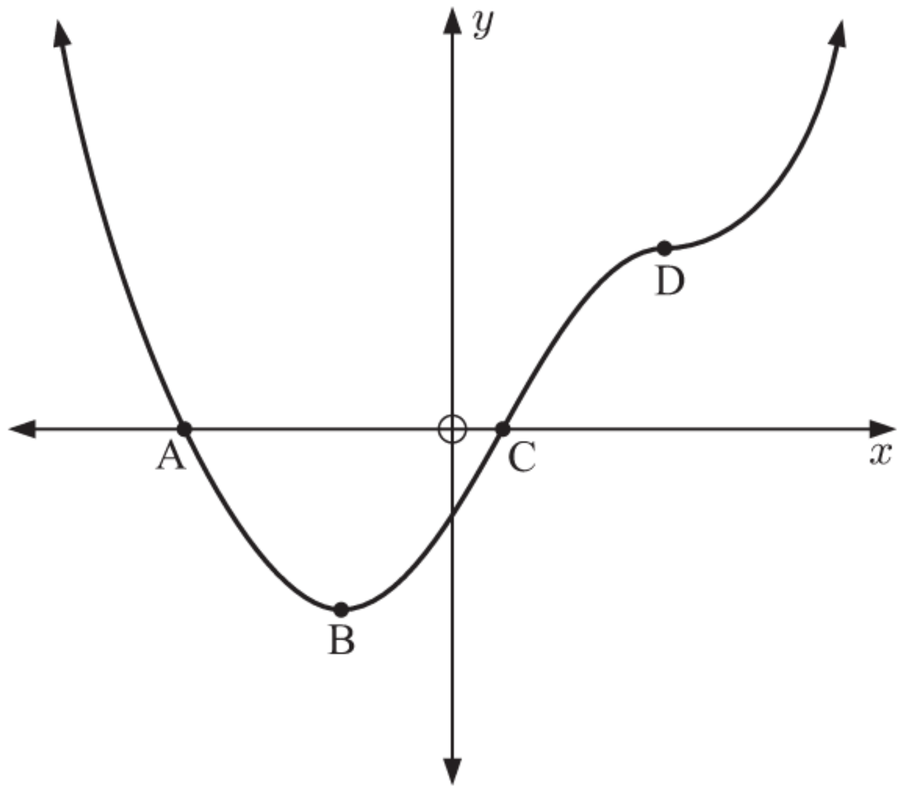
1 In the diagram alongside, each labelled point corresponds to a zero of $f(x)$, $f'(x)$, or $f''(x)$.

a Complete the table by indicating whether each value is zero, positive, or negative:

Point	$f(x)$	$f'(x)$	$f''(x)$
A			
B			
C			
D			

b Describe the turning point of $y = f(x)$.

c Describe the inflection points of $y = f(x)$.



2 Find and classify all points of inflection of:

a $f(x) = x^2 + 3$ **b** $f(x) = 2 - x^3$

c $f(x) = x^3 - 6x^2 + 9x + 1$ **d** $f(x) = -3x^4 - 8x^3 + 2$

e $f(x) = 3 - \dfrac{1}{\sqrt{x}}$ **f** $f(x) = x^3 + 6x^2 + 12x + 5$

g $f(x) = x^2 + 8\sqrt{x}$ **h** $f(x) = x^4 - 6x^2 + 10$

3 For each of the following functions:

i Find and classify all turning points.
ii Find and classify all points of inflection.
iii Find intervals where the function is increasing or decreasing.
iv Find intervals where the function is concave up or down.
v Sketch the function showing the features you have found.

a $f(x) = x^2 - 5x + 4$ **b** $f(x) = x^3 + 4x^2$ **c** $f(x) = x^3 - 3x^2 - 24x + 1$

d $f(x) = \sqrt{x}$ **e** $f(x) = (x-1)^4$ **f** $f(x) = 3x^4 + 4x^3 - 2$

g $f(x) = x^4 - 4x^2 + 3$ **h** $f(x) = 3 - \dfrac{4}{\sqrt{x}}$

4 A function commonly used in statistics is the *standard normal curve* $f(x) = \dfrac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}$.

a Find the turning points of the function.
b Find the intervals where the function is increasing and decreasing.
c Find all points of inflection.
d Discuss $f(x)$ as $x \to \infty$ and as $x \to -\infty$.
e Sketch the graph of $y = f(x)$, showing all important features.

5 Consider the function $f(x) = \cos x$.

a Show that $f''(x) = -f(x)$. What does this tell us about the location of the inflection points?
b Find and classify the inflection points of $f(x)$ on $0 \leqslant x \leqslant 2\pi$.
c Find the intervals on $0 \leqslant x \leqslant 2\pi$ where $f(x)$ is:
i increasing **ii** decreasing **iii** concave up **iv** concave down.
d Sketch the graph of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$, showing all important features.

6 Consider the *surge function* $f(t) = Ate^{-bt}$, $t \geqslant 0$, where A and b are positive constants.

a Prove that the function has:
i a local maximum at $t = \dfrac{1}{b}$ **ii** a point of inflection at $t = \dfrac{2}{b}$.
b Sketch the function, showing the features you have found.

7 Consider the *logistic function* $f(t) = \dfrac{L}{1 + Ce^{-kt}}$, $t \geqslant 0$, where L, C, and k are positive constants.

a Find the y-intercept.
b Explain why $y = L$ is a horizontal asymptote.
c Show that if $C > 1$, there is a point of inflection with y-coordinate $\dfrac{L}{2}$.
d Sketch the function, showing the features you have found.

REVIEW SET 19A

1 Find the equation of the tangent to:

a $y = -2x^2$ at the point where $x = -1$

b $y = x^3 - 5x + 2$ at $(2, 0)$

c $y = \dfrac{1 - 2x}{x^2}$ at $(1, -1)$

d $f(x) = e^{3x-1}$ at the point where $x = 0$

e $f(x) = \ln(x^2)$ at the point where $x = e$.

2 Find the equation of the normal to:

a $y = \sqrt{3x + 4}$ at $(4, 4)$

b $y = 3e^{2x}$ at the point where $x = 1$.

3 At the point where $x = 0$, the tangent to $f(x) = e^{4x} + px + q$ has equation $y = 5x - 7$. Find p and q.

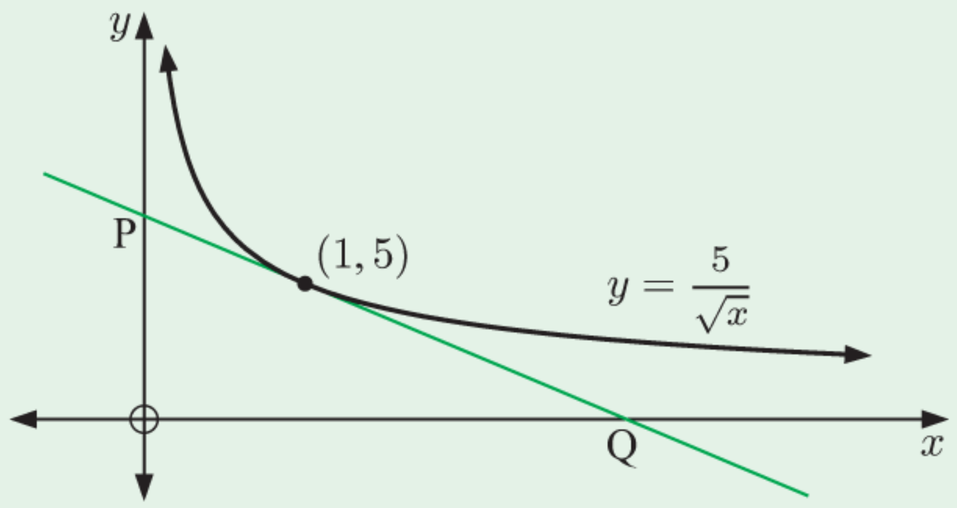
4 Find all points on the curve $y = 2x^3 + 3x^2 - 10x + 3$ where the gradient of the tangent is 2.

5 Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ meets the curve again.

6 **a** Find the equation of the normal to $y = e^{2x}$ at the point where $x = a$.

b Hence find the equation of the normal to $y = e^{2x}$ which passes through the origin.

7 Find the coordinates of P and Q if (PQ) is the tangent to $y = \dfrac{5}{\sqrt{x}}$ at $(1, 5)$.



8 The tangent to $y = x^2\sqrt{1 - x}$ at $x = -3$ cuts the axes at points A and B. Determine the area of triangle OAB.

9 Find intervals where $f(x) = -x^3 - 6x^2 + 36x - 17$ is:

a increasing

b decreasing.

10 Consider the function $f(x) = 2x^3 - 3x^2 - 36x + 7$.

a Find and classify all stationary points.

b Find intervals where the function is increasing and decreasing.

c Describe the behaviour of the function as $x \to \infty$ and as $x \to -\infty$.

d Sketch the graph of $y = f(x)$ showing the features you have found.

11 Consider the function $f(x) = \dfrac{3x - 2}{x + 3}$.

a State the domain of $f(x)$.

b Find the axes intercepts.

c Find $f'(x)$ and draw its sign diagram.

d Does $f(x)$ have any stationary points?

12 Find the greatest and least values of $x + \dfrac{32}{x^2}$ for $2 \leqslant x \leqslant 10$.

13 Consider $f(x) = xe^{1-2x}$.

a Show that $f'(x) = e^{1-2x}(1-2x)$.

b Find values of x for which:

i $f(x) > 0$ **ii** $f'(x) > 0$

c Find the stationary point of $y = f(x)$, and determine its nature.

14 Find intervals where $f(x)$ is increasing or decreasing:

a $f(x) = x^3 - 6x$ **b** $f(x) = e^x(x-2)$ **c** $f(x) = 2x - \sin x$

15 Find and classify the stationary points of:

a $f(x) = -x^3 + 2x^2 - x + 3$ **b** $f(x) = \dfrac{x^2}{x+3}$

16 For each of the following, determine the position and nature of the stationary points on the interval $-\pi \leqslant x \leqslant \pi$, then show them on a graph of the function.

a $y = \sin\frac{x}{2}$ **b** $y = \cos^2 x$ **c** $y = \tan^2 x$

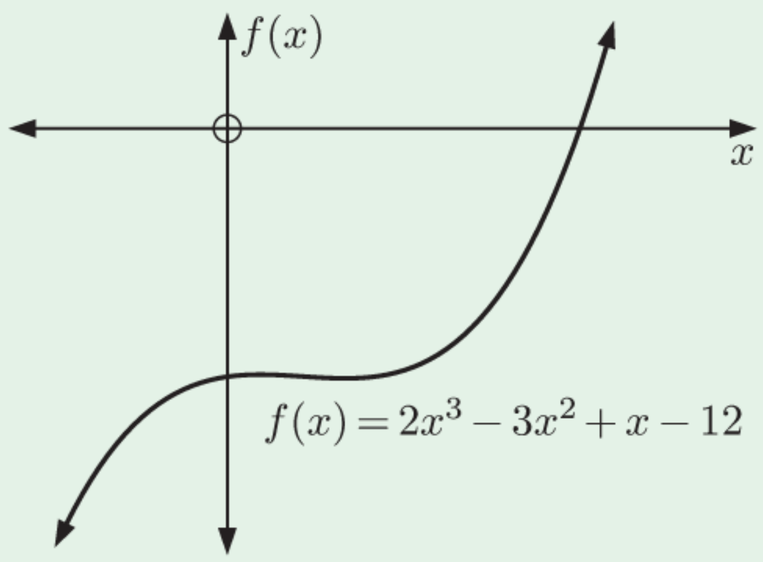
17 The graph of $f(x) = 2x^3 - 3x^2 + x - 12$ is shown alongside.

a Find $f'(x)$ and $f''(x)$.

b Draw the sign diagram of $f''(x)$.

c State the interval on which the function is concave down.

d Find the point at which the shape of $f(x)$ changes.



18 Find intervals where the curve is concave up or concave down:

a $y = x^3 - 4x^2 + 11$ **b** $y = -\dfrac{x+1}{x^2}$

19 Consider the function $f(x) = x + \ln x$.

a Find the values of x for which $f(x)$ is defined.

b Draw the sign diagrams of $f'(x)$ and $f''(x)$, and give a geometrical interpretation of each.

c Sketch the graph of $y = f(x)$.

20 Consider the function $f(x) = e^{x\sqrt{3}} \sin x$.

a Find $f'(x)$.

b Find x on $0 \leqslant x \leqslant 2\pi$ such that $f'(x) = 0$.

c Draw the sign diagram for $f'(x)$ on $0 \leqslant x \leqslant 2\pi$.

d Determine the intervals on $0 \leqslant x \leqslant 2\pi$ for which $f(x)$ is:

i increasing **ii** decreasing.

21 Consider the function $f(x) = \ln(x^2 + 5)$.

a Find and classify any turning points. **b** Find and classify any points of inflection.

c Find intervals where the function is increasing or decreasing.

d Find intervals where the function is concave up or down.

e Sketch the function, showing the features you have found.

REVIEW SET 19B

1 Find the equation of the tangent to:

a $f(x) = x^4 - 2x^2 + 7x - 3$ at $(2, 19)$ **b** $f(x) = \dfrac{1}{\sqrt{x+7}}$ at $(9, \frac{1}{4})$

c $f(x) = 3\sin 2x$ when $x = \frac{\pi}{6}$ **d** $f(x) = \dfrac{e^x}{2-x}$ when $x = 0$.

2 Find the equation of the normal to:

a $y = \dfrac{1}{x^2} - \dfrac{2}{x}$ at the point where $x = 1$ **b** $y = x\sin x$ at the origin.

3 Show that the equation of the tangent to $y = x\tan x$ at $x = \frac{\pi}{4}$ is $(2+\pi)x - 2y = \frac{\pi^2}{4}$.

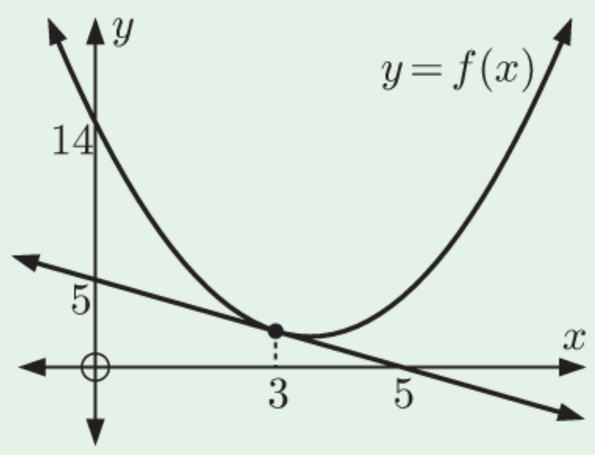
4 The curve $y = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of a and b.

5 Show that $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

6 $y = f(x)$ is the parabola shown.

a Find $f(3)$ and $f'(3)$.

b Hence find $f(x)$ in the form $f(x) = ax^2 + bx + c$.



7 The tangent to $y = x^3 + ax^2 - 4x + 3$ at $x = 1$ is parallel to the line $y = 3x$.

a Find a. **b** Find the equation of the tangent at $x = 1$.

c Where does the tangent meet the curve again?

8 Find the point where the normal to $y = x^2 - 4x + 2$ at $x = 3$, meets the curve again.

9 Find the equation of the tangent to:

a $y = \dfrac{1}{\sin x}$ at the point where $x = \frac{\pi}{3}$ **b** $y = \cos\frac{x}{2}$ at the point where $x = \frac{\pi}{2}$.

10 The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at $x = 1$ has equation $2x - y = 1$. Find a and b.

11 Find intervals where $f(x) = x^4 - 4x^3 - 8x^2 + 5$ is:

a increasing **b** decreasing.

12 $f(x) = x^3 - 3x^2 + ax + 50$ has a stationary point at $x = 3$.

a Find a. **b** Find the position and nature of all stationary points.

13 Consider the function $f(x) = e^x - x$.

a Find and classify any stationary points of $y = f(x)$.

b Discuss what happens to $f(x)$ as $x \to \infty$.

c Find $f''(x)$ and draw its sign diagram. Interpret the sign of $f''(x)$.

d Sketch the graph of $y = f(x)$.

e Deduce that $e^x \geqslant x + 1$ for all x.

14 Suppose $f(x) = \dfrac{x+1}{x^2 - 2x - 8}$.

a Show that $f'(x) = -\dfrac{x^2 + 2x + 6}{(x^2 - 2x - 8)^2}$ and draw its sign diagram.

b Hence show that $f(x)$ is never increasing.

15 Find and describe the stationary point of $y = \dfrac{x+a}{e^x}$, where a is a constant.

16 $f(x) = \dfrac{\ln(ax)}{bx}$ has a stationary point at $\left(\dfrac{e}{2}, \dfrac{2}{3e}\right)$. Find a and b.

17 Find and classify the inflection points of $y = x^4 - 3x^3 + 9$.

18 Consider $f(x) = \sqrt{\cos x}$ for $0 \leqslant x \leqslant 2\pi$.

a For what values of x in this interval is $f(x)$ defined?

b Find $f'(x)$ and hence find intervals where $f(x)$ is increasing or decreasing.

c Sketch the graph of $y = f(x)$ on $0 \leqslant x \leqslant 2\pi$.

19 Consider the function $f(x) = x^4 - 4x^3 + 7$.

a Find $f'(x)$ and $f''(x)$, and draw their sign diagrams.

b Find and classify any turning points.

c Find and classify any points of inflection.

d Find intervals where the function is:

i increasing **ii** decreasing **iii** concave up **iv** concave down.

e Sketch the function, showing all important features.

20 For the function $f(x) = \cos^2 x$, $0 \leqslant x \leqslant 2\pi$:

a Find and classify all turning points.

b Find and classify all points of inflection.

c Sketch the function, showing the features you have found.

21 Consider the function $f(x) = \dfrac{e^x}{x-1}$.

a Find the y-intercept of the function.

b For what values of x is $f(x)$ defined?

c Find the signs of $f'(x)$ and $f''(x)$ and comment on the geometrical significance of each.

d Sketch the graph of $y = f(x)$.

e Find the equation of the tangent at the point where $x = 2$.

Chapter 20

Applications of differentiation

Contents:

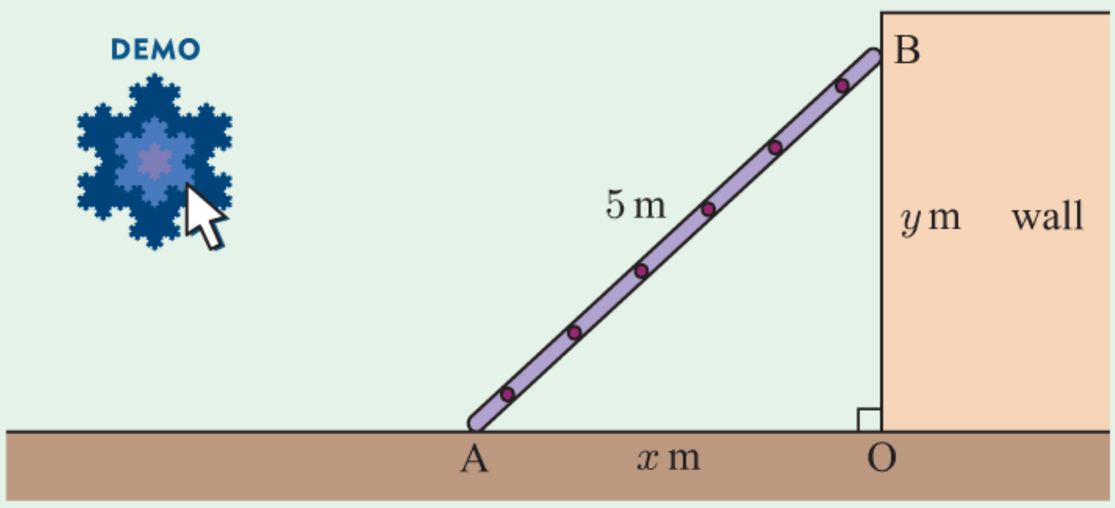
A Rates of change
B Optimisation
C Modelling with calculus
D Related rates



OPENING PROBLEM

A 5 m ladder rests against a vertical wall at point B. Its feet are at point A on horizontal ground.

Suppose the ladder slips and slides down the wall. As this happens, the value of x increases.



Things to think about:

a Why *must* the value of y change when the value of x changes?

b As x increases will y increase or decrease?

c What is the relationship between x and y?

d How can we establish a relationship between the *rate of change* in x and the *rate of change* in y?

We have already seen that if $y = f(x)$ then $f'(x)$ or $\frac{dy}{dx}$ gives the gradient of the tangent to $y = f(x)$ for any value of x.

In this Chapter we consider some real-world applications of differential calculus, using derivatives to tell us how one variable changes relative to another.

A RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:

- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

$\frac{dy}{dx}$ gives the **rate of change in y with respect to x.**

We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\frac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or $\text{m}\,\text{s}^{-1}$.
- $\frac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or $\text{L}\,\text{s}^{-1}$.



Example 1 **Self Tutor**

According to a psychologist, the ability of a child to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a child is:

i 9 years old **ii** 16 years old.

b Show that $\frac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Show that $\frac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$

$\therefore \frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}$

i When $t = 9$, $\frac{dA}{dt} = \frac{1}{18}$

$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old child.

ii When $t = 16$, $\frac{dA}{dt} = \frac{1}{24}$

$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old child.

b Since $\sqrt{t}$ is never negative, $\frac{1}{6\sqrt{t}}$ is never negative

$\therefore \frac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$

$\therefore \frac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\frac{1}{12t\sqrt{t}}$

$\therefore \frac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with age, the rate of increase slows down with age.

You are encouraged to use technology to graph each function you need to consider. This is often useful in interpreting results.

GRAPHING PACKAGE

EXERCISE 20A

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

a What is the current annual profit? **b** Find $\frac{dP}{dt}$ and state its units.

c Find $\frac{dP}{dt}$ when $t = 8$. Explain what this value means.

2 In a hot, dry summer, water is evaporating from a desert oasis. The volume of water remaining after t days is $V = 2(50 - t)^2$ m^3. Find:

a the average rate at which the water evaporates in the first 5 days

b the instantaneous rate at which the water is evaporating at $t = 5$ days.

3 The quantity of a chemical in human skin which is responsible for its "elasticity" is given by $Q(t) = 100 - 10\sqrt{t}$ where t is the age of a person in years.

a Find $Q(t)$ when:

i $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.

b At what rate is the quantity of the chemical changing when the person is aged:

i 25 years **ii** 50 years?

c Show that the quantity of the chemical is decreasing for all $t > 0$.

4 **a** A palm tree is grown from a seed in a pot. Its height after t years is given by $h = 0.1t^2 + 0.15t$ metres, where $0 \leqslant t \leqslant 4$.

i Find $\frac{dh}{dt}$ and state its units. **ii** How fast was the tree growing after 1 year?

iii At what time was the tree growing at 75 cm per year?

iv How tall was the tree after 4 years?

b After 4 years, the palm tree is taken out of its pot and planted in the ground. Its height over time can now be modelled by $H = 20 - \frac{k}{t}$ metres, where k is a constant at $t \geqslant 4$.

i Find the value of k. **ii** Find $\frac{dH}{dt}$.

iii Does the rate at which the tree is growing change when it is taken out of its pot and planted in the ground? Explain your answer.

iv Show that $\frac{dH}{dt} > 0$ for all $t \geqslant 4$ and explain what this means.

v Find the height of the tree after 10, 20, and 50 years. Comment on your answer.

c Graph the height of the tree for $0 \leqslant t \leqslant 10$ years.

Example 2 **Self Tutor**

The cost in dollars of producing x items in a factory each day is given by $C(x) = 9500 + 12x + 8x^{0.8}$.

a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

c Find $C(151) - C(150)$. Compare this with the answer in **b**.

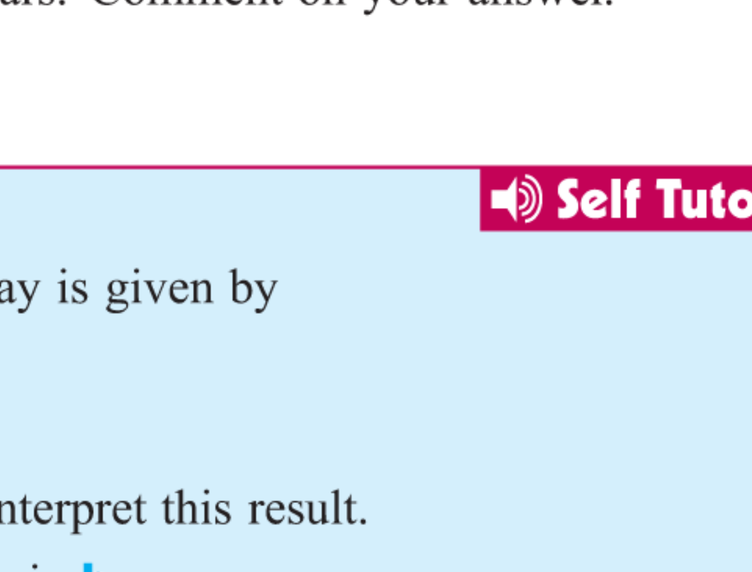
a The marginal cost function is
$C'(x) = 12 + 6.4x^{-0.2}$ dollars per item.

b $C'(150) \approx \$14.35$

This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day.

It gives an estimate of the cost of making the 151st item each day.

c $C(151) - C(150) \approx \$11\,754.87 - \$11\,740.52$
$\approx \$14.35$



This is the actual cost of making the 151st item each day, so the answer in **b** gives a very good estimate.

5 Seablue make denim jeans. The cost model for making x pairs per day is

$$C(x) = 7800 + 6x + 12x^{0.7} \text{ dollars.}$$

a Find the marginal cost function $C'(x)$. **b** Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. Discuss your answer.

6 The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \frac{200\,000}{v}$ euros where v is the average speed of the train in km h^{-1}.



a Find the total cost of the journey if the average speed is:

i 50 km h^{-1} **ii** 100 km h^{-1}.

b Find the rate of change in the cost of running the train for the average speed:

i 30 km h^{-1} **ii** 90 km h^{-1}.

c At what speed will the cost be a minimum?

7 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

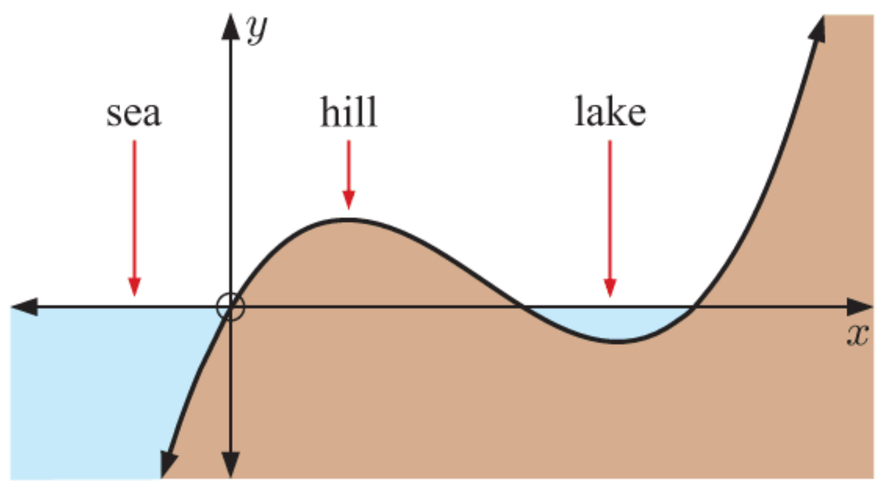
a Find $\frac{dV}{dt}$, and draw the graph of $\frac{dV}{dt}$ against t.

b At what time was the outflow fastest?

c Show that $\frac{d^2V}{dt^2}$ is always constant and positive. Interpret this result.

8 Alongside is a land and sea profile where the x-axis is sea level.

The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance x km from the shore line.



a Find where the lake is located relative to the shore line of the sea.

b Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.

c Find the deepest point of the lake, and the depth at this point.

9 A radioactive substance decays according to the formula $W = 20e^{-0.01t}$ grams where t is the time in hours.

a Find the weight of radioactive substance present:

i initially **ii** after 24 hours **iii** after 1 week.

b How long will it take for the weight to reach 1 gram?

c Show that $\frac{dW}{dt} = bW$ for some constant b.

d Find the rate of radioactive decay after:

i 100 hours **ii** 1000 hours.

10 The velocity of a car after t seconds is given by $v = 5\sqrt{t+1} - 5$ m s^{-1}. The momentum of the car is measured in kg m s^{-1}, and is given by $P = 1600v$.

a Find P as a function of t. **b** Find $\frac{dP}{dt}$ and interpret its value when $t = 3$.

c At what time is the momentum of the car increasing by 800 kg m s^{-1} per second?

11 The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-0.12t}$ °C where t is the time in minutes.

a What is the temperature of the liquid after 15 minutes?

b Show that $\frac{dT}{dt} = c(T - 5)$ for some constant c.

c At what rate is the temperature changing:

i initially **ii** after 10 minutes **iii** after 20 minutes?

12 The height of a shrub t years after it was planted is given by $H(t) = 20\ln(3t+2) + 30$ cm, $t \geqslant 0$.

a How high was the shrub when it was planted?

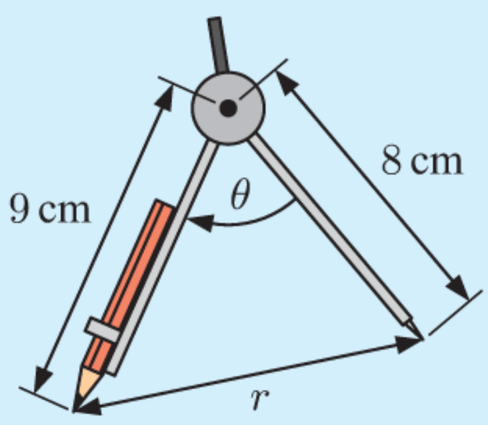
b How long will it take for the shrub to reach a height of 1 m?

c At what rate is the shrub's height changing:

i 3 years after being planted **ii** 10 years after being planted?

Example 3 Self Tutor

Cathy is using a compass to draw a circle. The arm with the needle is 8 cm long, and the arm with the pencil is 9 cm in total. The angle between the arms is θ.



a Write the radius of the circle to be drawn in terms of θ.

b Hence find the rate of change in r with respect to θ when $\theta = 60°$.

a Using the cosine rule, $r^2 = 9^2 + 8^2 - 2 \times 9 \times 8 \times \cos\theta$

$\therefore\ r = \sqrt{145 - 144\cos\theta}$ {since $r > 0$}

b $r = (145 - 144\cos\theta)^{\frac{1}{2}}$

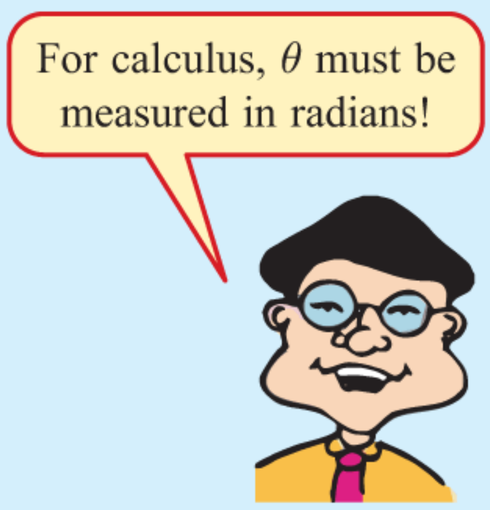
$\therefore\ \frac{dr}{d\theta} = \frac{1}{2}(145 - 144\cos\theta)^{-\frac{1}{2}}(144\sin\theta)$

$= \frac{72\sin\theta}{\sqrt{145 - 144\cos\theta}}$

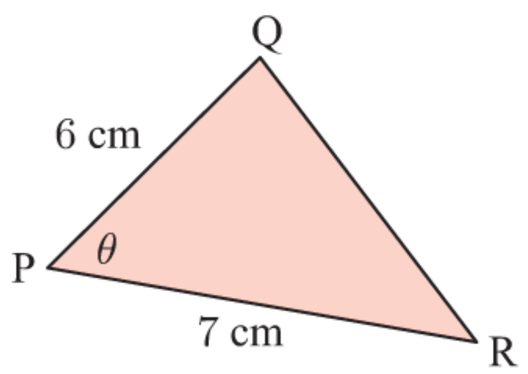
When $\theta = \frac{\pi}{3}$, $\frac{dr}{d\theta} = \frac{72\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{145 - 144\left(\frac{1}{2}\right)}}$

≈ 7.30 cm per radian

≈ 0.127 cm per degree



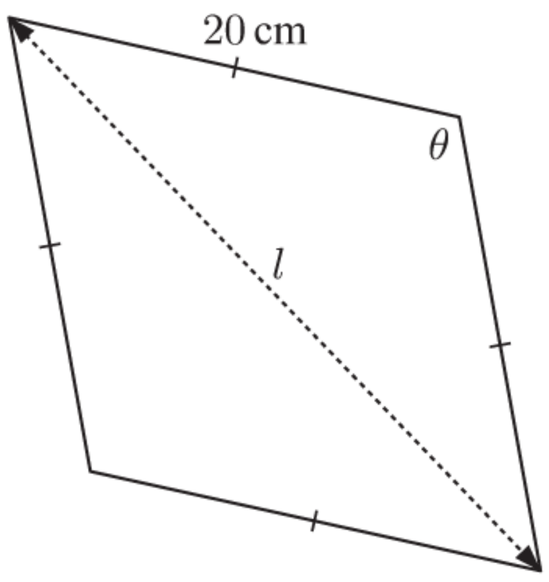
13 Find exactly the rate of change in the area of triangle PQR as θ changes, at the time when $\theta = 45°$.

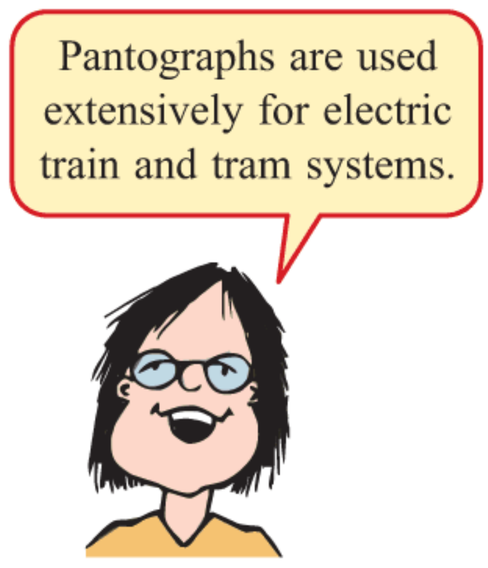


14



A set of retractable stairs is used to gain access to an attic. The frame uses a pantograph mechanism which is a set of rhombuses with variable angles to control the retraction. Each rhombus has side length 20 cm, and the angle where the arms meet is θ, as shown.





a Find a formula for the length l between the pivots, in terms of θ.

b Hence find the rate of change in l at the time when $\theta = 120°$.

15 The voltage in a circuit is given by $V(t) = 340\sin(100\pi t)$ volts where t is the time in seconds.

a Find the voltage in the circuit:

i initially **ii** after 0.125 seconds.

b At what rate is the voltage changing:

i when $t = 0.01$ **ii** when $V(t)$ is a maximum?

16 The number of bees in a hive after t months is modelled by $B(t) = \dfrac{3000}{1 + 0.5e^{-1.73t}}$.

a Find the initial bee population.

b Find the percentage increase in the population after 1 month.

c Is there a limit to the population size? If so, what is it?

d Find $B'(t)$, and use it to explain why the population is increasing over time.

e Find the rate at which the population is increasing after 6 months.

f Sketch the graph of $B(t)$.

B OPTIMISATION

Optimisation is the process of finding the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimal solution**.

We can find optimal solutions in several ways:

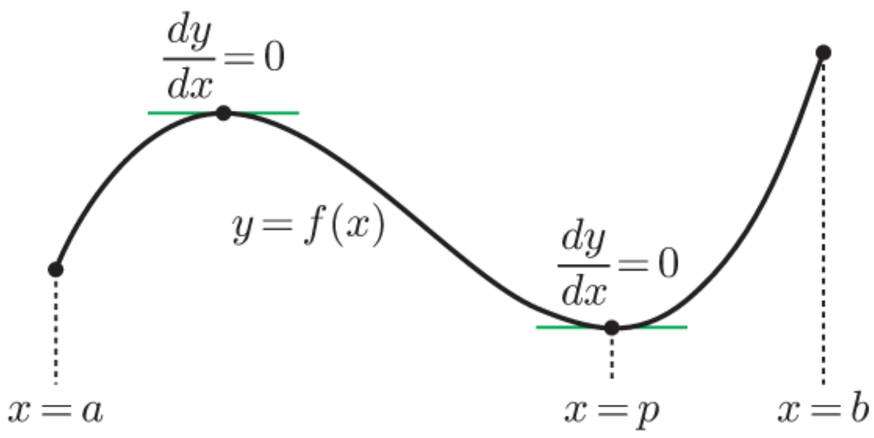
- using technology to graph the function and search for the maximum or minimum value
- using analytical methods such as the formula $x = -\dfrac{b}{2a}$ for the vertex of a parabola
- using differential calculus to locate the turning points of a function.

These last two methods are useful especially when exact solutions are required.

You should always be aware that:

> The maximum or minimum value does not always occur when the first derivative is zero.
>
> It is essential to also examine the values of the function at the end point(s) of the interval under consideration for global maxima and minima.

For example:



The maximum value of y occurs at the end point $x = b$.

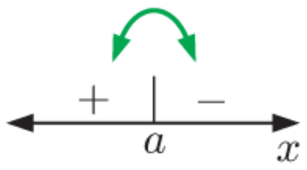
The minimum value of y occurs at the local minimum $x = p$.

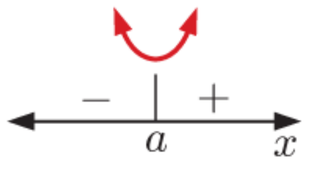
TESTING FOR LOCAL MAXIMA AND MINIMA

If we find a value $x = a$ such that $f'(a) = 0$, there are several tests we can use to see whether we have a local maximum or a local minimum at this point.

SIGN DIAGRAM TEST

If, near to $x = a$, the sign diagram is:

- 

we have a **local maximum**
- 

we have a **local minimum**.

SECOND DERIVATIVE TEST

- If $f''(a) < 0$ we have  shape, which indicates a **local maximum**.
- If $f''(a) > 0$ we have  shape, which indicates a **local minimum**.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a **formula** with the variable to be optimised as the subject. It should be written in terms of one convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the value(s) of x which make the first derivative **zero**.

Step 4: For each stationary point, use the **sign diagram test** or **second derivative test** to determine whether you have a local maximum or local minimum.

Step 5: Identify the optimal solution, also considering end points where appropriate.

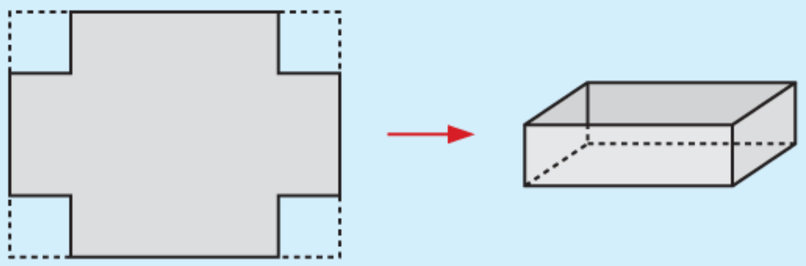
Step 6: Write your answer in a sentence, making sure you specifically answer the question.

Example 4

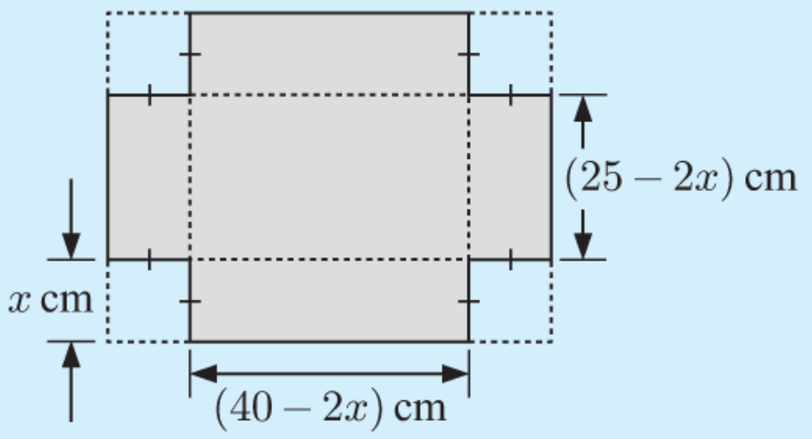
Self Tutor

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



Step 1: Let x cm be the side lengths of the squares that are cut out.



Step 2: Volume = length × width × depth

$$= (40 - 2x)(25 - 2x)x$$
$$= (1000 - 80x - 50x + 4x^2)x$$
$$= 1000x - 130x^2 + 4x^3 \text{ cm}^3$$

Since the side lengths must be positive, $x > 0$ and $25 - 2x > 0$.

$\therefore \quad 0 < x < 12.5$

Step 3:
$$\frac{dV}{dx} = 12x^2 - 260x + 1000$$
$$= 4(3x^2 - 65x + 250)$$
$$= 4(3x - 50)(x - 5)$$

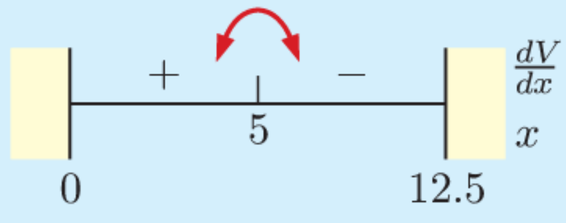
$\therefore \quad \frac{dV}{dx} = 0$ when $x = \frac{50}{3} = 16\frac{2}{3}$ or $x = 5$

$\therefore \quad x = 5$ as $0 < x < 12.5$

DEMO



Step 4: $\frac{dV}{dx}$ has sign diagram:



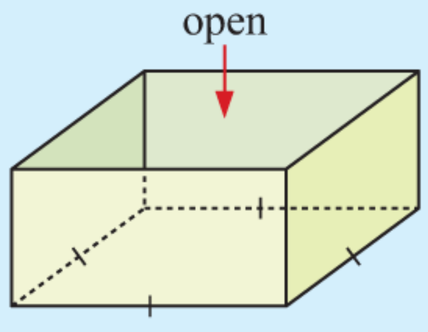
Step 5: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

Step 6: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

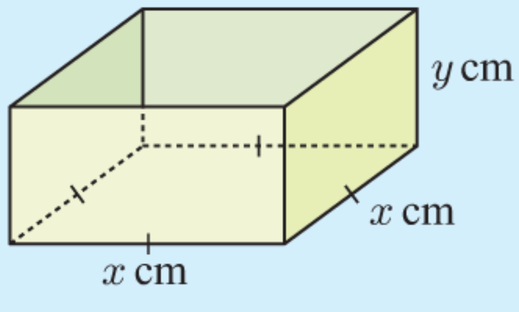
Example 5

Self Tutor

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.



Step 1:



Let the base lengths be x cm and the depth be y cm.

The volume V = length × width × depth

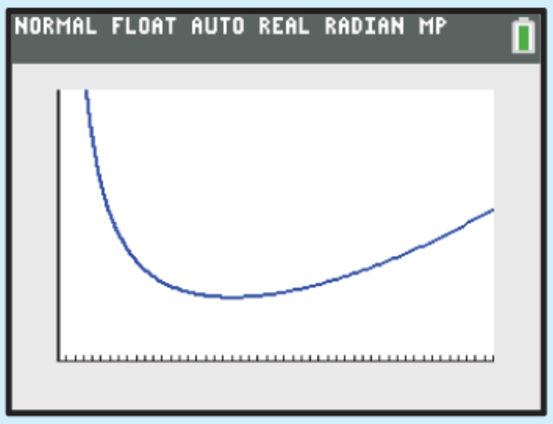
$\therefore \quad V = x^2 y$

$\therefore \quad 4000 = x^2 y \quad \ldots. (1) \quad \{1 \text{ litre} \equiv 1000 \text{ cm}^3\}$

Step 2: The total surface area

$$\begin{aligned} A &= \text{area of base} + 4(\text{area of one side}) \\ &= x^2 + 4xy \\ &= x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using (1)}\} \end{aligned}$$

$$\therefore \quad A(x) = x^2 + 16\,000x^{-1} \quad \text{where} \quad x > 0$$



Step 3: $\therefore \quad A'(x) = 2x - 16\,000x^{-2}$

$$\therefore \quad A'(x) = 0 \quad \text{when} \quad 2x = \frac{16\,000}{x^2}$$

$$\therefore \quad 2x^3 = 16\,000$$

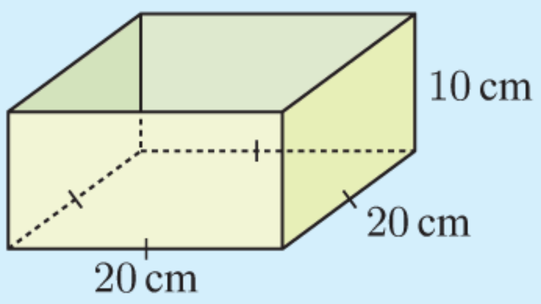
$$\therefore \quad x = \sqrt[3]{8000} = 20$$

Step 4: $A''(x) = 2 + 32\,000x^{-3}$

$$\therefore \quad A''(20) = 2 + \frac{32\,000}{20^3} = 6$$

Since $A''(20) > 0$, there is a local minimum at $x = 20$.

Step 5: The minimum material is used to make the container when $x = 20$ and $y = \dfrac{4000}{20^2} = 10$.



Step 6: The most economical shape has a square base 20 cm × 20 cm, and height 10 cm.

Use **calculus techniques** to answer the following problems. In cases where finding the zeros of the derivatives is difficult you may use technology to help you.

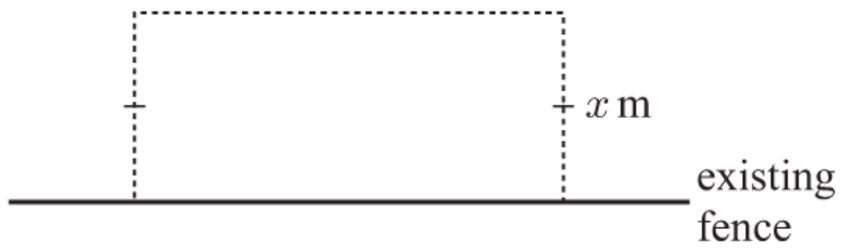
EXERCISE 20B

1 When a manufacturer makes x items per day, the profit function is $P(x) = -0.022x^2 + 11x - 720$ pounds. Find the production level that will maximise profits.

2 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ pounds, and for this production level each blanket may be sold for $\left(23 - \frac{1}{2}x\right)$ pounds.

How many blankets should be produced per day to maximise the total profit?

3 60 metres of fencing is used to build a rectangular enclosure along an existing fence. Suppose the sides adjacent to the existing fence are x m long.



a Show that the area A of the enclosure is given by $A(x) = x(60 - 2x)$ m^2.

b Find the dimensions which maximise the area of the enclosure.

4 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

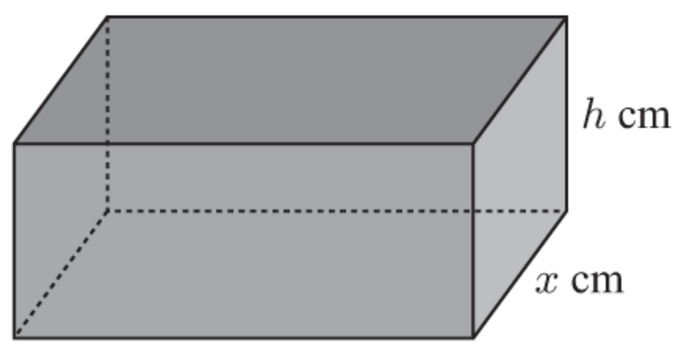


a If the sides adjacent to the existing fence have length x m, show that the required length of wire netting to be purchased is $L = 2x + \dfrac{100}{x}$.

b Find the minimum value of L and the corresponding value of x when this occurs.

c Sketch the optimal situation, showing all dimensions.

5 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of a box has dimensions in the ratio $2 : 1$.



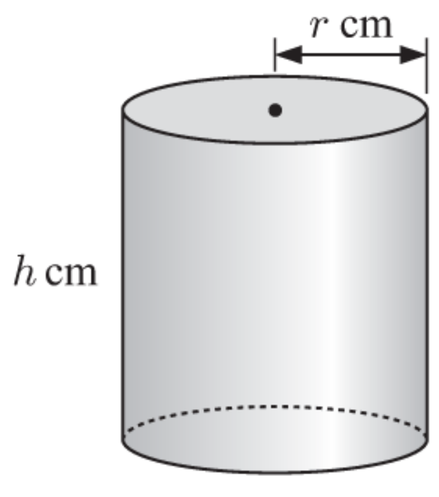
a Show that $x^2h = 100$.

b Show that the inner surface area of the box is given by $A(x) = 4x^2 + \dfrac{600}{x}$ cm^2.

c Find the minimum inner surface area of the box and the corresponding value of x.

d Sketch the optimal box shape, showing all dimensions.

6 Brenda is designing a cylindrical tin can for a canned fruit company. The cans must have capacity 1 litre, and they must use as little metal as possible.

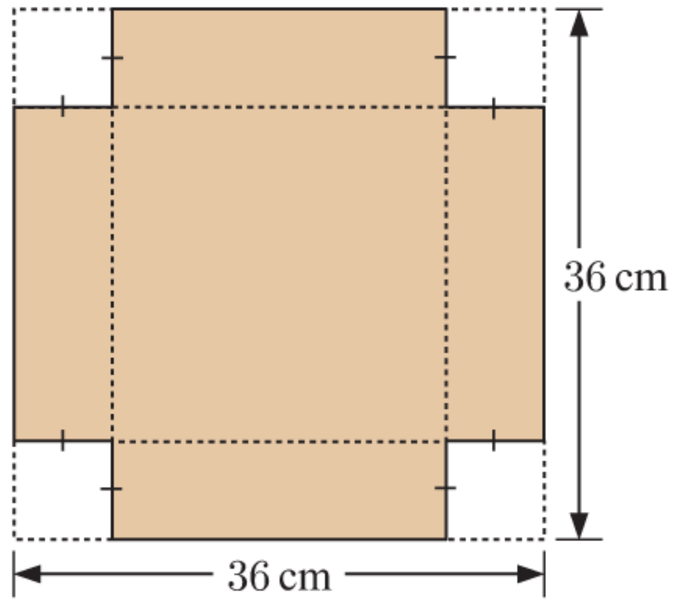


a Explain why the height h is given by $h = \dfrac{1000}{\pi r^2}$ cm.

b Show that the total surface area A is given by $A = 2\pi r^2 + \dfrac{2000}{r}$ cm^2.

c Find the dimensions of the can which make A as small as possible.

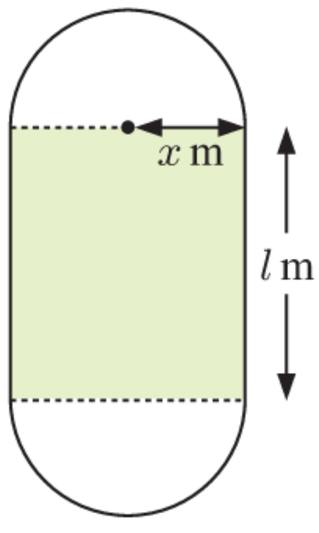
7 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.



a Show that the volume of the container is given by $V(x) = x(36 - 2x)^2$ cm^3.

b What sized squares should be cut out to produce the container of greatest capacity?

8 An athletics track has two "straights" of length l m, and two semi-circular ends of radius x m. The perimeter of the track is 400 m.



a Show that $l = 200 - \pi x$ and write down the possible values that x may have.

b What values of l and x maximise the shaded rectangle inside the track? What is this maximum area?

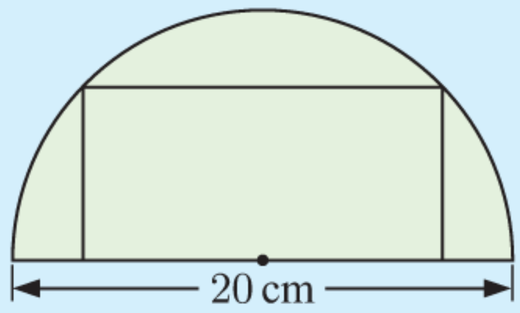
9 A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by $C(x) = 4 \ln x + \left(\dfrac{30 - x}{10}\right)^2$ pounds with a minimum production capacity of 10 kettles per day.

How many kettles should be manufactured to keep the cost per kettle to a minimum?

Example 6 **Self Tutor**

Infinitely many rectangles can be inscribed in a semi-circle of diameter 20 cm.

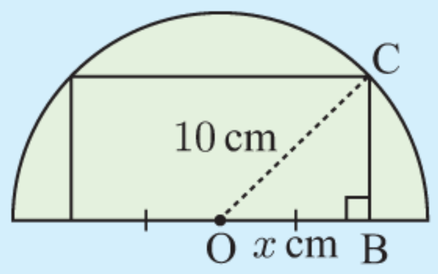
Find the shape of the largest rectangle which can be inscribed.



Step 1: Let OB $= x$ cm, $0 < x < 10$

In $\triangle$OBC, $BC^2 + x^2 = 10^2$ {Pythagoras}

$\therefore$ $BC = \sqrt{100 - x^2}$ {as $BC > 0$}



Step 2: The rectangle has area $A =$ length $\times$ width

$\therefore$ $A = 2x\sqrt{100 - x^2}$

$\therefore$ $A^2 = 4x^2(100 - x^2)$

$= 400x^2 - 4x^4$

Since $A > 0$, we can maximise A by maximising A^2. This makes the calculations easier!

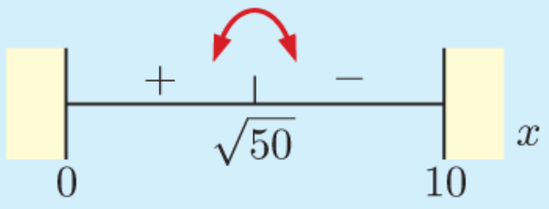


Step 3: $\frac{d}{dx}(A^2) = 800x - 16x^3$

$= 16x(50 - x^2)$

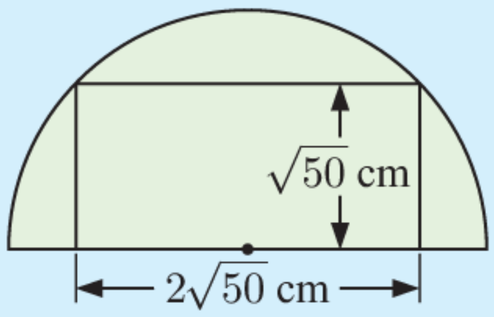
So, $\frac{d}{dx}(A^2) = 0$ when $x = 0$ or $\pm\sqrt{50}$.

Step 4: $\frac{d}{dx}(A^2)$ has sign diagram:



Step 5: The area is maximised when $x = \sqrt{50}$ and $BC = \sqrt{100 - 50} = \sqrt{50}$ cm.

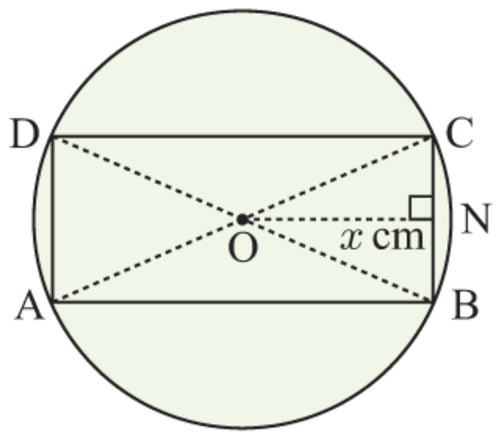
Step 6: The largest rectangle which can be inscribed is $2\sqrt{50}$ cm long and $\sqrt{50}$ cm wide.



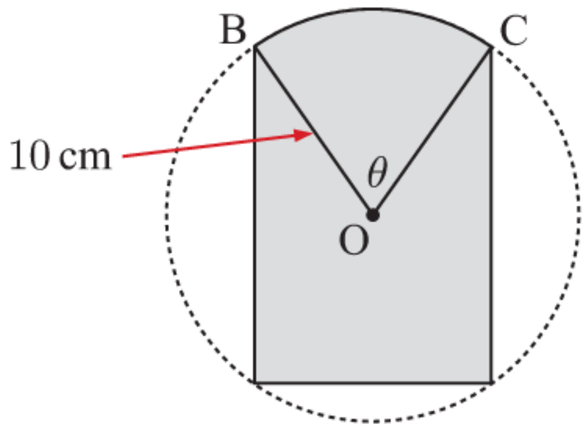
10 Infinitely many rectangles can be inscribed in a circle of diameter 10 cm. In the diagram alongside, suppose ON $= x$ cm.

a Find the area of ABCD in terms of x only.

b Find the dimensions of ABCD which maximises its area.



11

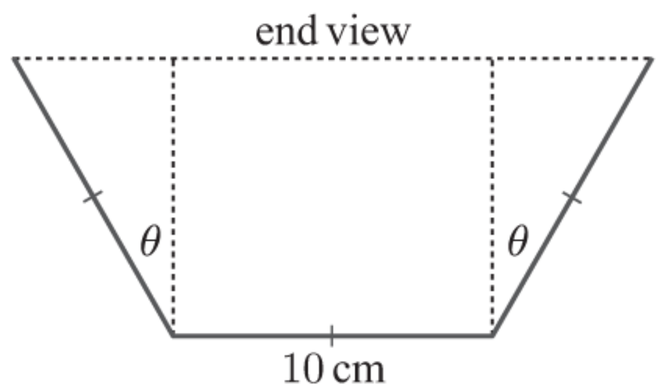


A circular piece of tin-plate with radius 10 cm has 3 segments removed as illustrated. The angle θ is measured in radians.

a Show that the remaining area is given by $A = 50(\theta + 3\sin\theta)$ cm^2.

b Find θ such that the area A is a maximum, and find the area A in this case.

12 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.



a Deduce that the cross-sectional area of the gutter is given by $A = 100\cos\theta(1 + \sin\theta)$ cm^2.

b Show that $\frac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.

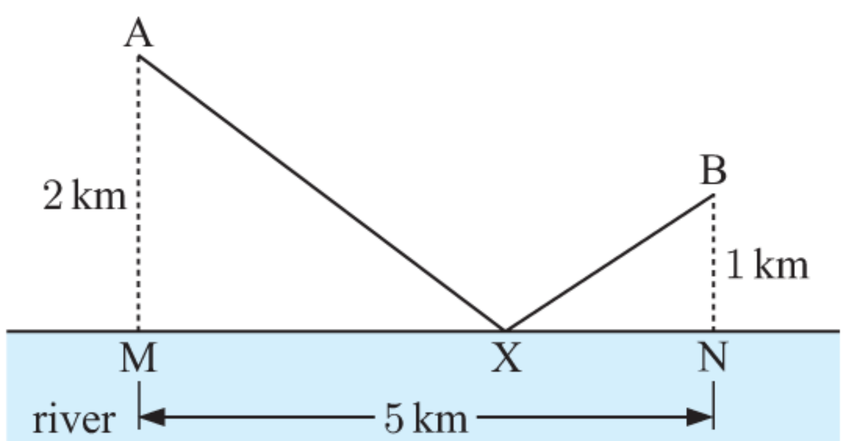
c For what value of θ does the gutter have maximum carrying capacity? Find the cross-sectional area for this value of θ.

13 When a new anaesthetic is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.

a Find $E'(t)$.

b At what time is the anaesthetic most effective?

14

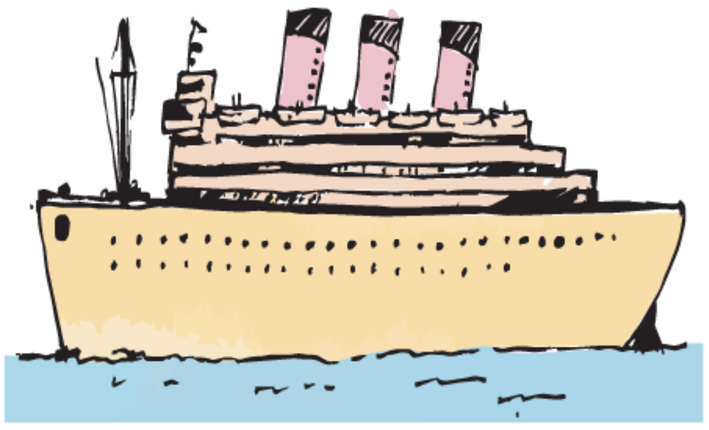


A pumphouse is to be placed at some point X along a river.

Two pipelines will then connect the pumphouse to homesteads A and B.

How far should point X be from M so that the total length of pipeline is minimised?

15 At 1:00 pm ship A leaves port P. It sails in the direction 30° east of north at 12 km h^{-1}. At the same time, ship B is 100 km due east of P, and is sailing at 8 km h^{-1} towards P.

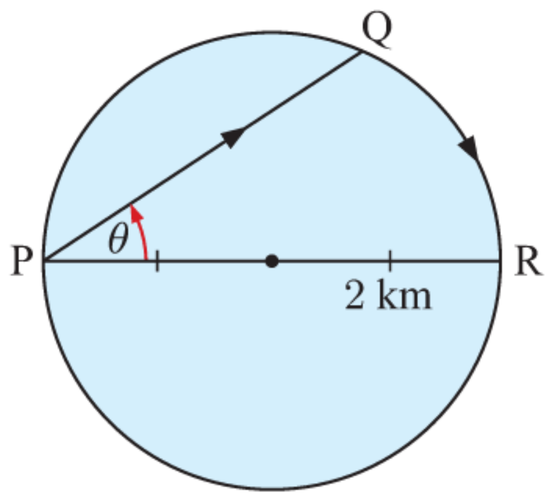


a Show that the distance between the two ships is given by $D(t) = \sqrt{304t^2 - 2800t + 10\,000}$ km, where t is the number of hours after 1:00 pm.

b Find the minimum value of D^2 for all $t \geqslant 0$.

c At what time, to the nearest minute, are the ships closest?

16 Hieu can row a boat at 3 km h^{-1}, and can walk at 6 km h^{-1}. He is currently at point P on the shore of a lake 2 km in radius. He will row to point Q, then walk around the shore to point R which is opposite P.



a Show that PQ $= 4\cos\theta$ km.

b Show that the time taken for Hieu's journey is given by $T = \frac{4}{3}\cos\theta + \frac{2\theta}{3}$ hours where $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

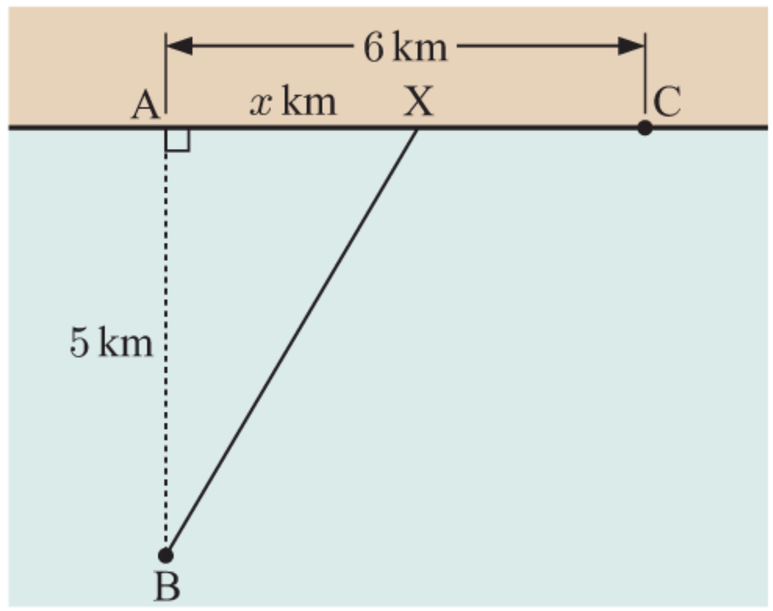
c Find θ such that $\frac{dT}{d\theta} = 0$ on $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

d Draw a sign diagram for $\frac{dT}{d\theta}$.

e What route should Hieu take to travel from P to R in:

i the longest time **ii** the shortest time?

17 B is a boat 5 km out at sea from A. [AC] is a straight sandy beach, 6 km long. Peter can row the boat at 8 km h^{-1} and run along the beach at 17 km h^{-1}. Suppose Peter rows directly from B to point X on [AC] such that AX $= x$ km.



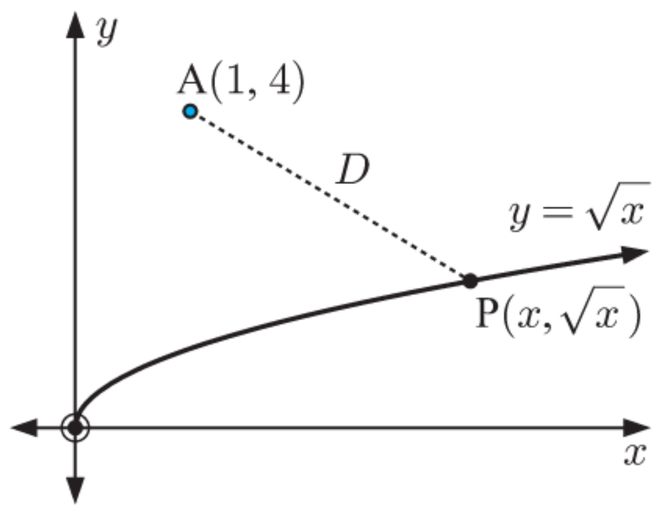
a Explain why $0 \leqslant x \leqslant 6$.

b Show that the *total time* Peter takes to row to X and then run along the beach to C, is given by

$$T = \frac{\sqrt{x^2 + 25}}{8} + \frac{6 - x}{17} \text{ hours}, \quad 0 \leqslant x \leqslant 6.$$

c Find x such that $\dfrac{dT}{dx} = 0$. Explain the significance of this value.

18 Suppose the point A$(1, 4)$ is distance D from a general point P on the curve $y = \sqrt{x}$.



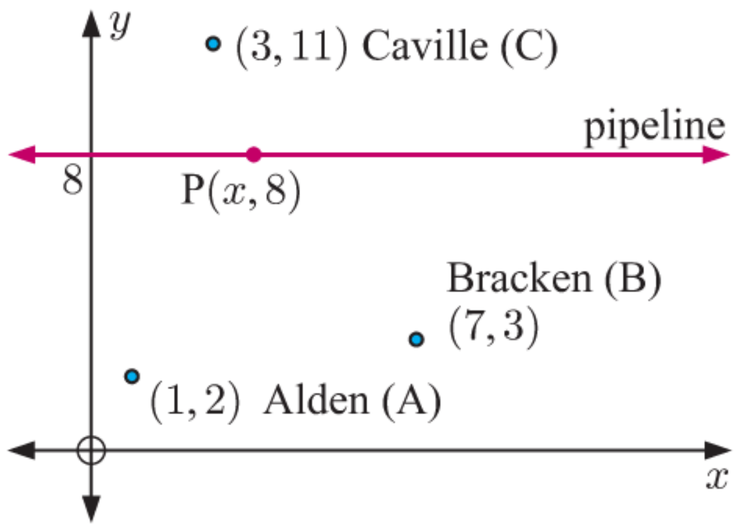
a Explain why $D = \sqrt{(x-1)^2 + (\sqrt{x} - 4)^2}$ units.

b Use technology to sketch the graph of D against x for $0 \leqslant x \leqslant 8$.

c Find the smallest value of D and the value of x where it occurs.

d What does this tell you about $\dfrac{dD}{dx}$?

19 Three towns and their grid references are marked on the diagram. A pumping station is to be located at P on the pipeline, to pump water to the three towns. The grid units are kilometres.



a Show that the distance PC is given by $\text{PC} = \sqrt{(x-3)^2 + 9}$.

b Find formulae for the distances PA and PB.

c Write a formula for the sum of the distances $S = \text{PA} + \text{PB} + \text{PC}$ in terms of x.

d Use technology to draw the graph of S against x.

e *Hence* draw a sign diagram for $S'(x)$.

f Where should P be located to minimise the total length of connecting pipe needed?

20 A small population of wasps is observed. After t weeks the population is modelled by

$$P(t) = \frac{50\,000}{1 + 1000e^{-0.5t}} \text{ wasps, where } 0 \leqslant t \leqslant 25.$$

a Find $P'(t)$.

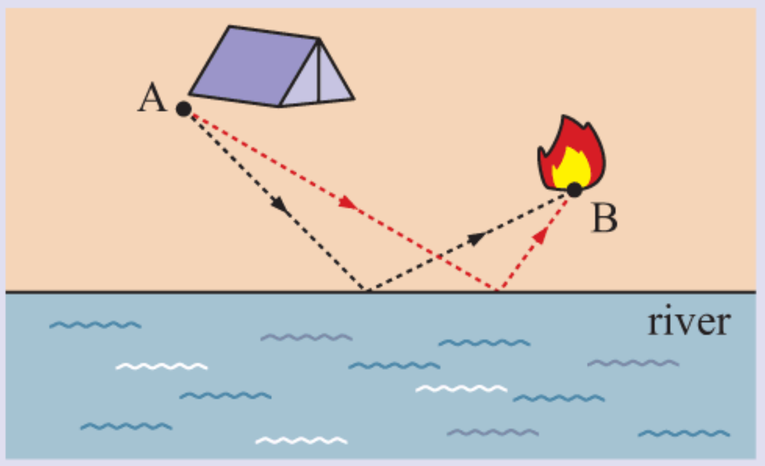
b Find the rate at which the population is increasing after 10 weeks.

c Use technology to determine the time(s) at which the population is increasing at 2000 wasps per week.

d At what time is the wasp population growing fastest?

THEORY OF KNOWLEDGE

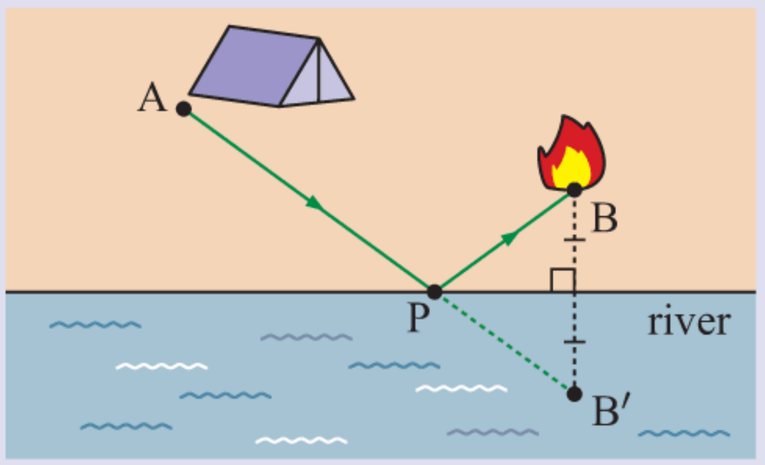
Suppose you are camping next to a river, and you pitch your tent at A. Suddenly you notice your campfire at B is out of control. You pick up your bucket and run to the river to get some water, then head to the fire to put it out.



Time is critical so you must take the shortest path. Where, on the river bank, should you fill your bucket?

To solve this problem, we reflect the point B in the river, creating the point B′. Each point on the river is equidistant from B and B′.

The shortest path from A to B′ is clearly the line [AB′]. Therefore, you should fill your bucket at the point P where the line [AB′] intersects the river.



1 Is optimisation a mathematical principle?

2 Is mathematics an intrinsic or natural part of other subjects?

C MODELLING WITH CALCULUS

We have seen how derivatives can be interpreted as rates of change, and how zeros of a derivative correspond to stationary points of a function. We can use knowledge of rates and turning points to help construct mathematical models.

Example 7 **Self Tutor**

The cost of manufacturing the parts for x robots each day is given by $C(x) = ax^3 + bx^2 + cx + d$ dollars, where the maximum output for the factory each day is 200 robots. The fixed costs for keeping the factory running amount to \$3400 each day. The parts for the first robot each day cost \$449. The cost of manufacturing parts for 100 robots each day is \$38 300, and the cost of the 101st set of parts is \$246.



a Find the cost function for producing the robot parts, using $C'(0) \approx C(1) - C(0)$ and $C'(100) \approx C(101) - C(100)$.

b Hence find the cost for producing the parts for 150 robots each day.

a The maximum output is 200 sets of parts, so the domain of $C(x)$ is $0 \leqslant x \leqslant 200$.

The fixed costs are $C(0) = \$3400$, so $d = 3400$.

Differentiating with respect to x, $C'(x) = 3ax^2 + 2bx + c$.

Now $C'(0) \approx 449$, so $c \approx 449$

and $C'(100) \approx 246$, so $30\,000a + 200b + 449 = 246$

$$\therefore \quad 30\,000a + 200b = -203 \quad \ldots. \ (1)$$

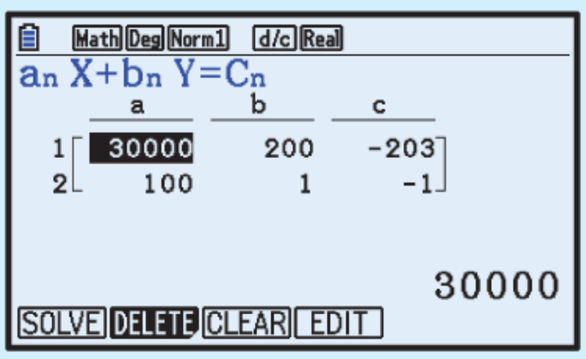
$C(100) = 38\,300$, so $1\,000\,000a + 10\,000b + 100c + 3400 = 38\,300$

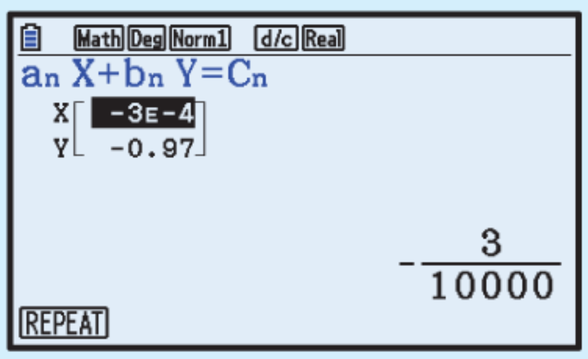
$$\therefore \quad 1\,000\,000a + 10\,000b = -10\,000$$

$$\therefore \quad 100a + b = -1 \quad \ldots. \ (2)$$

We solve (1) and (2) simultaneously using technology.

$\therefore$ $a \approx -0.0003$ and $b \approx -0.97$





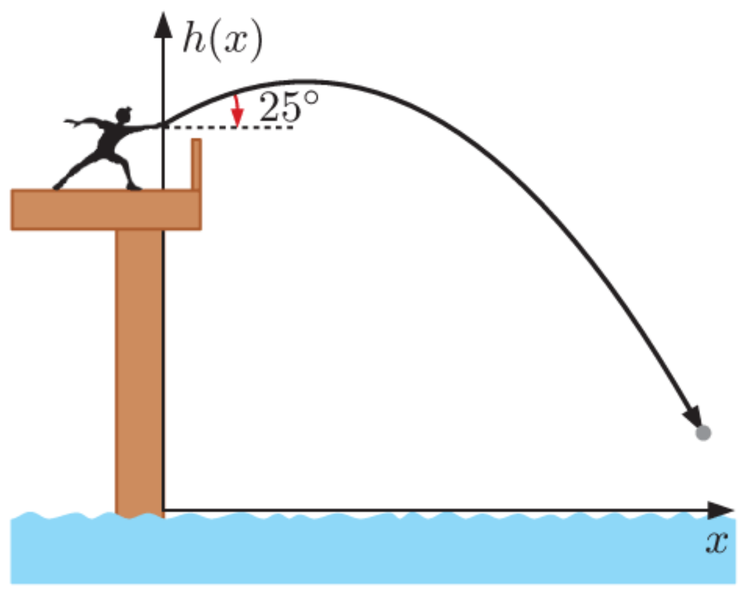
So, $C(x) = -0.0003x^3 - 0.97x^2 + 449x + 3400, \quad 0 \leqslant x \leqslant 200$.

b $C(150) = -0.0003(150)^3 - 0.97(150)^2 + 449(150) + 3400 = 47\,912.5$

$\therefore$ it costs about \$47 900 to produce the parts for 150 robots each day.

EXERCISE 20C

1 A stone is thrown from a bridge with initial trajectory 25° above horizontal. At the time when it has travelled x m horizontally, the height of the stone above the water under the bridge is given by $h(x) = ax^2 + bx + c$ metres. The stone is thrown from 3 m above the water, and it reaches its maximum height when $x = 5$.



a State $h(0)$ and hence find c.

b Find $h'(x)$.

c Explain why $h'(0) = \tan 25°$, and hence find b.

d Find the value of x when the stone lands in the water.

2 The cost of manufacturing the parts for x cars each day is given by $C(x) = ax^3 + bx^2 + cx + d$ euros. The maximum output for the factory each day is 140 cars. The fixed costs for keeping the factory running total €24 500 each day. The parts for the first car each day cost €3280. The cost of manufacturing parts for 80 cars each day is €294 000, and the cost of the 81st set of car parts is €2320.

a Find the cost function for producing the car parts, using $C'(0) \approx C(1) - C(0)$ and $C'(80) \approx C(81) - C(80)$.

b Hence find the cost of producing the parts for 120 cars each day.

3 The introduction of the automobile in the early 20th century had a dramatic effect on the population of domesticated horses, which had been used extensively for agriculture and transportation. In the United States of America there were 21.5 million horses in 1900, and the population peaked at 26.5 million in 1915. By 1950, the population had decreased to only 7.6 million.

Let $P(t) = at^3 + bt^2 + ct + d$ describe the population in millions of horses, in the United States of America t years after 1900.



a State $P(0)$ and hence find d.

b Find $P'(t)$.

c State the value of t for which $P'(t) = 0$.

d Construct three linear equations for a, b, and c. Hence find $P(t)$.

e Estimate the population of horses in:

i 1930 **ii** 1960.

f Discuss the limitations of your model.

4 Concerned with parking, transportation, catering, and all aspects of the local fair, the town mayor decided to model the attendance at the fair using the function $A(t) = at^3 + bt^2 + ct + d$, where t is the number of hours after 8 am.

The fair is open from 8 am to 6 pm. At 10 am, the attendance was increasing at the rate of 1770 people per hour. At 4 pm, the attendance was decreasing at the rate of 1800 people per hour.

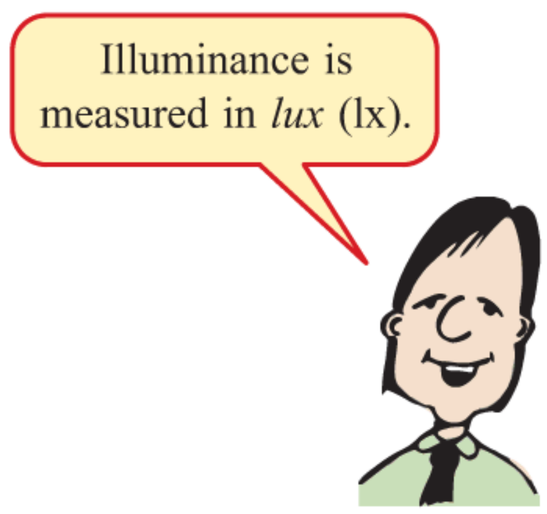
a State *four* conditions for $A(t)$ and $A'(t)$ you can deduce from the information given.

b Hence find the function $A(t)$.

c Graph $A(t)$ and discuss whether the mayor's model is reasonable.

d Predict the maximum attendance at the fair, and when this occurred.

5 Tia wanted to test how much light is in her office. She took measurements of the illuminance at different distances away from her light, as shown in the table:



Distance (r cm)	20	40	80
Illuminance (I lx)	450	112.3	28.1

a Which of the following models do you think will best fit the data? Explain your answer.

A $I = \frac{k}{r}$ **B** $I = \frac{k}{r^2}$ **C** $I = \frac{k}{r^3}$

b Use technology to find the power model which best fits the data, and hence check your answer to **a**.

c Find $\frac{dI}{dr}$.

d Hence find the rate at which the illuminance is decreasing at the distance 1 m from the light.

e Can you explain how this model tells us that the *total* illuminance from the light at any distance r cm will be the same, no matter what the value of r is?

ACTIVITY CUBIC SPLINES

A *cubic spline* is a piecewise function which uses several cubic polynomials to draw a smooth curve through a set of data points. Click on the icon to access this Activity.



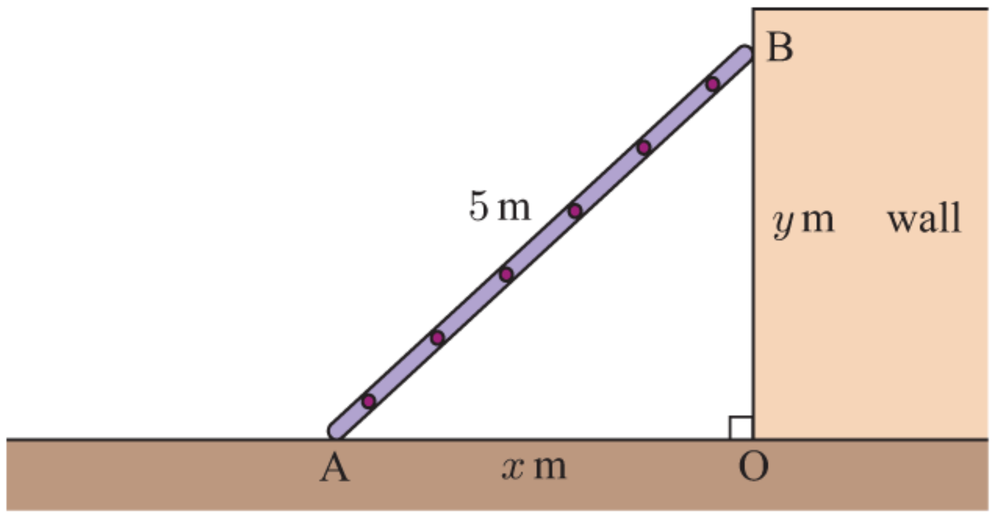
D RELATED RATES

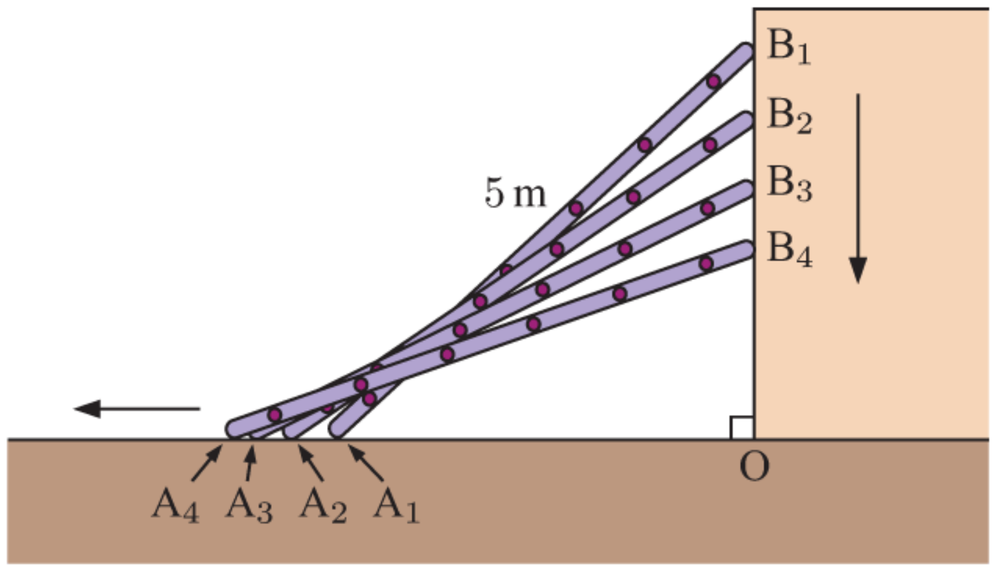
In the **Opening Problem** on page **524**, we observed a ladder slide down a wall.

Because the ladder has a fixed length, when the distance x m from the wall to the foot of the ladder increases, the height y m that the ladder reaches up the wall must decrease. The variables x and y are *related*.

However, we can also consider the *rates* at which x and y change with respect to time t.

The variables $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are **related rates**.





The method for solving related rates problems is:

Step 1: Draw a large, clear **diagram** of the situation.

Step 2: Write down the information, label the diagram, and make sure you distinguish between the **variables** and the **constants**.

Step 3: Write an **equation** connecting the variables. You may need to use:

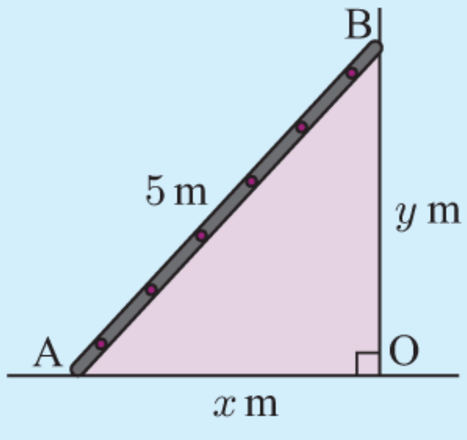
- Pythagoras' theorem
- similar triangles where corresponding sides are in proportion
- right angled triangle trigonometry
- sine and cosine rules.

Step 4: Use the chain rule to **differentiate** the equation with respect to time t.

Step 5: Substitute the values for the **particular case** corresponding to some instant in time, and solve to find the required unknown.

Example 8 Self Tutor

A 5 m long ladder rests against a vertical wall with its feet on horizontal ground. The feet on the ground slip, and at the instant when they are 3 m from the wall, they are moving at 10 m s^{-1}. At what speed is the other end of the ladder moving at this instant?



Let $\text{OA} = x$ m and $\text{OB} = y$ m

$\therefore\ x^2 + y^2 = 5^2$ {Pythagoras}

$\therefore\ y^2 = 25 - x^2$

$\therefore\ y = \sqrt{25 - x^2} = (25 - x^2)^{\frac{1}{2}}$ {as $y \geqslant 0$}

Now $\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$ {chain rule}

$$= \left[\tfrac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)\right] \times \frac{dx}{dt}$$

$$= -\frac{x}{\sqrt{25 - x^2}} \frac{dx}{dt}$$

Particular case:

At the instant when $x = 3$, $\dfrac{dx}{dt} = 10 \text{ m s}^{-1}$.

$$\therefore \quad \frac{dy}{dt} = -\frac{3}{\sqrt{25 - 3^2}} \times 10$$

$$= -\tfrac{3}{4} \times 10$$

$$= -7.5 \text{ m s}^{-1}$$

The other end of the ladder is moving down the wall at 7.5 m s^{-1} at this instant.

Do not substitute values for the particular case too early. Otherwise you will incorrectly treat variables as constants. The differentiated equation in fully generalised form must be established first.



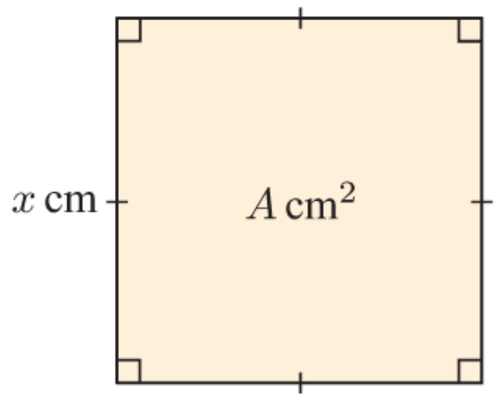
EXERCISE 20D

1 x and y are variables related by the equation $y = 2x^3$.

a Differentiate this equation with respect to t.

b At the instant when $x = 3$, x is increasing at 1 unit per second. What is happening to y at this instant?

2 A square has side length x cm and area A cm^2. The side length of the square is increasing at 2 cm per second.



a State the relationship between A and x.

b Differentiate the relationship in **a** with respect to t.

c At what rate is the area increasing when the side length is 6 cm?

3 The length of a rectangle is decreasing at 1 cm per minute. However, the area of the rectangle remains constant at 100 cm^2.

Let x cm be the length of the rectangle, and y cm be the width of the rectangle.

a Write y in terms of x.

b Differentiate the relationship in **a** with respect to t.

c At what rate is the width of the rectangle increasing at the instant when:

i the length is 20 cm **ii** the width is 4 cm?

4 A stone is thrown into a lake, and a circular ripple moves out at a constant speed of 1 m s^{-1}. Find the rate at which the circle's area is increasing at the instant when:

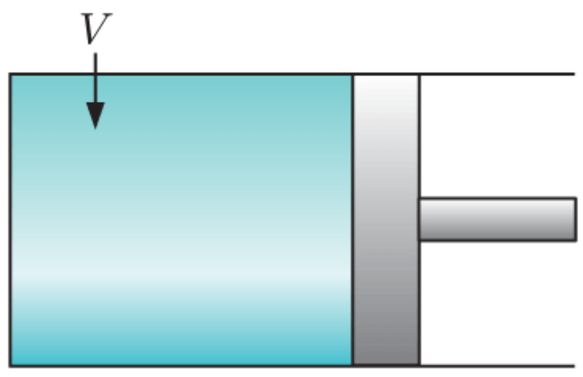
a $t = 2$ seconds **b** $t = 4$ seconds.

5 A spherical weather balloon has radius r m and volume V m^3.

a Write V in terms of r.

b Air is pumped into the balloon at a constant rate of 6π m^3 per minute. Find the rate of change in the balloon's radius at the instant when the radius is 2 m.

6



For a given mass of gas in a piston, $p = \dfrac{400}{V^{1.5}}$ where p is the pressure in N m^{-2}, and V is the volume in m^3.

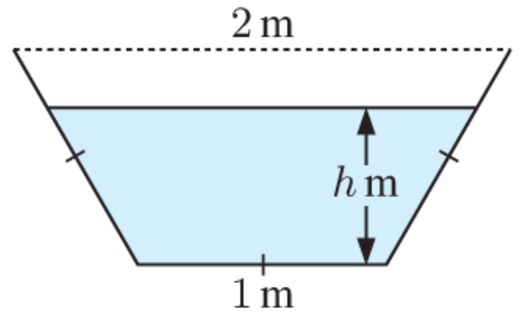
Suppose the pressure increases at a constant rate of $3\ \text{N m}^{-2}$ per minute. Find the rate at which the volume is changing at the instant when:

a the volume is $1\ \text{m}^3$ **b** the pressure is $50\ \text{N m}^{-2}$.

7 Wheat runs from a hole in a silo at a constant rate and forms a conical heap whose base radius is treble its height. After 1 minute, the height of the heap is 20 cm. Find the rate at which the height is rising at this instant.

8 Two jet aeroplanes fly on parallel courses which are 12 km apart. Their air speeds are $200\ \text{m s}^{-1}$ and $250\ \text{m s}^{-1}$. How fast is the distance between them changing at the instant when the slower jet is 5 km ahead of the faster one?

9 A trough of length 6 m has a uniform cross-section which is a trapezium with side lengths shown.



a Show that if the water is h m deep then the volume of water in the trough is given by $V = 6h + 2\sqrt{3}h^2\ \text{m}^3$.

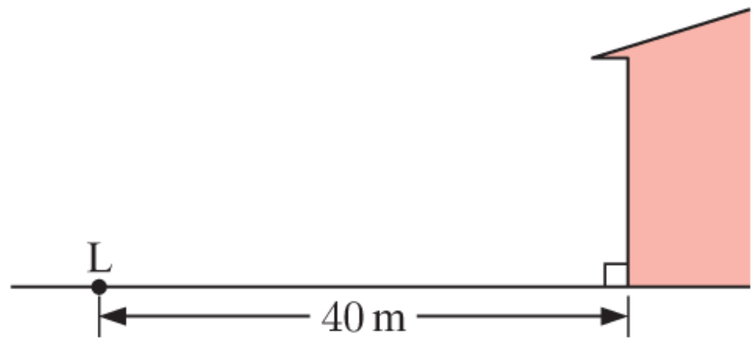
b Water leaks from the bottom of the trough at a constant rate of $0.1\ \text{m}^3$ per minute. Find the rate at which the water level is falling at the instant when the water is 20 cm deep.

10 A ground-level floodlight located 40 m from the foot of a building shines in the direction of the building.

A 2 m tall person walks directly from the floodlight towards the building at $1\ \text{m s}^{-1}$.

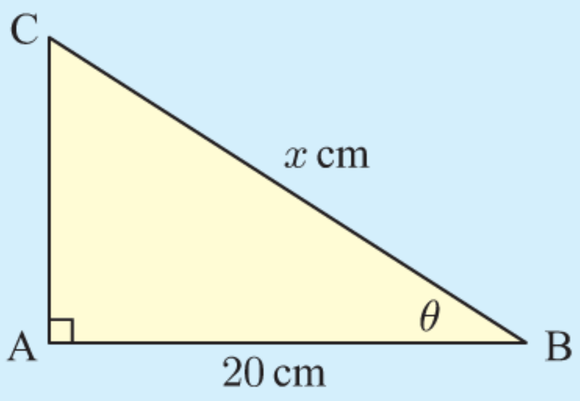
How fast is the person's shadow on the building shortening at the instant when the person is:

a 20 m **b** 10 m from the building?



Example 9 **Self Tutor**

Triangle ABC is right angled at A, and $AB = 20$ cm. $A\widehat{B}C$ increases at a constant rate of $1°$ per minute. At what rate is BC changing at the instant when $A\widehat{B}C$ measures $30°$?



Let $A\widehat{B}C = \theta$ and $BC = x$ cm

Now $\cos\theta = \dfrac{20}{x}$

$\therefore\ x = \dfrac{20}{\cos\theta} = 20(\cos\theta)^{-1}$

Now $\dfrac{dx}{dt} = \dfrac{dx}{d\theta} \times \dfrac{d\theta}{dt}$ {chain rule}

$= -20(\cos\theta)^{-2}(-\sin\theta) \times \dfrac{d\theta}{dt}$

$= \dfrac{20\sin\theta}{\cos^2\theta}\ \dfrac{d\theta}{dt}$

Particular case:

When $\theta = 30°$, $\dfrac{d\theta}{dt} = 1°$ per minute

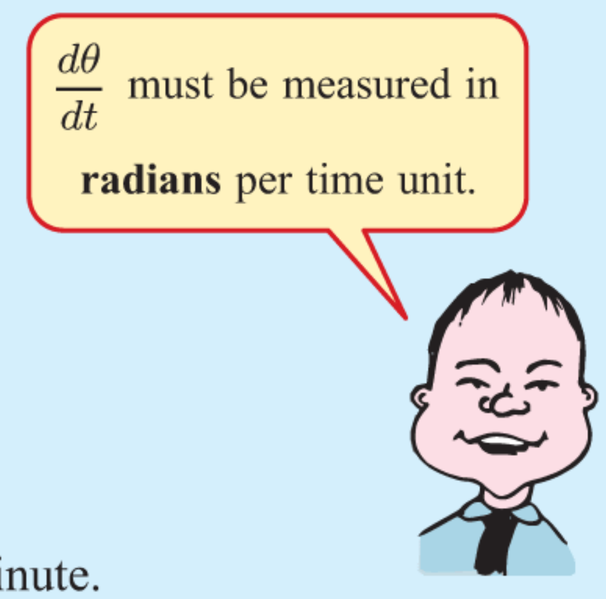
$= \frac{\pi}{180}$ radians per minute

$\therefore \quad \dfrac{dx}{dt} = \dfrac{20 \sin 30°}{\cos^2 30°} \times \dfrac{\pi}{180}$

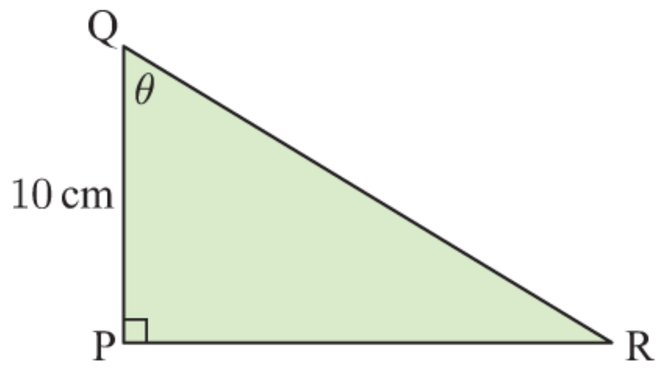
$= \dfrac{20 \times \frac{1}{2}}{\frac{3}{4}} \times \dfrac{\pi}{180}$

≈ 0.233 cm per minute

$\therefore$ BC is increasing at approximately 0.233 cm per minute.



11 Consider a right angled triangle PQR in which QP $= 10$ cm and $\widehat{PQR} = \theta$. $\widehat{PQR}$ is increasing at $2°$ per minute.



a As $\widehat{PQR}$ increases, would you expect QR to increase or decrease?

b At what rate is QR changing when $\widehat{PQR}$ measures $60°$?

12 A right angled triangle ABC has a fixed hypotenuse [AC] of length 10 cm. Side [AB] increases in length at 0.1 cm per second. At what rate is $\widehat{CAB}$ decreasing at the instant when the triangle is isosceles?

13 An aeroplane passes directly overhead, then flies horizontally away from an observer with altitude of 5000 m and air speed 200 m s^{-1}.

a Draw a diagram to illustrate the situation.

b At what rate is its angle of elevation to the observer changing at the instant when the angle of elevation is:

i $60°$ **ii** $30°$?

14 [AB] is a fixed diameter of a circle of radius 5 cm. Point P moves around the circle at a constant rate of 1 revolution in 10 seconds. Find the rate at which the distance AP is changing at the instant when:

a AP $= 5$ cm and increasing **b** P is at B.

REVIEW SET 20A

1 The height of a tree t years after it was planted is $H(t) = 60 + 40 \ln(2t + 1)$ cm, $t \geqslant 0$.

a How tall was the tree when it was planted?

b How long will it take for the tree to reach:

i 150 cm **ii** 300 cm?

c At what rate is the tree's height increasing after:

i 2 years **ii** 20 years?

2 The value of a car t years after its purchase is given by $V = 20\,000e^{-0.4t}$ pounds. Calculate:

a the purchase price of the car

b the rate at which the value of the car is decreasing 10 years after it was purchased.

3 The cost per hour of running a freight train is given by $C(v) = \frac{v^2}{20} + \frac{50\,000}{v}$ dollars where v is the average speed of the train in km h^{-1}.

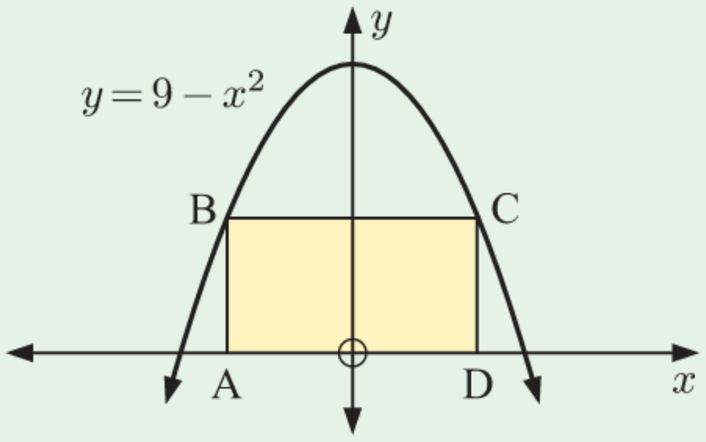
a Find the cost of running the train for 5 hours at $64\ \text{km h}^{-1}$.

b Find the rate of change in the hourly cost of running the train at speeds of:

i $75\ \text{km h}^{-1}$ **ii** $90\ \text{km h}^{-1}$.

c At what speed will the cost per hour be a minimum?

4



Rectangle ABCD is inscribed within the parabola $y = 9 - x^2$ and the x-axis, as shown.

a If $\text{OD} = x$, show that the rectangle ABCD has area function $A(x) = 18x - 2x^3$.

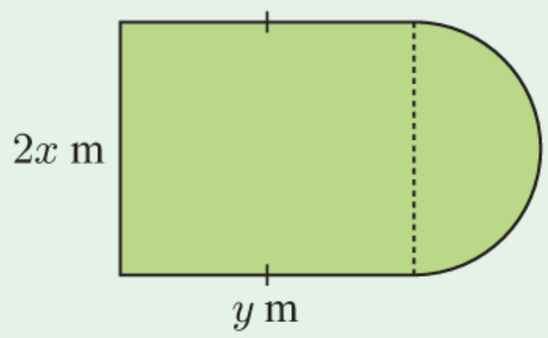
b Find the coordinates of C when rectangle ABCD has maximum area.

5 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.

a Using the dimensions shown on the figure, show that $y = 100 - x - \frac{\pi}{2}x$.

b Find the area of the lawn A in terms of x only.

c Find the dimensions of the lawn of maximum area.

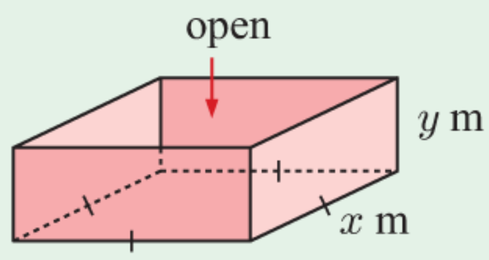


6 A manufacturer of open steel boxes has to make one with a square base and a capacity of 1 kL. The steel costs £2 per square metre.

a If the base measures x m by x m and the height is y m, find y in terms of x.

b Hence show that the total cost of the steel is

$C(x) = 2x^2 + \frac{8}{x}$ pounds.



c Find the dimensions of the steel box which would cost the least to make.

7 When Max throws a stone into the air, it flies for 5.9 seconds before landing. Over the course of its flight, the distance between the stone and Max after t seconds is given by

$D(t) = \sqrt{24.01t^4 - 294t^3 + 936t^2}$ metres.

a State the domain of $D(t)$.

b Find the time at which D^2 is maximised.

c Hence find the maximum distance between the stone and Max.

d How far away from Max does the stone land?

8 A ball is thrown into the air from height 1.6 m above the ground. It initially gains height at $16.4\ \text{m s}^{-1}$. After 2 seconds, it is falling at $3.2\ \text{m s}^{-1}$. The height of the ball over time can be modelled by the quadratic function $h(t) = at^2 + bt + c$ metres, where t is the time in seconds.

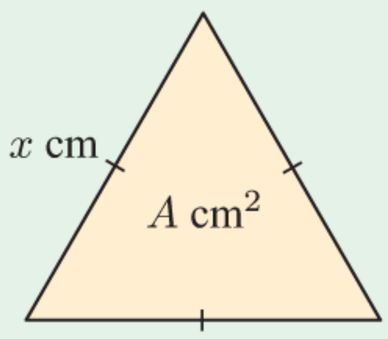
a State $h(0)$ and hence find c.

b State $h'(0)$ and hence find b.

c Use the remaining information to find a.

d What was the maximum height reached by the ball, and when did it reach this height?

9



An equilateral triangle has side length x cm and area A cm^2. The side length of the triangle is increasing at 3 cm per minute.

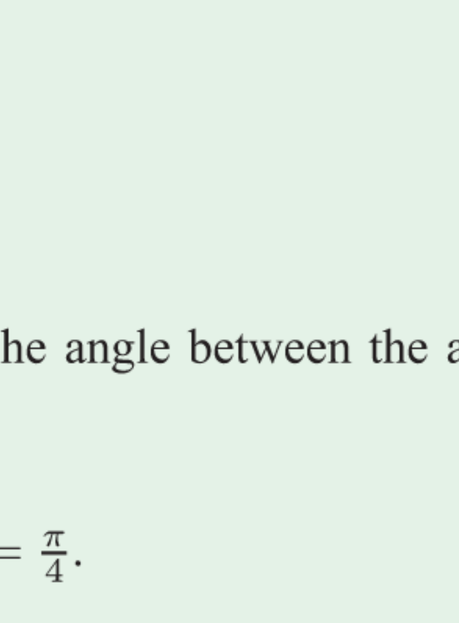
a State the relationship between A and x.

b At what rate is the area increasing when the side length is 15 cm?

10 Water exits a conical tank at a constant rate of 0.2 m^3 per minute. The surface of the water has radius r m.

a Find $V(r)$, the volume of the water remaining.

b Find the rate at which the surface radius is changing at the instant when the water is 5 m deep.



REVIEW SET 20B

1 The cost in euros of producing x items in a factory each day is given by $C(x) = 850 + 3.3x^{0.85} + 2.8x^{0.5}$.

a Find the marginal cost function $C'(x)$.

b Find $C'(1000)$ and explain what this result estimates.

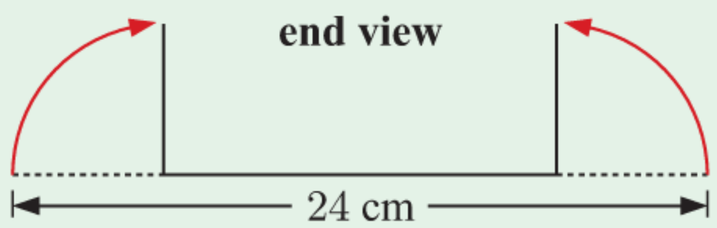
c Find $C(1001) - C(1000)$. Discuss your answer.

2 Mark has set his compass so that both arms have length 11 cm. The angle between the arms is θ.

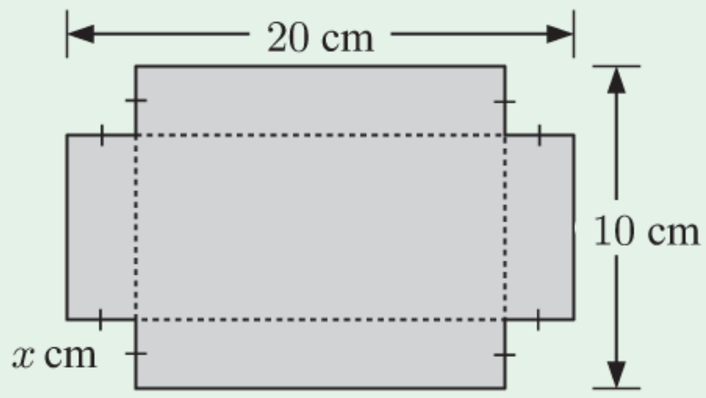
a Write the area A of the circle Mark will draw in terms of θ.

b Hence find the rate of change in A with respect to θ when $\theta = \frac{\pi}{4}$.

3 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown.
Where must the bends be made in order to maximise the capacity of the gutter?



4



A rectangular sheet of tin-plate is 20 cm by 10 cm. Four squares, each with sides x cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of x which will maximise the capacity of the container.

5 On the Indonesian coast, the depth of water t hours after midnight is given by $D = 9.3 + 6.8\cos(0.507t)$ metres.

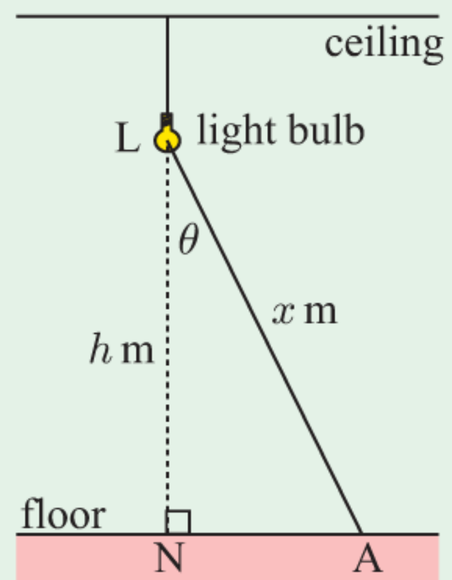
a Find the derivative function $\frac{dD}{dt}$ and explain what it tells us.

b What is the depth of water at 8 am?

c Is the tide rising or falling at 8 am? Explain your answer.

d At what time(s) is the tide highest on this day? What is the maximum depth of water?

6 A light bulb hangs from the ceiling at height h metres above the floor, directly above point N. At any point A on the floor which is x metres from the light bulb, the illumination I is given by $I = \dfrac{\sqrt{8}\cos\theta}{x^2}$ units.



a If NA $= 1$ metre, show that at A, $I = \sqrt{8}\cos\theta\sin^2\theta$.

b The light bulb may be lifted or lowered to change the intensity at A.

i Use technology to graph I against θ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

ii Hence draw a sign diagram for $\dfrac{dI}{d\theta}$ on $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

iii Find the height the bulb should be above the floor to provide the greatest illumination at A.

7 At time t years after mining begins on a mountain of iron ore, the rate of iron ore output is given by $R(t) = \dfrac{100 \times 3^{0.03t}}{25 + 2^{0.25t-10}}$ million tonnes per year, $t \geqslant 0$.

a Use technology to graph $R(t)$ against t for $0 \leqslant t \leqslant 100$.

b *Hence* draw a sign diagram for $R'(t)$ on $0 \leqslant t \leqslant 100$.

c At what rate will the ore be mined after $t = 20$ years?

d When will the rate of mining be 10 million tonnes per year?

e What will be the maximum rate of mining, and at what time will it occur?

8 The share price for a business listed on the New York Stock Exchange is given by $S(t) = at^3 + bt^2 + ct + d$ dollars, where t is the number of hours after 9:30 am, $0 \leqslant t \leqslant 6.5$. When the market opened at 9:30 am, the share price was \$22.81, but it immediately began to fall with instantaneous rate 16 cents per hour. At 12:30 pm the share price was \$22.49, and by 2:30 pm it had increased to \$22.60.

a State $S(0)$ and hence find d.

b State $S'(0)$ and hence find c.

c Use the remaining information to find a and b.

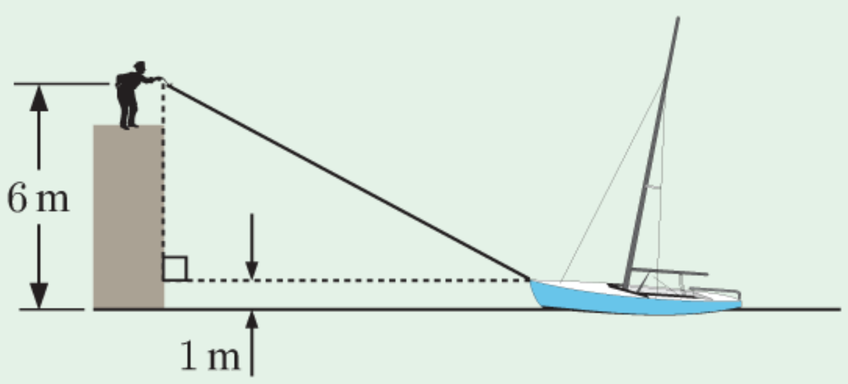
d Use technology to graph $S(t)$.

e Predict the share price at the 4 pm market close.

f Find the minimum and maximum values of the share price during the day.

9 Rectangle PQRS has [PQ] of fixed length 20 cm, and [QR] increases in length at a constant rate of 2 cm s^{-1}. At what rate is the acute angle between the diagonals of the rectangle changing at the instant when [QR] is 15 cm long?

10



A man on a jetty pulls a boat towards him by hauling a rope at the rate 20 metres per minute. The rope is attached to the boat 1 m above water level, and the man's hands are 6 m above the water level. How fast is the boat approaching the jetty at the instant when it is 15 m from the jetty?

Chapter 21

Introduction to integration

Contents:

A Approximating the area under a curve
B The Riemann integral
C Antidifferentiation
D The Fundamental Theorem of Calculus

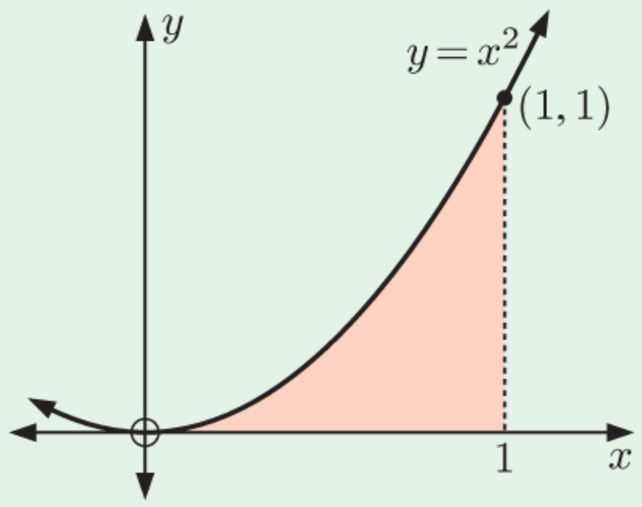


OPENING PROBLEM

Another of **Archimedes'** achievements was devising a method for calculating the area under a curve.

In an article containing 24 propositions, he provided essential theory for what, over 1800 years later, would be developed into **integral calculus**.

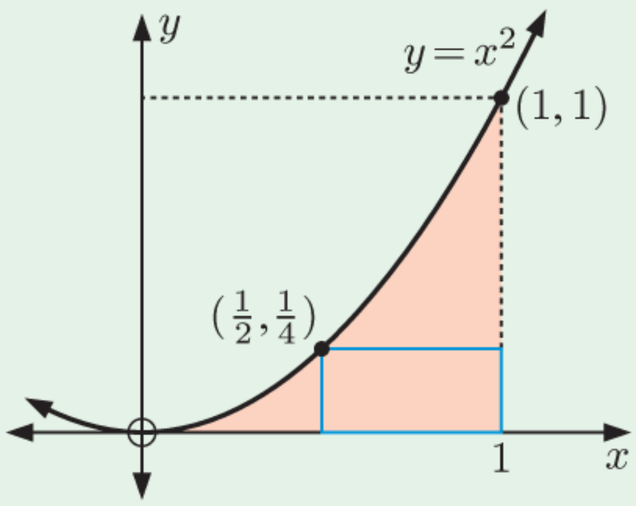
In the process, Archimedes found the exact area A between the curve $y = x^2$ and the x-axis, on the interval $0 \leqslant x \leqslant 1$.



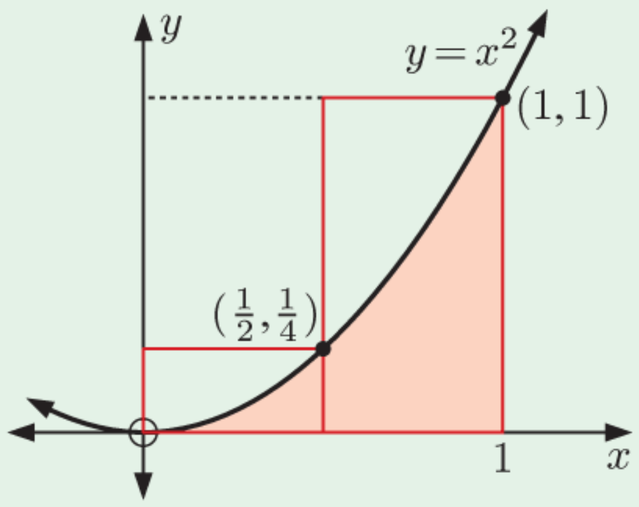
Things to think about:

a Can you use the:

i blue rectangle to explain why $A > \frac{1}{8}$



ii red rectangles to explain why $A < \frac{5}{8}$?



b How can we obtain a better estimate for A?

c What function has x^2 as its derivative?

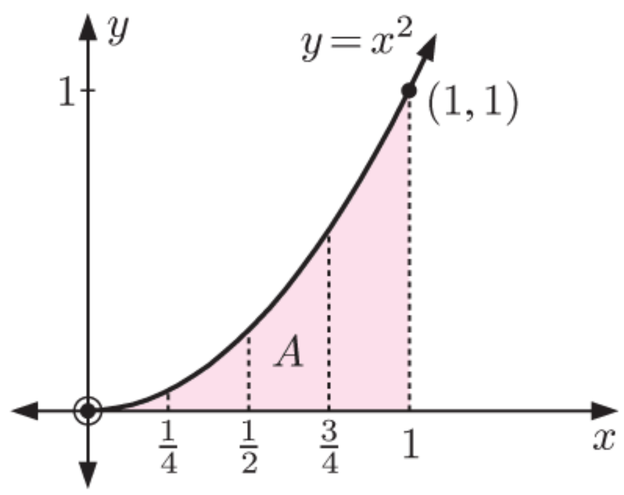
In this Chapter we consider **integral calculus**. This involves **antidifferentiation**, which is the reverse process of differentiation.

A APPROXIMATING THE AREA UNDER A CURVE

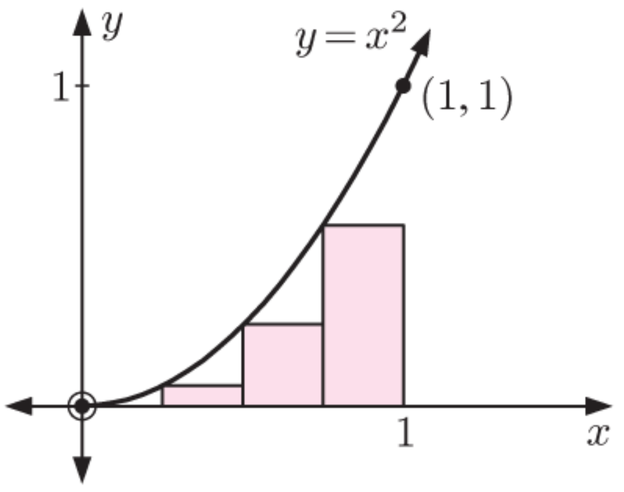
Consider the function $f(x) = x^2$ in the **Opening Problem**.

We wish to estimate the area A enclosed by $y = f(x)$, the x-axis, and the vertical line $x = 1$.

Suppose we divide the interval $0 \leqslant x \leqslant 1$ into 4 strips of width $\frac{1}{4}$ unit as shown. We obtain 4 subintervals of equal width.



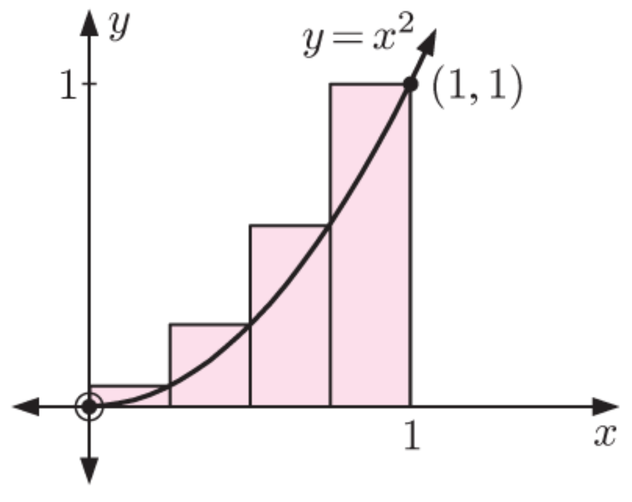
The diagram alongside shows **lower rectangles**, which are rectangles with height equal to the *lower* value of the function at the endpoints of the subinterval.



The total area of the lower rectangles is

$$\begin{aligned} A_L &= \tfrac{1}{4} \times f(0) + \tfrac{1}{4} \times f(\tfrac{1}{4}) + \tfrac{1}{4} \times f(\tfrac{1}{2}) + \tfrac{1}{4} \times f(\tfrac{3}{4}) \\ &= \tfrac{1}{4}(0)^2 + \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 \\ &= 0.218\,75 \end{aligned}$$

The next diagram shows **upper rectangles**, which are rectangles with height equal to the *upper* value of the function at the endpoints of the subinterval.



The total area of the upper rectangles is

$$\begin{aligned} A_U &= \tfrac{1}{4} \times f(\tfrac{1}{4}) + \tfrac{1}{4} \times f(\tfrac{1}{2}) + \tfrac{1}{4} \times f(\tfrac{3}{4}) + \tfrac{1}{4} \times f(1) \\ &= \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{4}(1)^2 \\ &= 0.468\,75 \end{aligned}$$

Clearly, for increasing functions such as $f(x) = x^2$, $A_L < A < A_U$, so the area A lies between $0.218\,75$ units^2 and $0.468\,75$ units^2.

If the interval $0 \leqslant x \leqslant 1$ was divided into 8 subintervals instead, each of width $\frac{1}{8}$, then

$$\begin{aligned} A_L &= \tfrac{1}{8}\left[f(0) + f(\tfrac{1}{8}) + f(\tfrac{1}{4}) + f(\tfrac{3}{8}) + f(\tfrac{1}{2}) + f(\tfrac{5}{8}) + f(\tfrac{3}{4}) + f(\tfrac{7}{8})\right] \\ &= \tfrac{1}{8}\left[0 + \tfrac{1}{64} + \tfrac{1}{16} + \tfrac{9}{64} + \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{9}{16} + \tfrac{49}{64}\right] \\ &\approx 0.273\,44 \\ A_U &= \tfrac{1}{8}\left[f(\tfrac{1}{8}) + f(\tfrac{1}{4}) + f(\tfrac{3}{8}) + f(\tfrac{1}{2}) + f(\tfrac{5}{8}) + f(\tfrac{3}{4}) + f(\tfrac{7}{8}) + f(1)\right] \\ &= \tfrac{1}{8}\left[\tfrac{1}{64} + \tfrac{1}{16} + \tfrac{9}{64} + \tfrac{1}{4} + \tfrac{25}{64} + \tfrac{9}{16} + \tfrac{49}{64} + 1\right] \\ &\approx 0.398\,44 \end{aligned}$$

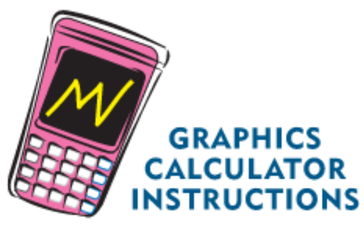
From this refinement we conclude that the area A lies between $0.273\,44$ units^2 and $0.398\,44$ units^2.

As we create more subintervals, the estimates A_L and A_U will become more and more accurate. In fact, as the subinterval width is reduced further and further, both A_L and A_U will **converge** to A.

Now suppose there are n subintervals between $x = 0$ and $x = 1$, each of width $\frac{1}{n}$.

You can use the **area finder** software or your **graphics calculator** to help calculate A_L and A_U for large values of n.





The table alongside summarises the results you should obtain for $n = 4$, 8, 16, 50, 200, 1000, and 10 000.

From the table, it appears that both A_L and A_U are *converging* to $\frac{1}{3}$ as n increases.

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
8	0.273 44	0.398 44	0.335 50
16	0.302 73	0.365 23	0.333 98
50	0.323 40	0.343 40	0.333 40
200	0.330 84	0.335 84	0.333 38
1000	0.332 83	0.333 83	0.333 33
10 000	0.333 28	0.333 38	0.333 33

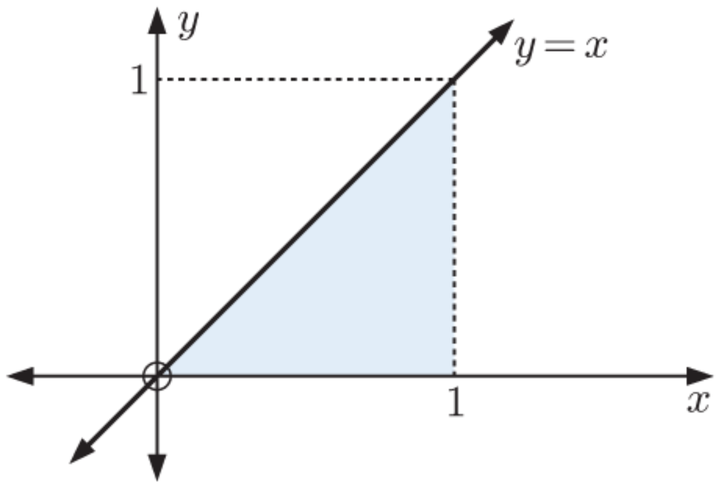
EXERCISE 21A.1

1 Consider the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$.

a Divide the interval into 5 subintervals of equal width, then estimate the area using:

i lower rectangles **ii** upper rectangles.

b Calculate the actual area and compare it with your answers in **a**.



2 Consider the area between $y = \frac{1}{x}$ and the x-axis from $x = 2$ to $x = 4$. Divide the interval into 6 subintervals of equal width, then estimate the area using:

a lower rectangles **b** upper rectangles.

3 Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the x-axis for $1 \leqslant x \leqslant 2$. Use $n = 10$, 25, 50, 100, and 500. Give your answers to 4 decimal places.

As n gets larger, both A_L and A_U converge to the same rational number. What is it?



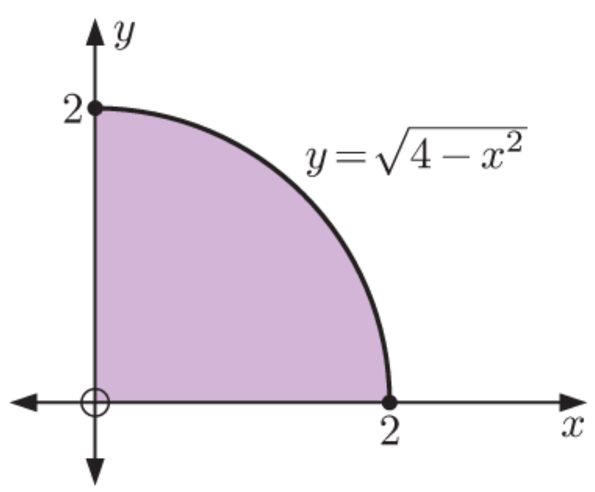
4 **a** Use lower and upper rectangle sums to estimate the area between each of the following functions and the x-axis for $0 \leqslant x \leqslant 1$. Use $n = 5$, 10, 50, 100, 500, 1000, and 10 000. Give your answer to 5 decimal places in each case.

i $y = x^3$ **ii** $y = x$ **iii** $y = x^{\frac{1}{2}}$ **iv** $y = x^{\frac{1}{3}}$

b For each case in **a**, write down the value to which A_L and A_U converge.

c Using your answer to **b**, predict the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ and any number $a > 0$.

5 Consider the quarter circle with centre $(0, 0)$ and radius 2 units illustrated.



Its area is $\frac{1}{4}$(full circle with radius 2 units) $= \frac{1}{4} \times \pi \times 2^2$
$= \pi$ units^2

a Estimate the area using lower and upper rectangles for $n = 10$, 50, 100, 200, 1000, and 10 000. Hence find rational bounds for π.

b Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

For what value of n is your estimate for π better than that of Archimedes?

INVESTIGATION 1 THE AREA UNDER $f(x) = x^2$

For the curve $f(x) = x^2$, we already have the tools necessary to calculate the area A between $y = f(x)$ and the x-axis on the interval $0 \leqslant x \leqslant 1$ exactly.

Suppose we divide the interval $0 \leqslant x \leqslant 1$ into n subintervals, each of width $\frac{1}{n}$.

What to do:

1 **a** Explain why the total area of lower rectangles can be written as $A_L = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right)$.

b Use the sum of series formulae $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ to show that $A_L = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$.

c What value does A_L approach as $n \to \infty$?

2 **a** Explain why the total area of upper rectangles can be written as $A_U = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right)$.

b Hence show that $A_U = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$.

c What value does A_U approach as $n \to \infty$?

3 Use the results from **1** and **2** to explain why $A = \frac{1}{3}$ units^2.

THE TRAPEZOIDAL RULE

For a function which is either increasing on the whole interval, or decreasing on the whole interval, the actual area under the curve is guaranteed to lie between the lower rectangle and upper rectangle estimates: $A_L \leqslant A \leqslant A_U$.

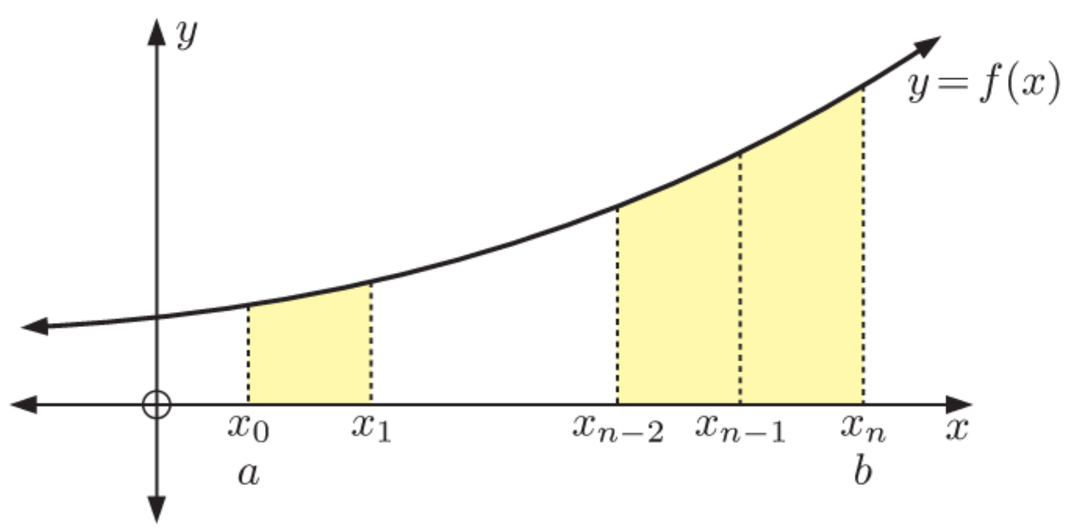
For functions with turning points on the interval, we no longer have that guarantee, and it is inefficient to ask a computer to calculate A_L and A_U. This is because the computer has to make extra decisions about which is the lower and which is the upper value for each subinterval.

So, while we have seen that increasing the number of subintervals will improve the accuracy of A_L and A_U, we really want a more efficient way to numerically approximate the area under a curve.

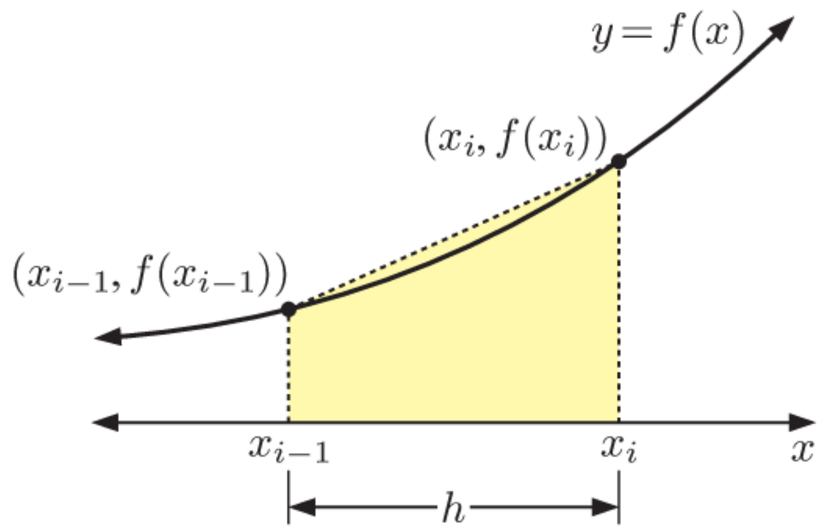
Suppose we use a trapezium to approximate the area under the curve on each subinterval. In doing this, we use the *average* of the values of the function at the left and right endpoints. The value calculated for each subinterval will always lie between the area of the lower rectangle and the area of the upper rectangle, and generally be a better approximation of the actual area.

Suppose we divide the interval $a \leqslant x \leqslant b$ into n subintervals of equal width $h = \dfrac{b-a}{n}$.

We let $x_i = a + ih$ for $i = 0, 1, 2, ..., n$, so the ith subinterval is $x_{i-1} \leqslant x \leqslant x_i$.



The area for the ith subinterval is the shaded trapezium below:

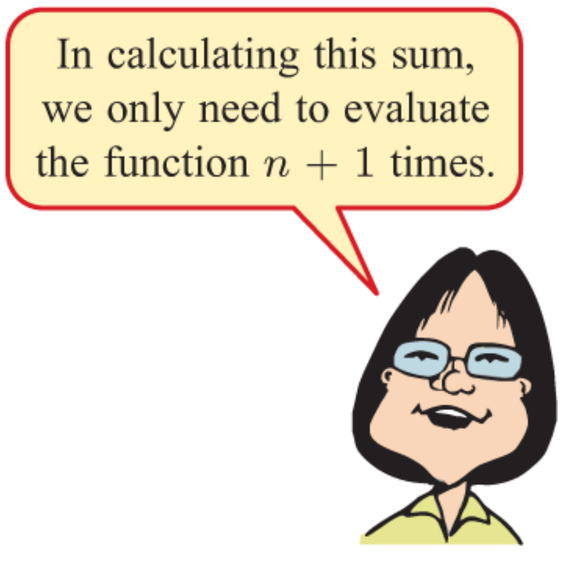


Our approximation for the area under the curve is therefore

$$A \approx h\left(\frac{f(x_0)+f(x_1)}{2}\right) + h\left(\frac{f(x_1)+f(x_2)}{2}\right) + ... + h\left(\frac{f(x_{n-1})+f(x_n)}{2}\right)$$

$$\approx \frac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n))$$

$$\approx \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$$



Example 1 **Self Tutor**

Use the trapezoidal rule with 6 subintervals to estimate the area between $f(x) = \sqrt{6 - x^2}$ and the x-axis from $x = 1$ to $x = 2$.

$n = 6,\ \ a = 1,\ \ b = 2,\ \ f(x) = \sqrt{6 - x^2}$

$\therefore\ \ h = \dfrac{b-a}{n} = \dfrac{1}{6}$

$\therefore\ \ x_i = 1 + \frac{1}{6}i$

Using the trapezoidal rule, the area

$\approx \dfrac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_5) + f(x_6))$

≈ 1.8983 units^2

i	x_i	$f(x_i)$
0	1	2.236 068
1	$1\frac{1}{6}$	2.153 808
2	$1\frac{1}{3}$	2.054 805
3	$1\frac{1}{2}$	1.936 492
4	$1\frac{2}{3}$	1.795 055
5	$1\frac{5}{6}$	1.624 466
6	2	1.414 214

EXERCISE 21A.2

For questions **1** to **3**, construct a table of values manually to answer the question. You can then check your answers by constructing the table of values directly on your calculator or using the graphing package.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

1 Use the trapezoidal rule with 4 subintervals to approximate the area between the x-axis and:

a $f(x) = \dfrac{2}{\sqrt{x}}$ from $x = 2$ to $x = 4$ **b** $f(x) = -x^2 + 6x - 4$ from $x = 1$ to $x = 3$.

2 **a** Use the trapezoidal rule with 6 subintervals to calculate the area between the x-axis and $f(x) = 3 - x$ from $x = 0$ to $x = 3$.

b Explain why your calculation is exact for this function.

3 Use the trapezoidal rule with 8 subintervals to approximate the area between the x-axis and:

a $f(x) = \sqrt{x}$ from $x = 0$ to $x = 4$ **b** $f(x) = \sqrt{x}e^{-\pi x}$ from $x = 0$ to $x = 1$

c $f(x) = x^3 - 2x^2 + 1$ from $x = -0.6$ to $x = 1$.

4 **a** Apply the trapezoidal rule with technology to estimate the area under $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$, with $n = 8$, 40, 100, and 1000 subintervals.

Record your results in a table like the one alongside.

n	*Area estimate*
8	
40	
100	
1000	

b For what value of n does the trapezoidal rule estimate π more accurately than Archimedes (see **Exercise 21A.1** question **5 b**)?

B THE RIEMANN INTEGRAL

We have seen that for the special case of a quadratic, we can use series formulae to evaluate the area under the curve exactly.

However, for most functions we do not have such formulae. We therefore need a more general method for finding the area under a curve.

HISTORICAL NOTE

Italian Mathematician **Bonaventura Cavalieri** (1598 - 1647) became Professor of Mathematics at Bologna in 1629. He published tables for many trigonometric and logarithmic functions. However, his best known contribution to mathematics was the invention of **indivisibles**.



Bonaventura Cavalieri

In his Method of Indivisibles, Cavalieri considered that a moving point could be used to sketch a curve. The curve could therefore be considered as the set of an infinite number of points, each with no length.

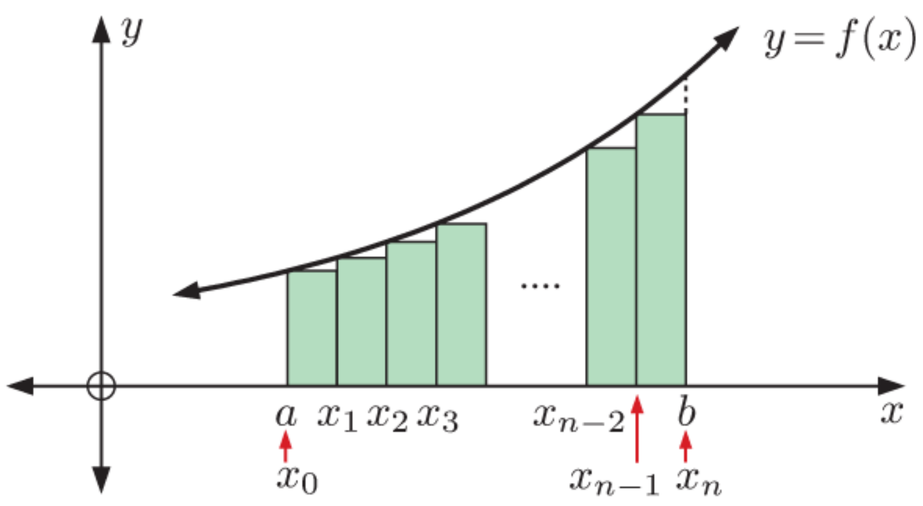
In a similar way, the "indivisibles" that made up a surface were an infinite number of lines. Almost every introduction to integral calculus starts with the division of an area into a number of rectangular strips with finite width.

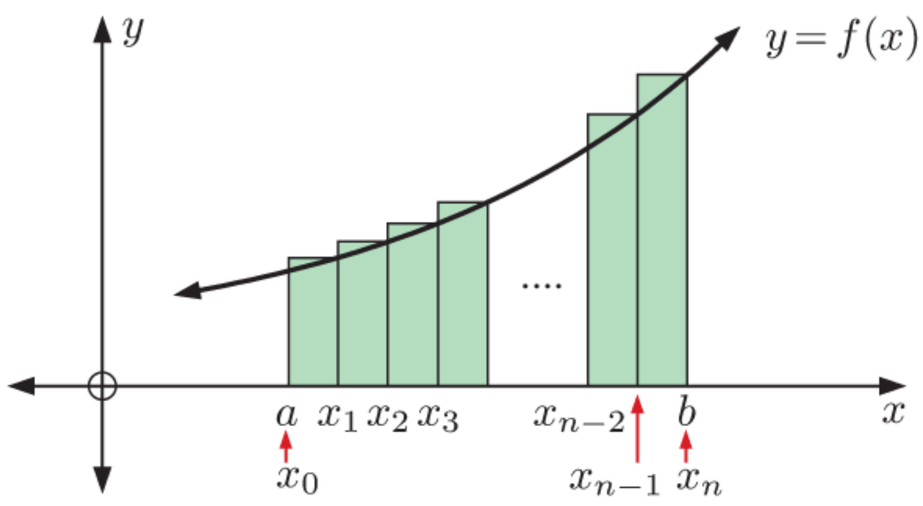
Cavalieri's important step was to make the strips narrower and narrower until they were infinitely thin lines. This reduces the "jagged" steps of the strips until they exactly define the curved boundary of the area.

It was not until Englishman **Sir John Wallis** (1616 - 1703) formally introduced the idea of a **limit** in 1656 that Cavalieri's Method of Indivisibles progressed into the foundation for Integral Calculus.

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into n subintervals, each of width $w = \dfrac{b-a}{n}$.





Since the function is increasing:

$$A_L = w\,f(x_0) + w\,f(x_1) + \ldots. + w\,f(x_{n-2}) + w\,f(x_{n-1}) = w\sum_{i=0}^{n-1} f(x_i)$$

$$A_U = w\,f(x_1) + w\,f(x_2) + \ldots. + w\,f(x_{n-1}) + w\,f(x_n) \quad = w\sum_{i=1}^{n} f(x_i)$$

$$\therefore \quad A_U - A_L = w\,(f(x_n) - f(x_0))$$

$$= \frac{1}{n}(b-a)\,(f(b) - f(a))$$

Following Cavalieri's suggestion, we allow there to be infinitely many subintervals, so $n \to \infty$.

In this case $\lim\limits_{n\to\infty} (A_U - A_L) = 0$ $\quad \{\text{since } \lim\limits_{n\to\infty} \dfrac{1}{n} = 0\}$

$$\therefore \quad \lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U \quad \{\text{provided both limits exist}\}$$

$\therefore$ since $A_L < A < A_U$ for all values of n, it follows that $\lim\limits_{n\to\infty} A_L = A = \lim\limits_{n\to\infty} A_U$.

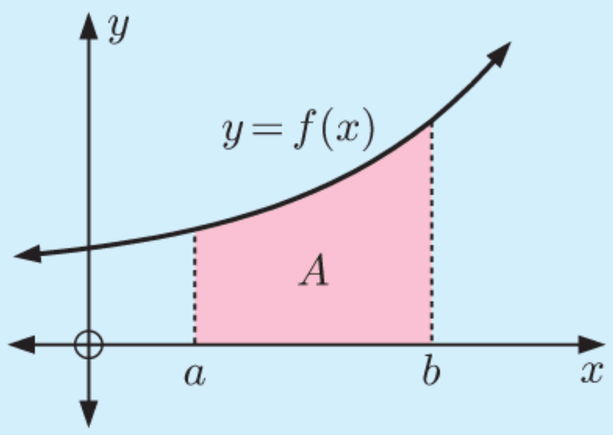
We can obtain a result like this for every increasing and decreasing interval of a positive function provided the function is *continuous*. This means that the function must have a defined value $f(k)$ for all $a \leqslant k \leqslant b$, and that $\lim_{x \to k} f(x) = f(k)$ for all $a \leqslant k \leqslant b$.

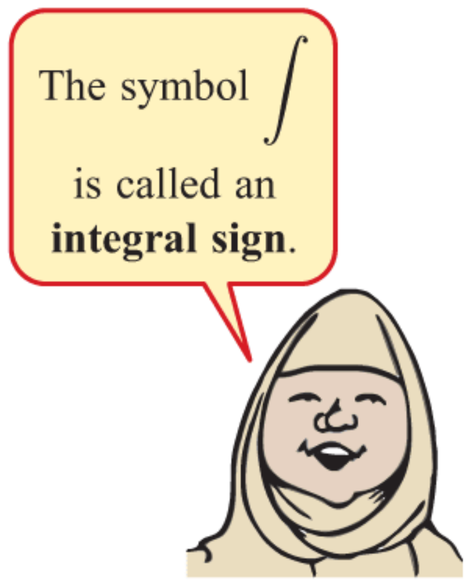
If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$\int_a^b f(x)\,dx$ is defined as the shaded area A.

This is known as the **Riemann integral**.

We would say "the integral of $f(x)$ from a to b with respect to x".



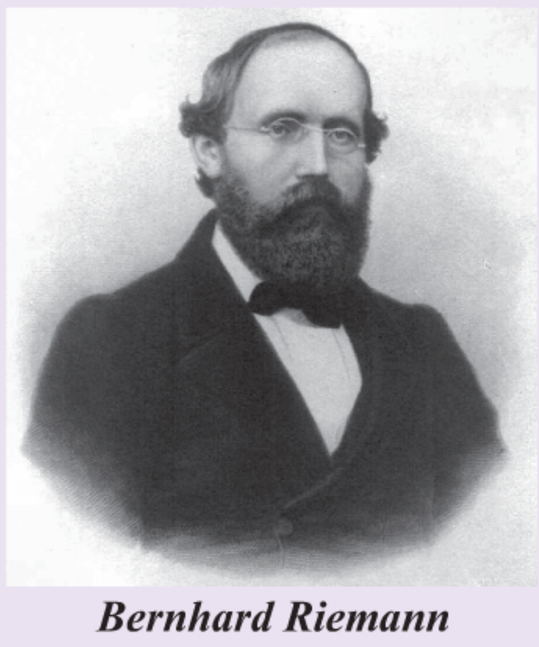


HISTORICAL NOTE

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas of the thin rectangular strips are put together into one whole area.

The theory of integration was developed independently by **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**.

It was rigorously formalised using limits by the German mathematician **Bernhard Riemann** (1826 - 1866), whose name is given to the integral which calculates the area under a curve.



Bernhard Riemann

Example 2 Self Tutor

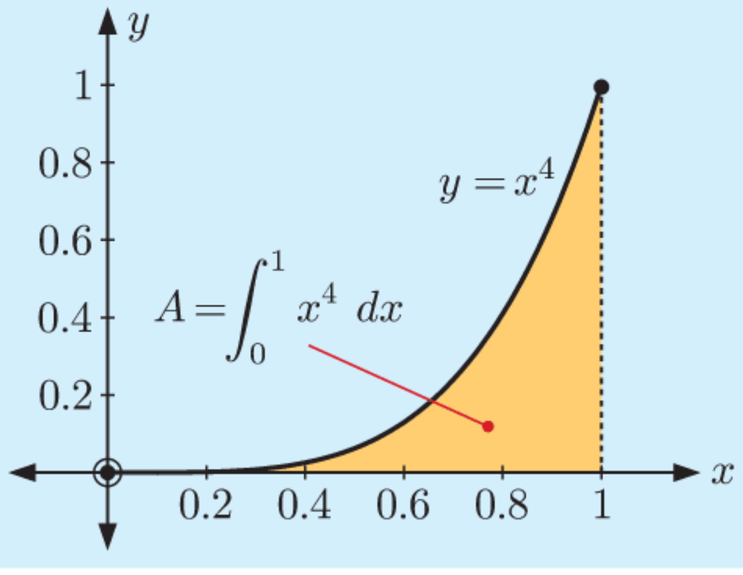
a Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4\,dx$.

b Use technology to calculate the lower and upper rectangle sums for n equal subintervals where $n = 5$, 10, 50, 100, and 500.

c Hence evaluate $\int_0^1 x^4\,dx$ to 2 significant figures.

d Approximate $\int_0^1 x^4\,dx$ using the trapezoidal method with 10 subintervals. Comment on your answer.

a



b

n	A_L	A_U
5	0.1133	0.3133
10	0.1533	0.2533
50	0.1901	0.2101
100	0.1950	0.2050
500	0.1990	0.2010

c When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

$\therefore$ since $A_L < \int_0^1 x^4\,dx < A_U$, $\int_0^1 x^4\,dx \approx 0.20$

d $n = 10$, $a = 0$, $b = 1$, $f(x) = x^4$

$\therefore\ h = \dfrac{b-a}{n} = \dfrac{1}{10}$ and hence $x_i = \frac{1}{10}i$

Using the trapezoidal rule, the area

$\approx \dfrac{h}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_9) + f(x_{10}))$

$\approx 0.203\,33$

With just 10 subintervals, the trapezoidal method is more accurate than lower and upper rectangles were with $n = 100$.

i	x_i	$f(x_i)$
0	0	0
1	0.1	0.0001
2	0.2	0.0016
3	0.3	0.0081
4	0.4	0.0256
5	0.5	0.0625
6	0.6	0.1296
7	0.7	0.2401
8	0.8	0.4096
9	0.9	0.6561
10	1	1

EXERCISE 21B

1 **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$.

Shade the area described by $\int_0^1 \sqrt{x}\,dx$.

AREA FINDER



b Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100, and 500.

c Hence evaluate $\int_0^1 \sqrt{x}\,dx$ to 2 significant figures.

d Approximate $\int_0^1 \sqrt{x}\,dx$ using the trapezoidal method with 8 subintervals. Comment on your answer.

2 Consider the region enclosed by $y = \sqrt{1+x^3}$ and the x-axis for $0 \leqslant x \leqslant 2$.

GRAPHING PACKAGE



a Write expressions for the lower and upper rectangle sums using n subintervals where $n \in \mathbb{Z}^+$.

b Find the lower and upper rectangle sums for $n = 50$, 100, and 500.

c Hence estimate $\int_0^2 \sqrt{1+x^3}\,dx$.

d Approximate $\int_0^2 \sqrt{1+x^3}\,dx$ using the trapezoidal method with 10 subintervals.

3 The integral $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$ is of considerable interest to statisticians.

a Use the graphing package to help sketch $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.

b Calculate the lower and upper rectangle sums for the interval $0 \leqslant x \leqslant 3$ using $n = 2250$.

c Use the symmetry of $y = e^{-\frac{x^2}{2}}$ to estimate $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx$. Compare your answer with $\sqrt{2\pi}$.

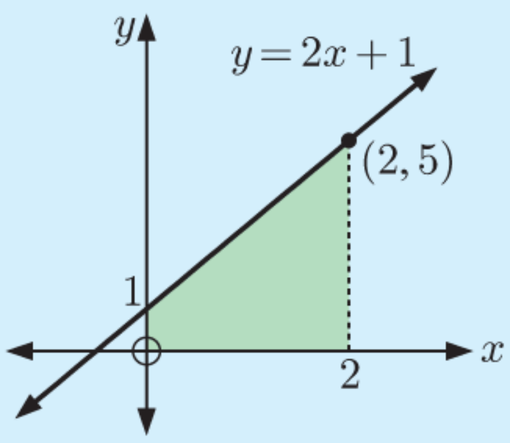
d How many subintervals are necessary with the trapezoidal rule to estimate $\int_0^3 e^{-\frac{x^2}{2}}\,dx$ more accurately than your lower and upper rectangle sums in **b**?

Example 3 **Self Tutor**

Use graphical evidence and known area facts to find:

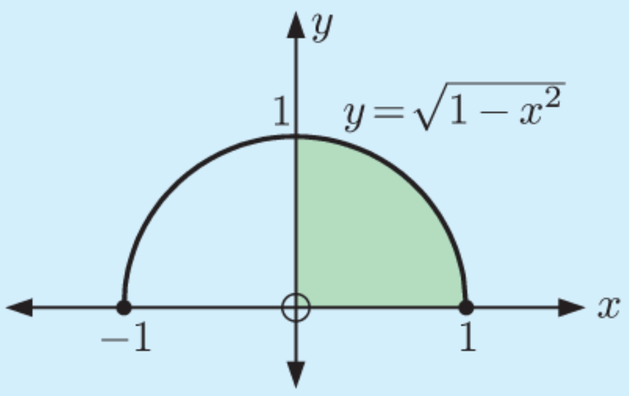
a $\int_0^2 (2x+1)\,dx$ **b** $\int_0^1 \sqrt{1-x^2}\,dx$

a



$$\int_0^2 (2x+1)\,dx = \text{shaded area}$$
$$= \left(\frac{1+5}{2}\right) \times 2$$
$$= 6$$

b If $y = \sqrt{1-x^2}$ then $y^2 = 1-x^2$ and so $x^2 + y^2 = 1$. This is the equation of a circle with radius 1 unit, and $y = \sqrt{1-x^2}$ is the upper half.



$$\int_0^1 \sqrt{1-x^2}\,dx = \text{shaded area}$$
$$= \tfrac{1}{4} \times \pi \times 1^2$$
$$= \tfrac{\pi}{4}$$

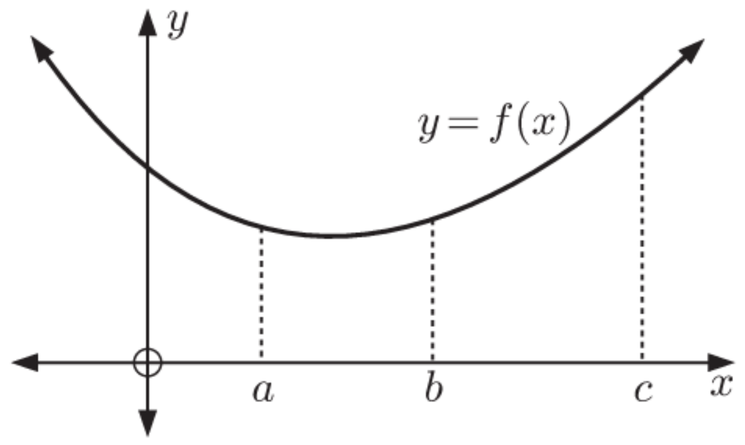
4 Use graphical evidence and known area facts to find:

a $\int_1^3 (1+4x)\,dx$ **b** $\int_{-1}^2 (2-x)\,dx$ **c** $\int_{-2}^2 \sqrt{4-x^2}\,dx$

5 **a** Use the diagram alongside to show that for any positive function $f(x)$:

i $\int_a^a f(x)\,dx = 0$

ii $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$



b For a positive function $f(x)$, $\int_2^5 f(x)\,dx = 10$, and $\int_5^9 f(x)\,dx = 12$. Find:

i $\int_5^5 f(x)\,dx$ **ii** $\int_2^9 f(x)\,dx$

C ANTIDIFFERENTIATION

In many problems in calculus, we know the rate of change of one variable with respect to another, but we do not have a formula which directly relates the variables. In other words, we know $\frac{dy}{dx}$, but we need to know y in terms of x.

The process of finding y from $\frac{dy}{dx}$, or $f(x)$ from $f'(x)$, is the reverse process of differentiation. We call it **antidifferentiation**.



Consider $\frac{dy}{dx} = x^2$.

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that y must involve x^3.

Now if $y = x^3$ then $\frac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\frac{dy}{dx} = x^2$.

This is the correct result. However, for *all* of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$, and $y = \frac{1}{3}x^3 - 7$, we find that $\frac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where c is an arbitrary constant, which will give $\frac{dy}{dx} = x^2$. Ignoring the arbitrary constant, we say that $\frac{1}{3}x^3$ is the **antiderivative** of x^2. It is the simplest function which, when differentiated, gives x^2.

If $F(x)$ is the simplest function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$
- the **antiderivative** of $f(x)$ is $F(x)$.

Example 4 **Self Tutor**

Find the antiderivative of:

a x^3 **b** e^{2x} **c** $\frac{1}{\sqrt{x}}$

a $\frac{d}{dx}(x^4) = 4x^3$

$\therefore \frac{d}{dx}(\frac{1}{4}x^4) = x^3$

$\therefore$ the antiderivative of x^3 is $\frac{1}{4}x^4$.

b $\frac{d}{dx}(e^{2x}) = e^{2x} \times 2 = 2e^{2x}$

$\therefore \frac{d}{dx}(\frac{1}{2}e^{2x}) = \frac{1}{2} \times 2e^{2x} = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

c $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Now $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

EXERCISE 21C

1 **a** Find the antiderivative of:

i x **ii** x^2 **iii** x^5 **iv** x^{-2}

v x^{-4} **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{3}}$ **viii** $x^{\frac{2}{3}}$

b Predict a general rule for the antiderivative of x^n, for $n \neq -1$.

2 **a** Find the antiderivative of:

i e^{3x} **ii** e^{5x} **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

b Predict a general rule for the antiderivative of e^{kx} where $k \neq 0$ is a constant.

3 Find the antiderivative of:

a $6x^2 + 4x$ by first differentiating $x^3 + x^2$ **b** $\sqrt{x}$ by first differentiating $x\sqrt{x}$

c $\dfrac{1}{x\sqrt{x}}$ by first differentiating $\dfrac{1}{\sqrt{x}}$.

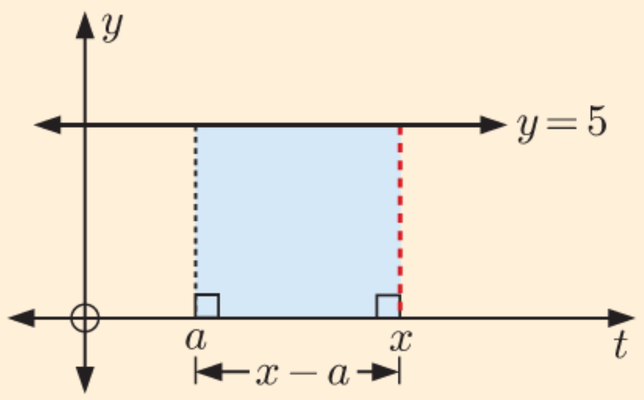
D THE FUNDAMENTAL THEOREM OF CALCULUS

We can now use the Riemann integral to explain the link between differential calculus and the definite integral or limit of an area sum we saw in **Section B**. This link is called the **Fundamental Theorem of Calculus**.

INVESTIGATION 2 THE AREA FUNCTION

Consider the constant function $f(t) = 5$.

We wish to find an **area function** which will give the area under the function between $t = a$ and some other value of t which we will call x.



The area function is $A(x) = \int_a^x 5 \, dt$

$= \text{shaded area}$

$= (x - a)5$

$= 5x - 5a$

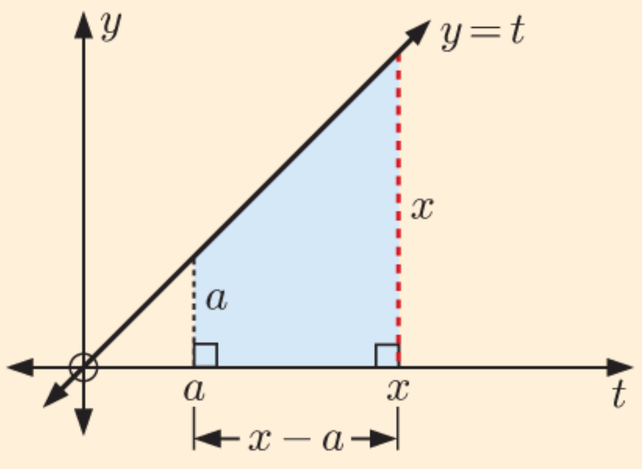
Since $f(t) = 5$ has the antiderivative $F(t) = 5t$, we can write $A(x)$ in the form $F(x) - F(a)$.

What to do:

1 What is the derivative $F'(t)$ of the function $F(t) = 5t$? How is this related to $f(t)$?

2 Consider the simplest linear function $f(t) = t$.
The corresponding area function is

$$A(x) = \int_a^x t \, dt$$
$$= \text{shaded area}$$
$$= \left(\frac{x + a}{2}\right)(x - a)$$

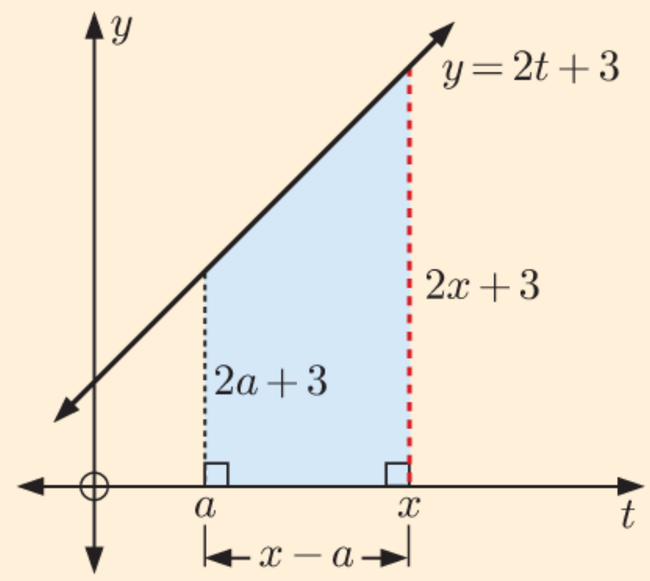


a Write $A(x)$ in the form $F(x) - F(a)$.

b What is the derivative $F'(t)$? How is this related to $f(t)$?

3 Consider $f(t) = 2t + 3$. The corresponding area function is

$$A(x) = \int_a^x (2t+3)\,dt$$
$$= \text{shaded area}$$
$$= \left(\frac{2x+3+2a+3}{2}\right)(x-a)$$



a Write $A(x)$ in the form $F(x) - F(a)$.

b What is the derivative $F'(t)$?
How is this related to $f(t)$?

4 Repeat the procedure in **2** and **3** to find area functions for:

a $f(t) = \frac{1}{2}t + 3$ **b** $f(t) = 5 - 2t$

Do your results fit with your earlier observations?

5 If $f(t) = 3t^2 + 4t + 5$, predict what $F(t)$ will be without performing the algebraic procedure.

From the **Investigation** you should have found that, for $f(t) \geqslant 0$,

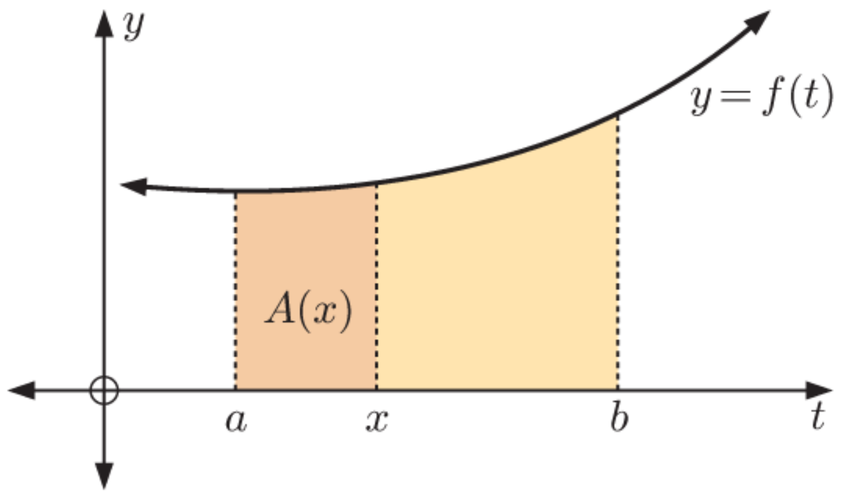
$$\int_a^x f(t)\,dt = F(x) - F(a) \quad \text{where} \quad F'(t) = f(t).$$

$F(t)$ is the **antiderivative** of $f(t)$.

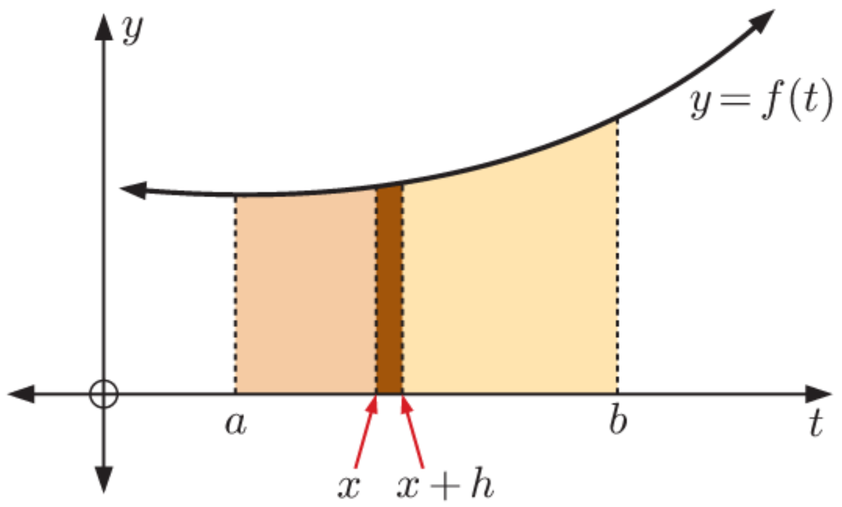
The following argument shows why this is true for all functions $f(t) \geqslant 0$.

Consider a function $y = f(t)$ which has antiderivative $F(t)$ and an area function $A(x) = \int_a^x f(t)\,dt$ which is the area from $t = a$ to $t = x$.

$A(x)$ is an increasing function since $f(x) \geqslant 0$, and $A(a) = 0$ .... (1)



Consider the narrow strip between $t = x$ and $t = x+h$. The area of this strip is $A(x+h) - A(x)$, but we also know it must lie between a lower and upper rectangle on the interval $x \leqslant t \leqslant x + h$ of width h.



$$\text{area of lower rectangle} \leqslant A(x+h) - A(x) \leqslant \text{area of upper rectangle}$$

If $f(t)$ is increasing on this interval then

$$hf(x) \leqslant A(x+h) - A(x) \leqslant hf(x+h)$$
$$\therefore \quad f(x) \leqslant \frac{A(x+h) - A(x)}{h} \leqslant f(x+h)$$

Equivalently, if $f(t)$ is decreasing on this interval then $f(x+h) \leqslant \dfrac{A(x+h) - A(x)}{h} \leqslant f(x)$.

Taking the limit as $h \to 0$ gives

$$f(x) \leqslant A'(x) \leqslant f(x)$$
$$\therefore \quad A'(x) = f(x)$$

So, the area function $A(x)$ must only differ from the antiderivative of $f(x)$ by a constant.

$$\therefore \quad A(x) = F(x) + c$$

Letting $x = a$, $\quad A(a) = F(a) + c$

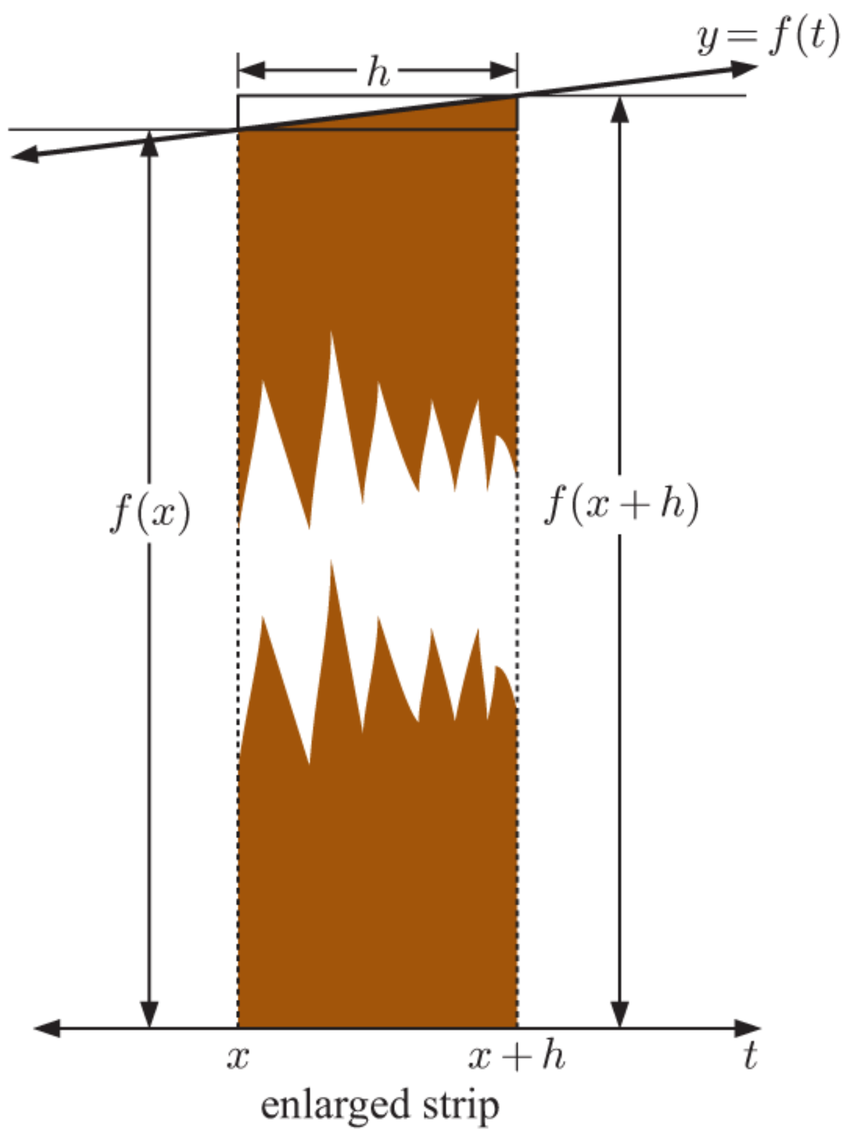
But from (1), $\quad A(a) = 0$

$$\therefore \quad c = -F(a)$$

$$\therefore \quad A(x) = F(x) - F(a)$$

and so $\displaystyle\int_a^x f(t)\,dt = F(x) - F(a)$

Letting $x = b$, $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$



This result is in fact true for *all* continuous functions $f(t)$, even if they are negative.

However, in situations where a function is negative, the area between the curve and the x-axis is counted as negative. We therefore refer to $A(x)$ as a **signed area function**.

THE FUNDAMENTAL THEOREM OF CALCULUS

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\,dx = F(b) - F(a)}$.

In general, $\displaystyle\int_a^b f(x)\,dx$ is called a **definite integral**.

PROPERTIES OF DEFINITE INTEGRALS

The following properties of definite integrals can all be deduced from the Fundamental Theorem of Calculus:

- $\displaystyle\int_a^a f(x)\,dx = 0$
- $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\displaystyle\int_a^b k\,dx = k(b-a)$ $\quad \{k \text{ is a constant}\}$
- $\displaystyle\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

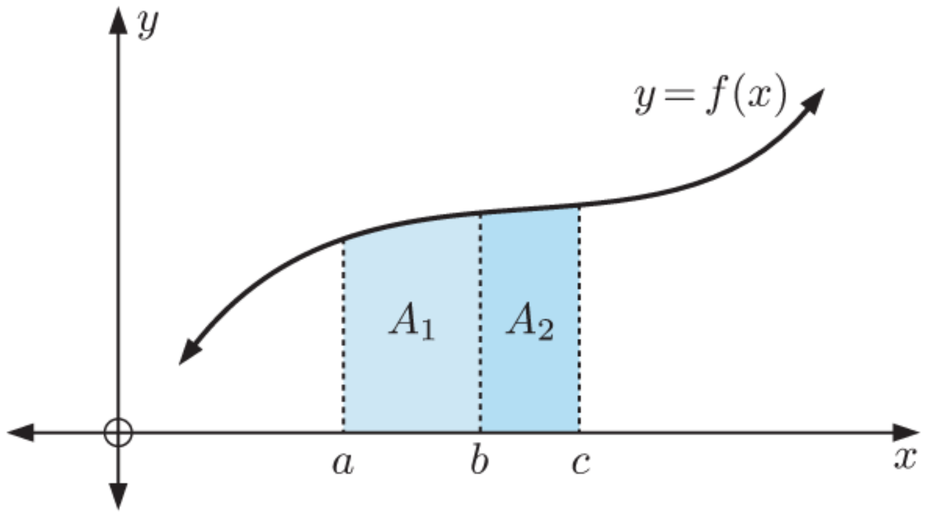
Example proof:

$$\begin{aligned}\int_a^b f(x)\,dx + \int_b^c f(x)\,dx &= F(b) - F(a) + F(c) - F(b)\\ &= F(c) - F(a)\\ &= \int_a^c f(x)\,dx\end{aligned}$$

The properties of definite integrals all have geometric interpretations.

For example, for the case where $a \leqslant b \leqslant c$ and $f(x) \geqslant 0$ for $a \leqslant x \leqslant c$, we observe that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$
$$= A_1 + A_2$$
$$= \int_a^c f(x)\,dx$$



The Fundamental Theorem of Calculus allows us to calculate areas under curves that we could previously only estimate.

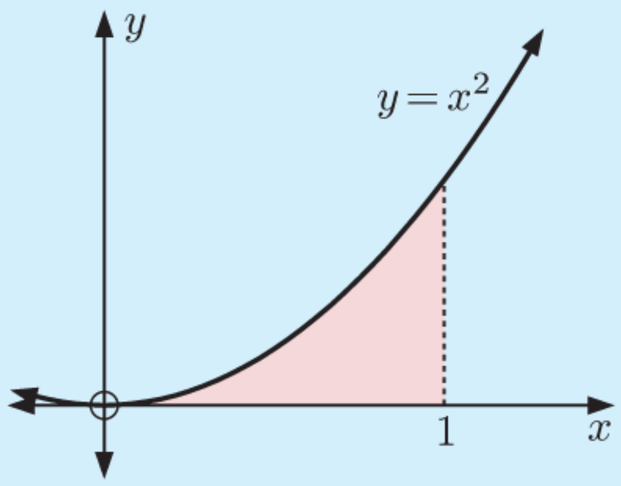
Example 5 — Self Tutor

Use the Fundamental Theorem of Calculus to find the area between:

a the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

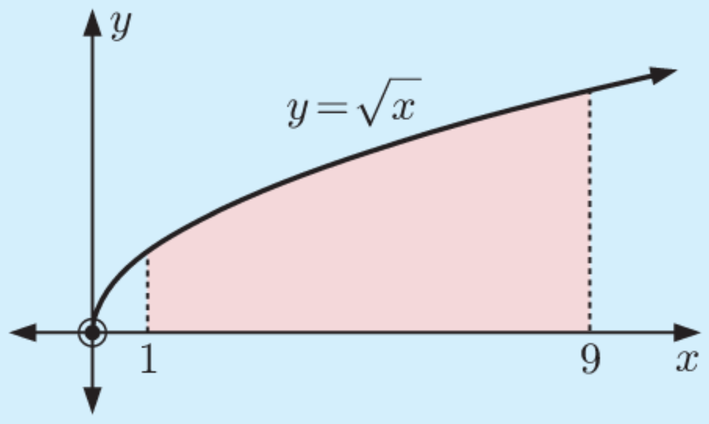
a



$f(x) = x^2$ has antiderivative $F(x) = \dfrac{x^3}{3}$

$$\therefore \text{ shaded area} = \int_0^1 x^2\,dx$$
$$= F(1) - F(0)$$
$$= \tfrac{1}{3} - 0$$
$$= \tfrac{1}{3} \text{ units}^2$$

b

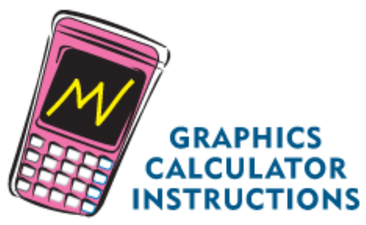


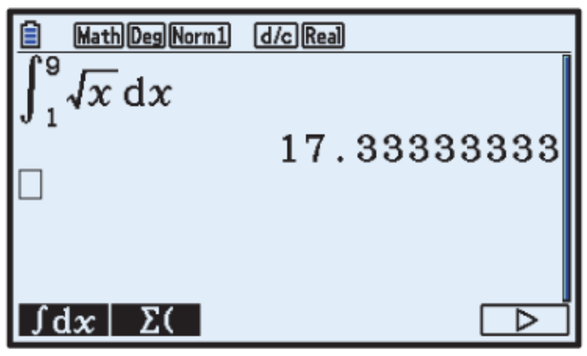
$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} = \tfrac{2}{3}x\sqrt{x}$

$$\therefore \text{ shaded area} = \int_1^9 x^{\frac{1}{2}}\,dx$$
$$= F(9) - F(1)$$
$$= \tfrac{2}{3} \times 27 - \tfrac{2}{3} \times 1$$
$$= 17\tfrac{1}{3} \text{ units}^2$$

Instructions for evaluating definite integrals on your calculator can be found by clicking on the icon.



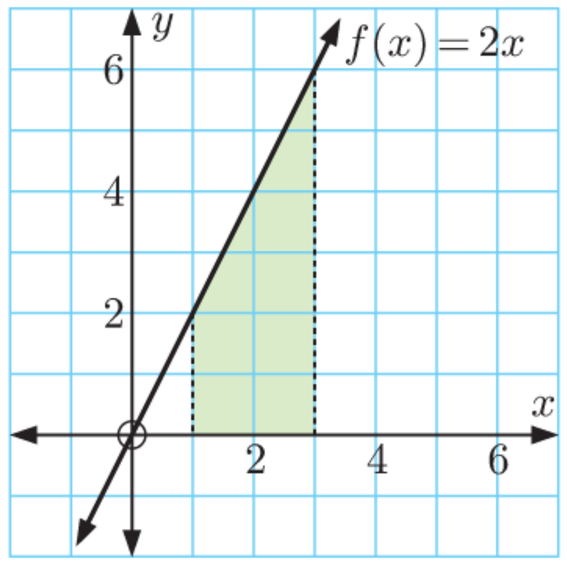


EXERCISE 21D

1 **a** Differentiate x^2, and hence find the antiderivative of $2x$.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $f(x) = 2x$ from $x = 1$ to $x = 3$.

c Use graphical methods to check your answer.



2 **a** Find the antiderivative of $\sqrt{x}$.

b Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

c Compare your answer to **Exercise 21B** question **1**.

3 **a** Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = x^3$ from:

i $x = 0$ to $x = 2$ **ii** $x = 2$ to $x = 3$ **iii** $x = 0$ to $x = 3$.

b Comment on your answers in **a**.

4 Use the Fundamental Theorem of Calculus to find the area between the x-axis and:

a $y = x^3$ from $x = 1$ to $x = 2$ **b** $y = x^2$ from $x = 1$ to $x = 3$

c $y = \sqrt{x}$ from $x = 1$ to $x = 2$ **d** $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$.

Check your answers using technology.

5 Use the Fundamental Theorem of Calculus to show that:

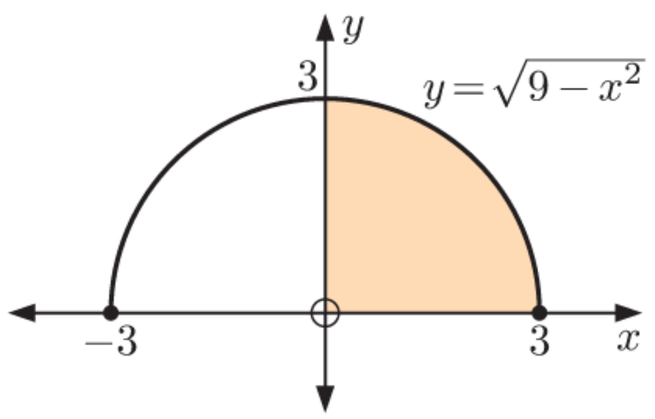
a $\int_a^a f(x)\,dx = 0$ **b** $\int_a^b k\,dx = k(b-a)$ for $k \in \mathbb{R}$

c $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ **d** $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$ for $k \in \mathbb{R}$

e $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

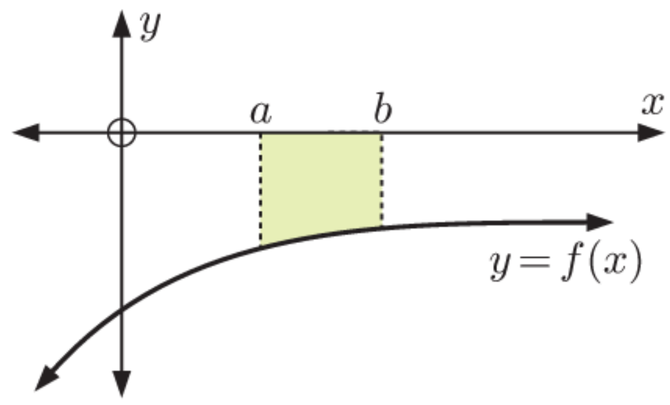
6 Use technology to find the area between the x-axis and $y = \sqrt{9 - x^2}$ from $x = 0$ to $x = 3$.

Check your answer by direct calculation of the area.



7 **a** Use the Fundamental Theorem of Calculus to show that

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$$



b Hence show that if $f(x) \leqslant 0$ for all x on $a \leqslant x \leqslant b$ then the shaded area $= -\int_a^b f(x)\,dx$ units^2.

c Calculate the following integrals, and give graphical interpretations of your answers:

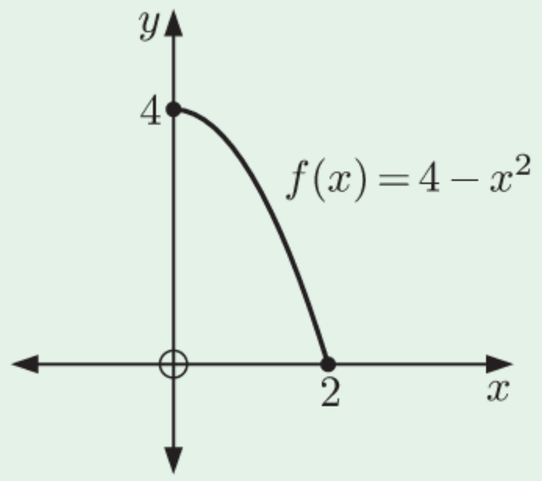
i $\int_0^1 (-x^2)\,dx$ **ii** $\int_0^1 (x^2 - x)\,dx$ **iii** $\int_{-2}^0 3x\,dx$

d Use graphical evidence and known area facts to find $\int_0^2 \left(-\sqrt{4-x^2}\right) dx$.

REVIEW SET 21A

1 **a** Use *four* lower and upper rectangles to find rational numbers A and B such that:

$$A < \int_0^2 (4 - x^2)\,dx < B.$$



b Estimate $\int_0^2 (4 - x^2)\,dx$ using the trapezoidal method with 8 subintervals.

c Compare your answers with the exact value of $\int_0^2 (4 - x^2)\,dx$.

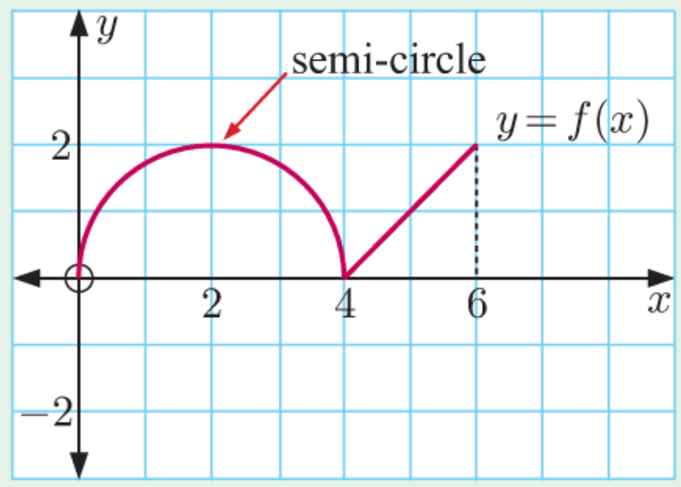
2 **a** Sketch the region between $y = \sin x$ and the x-axis for $0 \leqslant x \leqslant \frac{\pi}{2}$.

b Divide the interval into 3 equal parts and display the 3 upper and 3 lower rectangles. Hence find boundaries for the value of $\int_0^{\frac{\pi}{2}} \sin x\,dx$.

3 Find the antiderivative of:

a x^4 **b** $\dfrac{1}{2x^2}$ **c** $e^{-\frac{1}{2}x}$ **d** $\cos x$

4 The graph of $y = f(x)$ is illustrated.



Evaluate the following using area interpretation:

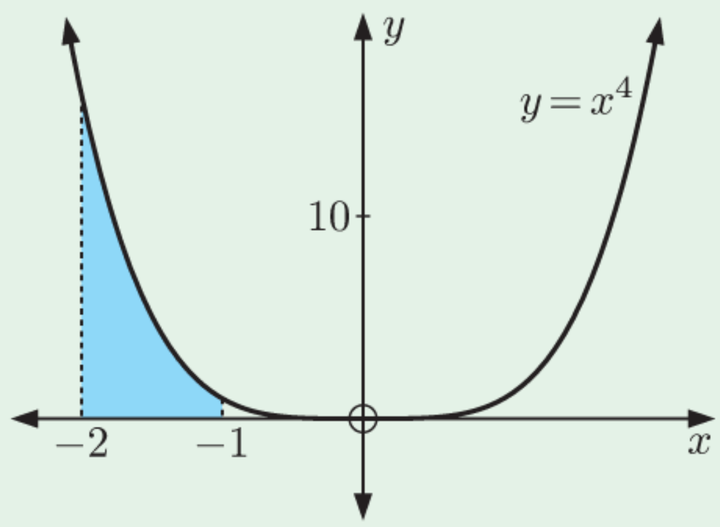
a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

5 **a** Use the Fundamental Theorem of Calculus to find the area between the x-axis and $y = x^2$ from:

i $x = 0$ to $x = 1$ **ii** $x = 1$ to $x = 2$ **iii** $x = 0$ to $x = 2$.

b Comment on your answers in **a**.

6 The graph of $y = x^4$ is shown alongside.



a Use the trapezoidal rule with 8 subintervals to estimate $\int_{-2}^{-1} x^4\,dx$.

b Use the graph to explain whether your estimate is an over estimate or an under estimate.

c Check your answer to **b** by performing the integration.

REVIEW SET 21B

1 **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the x-axis for $0 \leqslant x \leqslant 1$.

Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

b Use the area finder software to find the lower and upper rectangle sums for $n = 5$, 50, 100, and 500.



c Give your best estimate for $\int_0^1 \dfrac{4}{1+x^2}\,dx$.

d Estimate $\int_0^1 \dfrac{4}{1+x^2}\,dx$ using the trapezoidal method with 10 subintervals.

e Compare your results with the exact answer π.

2 **a** Sketch the region between $y = e^x - 1$ and the x-axis for $0 \leqslant x \leqslant 1$.

b With 10 subintervals, estimate $\int_0^1 (e^x - 1)\,dx$ using:

i lower rectangles **ii** upper rectangles **iii** the trapezoidal rule.

c Comment on your results.

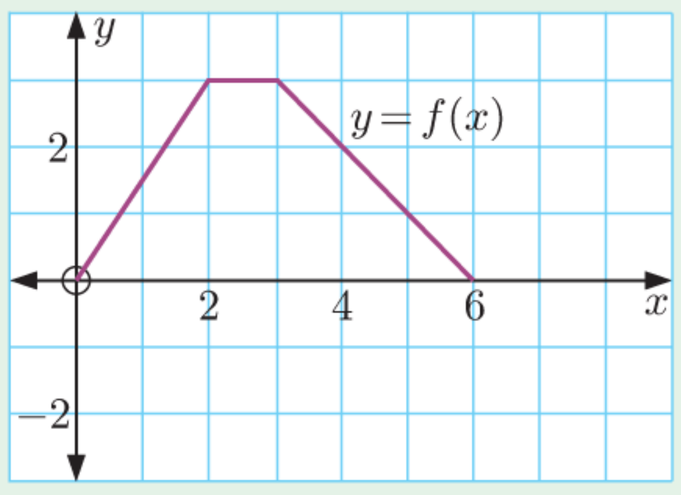
3 Use graphical evidence and known area facts to find:

a $\int_2^4 (2x - 1)\,dx$ **b** $\int_{-1}^1 \sqrt{1-x^2}\,dx$

4 Find the antiderivative of:

a $3x^2 - 2$ by first differentiating $x^3 - 2x$ **b** $\sqrt[3]{x}$ by first differentiating $x^{\frac{4}{3}}$.

5 The graph of $y = f(x)$ is illustrated.



Evaluate the following using area interpretation:

a $\int_0^2 f(x)\,dx$ **b** $\int_2^6 f(x)\,dx$

6 Use the Fundamental Theorem of Calculus to find the area between the x-axis and:

a $y = 4x$ from $x = 0$ to $x = 3$ **b** $y = \sqrt{x}$ from $x = 0$ to $x = 9$.

Chapter 22

Techniques for integration

Contents:

A Discovering integrals
B Rules for integration
C Particular values
D Integrating $f(ax+b)$
E Integration by substitution



OPENING PROBLEM

The Fundamental Theorem of Calculus developed by Newton and Leibniz identifies the link between differential calculus and a definite integral.

If we are to use the Fundamental Theorem of Calculus to calculate areas, we need to be able to identify an area function from its derivative.

Things to think about:

a Can you identify the function which has derivative:

i $3x^2$ **ii** $x^3 - x + 1$ **iii** $\cos 2x$?

b What can the rules of differentiation teach us about the reverse process of integration?

In our previous study of the Fundamental Theorem of Calculus, we showed that the **antiderivative** of x^2 is $\frac{1}{3}x^3$, and that any function of the form $\frac{1}{3}x^3 + c$ where c is a constant, has derivative x^2.

We say that the **indefinite integral** or **integral** of x^2 is $\frac{1}{3}x^3 + c$, and write $\int x^2\,dx = \frac{1}{3}x^3 + c$.

We read this as "the integral of x^2 with respect to x is $\frac{1}{3}x^3 + c$, where c is a constant".

If $F'(x) = f(x)$ then $\int f(x)\,dx = F(x) + c$.

The constant c is called the **constant of integration**.

This process of finding an indefinite integral is called **indefinite integration**.

A DISCOVERING INTEGRALS

Just as we did in antidifferentiation, we can sometimes discover integrals by differentiation.

The following rules will prove useful:

- Any constant within the integral may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

Proof: Consider differentiating $kF(x)$ where $F'(x) = f(x)$.

$$\frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$

$$\therefore \int k\,f(x)\,dx = k\,F(x) + c$$

$$= k\int f(x)\,dx$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 1 **Self Tutor**

Find the derivative of x^4+2x^3, and hence find $\int (2x^3+3x^2)\,dx$.

$$\frac{d}{dx}(x^4+2x^3) = 4x^3+6x^2$$

$$\therefore \int (4x^3+6x^2)\,dx = x^4+2x^3+c$$

$$\therefore \int 2(2x^3+3x^2)\,dx = x^4+2x^3+c$$

$$\therefore 2\int (2x^3+3x^2)\,dx = x^4+2x^3+c$$

$$\therefore \int (2x^3+3x^2)\,dx = \tfrac{1}{2}x^4+x^3+c$$

c represents an arbitrary constant, so is simply any value $c \in \mathbb{R}$. Instead of writing $\frac{c}{2}$, we can therefore still write just c.



EXERCISE 22A

1 **a** Find the derivative of x^7, and hence find $\int x^6\,dx$.

b Find the derivative of $x^{\frac{3}{2}}$, and hence find $\int \sqrt{x}\,dx$.

c Find the derivative of $x^{-\frac{1}{2}}$, and hence find $\int x^{-\frac{3}{2}}\,dx$.

d Find the derivative of x^{n+1}, $n \neq -1$. Hence find $\int x^n\,dx$, $n \neq -1$.

We can check that an integral is correct by differentiating the answer. It should give us the **integrand**, the function we originally integrated.



2 **a** Find the derivative of e^{4x}, and hence find $\int e^{4x}\,dx$.

b Find the derivative of $e^{-\frac{x}{2}}$, and hence find $\int e^{-\frac{x}{2}}\,dx$.

c Find the derivative of e^{kx}, $k \neq 0$. Hence find $\int e^{kx}\,dx$, $k \neq 0$.

3 **a** Find the derivative of $\sin x$, and hence find $\int \cos x\,dx$.

b Find the derivative of $\cos x$, and hence find $\int \sin x\,dx$.

c Find the derivative of $\tan x$. Hence find $\int \frac{1}{\cos^2 x}\,dx$.

4 Find the derivative of x^3+x^2, and hence find $\int (3x^2+2x)\,dx$.

5 Find the derivative of $3x^4-2x^2$, and hence find $\int (3x^3-x)\,dx$.

6 Suppose $F(x)$ and $G(x)$ have the derivative functions $f(x)$ and $g(x)$ respectively.

a Find the derivative of $F(x) + G(x)$.

b Show that $\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$.

Example 2 **Self Tutor**

Find the derivative of $\cos\left(2x + \frac{\pi}{3}\right)$, and hence find $\int \sin\left(2x + \frac{\pi}{3}\right) dx$.

$$\frac{d}{dx}\left(\cos\left(2x + \tfrac{\pi}{3}\right)\right) = -\sin\left(2x + \tfrac{\pi}{3}\right)(2)$$
$$= -2\sin\left(2x + \tfrac{\pi}{3}\right)$$
$$\therefore \int \left(-2\sin\left(2x + \tfrac{\pi}{3}\right)\right) dx = \cos\left(2x + \tfrac{\pi}{3}\right) + c$$
$$\therefore -2\int \sin\left(2x + \tfrac{\pi}{3}\right) dx = \cos\left(2x + \tfrac{\pi}{3}\right) + c$$
$$\therefore \int \sin\left(2x + \tfrac{\pi}{3}\right) dx = -\tfrac{1}{2}\cos\left(2x + \tfrac{\pi}{3}\right) + c$$

7 **a** Find the derivative of $\sin 3x$, and hence find $\int \cos 3x\,dx$.

b Find the derivative of $\cos\left(\frac{\pi}{3} - x\right)$, and hence find $\int \sin\left(\frac{\pi}{3} - x\right) dx$.

c Find the derivative of e^{3x+1}, and hence find $\int e^{3x+1}\,dx$.

d Find the derivative of $\sqrt{5x-1}$, and hence find $\int \frac{1}{\sqrt{5x-1}}\,dx$.

e Find the derivative of $(2x+1)^4$, and hence find $\int (2x+1)^3\,dx$.

f Find the derivative of $(x^2 - x)^3$, and hence find $\int (2x-1)(x^2-x)^2\,dx$.

8 **a** Find the derivative of $\ln x$, $x > 0$. **b** Find the derivative of $\ln(-x)$, $x < 0$.

c Hence explain why $\int \frac{1}{x}\,dx = \ln|x| + c$, $x \neq 0$.

B RULES FOR INTEGRATION

In **Chapter 18** we developed a set of rules to help us differentiate functions more efficiently. These rules or combinations of them can be used to differentiate all of the functions we consider in this course.

However, the task of finding **antiderivatives** is not so easy. Many functions simply do not have antiderivatives which can be expressed easily using standard functions.

HISTORICAL NOTE

Robert Henry Risch (1939 -) is an American mathematician. He studied at the University of California, Berkeley.

In his doctorate studies in 1968, Risch devised a method for deciding if a function has an elementary antiderivative, and if it does, finding it. The original summary of his method took over 100 pages. Later developments from this are now used in all computer algebra systems.

After completing his doctorate, he worked at the IBM Thomas Watson Research Centre.

We *can* construct some rules which will allow us to integrate most of the function types we consider in this course.

INTEGRATING BASIC FUNCTION TYPES

For k a constant, $\dfrac{d}{dx}(kx+c) = k$ $\quad\therefore\quad$ $\boxed{\int k\,dx = kx + c}$

If $n \neq -1$, $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1} + c\right) = \dfrac{(n+1)x^n}{n+1} = x^n$ $\quad\therefore\quad$ $\boxed{\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c,\ n \neq -1}$

$\dfrac{d}{dx}(e^x + c) = e^x$ $\quad\therefore\quad$ $\boxed{\int e^x\,dx = e^x + c}$

For $x > 0$, $\dfrac{d}{dx}(\ln x + c) = \dfrac{1}{x}$

For $x < 0$, $\dfrac{d}{dx}(\ln(-x) + c) = \dfrac{-1}{-x} = \dfrac{1}{x}$

$$\therefore \int \frac{1}{x}\,dx = \begin{cases} \ln x + c & \text{if } x > 0 \\ \ln(-x) + c & \text{if } x < 0 \end{cases}$$

$\therefore$ $\boxed{\int \dfrac{1}{x}\,dx = \ln|x| + c,\ x \neq 0}$

$|x|$ is the absolute value of x, which was studied in **Chapter 5**.

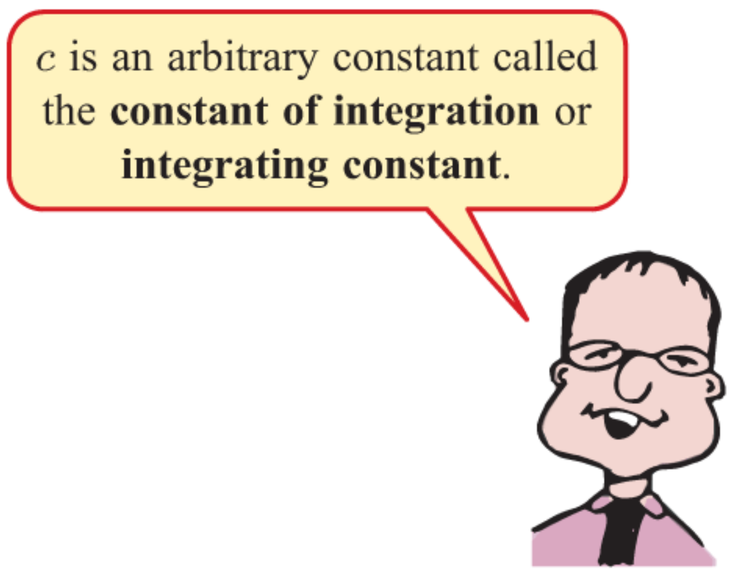


$\dfrac{d}{dx}(\sin x + c) = \cos x$ $\quad\therefore\quad$ $\boxed{\int \cos x\,dx = \sin x + c}$

$\dfrac{d}{dx}(-\cos x + c) = \sin x$ $\quad\therefore\quad$ $\boxed{\int \sin x\,dx = -\cos x + c}$

$\dfrac{d}{dx}(\tan x + c) = \dfrac{1}{\cos^2 x}$ $\quad\therefore\quad$ $\boxed{\int \dfrac{1}{\cos^2 x}\,dx = \tan x + c}$

Function	*Integral*
k, a constant	$kx + c$
$x^n, \quad n \neq -1$	$\dfrac{x^{n+1}}{n+1} + c$
e^x	$e^x + c$
$\dfrac{1}{x}$	$\ln \lvert x \rvert + c, \quad x \neq 0$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
$\dfrac{1}{\cos^2 x}$	$\tan x + c$



GENERAL RULES FOR INTEGRATION

These rules can be combined with the results we saw in the previous Section:

- Any constant within the integral may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 3 — Self Tutor

Find:

a $\int (-2x^3 + 5x - 2)\,dx$ **b** $\int \left(5e^x + \dfrac{2}{\sqrt{x}}\right) dx$ **c** $\int (2\sin x - \cos x)\,dx$

a $\int (-2x^3 + 5x - 2)\,dx$

$= \dfrac{-2x^4}{4} + \dfrac{5x^2}{2} - 2x + c$

$= -\frac{1}{2}x^4 + \frac{5}{2}x^2 - 2x + c$

b $\int \left(5e^x + \dfrac{2}{\sqrt{x}}\right) dx$

$= \int (5e^x + 2x^{-\frac{1}{2}})\,dx$

$= 5e^x + \dfrac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= 5e^x + 4\sqrt{x} + c$

c $\int (2\sin x - \cos x)\,dx$

$= 2(-\cos x) - \sin x + c$

$= -2\cos x - \sin x + c$

There are no power or quotient rules for integration, so we often have to perform the multiplication or division first and then integrate.

EXERCISE 22B

1 Find:

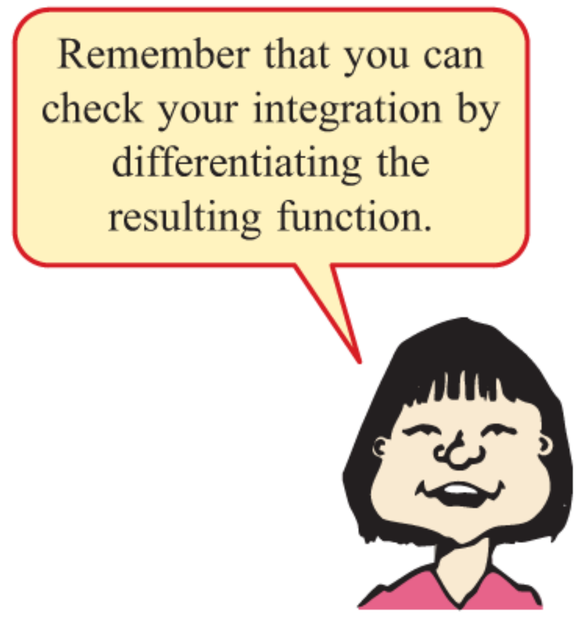
a $\int (x^2 + 3x - 2)\,dx$ **b** $\int (2x^2 - 3x + 1)\,dx$

c $\int (-x^3 + 4x^2 - 3)\,dx$ **d** $\int \left(\frac{1}{2}x + x^2 + x^3\right)dx$

e $\int (x^4 - x^2 - x + 2)\,dx$ **f** $\int \left(4x^2 + \frac{1}{x}\right)dx$

g $\int \left(2\sqrt{x} - \frac{3}{\sqrt{x}}\right)dx$ **h** $\int \left(\frac{1}{3x} - \frac{2}{x^2}\right)dx$

i $\int (x\sqrt{x} - 9)\,dx$ **j** $\int (3x^{-\frac{3}{2}} + x^{\frac{1}{4}})\,dx$



2 Integrate with respect to x:

a $2e^x - 3x$ **b** $\frac{4}{x} + x^2 - e^x$ **c** $5e^x + \frac{1}{2}x^2$

d $3\sin x - 2$ **e** $4x - 2\cos x$ **f** $\sin x - 2\cos x + e^x$

g $x^2\sqrt{x} - 10\sin x$ **h** $\frac{x(x-1)}{3} + \frac{1}{\cos^2 x}$ **i** $-\sin x + 2\sqrt{x}$

3 Find y if:

a $\frac{dy}{dx} = 6$ **b** $\frac{dy}{dx} = 4x^2$ **c** $\frac{dy}{dx} = \frac{1}{x^2}$

d $\frac{dy}{dx} = \frac{2}{\sqrt[3]{x}}$ **e** $\frac{dy}{dx} = 2x^3 - 4$ **f** $\frac{dy}{dx} = 4x^3 + 3x^2$

g $\frac{dy}{dx} = 2 - \frac{1}{x}$ **h** $\frac{dy}{dx} = \sin x + 2\cos x$ **i** $\frac{dy}{dx} = 2e^x - 5 + x$

4 Find:

a $\int (2e^t - 4\sin t)\,dt$ **b** $\int \left(3\cos t - \frac{1}{t}\right)dt$ **c** $\int (5\sin t - \sqrt{t})\,dt$

d $\int \left(\frac{1}{\cos^2 x} + 2\sin x\right)dx$ **e** $\int (\theta - \sin\theta)\,d\theta$ **f** $\int \left(\frac{2}{\theta} - \frac{1}{\cos^2\theta}\right)d\theta$

Example 4 **Self Tutor**

Find: **a** $\int \left(x - \frac{1}{\sqrt{x}}\right)^2 dx$ **b** $\int \frac{3\sqrt{x} + x^2}{x}\,dx$

a
$$\int \left(x - \frac{1}{\sqrt{x}}\right)^2 dx$$
$$= \int \left(x^2 - 2\sqrt{x} + \frac{1}{x}\right)dx$$
$$= \int (x^2 - 2x^{\frac{1}{2}} + x^{-1})\,dx$$
$$= \tfrac{1}{3}x^3 - \tfrac{4}{3}x^{\frac{3}{2}} + \ln|x| + c$$

b
$$\int \frac{3\sqrt{x} + x^2}{x}\,dx$$
$$= \int (3x^{-\frac{1}{2}} + x)\,dx$$
$$= \frac{3x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^2}{2} + c$$
$$= 6\sqrt{x} + \tfrac{1}{2}x^2 + c$$

5 Find:

a $\int (2x+1)^2\,dx$ **b** $\int \left(x+\frac{1}{x}\right)^2 dx$ **c** $\int \frac{1-4x}{x\sqrt{x}}\,dx$

d $\int \frac{2x-1}{\sqrt{x}}\,dx$ **e** $\int \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx$ **f** $\int \frac{1-x^2}{x}\,dx$

g $\int \left(\frac{2}{x}+1\right)^2 dx$ **h** $\int \frac{x^2-4x+10}{x}\,dx$ **i** $\int \frac{3x^3-2x^2+5}{x^2}\,dx$

6 Find $f(x)$ if:

a $f'(x) = (1-2x)^2$ **b** $f'(x) = \sqrt{x} - \frac{2}{\sqrt{x}}$ **c** $f'(x) = \frac{x^2-5}{x^2}$

7 Find:

a $\int \left(\sqrt{x} + \frac{1}{2}\cos x\right) dx$ **b** $\int (2e^x - 4\sin x)\,dx$ **c** $\int (3\cos x - \sin x)\,dx$

8 Find:

a $\int \left(x^2 - \frac{1}{x}\right)^2 dx$ **b** $\int \frac{x^2-4x+2}{\sqrt{x}}\,dx$ **c** $\int \sqrt{x}(3x-1)^2\,dx$

DISCUSSION

In the rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$, why did we exclude the value $n = -1$?

C PARTICULAR VALUES

We can find the constant of integration c if we are given a particular value of the function.

Example 5 Self Tutor

Find $f(x)$ given that:

a $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$ **b** $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

a

$$f'(x) = x^3 - 2x^2 + 3$$

$$\therefore\ f(x) = \int (x^3 - 2x^2 + 3)\,dx$$

$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $c = 2$

Thus $f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + 2$.

b

$$f'(x) = 2\sin x - \sqrt{x}$$

$$\therefore\ f(x) = \int (2\sin x - x^{\frac{1}{2}})\,dx$$

$$\therefore\ f(x) = 2(-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore\ f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

But $f(0) = 4$, so $-2\cos 0 - 0 + c = 4$

$$\therefore\ c = 6$$

Thus $f(x) = -2\cos x - \frac{2}{3}x^{\frac{3}{2}} + 6$.

EXERCISE 22C

1 Find $f(x)$ given that:

a $f'(x) = 2x - 1$ and $f(0) = 3$

b $f'(x) = 3x^2 + 2x$ and $f(2) = 5$

c $f'(x) = 2 + \dfrac{1}{\sqrt{x}}$ and $f(1) = 1$

d $f'(x) = x - \dfrac{2}{\sqrt{x}}$ and $f(1) = 2$

e $f'(x) = \sqrt{x} - 2$ and $f(4) = 0$

f $f'(x) = \dfrac{1}{x}$ and $f(e) = 2$.

2 A curve has gradient function $\dfrac{dy}{dx} = x - 2x^2$ and passes through $(2, 4)$. Find the equation of the curve.

3 A curve has gradient function $\dfrac{dy}{dx} = 1 - e^x$ and passes through $(3, e^3)$. Find the equation of the curve.

4 Find $f(x)$ given that:

a $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$

b $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

c $f'(x) = \sqrt{x} - \dfrac{2}{\cos^2 x}$ and $f(\pi) = 0$

d $f'(x) = e^x + 3\cos x$ and $f(\pi) = 0$.

5 A curve has gradient function $f'(x) = ax + 1$ where a is a constant. Find $f(x)$ given that $f(0) = 3$ and $f(3) = -3$.

6 A curve has gradient function $f'(x) = ax^2 + bx$ where a, b are constants. Find $f(x)$ given that $f(-1) = -2$, $f(0) = 1$, and $f(1) = 4$.

Example 6 — Self Tutor

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$, and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

$$\therefore f'(x) = \int (12x^2 - 4)\,dx$$

$$= \frac{12x^3}{3} - 4x + c$$

$$= 4x^3 - 4x + c$$

But $f'(0) = -1$, so $c = -1$

Thus $f'(x) = 4x^3 - 4x - 1$

$$\therefore f(x) = \int (4x^3 - 4x - 1)\,dx$$

$$= \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$$

$$= x^4 - 2x^2 - x + d$$

But $f(1) = 4$, so $1 - 2 - 1 + d = 4$ and hence $d = 6$

Thus $f(x) = x^4 - 2x^2 - x + 6$

7 Find $f(x)$ given that:

a $f''(x) = 2x + 1$, $f'(1) = 3$, and $f(2) = 7$

b $f''(x) = 15\sqrt{x} + \dfrac{3}{\sqrt{x}}$, $f'(1) = 12$, and $f(0) = 5$

c $f''(x) = \cos x$, $f'\left(\frac{\pi}{2}\right) = 0$, and $f(0) = 3$

d $f''(x) = 2x$ and the points $(1, 0)$ and $(0, 5)$ lie on the curve $y = f(x)$.

8 Suppose $f''(x) = 3e^{-x}$, $f(1) = \dfrac{3}{e}$, and $f(3) = \dfrac{3}{e^3} - 2$. Find $f(x)$.

D INTEGRATING $f(ax + b)$

In this Section we deal with integrals of functions which are composite with the linear function $ax + b$.

Notice that $\dfrac{d}{dx}(e^{ax+b}) = ae^{ax+b}$.

$$\therefore \quad \int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c \quad \text{for } a \neq 0.$$

Likewise if $n \neq -1$, $\dfrac{d}{dx}((ax+b)^{n+1}) = (n+1)(ax+b)^n \times a$

$$= a(n+1)(ax+b)^n$$

$$\therefore \quad \int (ax+b)^n\,dx = \frac{1}{a}\,\frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for } n \neq -1,\ a \neq 0.$$

We can also observe that $\dfrac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$.

So, $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c \quad \text{for } a \neq 0.$$

Likewise, $$\int (-\sin(ax+b))\,dx = \frac{1}{a}\cos(ax+b) + c \quad \text{for } a \neq 0.$$

and $$\int \frac{1}{\cos^2(ax+b)}\,dx = \frac{1}{a}\tan(ax+b) + c \quad \text{for } a \neq 0.$$

Finally, $\dfrac{d}{dx}\left(\frac{1}{a}\ln(ax+b)\right) = \frac{1}{a}\left(\frac{a}{ax+b}\right) = \frac{1}{ax+b}$ for $ax + b > 0$, $a \neq 0$

$$\therefore \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln(ax+b) + c \quad \text{for } ax + b > 0, \quad a \neq 0$$

We can similarly show that $\int \dfrac{1}{ax+b}\,dx = \dfrac{1}{a}\ln(-(ax+b)) + c$ for $ax + b < 0$, $a \neq 0$

$$\therefore \quad \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c \quad \text{for } a \neq 0.$$

For a, b constants with $a \neq 0$, we have:

Function	*Integral*
e^{ax+b}	$\frac{1}{a}\,e^{ax+b} + c$
$(ax+b)^n, \quad n \neq -1$	$\frac{1}{a}\,\frac{(ax+b)^{n+1}}{n+1} + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert + c$

Function	*Integral*
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$-\sin(ax+b)$	$\frac{1}{a}\cos(ax+b) + c$
$\frac{1}{\cos^2(ax+b)}$	$\frac{1}{a}\tan(ax+b) + c$

Example 7 **Self Tutor**

Find:

a $\int (2x+3)^4\,dx$ **b** $\int \frac{1}{\sqrt{1-3x}}\,dx$

a $\int (2x+3)^4\,dx = \frac{1}{2} \times \frac{(2x+3)^5}{5} + c$

$= \frac{1}{10}(2x+3)^5 + c$

b $\int \frac{1}{\sqrt{1-3x}}\,dx = \int (1-3x)^{-\frac{1}{2}}\,dx$

$= \frac{1}{-3} \times \frac{(1-3x)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= -\frac{2}{3}\sqrt{1-3x} + c$

EXERCISE 22D

1 Find:

a $\int (2x+5)^3\,dx$ **b** $\int (4x-3)^7\,dx$ **c** $\int 3(1-x)^4\,dx$

d $\int \frac{1}{(3-2x)^2}\,dx$ **e** $\int \frac{4}{(2x-1)^4}\,dx$ **f** $\int \sqrt{3x-4}\,dx$

g $\int \frac{10}{\sqrt{1-5x}}\,dx$ **h** $\int \frac{4}{\sqrt{3-4x}}\,dx$ **i** $\int \frac{5}{(3x-2)^3}\,dx$

2 Find $y = f(x)$ given $\frac{dy}{dx} = \sqrt{2x-7}$ and that $f(8) = 11$.

3 The function $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1-x}}$, and the curve $y = f(x)$ passes through the point $(-3, -11)$.

Find the point on the graph of $y = f(x)$ with x-coordinate -8.

4 Find:

a $\int 3(2x-1)^2\,dx$ **b** $\int (4x-5)^2\,dx$ **c** $\int (1-3x)^3\,dx$

d $\int (2-5x)^2\,dx$ **e** $\int 4\sqrt{5-x}\,dx$ **f** $\int (7x+1)^4\,dx$

5 Find an expression for y given that $\frac{dy}{dx} = x - \frac{5}{(1-x)^2}$ and that the curve passes through $(2, 0)$.

Example 8 — Self Tutor

Integrate with respect to x:

a $2e^{2x} - e^{-3x}$ **b** $2\sin 3x + \cos(4x + \pi)$

a
$$\int (2e^{2x} - e^{-3x})\,dx$$
$$= 2\left(\tfrac{1}{2}\right)e^{2x} - \left(\tfrac{1}{-3}\right)e^{-3x} + c$$
$$= e^{2x} + \tfrac{1}{3}e^{-3x} + c$$

b
$$\int (2\sin 3x + \cos(4x + \pi))\,dx$$
$$= 2 \times -\tfrac{1}{3}\cos 3x + \tfrac{1}{4}\sin(4x + \pi) + c$$
$$= -\tfrac{2}{3}\cos 3x + \tfrac{1}{4}\sin(4x + \pi) + c$$

6 Integrate with respect to x:

a $\sin 3x$ **b** $2\cos(-4x) + 1$ **c** $\dfrac{1}{\cos^2 2x}$

d $3\cos\frac{x}{2}$ **e** $3\sin 2x - e^{-x}$ **f** $2\sin\left(2x + \frac{\pi}{6}\right)$

g $-3\cos\left(\frac{\pi}{4} - x\right)$ **h** $e^{2x} - \dfrac{2}{\cos^2(\frac{x}{2})}$ **i** $\dfrac{4}{\cos^2(\frac{\pi}{3} - 2x)}$

j $\cos 2x + \sin 2x$ **k** $2\sin 3x + 5\cos 4x$ **l** $\frac{1}{2}\cos 8x - 3\sin x$

7 Find:

a $\displaystyle\int (2e^x + 5e^{2x})\,dx$ **b** $\displaystyle\int (3e^{5x-2})\,dx$ **c** $\displaystyle\int (e^{7-3x})\,dx$

d $\displaystyle\int (e^x + e^{-x})^2\,dx$ **e** $\displaystyle\int (e^{-x} + 2)^2\,dx$ **f** $\displaystyle\int \frac{(e^{2x} - 5)^2}{e^x}\,dx$

8 Find an expression for y given that $\dfrac{dy}{dx} = (1 - e^x)^2$, and that the graph has y-intercept 4.

9 Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

10 Suppose $f'(x) = p\sin\frac{x}{2}$, $f(0) = 1$, and $f(2\pi) = 0$. Find p and hence $f(x)$.

11 Consider a function g such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

12 A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

Example 9 — Self Tutor

Find:

a $\displaystyle\int \frac{1}{5x - 3}\,dx$ **b** $\displaystyle\int \frac{4}{1 - 2x}\,dx$

a
$$\int \frac{1}{5x - 3}\,dx$$
$$= \tfrac{1}{5}\ln|5x - 3| + c$$

b
$$\int \frac{4}{1 - 2x}\,dx$$
$$= 4\left(\tfrac{1}{-2}\right)\ln|1 - 2x| + c$$
$$= -2\ln|1 - 2x| + c$$

13 Find:

a $\int \frac{6}{x+4}\,dx$ **b** $\int \frac{1}{2x-1}\,dx$ **c** $\int \frac{5}{1-3x}\,dx$

d $\int \left(4+\frac{1}{5x-2}\right)dx$ **e** $\int \left(1-2x+\frac{4}{x-3}\right)dx$ **f** $\int \left(\sin 2x-\frac{3}{1-2x}\right)dx$

14 To find $\int \frac{1}{4x}\,dx$, Tracy's answer was $\int \frac{1}{4x}\,dx = \frac{1}{4}\ln|4x| + c$.

Nadine's answer was $\int \frac{1}{4x}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx = \frac{1}{4}\ln|x| + c$.

Which of them has found the correct answer? Prove your statement.

15 Show that $\frac{3x-1}{x+2}$ may be written in the form $3-\frac{7}{x+2}$. Hence find $\int \frac{3x-1}{x+2}\,dx$.

16 Find $f(x)$ given $f'(x) = 2x - \frac{2}{1-x}$ and $f(-1) = 3$.

E INTEGRATION BY SUBSTITUTION

Consider the integral $\int (x^2+3x)^4(2x+3)\,dx$.

The function we are integrating is the product of two expressions:

- The first expression is $(x^2+3x)^4$, which has the form $f(u(x))$ where $f(u) = u^4$ and $u(x) = x^2+3x$.
- The second expression is $2x+3$, and we notice that this is $\frac{du}{dx}$.

So, we can write the integral in the form $\int f(u(x))\frac{du}{dx}\,dx$ where $f(u) = u^4$ and $u(x) = x^2+3x$.

Integrals in this form can be found using a method called **integration by substitution**.

Integration by substitution is the reverse process of differentiating using the **chain rule**.

Suppose $F(u)$ is the antiderivative of $f(u)$, so $\frac{dF}{du} = f(u)$

$$\therefore \int f(u)\,du = F(u) + c \quad \ldots. \ (1)$$

But $\frac{dF}{dx} = \frac{dF}{du}\frac{du}{dx}$ {chain rule}

$$= f(u)\frac{du}{dx}$$

$$\therefore \int f(u)\frac{du}{dx}\,dx = F(u) + c \quad \ldots. \ (2)$$

Comparing (1) and (2), we conclude that: $\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$

Using this method,

$$\int (x^2+3x)^4(2x+3)\,dx$$

$$= \int u^4 \frac{du}{dx}\,dx \qquad \{u = x^2+3x,\ \frac{du}{dx} = 2x+3\}$$

$$= \int u^4\,du \qquad \{\text{replacing } \frac{du}{dx}\,dx \text{ by } du\}$$

$$= \frac{u^5}{5} + c$$

$$= \tfrac{1}{5}(x^2+3x)^5 + c$$



Example 10

Self Tutor

Use substitution to find:

a $\int \sqrt{x^3+2x}\,(3x^2+2)\,dx$ **b** $\int xe^{1-x^2}\,dx$

c $\int \cos^3 x \sin x\,dx$ **d** $\int \frac{\cos x}{\sin x}\,dx$

a

$$\int \sqrt{x^3+2x}\,(3x^2+2)\,dx$$

$$= \int \sqrt{u} \frac{du}{dx}\,dx \qquad \{u = x^3+2x,\ \frac{du}{dx} = 3x^2+2\}$$

$$= \int u^{\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \tfrac{2}{3}(x^3+2x)^{\frac{3}{2}} + c$$

b

$$\int xe^{1-x^2}\,dx$$

$$= -\tfrac{1}{2}\int (-2x)e^{1-x^2}\,dx$$

$$= -\tfrac{1}{2}\int e^u \frac{du}{dx}\,dx \qquad \{u = 1-x^2,\ \frac{du}{dx} = -2x\}$$

$$= -\tfrac{1}{2}\int e^u\,du$$

$$= -\tfrac{1}{2}e^u + c$$

$$= -\tfrac{1}{2}e^{1-x^2} + c$$

c

$$\int \cos^3 x \sin x\,dx$$

$$= \int (\cos x)^3 \sin x\,dx$$

$$= \int u^3\left(-\frac{du}{dx}\right)dx \qquad \{u = \cos x,\ \frac{du}{dx} = -\sin x\}$$

$$= -\int u^3\,du$$

$$= -\frac{u^4}{4} + c$$

$$= -\tfrac{1}{4}\cos^4 x + c$$

d

$$\int \frac{\cos x}{\sin x}\,dx$$

$$= \int \frac{1}{u}\frac{du}{dx}\,dx \qquad \{u = \sin x,\ \frac{du}{dx} = \cos x\}$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + c$$

$$= \ln|\sin x| + c$$



We need to choose our substitutions carefully.

For example, in **Example 10** part **d**, if we let $u = \cos x$, then $\frac{du}{dx} = -\sin x$ and we obtain $\int \frac{\cos x}{\sin x}\,dx = \int \frac{u}{-\frac{du}{dx}}\,dx$. This is not in the correct form to apply our theorem, so we have made the wrong substitution and we need to try another option.

EXERCISE 22E

1 Differentiate $(2x^2 - 5x)^3$, and hence find $\int (4x-5)(2x^2-5x)^2\,dx$.

2 Differentiate $\sin(x^2)$, and hence find $\int x\cos(x^2)\,dx$.

3 Differentiate $\ln(5 - 3x + x^2)$, and hence find $\int \frac{4x-6}{5-3x+x^2}\,dx$.

4 Use the substitution given to perform each integration:

a $\int 3x^2(x^3+1)^4\,dx$ using $u = x^3+1$

b $\int x^2 e^{x^3+1}\,dx$ using $u = x^3+1$

c $\int \sin^4 x \cos x\,dx$ using $u = \sin x$

d $\int 2x\cos(x^2-3)\,dx$ using $u = x^2-3$

5 Integrate by substitution:

a $4x^3(2+x^4)^3$

b $\frac{2x}{\sqrt{x^2+3}}$

c $\frac{6x^2}{(2x^3-1)^4}$

d $(x^3+2x+1)^4(3x^2+2)$

e $\frac{x}{(1-x^2)^5}$

f $\frac{x+2}{(x^2+4x-3)^2}$

6 Find:

a $\int -2e^{1-2x}\,dx$

b $\int 2xe^{x^2}\,dx$

c $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

7 Find:

a $\int \frac{2x}{x^2+1}\,dx$

b $\int \frac{x}{2-x^2}\,dx$

c $\int \frac{2x-3}{x^2-3x}\,dx$

8 Integrate with respect to x:

a $x^2(3-x^3)^2$

b $x\sqrt{1-x^2}$

c xe^{1-x^2}

d $\frac{(\ln x)^3}{x}$

e $(2x-1)e^{x-x^2}$

f $\frac{1-x^2}{x^3-3x}$

9 Integrate with respect to x:

a $\sin^7 x\cos x$

b $\cos^5 x\sin x$

c $\frac{\sin x}{\sqrt{\cos x}}$

d $\tan x$

e $\sqrt{\sin x}\cos x$

f $\frac{\cos x}{(2+\sin x)^2}$

g $\frac{\sin x}{1-\cos x}$

h $\frac{\cos 2x}{\sin 2x - 3}$

i $x\sin(x^2)$

10 Find:

a $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

b $\int \cos^3 x\,dx$

c $\int \sin^3 2x\cos 2x\,dx$

REVIEW SET 22A

1 Find the derivative of $x^4 - x^2$, and hence find $\int (2x^3 - x)\,dx$.

2 Find the derivative of $\sin\left(\frac{\pi}{3} - 2x\right)$, and hence find $\int \cos\left(\frac{\pi}{3} - 2x\right)\,dx$.

3 Find:

a $\int \left(\sqrt{x} - \frac{2}{x^2}\right)\,dx$ **b** $\int \left(2x - \frac{3}{\sqrt[3]{x}}\right)\,dx$ **c** $\int \frac{6x+5}{\sqrt{x}}\,dx$

4 Integrate with respect to x:

a $\frac{4}{\sqrt{x}}$ **b** $\frac{1}{3}x^3 + 2x$ **c** $\frac{1-2x}{x^3}$

5 Find:

a $\int (-3x^4 + 6x^2)\,dx$ **b** $\int \frac{3x^3 - x^2 - 1}{x^2}\,dx$ **c** $\int (2x - \sqrt{x})^2\,dx$

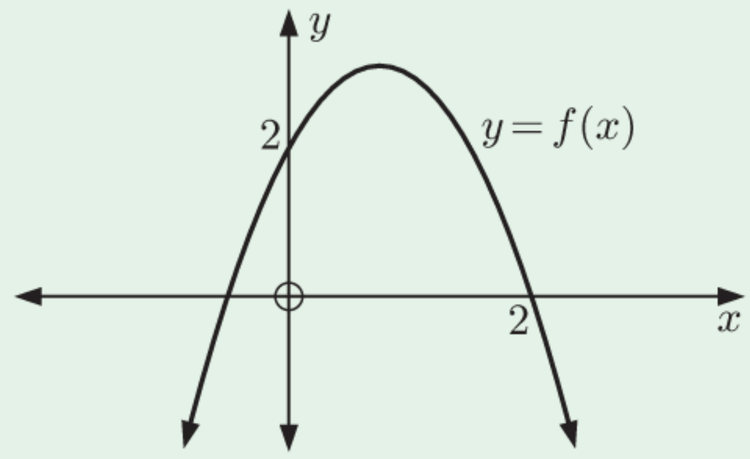
d $\int \left(4e^x - \frac{3}{x}\right)\,dx$ **e** $\int \sin(4x - 5)\,dx$ **f** $\int e^{4-3x}\,dx$

6 Find y if:

a $\frac{dy}{dx} = 3e^{-x} - 2\sin\left(\frac{\pi}{2} - x\right)$ **b** $\frac{dy}{dx} = \cos 4x - \frac{1}{2}x^2$

7 Given that $f'(x) = 3x^2 - 4x + 1$ and $f(0) = 2$, find $f(x)$.

8 The curve $y = f(x)$ shown alongside has gradient function $f'(x) = ax + 3$.
Find the equation of the curve.



9 Given that $f'(x) = 3e^{2x}$ and $f(0) = 2$, find $f(x)$.

10 Find:

a $\int \frac{x^2 - 7}{x}\,dx$ **b** $\int \left(e^{2x-3} - \frac{2}{3x-1}\right)\,dx$

c $\int \left((4 - 3x)^3 + \sin(-2x)\right)\,dx$

11 A curve has gradient function $f'(x) = a\cos 3x$ where a is a constant. $f(0) = -1$ and $f(\frac{\pi}{4}) = 1$. Find a and hence $f(x)$.

12 Find the derivative of $\sqrt{x^2 - 4}$, and hence find $\int \frac{x}{\sqrt{x^2-4}}\,dx$.

13 Find $\int x\sin\left(x^2 + \frac{\pi}{3}\right)\,dx$ using $u = x^2 + \frac{\pi}{3}$.

14 Integrate by substitution:

a $\int \frac{x+2}{x^2+4x}\,dx$ **b** $\int 2xe^{x^2-1}\,dx$

c $\int \sin^9 x \cos x\,dx$ **d** $\int \tan 2x\,dx$

REVIEW SET 22B

1 Find the derivative of $6e^{-2x}$, and hence find $\int e^{-2x}\,dx$.

2 Find the derivative of $\ln(2x+1)$, and hence find $\int \frac{1}{2x+1}\,dx$.

3 Integrate with respect to x:

a $\frac{x^2-2}{x^2}$ **b** $(3x-4)^2$ **c** $4-2x^2$

4 Find:

a $\int \left(x^{\frac{1}{3}}+3\right)dx$ **b** $\int (3x^2-2)\,dx$ **c** $\int (3+2x)^2\,dx$

5 Given that $f'(x)=x^2-3x+2$ and $f(1)=3$, find $f(x)$.

6 Find y if:

a $\frac{dy}{dx}=(x^2-1)^2$ **b** $\frac{dy}{dx}=400-20x^{-\frac{1}{2}}$

7 Find:

a $\int (2x^3-5x+7)\,dx$ **b** $\int \left(3x-\frac{1}{x}\right)dx$ **c** $\int (2e^{-x}+3)\,dx$

d $\int 4\cos 2x\,dx$ **e** $\int \left(3+e^{2x-1}\right)^2 dx$ **f** $\int \frac{1}{\cos^2(\frac{\pi}{4}-x)}\,dx$

8 Find $f(x)$ given $f'(x)=\frac{2}{x}-1$ and $f(2)=e$.

9 A curve has gradient function $\frac{dy}{dx}=ax^2+b\sqrt{x-1}$ where a, b are constants.
Find the equation of the curve given that it contains the points $(1,\,4)$, $(2,\,4)$, and $(5,\,1)$.

10 Find $f(x)$ given $f'(x)=\frac{3}{\sqrt{4-3x}}$ and $f(-4)=0$.

11 Find:

a $\int \frac{1}{3-2x}\,dx$ **b** $\int \frac{4}{5x+1}\,dx$

12 By differentiating $(3x^2+x)^3$, find $\int (3x^2+x)^2(6x+1)\,dx$.

13 Integrate by substitution:

a $\displaystyle\int \frac{2x}{\sqrt{x^2-5}}\,dx$

b $\displaystyle\int \frac{\sin x}{\cos^4 x}\,dx$

c $\displaystyle\int 4xe^{-x^2}\,dx$

d $\displaystyle\int \sin^3 x\,dx$

14 Find $\displaystyle\int \frac{x}{x^2-9}\,dx$ using the substitution:

a $u = x^2 - 9$

b $x = 3\sin t$

Chapter 23

Definite integrals

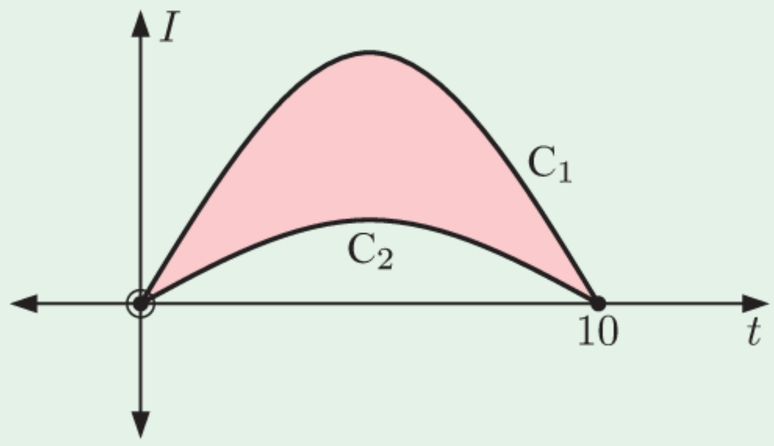
Contents:

A Definite integrals
B Definite integrals involving substitution
C The area under a curve
D The area above a curve
E The area between a curve and the y-axis
F Solids of revolution
G Problem solving by integration



OPENING PROBLEM

On August 8, the intensity of light entering a greenhouse is given by $I = 3\sin\frac{\pi t}{10}$ units per hour if the day is sunny, and $I = \sin\frac{\pi t}{10}$ units per hour if the day is overcast, where t is the number of hours after sunrise, $0 \leqslant t \leqslant 10$.



The graphs of these curves are shown alongside.

Things to think about:

a Can you identify each curve?

b Can you find the shaded area enclosed by C_1 and C_2 for $0 \leqslant t \leqslant 10$?

c If the intensity of light is regarded as its rate of energy transfer, what does the area in **b** mean about the energy entering the greenhouse?

We have already seen how the Fundamental Theorem of Calculus gives meaning to a **definite integral** over a particular domain:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a).}$

In this Chapter we explore definite integrals and how they are used to calculate areas.

A DEFINITE INTEGRALS

The definite integral

$\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x"
or "the integral from a to b of $f(x)$ with respect to x".

When calculating definite integrals we can omit the constant of integration c as this will always cancel out in the subtraction process.

We write $F(b) - F(a)$ as $[F(x)]_a^b$, and so $\boldsymbol{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).}$

For continuous functions, we can list the following properties of definite integrals:

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$, k is any constant
- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 1 **Self Tutor**

Find:

a $\int_1^2 (x^3 - 4x + 5)\,dx$ **b** $\int_1^4 \left(2\sqrt{x} + \frac{3}{x}\right) dx$ **c** $\int_0^{\frac{\pi}{3}} \sin 2x\,dx$

a
$$\int_1^2 (x^3 - 4x + 5)\,dx$$
$$= \left[\tfrac{1}{4}x^4 - 2x^2 + 5x\right]_1^2$$
$$= \left(\tfrac{2^4}{4} - 2(2)^2 + 5(2)\right) - \left(\tfrac{1^4}{4} - 2(1)^2 + 5(1)\right)$$
$$= 6 - 3\tfrac{1}{4}$$
$$= \tfrac{11}{4}$$

We omit the constant of integration c for definite integrals.



b
$$\int_1^4 \left(2\sqrt{x} + \frac{3}{x}\right) dx$$
$$= \left[\tfrac{4}{3}x^{\frac{3}{2}} + 3\ln|x|\right]_1^4$$
$$= \left(\tfrac{4}{3}(4)^{\frac{3}{2}} + 3\ln 4\right) - \left(\tfrac{4}{3}(1)^{\frac{3}{2}} + 3\ln 1\right)$$
$$= \tfrac{4}{3} \times 8 + 6\ln 2 - \tfrac{4}{3}$$
$$= \tfrac{28}{3} + 6\ln 2$$

c
$$\int_0^{\frac{\pi}{3}} \sin 2x\,dx = \left[-\tfrac{1}{2}\cos 2x\right]_0^{\frac{\pi}{3}}$$
$$= \left(-\tfrac{1}{2}\cos\tfrac{2\pi}{3}\right) - \left(-\tfrac{1}{2}\cos 0\right)$$
$$= -\tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \tfrac{1}{2}$$
$$= \tfrac{3}{4}$$

Some definite integrals are difficult or even impossible to evaluate analytically. In these cases you are expected to use a graphics calculator to evaluate the integral.

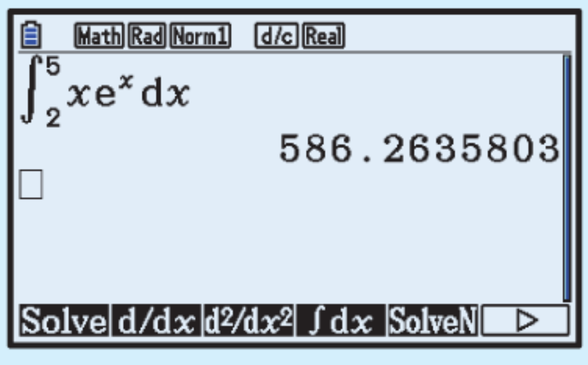


GRAPHICS CALCULATOR INSTRUCTIONS

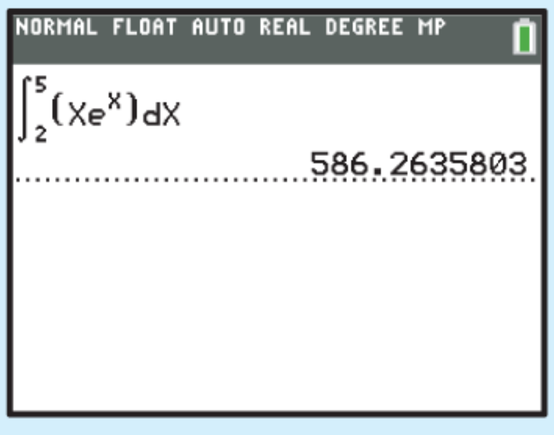
Example 2 **Self Tutor**

Evaluate $\int_2^5 xe^x\,dx$ correct to 4 significant figures.

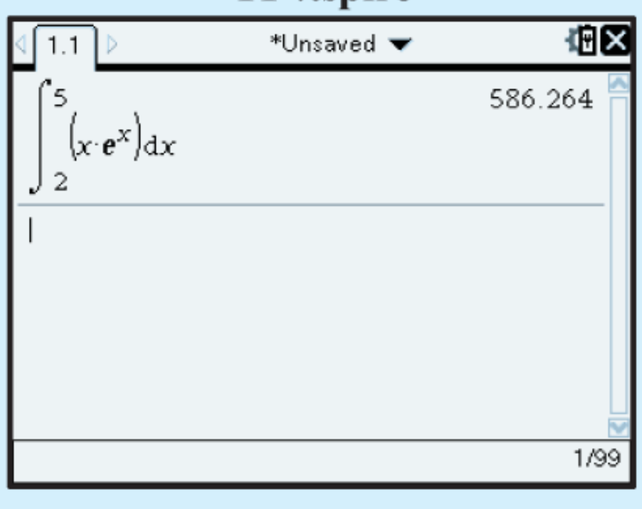
Casio fx-CG50



TI-84 Plus CE



TI-*n*spire



$$\int_2^5 xe^x\,dx \approx 586.3$$

EXERCISE 23A

1 Find:

a $\int_1^4 \sqrt{x}\,dx$ and $\int_1^4 (-\sqrt{x})\,dx$

b $\int_0^1 x^7\,dx$ and $\int_0^1 (-x^7)\,dx$

Use questions **1** to **4** to check the properties of definite integrals.



2 Find:

a $\int_0^1 x^2\,dx$ **b** $\int_1^2 x^2\,dx$ **c** $\int_0^2 x^2\,dx$ **d** $\int_0^1 3x^2\,dx$

3 Find:

a $\int_0^2 (x^3 - 4x)\,dx$ **b** $\int_2^3 (x^3 - 4x)\,dx$ **c** $\int_0^3 (x^3 - 4x)\,dx$

4 Find:

a $\int_0^1 2x^2\,dx$ **b** $\int_0^1 \sqrt{x}\,dx$ **c** $\int_0^1 (2x^2 + \sqrt{x})\,dx$

5 Evaluate:

a $\int_0^1 x^3\,dx$ **b** $\int_0^2 (x^2 - x)\,dx$ **c** $\int_0^2 (3x^2 - x + 6)\,dx$

d $\int_1^4 \left(x - \frac{3}{\sqrt{x}}\right) dx$ **e** $\int_1^4 (x + 2\sqrt{x})\,dx$ **f** $\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

g $\int_1^3 \frac{1}{x^2}\,dx$ **h** $\int_1^2 (x+3)^2\,dx$ **i** $\int_1^4 \left(x^2 + \frac{1}{x}\right) dx$

6 Find m such that $\int_m^{2m} (2x - 1)\,dx = 4$.

7 Find:

a $\int_0^1 (3x+1)^4\,dx$ **b** $\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **c** $\int_{-3}^0 \sqrt{1-x}\,dx$

8 Evaluate:

a $\int_0^1 e^x\,dx$ **b** $\int_0^3 (2e^x - 3)\,dx$ **c** $\int_0^2 e^{3x}\,dx$

d $\int_0^1 e^{1-x}\,dx$ **e** $\int_0^{\ln 4} e^x(e^x - 2)\,dx$ **f** $\int_1^2 (e^{-x} + 1)^2\,dx$

9 Evaluate:

a $\int_0^{\frac{\pi}{6}} \cos x\,dx$ **b** $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx$

d $\int_0^{\frac{\pi}{6}} \sin 3x\,dx$ **e** $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos\left(x - \frac{\pi}{3}\right) dx$ **f** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\left(2x - \frac{\pi}{4}\right) dx$

10 Evaluate: $\int_0^{\frac{\pi}{6}} (\sin 3x - \cos x)\,dx$

11 Write $\int_3^{12} \frac{1}{x}\,dx$ as a single logarithm.

12 Find:

a $\int_{-6}^{-2} \frac{1}{x}\,dx$ **b** $\int_{-1}^{5} \frac{1}{x+4}\,dx$ **c** $\int_1^8 \frac{2}{3x+4}\,dx$ **d** $\int_{-4}^{0} \frac{4}{5-2x}\,dx$

13 Show that $\frac{4x+1}{x-1}$ may be written in the form $4 + \frac{5}{x-1}$.

Hence show that $\int_3^5 \frac{4x+1}{x-1}\,dx = 8 + 5\ln 2$.

14 Evaluate using technology:

a $\int_1^3 \ln x\,dx$ **b** $\int_{-1}^{1} e^{-x^2}\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \sin(\sqrt{x})\,dx$

d $\int_2^3 \frac{1+\sqrt{x}}{x^2+1}\,dx$ **e** $\int_{-1}^{1} \tan(x^2)\,dx$ **f** $\int_{-2}^{1} (2^x - x^3)\,dx$

B DEFINITE INTEGRALS INVOLVING SUBSTITUTION

When we evaluate a definite integral using substitution, we need to make sure the endpoints are converted to the new variable.

Example 3 Self Tutor

Evaluate $\int_0^1 \frac{6x}{(x^2+1)^3}\,dx$.

Let $u = x^2 + 1$

$\therefore \frac{du}{dx} = 2x$

When $x = 0$, $u = 1$

When $x = 1$, $u = 2$

$$\therefore \int_0^1 \frac{6x}{(x^2+1)^3}\,dx = \int_0^1 \frac{1}{u^3}\left(3\,\frac{du}{dx}\right)dx$$

$$= 3\int_1^2 u^{-3}\,du$$

$$= 3\left[\frac{u^{-2}}{-2}\right]_1^2$$

$$= 3\left(\frac{2^{-2}}{-2} - \frac{1^{-2}}{-2}\right)$$

$$= \frac{9}{8}$$

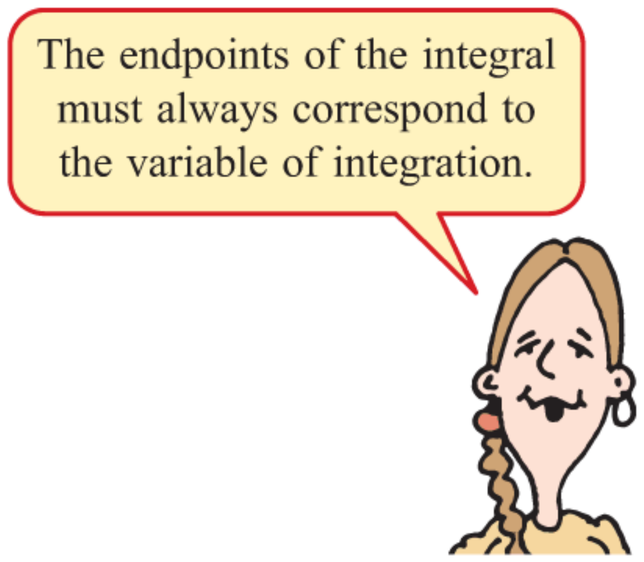
EXERCISE 23B

1 Evaluate:

a $\int_1^2 2x(x^2-1)^3\,dx$ **b** $\int_1^2 \frac{x}{(x^2+2)^2}\,dx$

c $\int_0^1 x^2 e^{x^3+1}\,dx$ **d** $\int_0^3 x\sqrt{x^2+16}\,dx$

e $\int_1^2 xe^{-2x^2}\,dx$ **f** $\int_2^3 \frac{x}{2-x^2}\,dx$



Example 4 **Self Tutor**

Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin x}\cos x\,dx$.

Let $u = \sin x$ $\therefore \frac{du}{dx} = \cos x$

When $x = \frac{\pi}{6}$, $u = \sin\frac{\pi}{6} = \frac{1}{2}$

When $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$

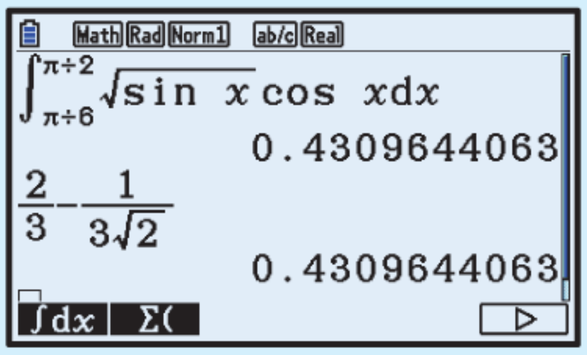
$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{\sin x}\cos x\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{u}\,\frac{du}{dx}\,dx$$

$$= \int_{\frac{1}{2}}^{1} u^{\frac{1}{2}}\,du$$

$$= \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{\frac{1}{2}}^{1}$$

$$= \tfrac{2}{3}(1)^{\frac{3}{2}} - \tfrac{2}{3}\left(\tfrac{1}{2}\right)^{\frac{3}{2}}$$

$$= \tfrac{2}{3} - \frac{1}{3\sqrt{2}}$$



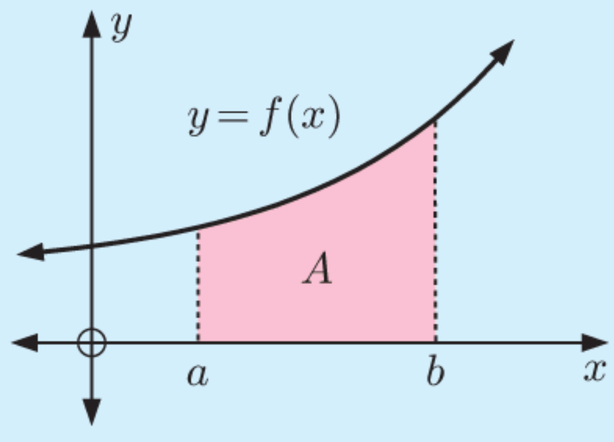
2 Evaluate:

a $\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{\cos x}}\,dx$ **b** $\int_0^{\frac{\pi}{6}} \sin^2 x\cos x\,dx$ **c** $\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \tan x\,dx$

C THE AREA UNDER A CURVE

We have already established in **Chapter 21** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.

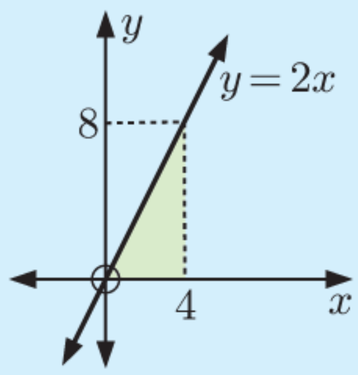


Example 5 Self Tutor

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$, and $x = 4$ using:

a a geometric argument **b** integration.

a

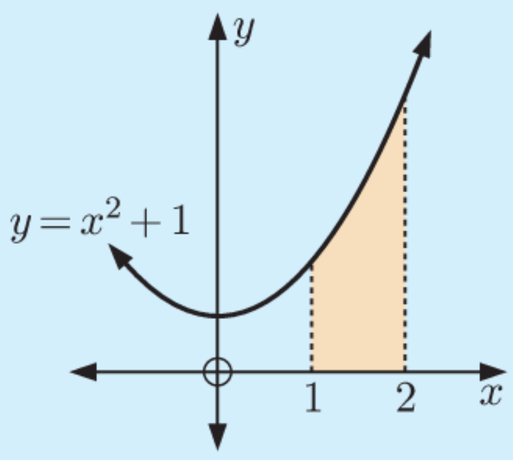


$$\text{Area} = \tfrac{1}{2} \times 4 \times 8$$
$$= 16 \text{ units}^2$$

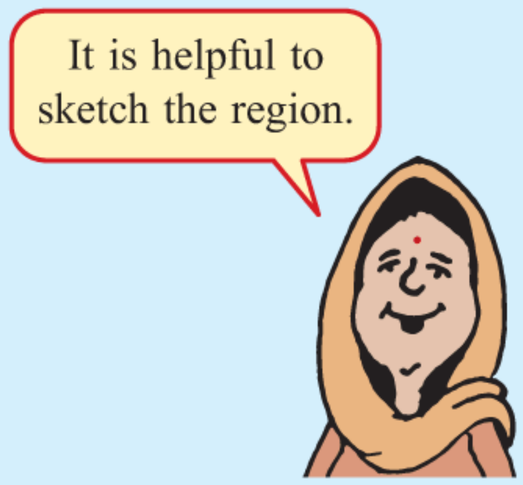
b
$$\text{Area} = \int_0^4 2x \, dx$$
$$= \left[x^2\right]_0^4$$
$$= 4^2 - 0^2$$
$$= 16 \text{ units}^2$$

Example 6 Self Tutor

Find the area of the region enclosed by $y = x^2 + 1$, the x-axis, $x = 1$, and $x = 2$.

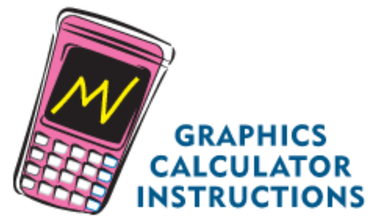


$$\text{Area} = \int_1^2 (x^2 + 1) \, dx$$
$$= \left[\frac{x^3}{3} + x\right]_1^2$$
$$= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$
$$= 3\tfrac{1}{3} \text{ units}^2$$



We can check this result using technology.



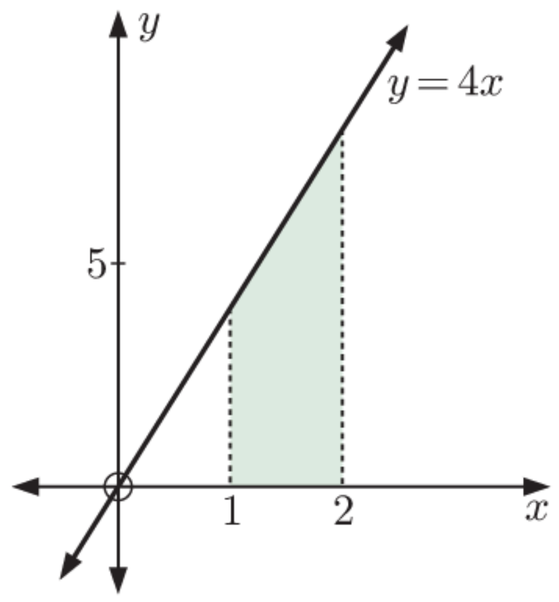


EXERCISE 23C

1 Find the shaded area using:

a a geometric argument

b integration.



2 Find the area of each region described below using:

i a geometric argument **ii** integration

a the region enclosed by $y = 5$, the x-axis, $x = -6$, and $x = 0$

b the region enclosed by $y = x$, the x-axis, $x = 4$, and $x = 5$

c the region enclosed by $y = -3x$, the x-axis, $x = -3$, and $x = 0$.

3 Find the area of the region bounded by:

a $y = x^2$, the x-axis, and $x = 1$

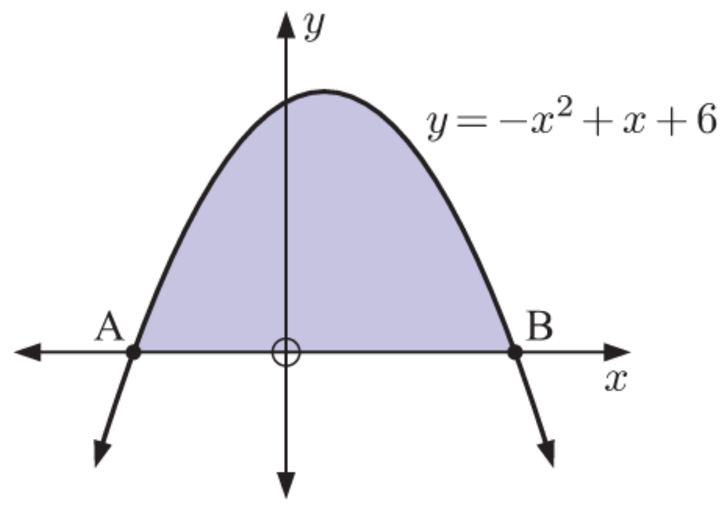
b $y = x^3$, the x-axis, $x = 1$, and $x = 4$

c $y = \frac{1}{2}x^2 - 1$, the x-axis, $x = 2$, and $x = 3$.

4 The graph of $y = -x^2 + x + 6$ is shown alongside.

a Find the coordinates of A and B.

b Hence find the shaded area.



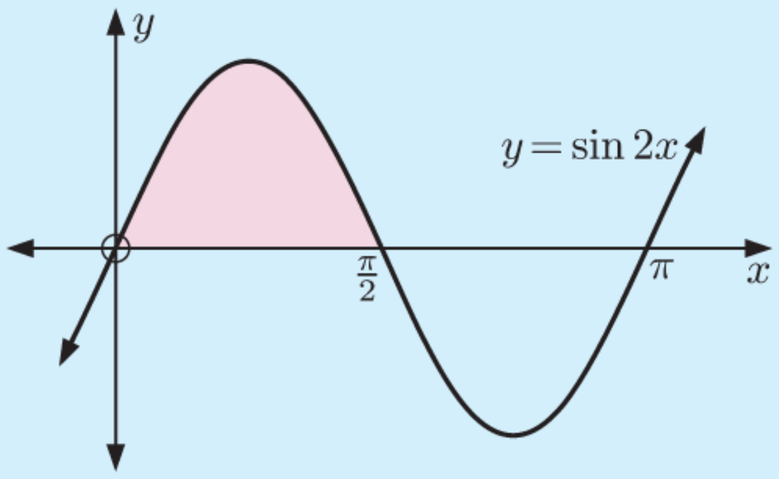
5 Find the area enclosed by each curve and the x-axis:

a $y = -x^2 + 7x - 10$ **b** $y = -2x^2 + 2x + 4$ **c** $y = 3 - x^2$

Example 7 Self Tutor

Find the area enclosed by one arch of the curve $y = \sin 2x$ and the x-axis.

The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$, so the first positive x-intercept is $\frac{\pi}{2}$.



$$\text{The required area} = \int_0^{\frac{\pi}{2}} \sin 2x \, dx$$
$$= \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}(\cos\pi - \cos 0)$$
$$= 1 \text{ unit}^2$$

6 Show that the area enclosed by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ is 2 units^2.

7 Find the area of the region bounded by $y = \cos x$, the x-axis, $x = 0$, and $x = \frac{\pi}{2}$.

8 Find the area enclosed by one arch of the curve $y = \cos 3x$ and the x-axis.

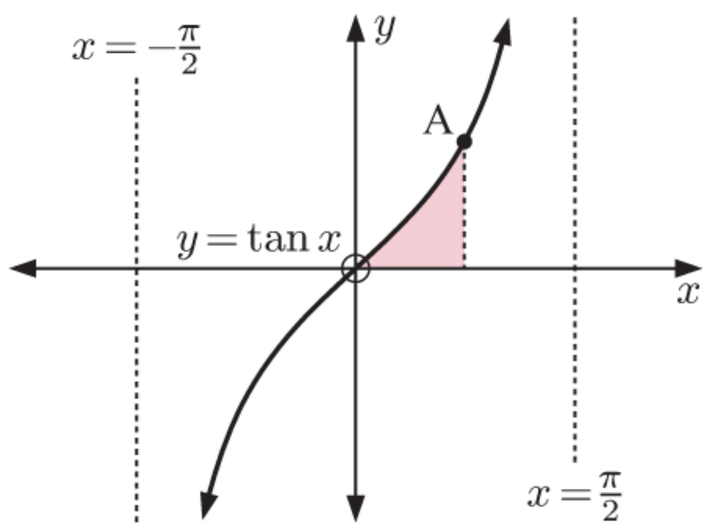
9 The graph alongside shows $y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

A is the point on the graph with y-coordinate 1.

a Find the x-coordinate of A.

b Find $\int \tan x \, dx$.

c Hence find the shaded area.



10 Find the area of the region bounded by:

a $y = \dfrac{1}{x^2}$, the x-axis, $x = 1$, and $x = 2$

b $y = e^x$, the axes, and $x = 1$

c $y = 2 - \dfrac{1}{\sqrt{x}}$, the x-axis, and $x = 4$

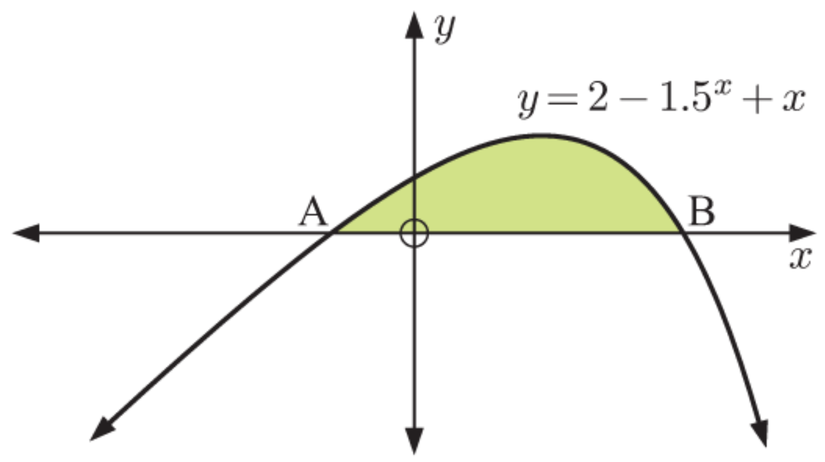
d the axes and $y = \sqrt{9 - x}$

e $y = e^x + e^{-x}$, the x-axis, $x = -1$, and $x = 1$.

11 Use technology to find, correct to 3 significant figures:

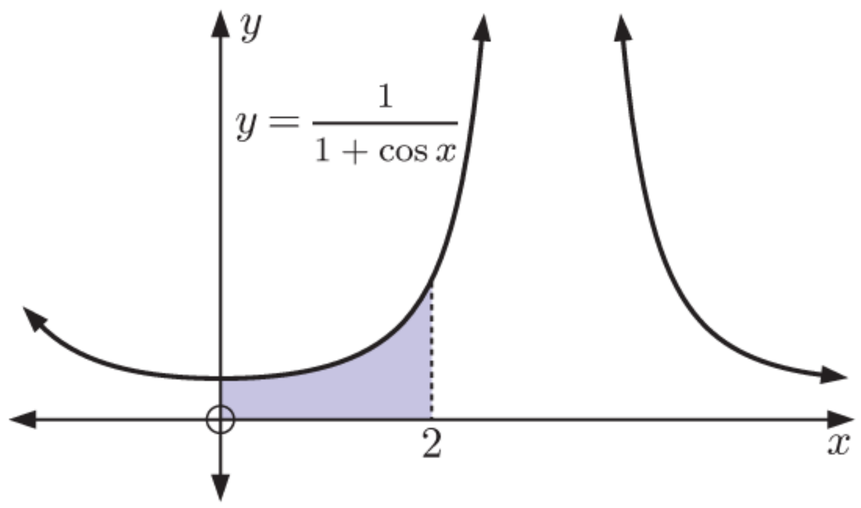
a the x-coordinates of A and B

b the shaded area.

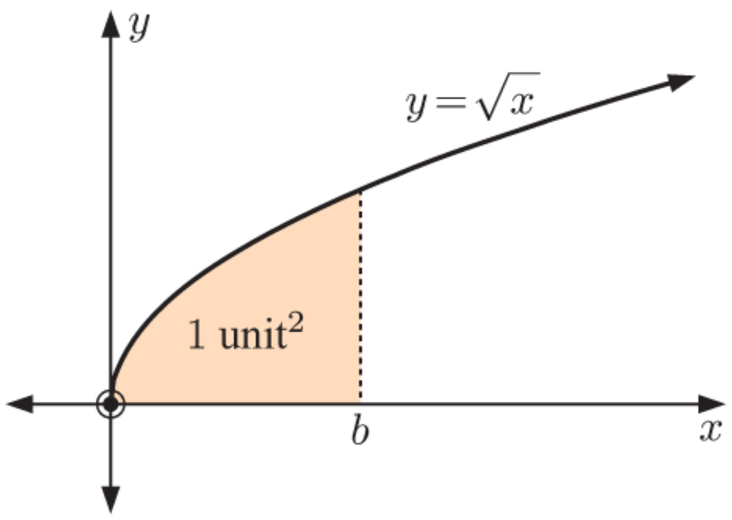


12 **a** Show that $\dfrac{d}{dx}\left(\dfrac{\sin x}{1+\cos x}\right) = \dfrac{1}{1+\cos x}$.

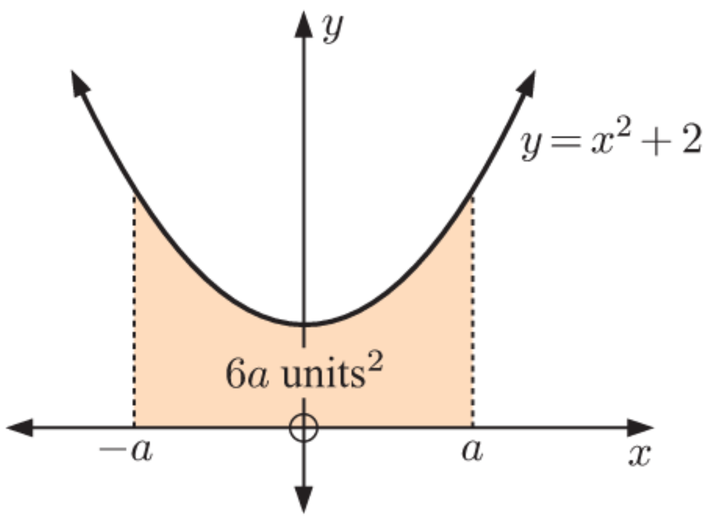
b Hence find the area of the region bounded by $y = \dfrac{1}{1+\cos x}$, the x-axis, $x = 0$, and $x = 2$.



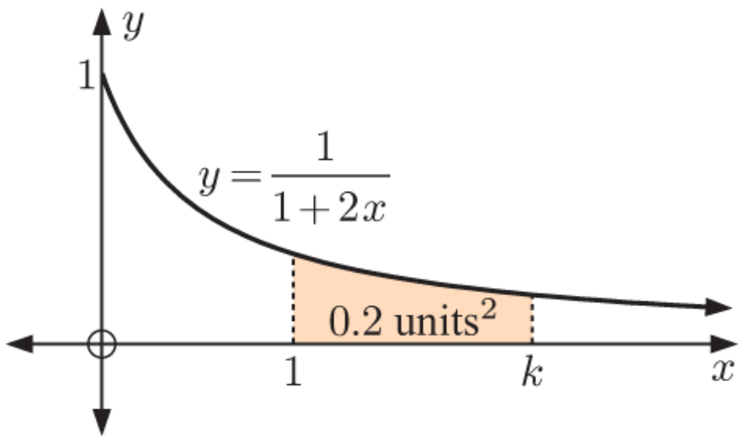
13 **a** Find b, correct to 4 decimal places.



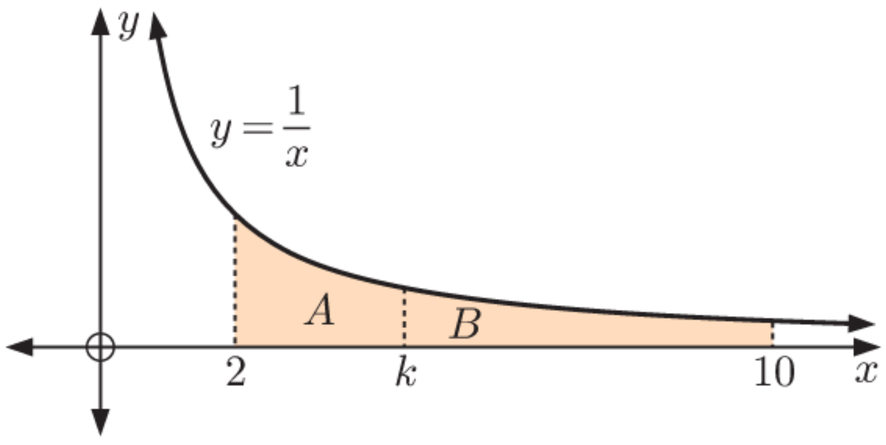
b Find the exact value of a.



c Find k, correct to 4 decimal places.



d Find k such that area A and area B are equal.



INVESTIGATION $\int_a^b f(x)\,dx$ AND AREAS

Can $\int_a^b f(x)\,dx$ always be interpreted as an area?

What to do:

1 **a** Find $\int_0^1 x^3\,dx$ and $\int_{-1}^1 x^3\,dx$.

b Using a graph, explain why the first integral in **a** gives an area, whereas the second integral does not.

c Find $\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.

d Show that $\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.

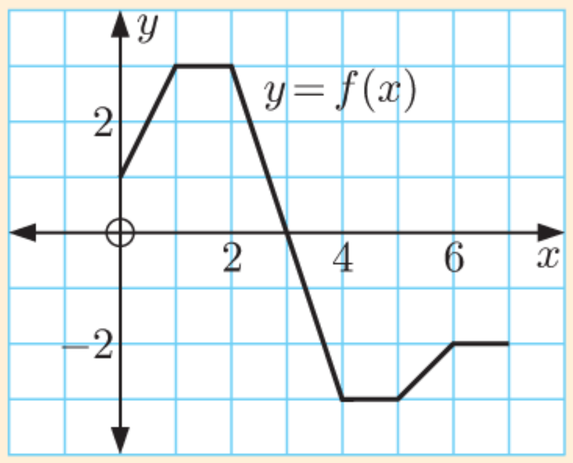
e Find $\int_0^{-1} x^3\,dx$ and interpret its meaning.

2 Suppose $f(x)$ is a function such that $f(x) \leqslant 0$ for all $a \leqslant x \leqslant b$. Write an expression for the area between the function and the x-axis for $a \leqslant x \leqslant b$.

3 Evaluate using area interpretation:

a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

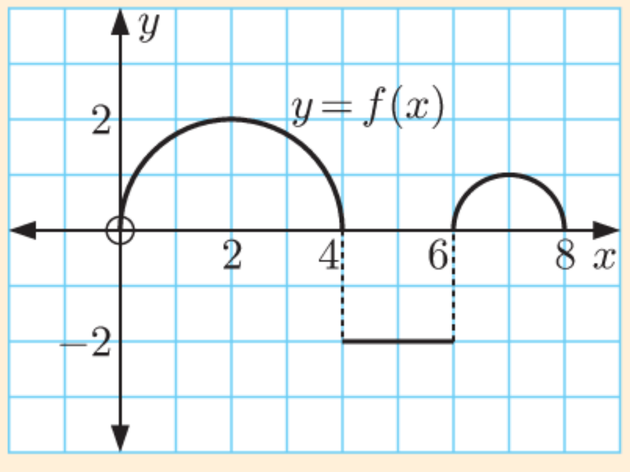
c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$



4 Evaluate using area interpretation:

a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

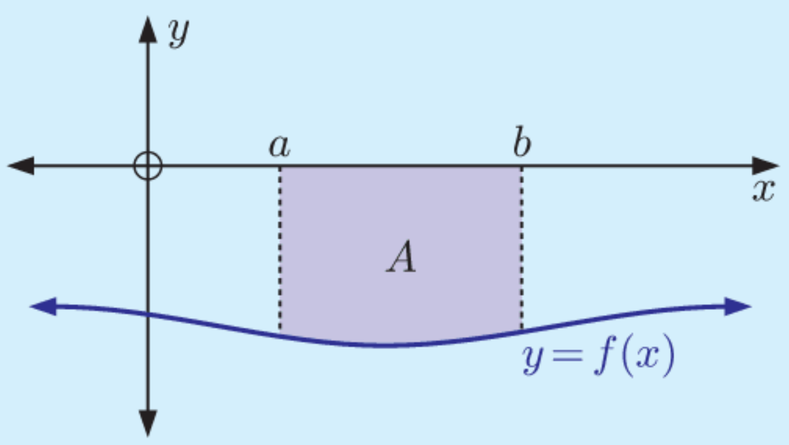
c $\int_6^8 f(x)\,dx$ **d** $\int_0^8 f(x)\,dx$



D THE AREA ABOVE A CURVE

If $f(x)$ is negative and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

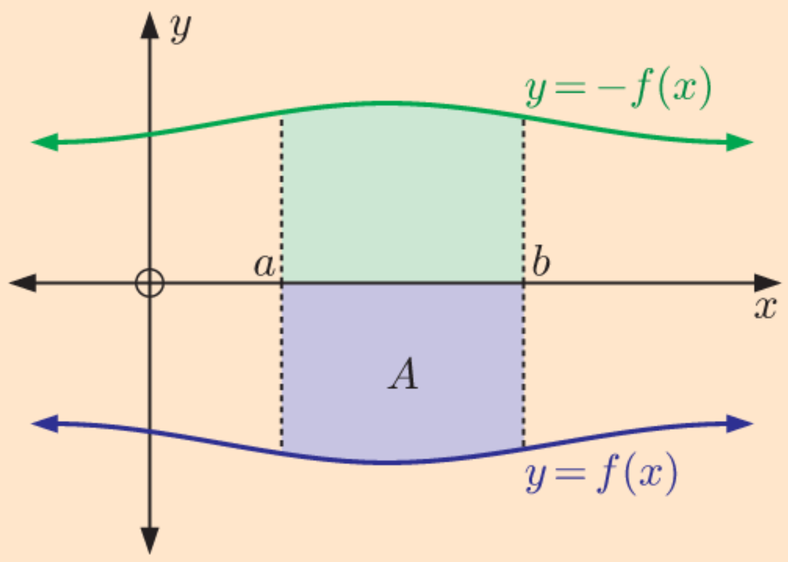
$$A = -\int_a^b f(x)\,dx \quad \text{or} \quad -\int_a^b y\,dx.$$



Proof:

The reflection of $y = f(x)$ in the x-axis is $y = -f(x)$.

By symmetry, $-f(x) \geqslant 0$ on the interval $a \leqslant x \leqslant b$, and the area bounded by $y = -f(x)$, the x-axis, $x = a$, and $x = b$ is also A.



$$\therefore \quad A = \int_a^b [-f(x)]\,dx$$

$$= -\int_a^b f(x)\,dx$$

Example 8 — Self Tutor

Find the area bounded by the x-axis and $y = x^2 - 2x$.

$y = x^2 - 2x$
$= x(x - 2)$

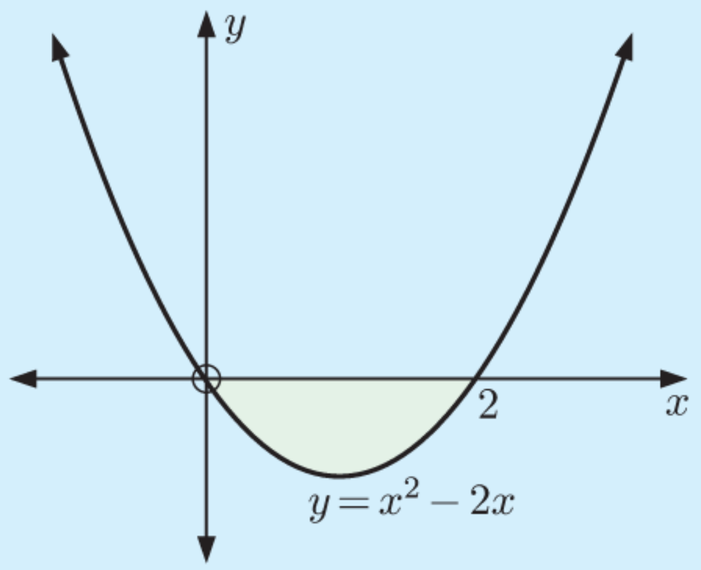
$\therefore$ the x-intercepts are 0 and 2.

$$\text{Area} = -\int_0^2 (x^2 - 2x)\,dx$$

$$= -\left[\frac{x^3}{3} - x^2\right]_0^2$$

$$= -\left[\left(\tfrac{8}{3} - 4\right) - (0)\right] = \tfrac{4}{3}$$

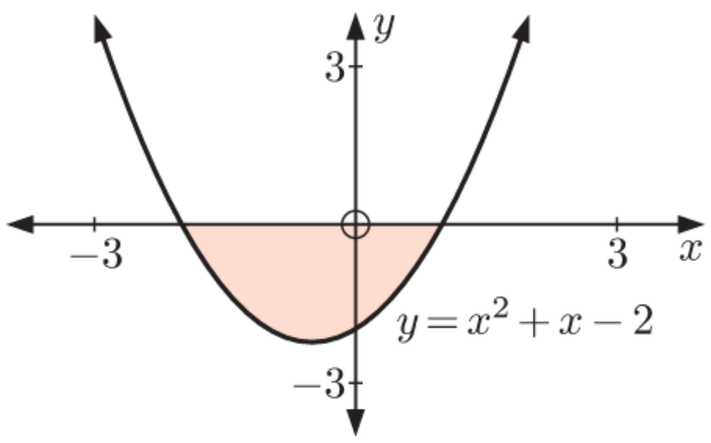
$\therefore$ the area is $\frac{4}{3}$ units^2.



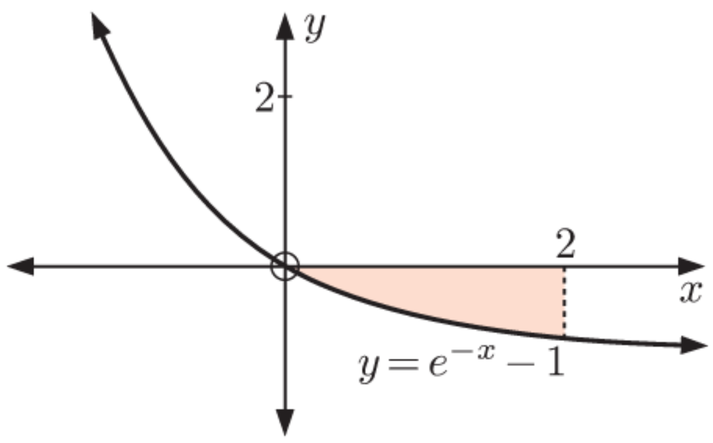
EXERCISE 23D

1 Find the shaded area:

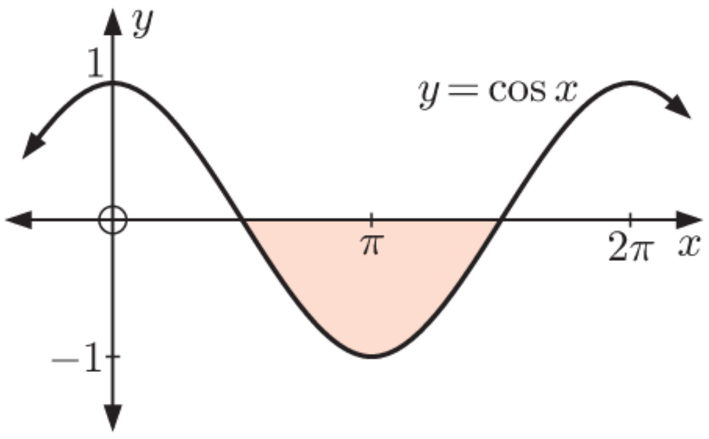
a



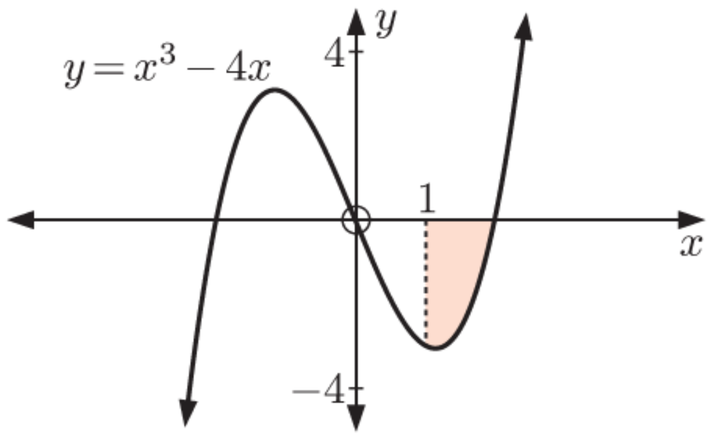
b



c



d

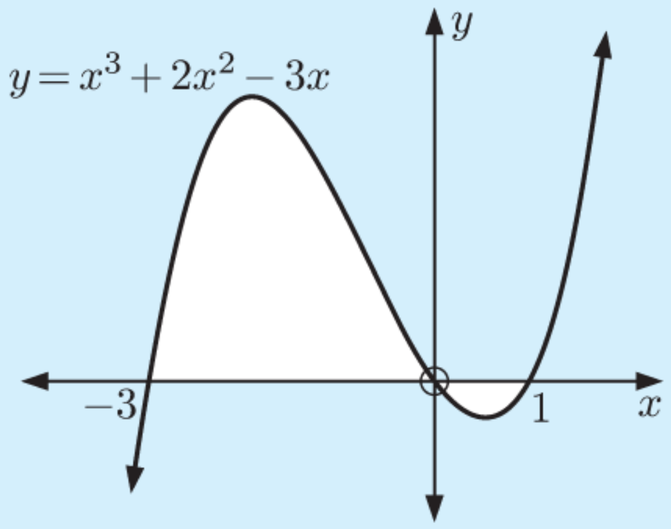


Example 9 **Self Tutor**

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 3x \\ &= x(x^2 + 2x - 3) \\ &= x(x-1)(x+3) \end{aligned}$$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 1, and -3.



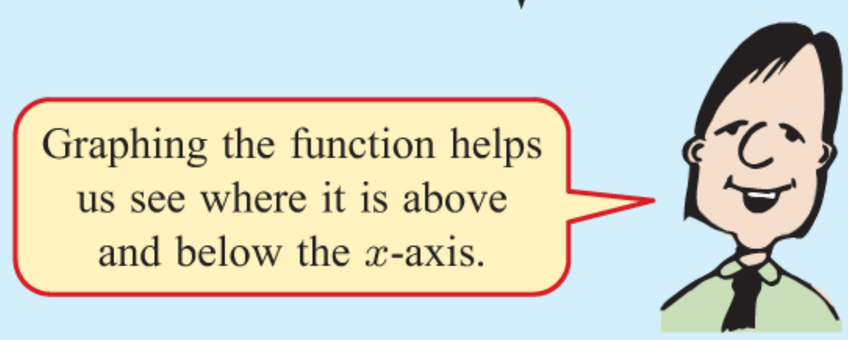
Total area

$$= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\,dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1}$$

$$= \left(0 - \left(-11\tfrac{1}{4}\right)\right) - \left(-\tfrac{7}{12} - 0\right)$$

$$= 11\tfrac{5}{6} \text{ units}^2$$

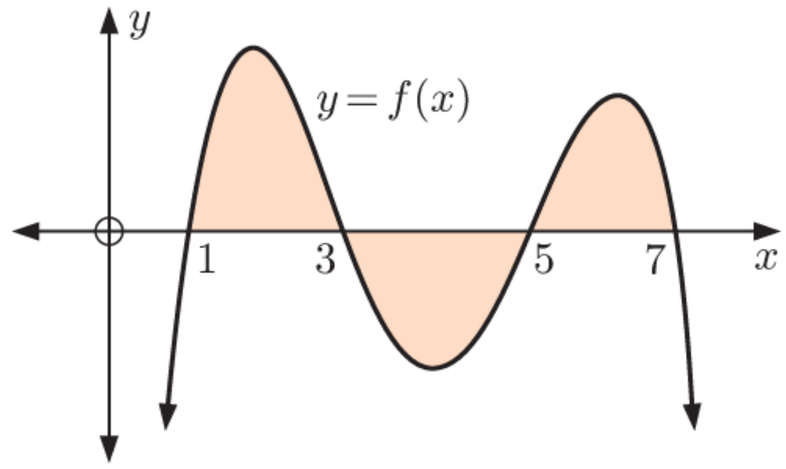


2 Find the total area enclosed by the function $y = f(x)$ and the x-axis for:

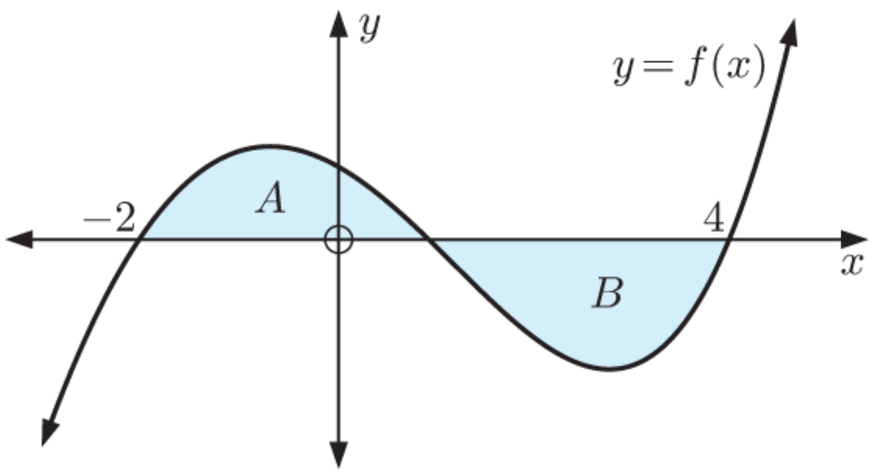
a $f(x) = x^3 - 9x$ **b** $f(x) = -x(x-2)(x-4)$.

3 **a** Explain why the total area shaded is *not* equal to $\int_1^7 f(x)\,dx$.

b Write an expression for the total shaded area in terms of integrals.



4



For the graph of $y = f(x)$ alongside, $\int_{-2}^{4} f(x)\,dx = -6$. Which region is larger, A or B? Explain your answer.

5 The area of the region bounded by $f(x) = -\dfrac{9}{x}$, the x-axis, $x = 3$, and $x = k$, is $9\ln 2$ units^2. Find the value of k.

6 **a** Sketch the graph of $y = 2\sin x + 1$ for $0 \leqslant x \leqslant 2\pi$.

b Find the area between the x-axis and the part of $y = 2\sin x + 1$ that is below the x-axis on $0 \leqslant x \leqslant 2\pi$.

7 Suppose $f(x) = \dfrac{1}{x^2 + 5x + 4}$. Use technology to:

a sketch the graph of $y = f(x)$

b find the area of the region bounded by $y = f(x)$, the x-axis, $x = -3$, and $x = -2$.

8 The cross-section of a roof gutter is defined by $f(x) = 0.02x^4 - 0.4x^2 - 2.5$, for $-5 \leqslant x \leqslant 5$ cm.

a Find the cross-sectional area of the gutter.

b The gutter is 20 m long. How much water can it hold in total?

ACTIVITY 1 CALCULATING AREAS USING TECHNOLOGY

We have seen that to find the total area between a function and the x-axis, we may need to add several integrals together.

This has required us to find every x-intercept, and take care as to whether we are adding or subtracting each integral.

If we are using technology, we can save some work by using an absolute value function.

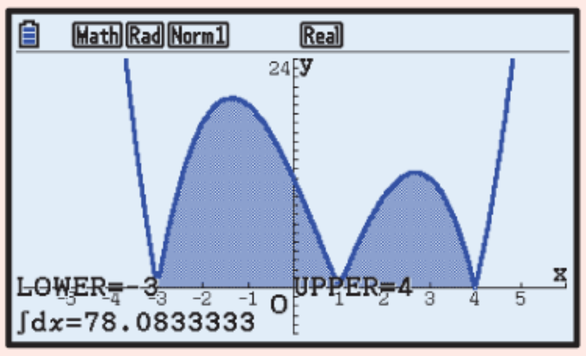
The area between $f(x)$ and the x-axis on the interval $a \leqslant x \leqslant b$ is $A = \int_a^b |f(x)|\,dx$.

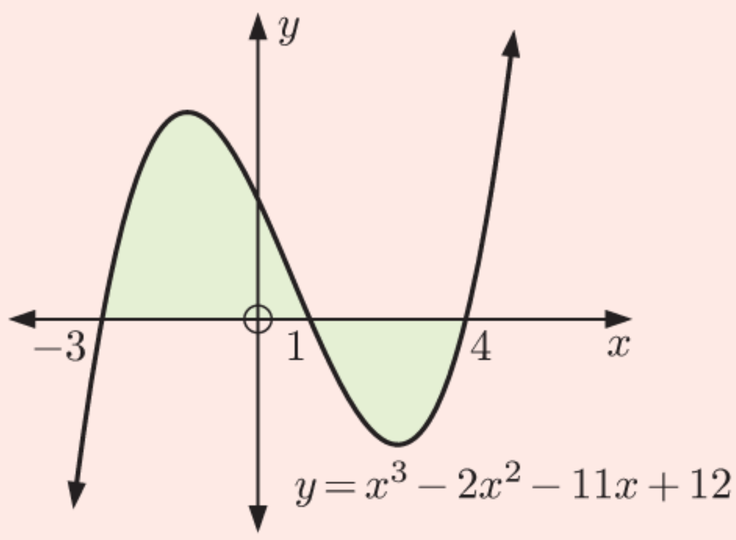
To draw the graph of $y = |f(x)|$, the parts of $y = f(x)$ which lie below the x-axis are reflected in the x-axis. This ensures that all of the components of the area lie above the x-axis, and are correctly added together.

For example, the shaded area alongside can be calculated as

$$\int_{-3}^{4} |x^3 - 2x^2 - 11x + 12|\,dx$$

≈ 78.1 units^2





What to do:

1 Use technology to find the area between:

a $y = x^2 - x - 2$ and the x-axis on the interval $0 \leqslant x \leqslant 3$

b $y = x^3 - 7x + 5$ and the x-axis on the interval $-2 \leqslant x \leqslant 4$

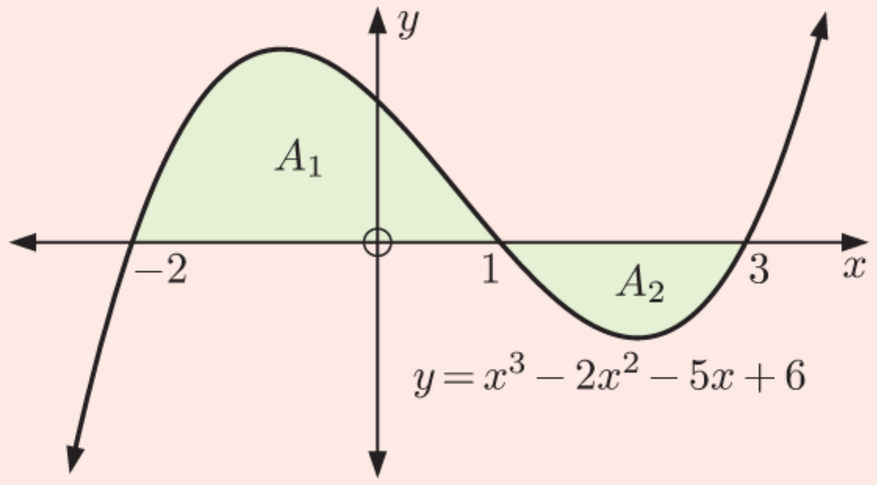
c $y = \sin x$ and the x-axis on the interval $0 \leqslant x \leqslant 5$.

2 **a** Use technology to find:

i $\int_{-2}^{3} (x^3 - 2x^2 - 5x + 6)\,dx$

ii $\int_{-2}^{3} |x^3 - 2x^2 - 5x + 6|\,dx$

b Interpret your answers in terms of the areas A_1 and A_2 alongside.

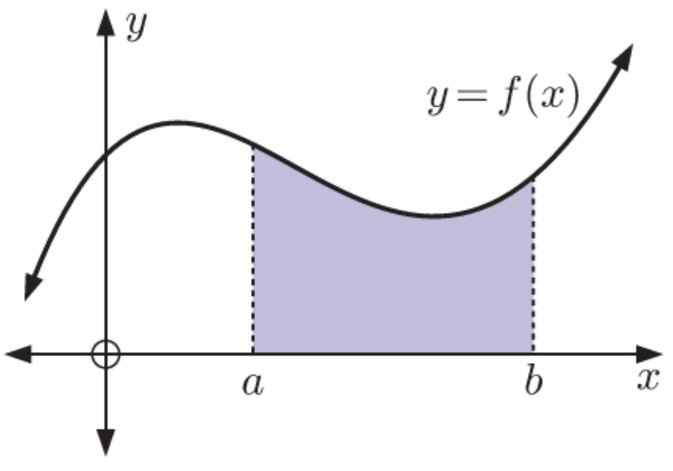


3 Use technology to check your answers to question **2** of **Exercise 23D**.

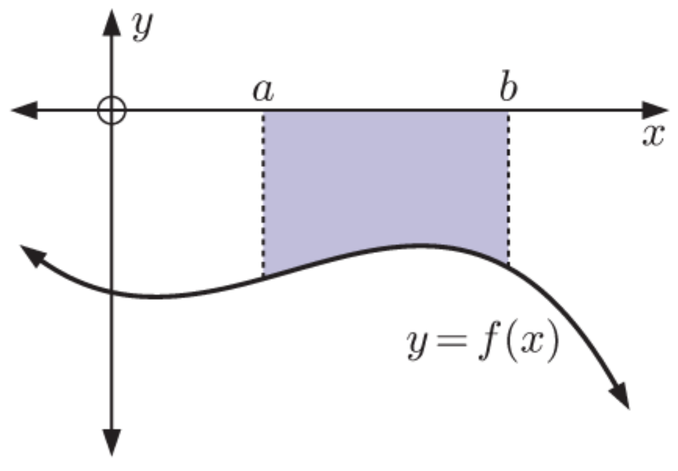
E THE AREA BETWEEN A CURVE AND THE y-AXIS

We have seen that the area between the curve $y = f(x)$ and the x-axis on the interval $a \leqslant x \leqslant b$ is:

- For $y = f(x) > 0$, area $= \int_a^b f(x)\,dx$.



- For $y = f(x) < 0$, area $= -\int_a^b f(x)\,dx$.

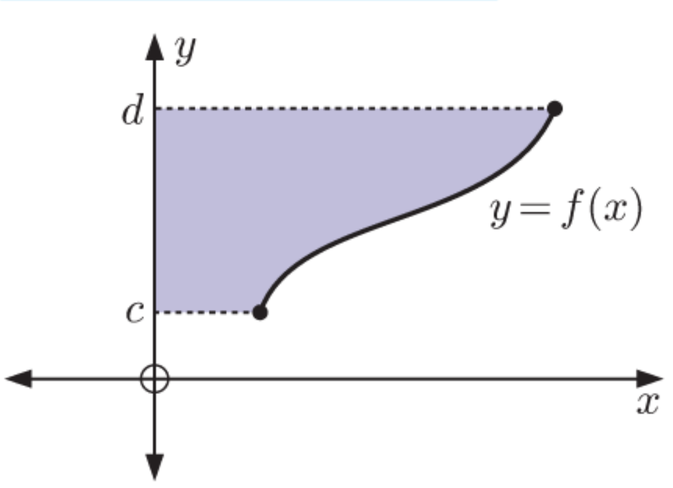


If f is invertible so that $x = f^{-1}(y)$, then we can write analogous integrals for the area between the curve and the y-axis.

The area between the curve $y = f(x)$ and the y-axis on the interval $c \leqslant y \leqslant d$ is:

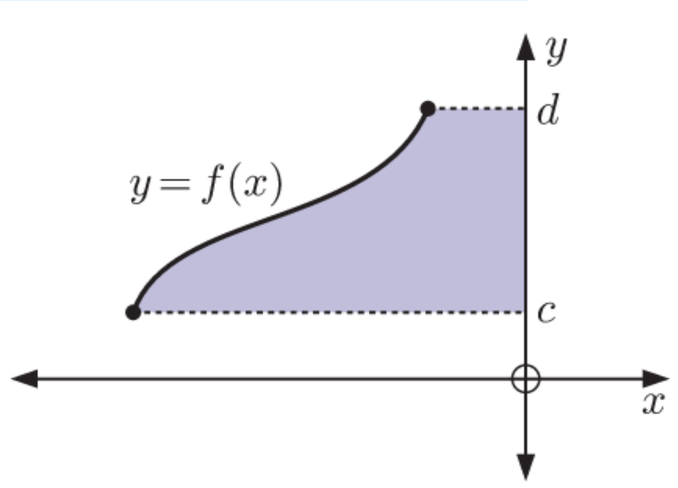
- For $x = f^{-1}(y) > 0$,

$$\text{area} = \int_c^d f^{-1}(y)\,dy.$$



- For $x = f^{-1}(y) < 0$,

$$\text{area} = -\int_c^d f^{-1}(y)\,dy.$$



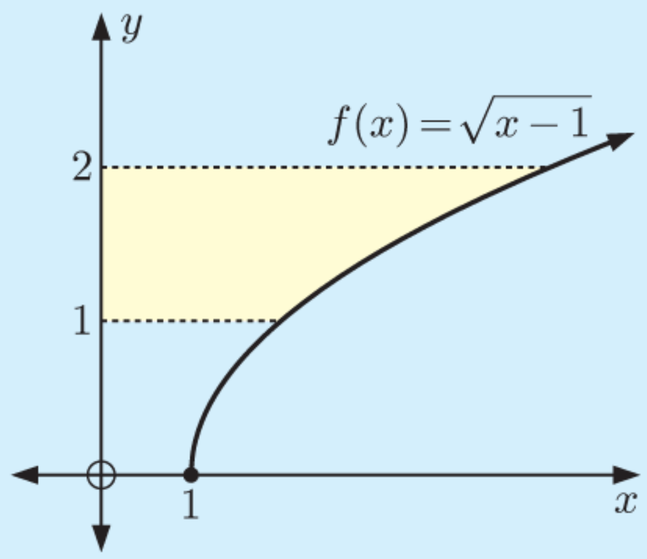
Example 10 — Self Tutor

Find the area between the curve:

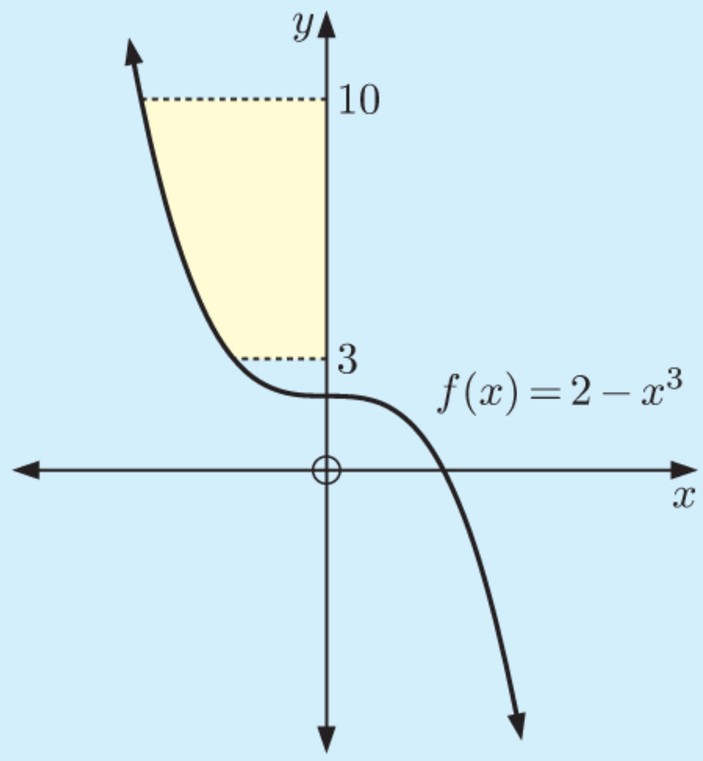
a $f(x) = \sqrt{x-1}$ and the y-axis between $y = 1$ and $y = 2$

b $f(x) = 2 - x^3$ and the y-axis between $y = 3$ and $y = 10$.

a
$$y = \sqrt{x-1}$$
$$\therefore\ y^2 = x - 1$$
$$\therefore\ x = y^2 + 1$$
$$\therefore\ f^{-1}(y) = y^2 + 1$$
$$\therefore\ \text{area} = \int_1^2 (y^2 + 1)\,dy$$
$$= \left[\frac{y^3}{3} + y\right]_1^2$$
$$= \tfrac{8}{3} + 2 - \tfrac{1}{3} - 1$$
$$= \tfrac{10}{3} \text{ units}^2$$



b

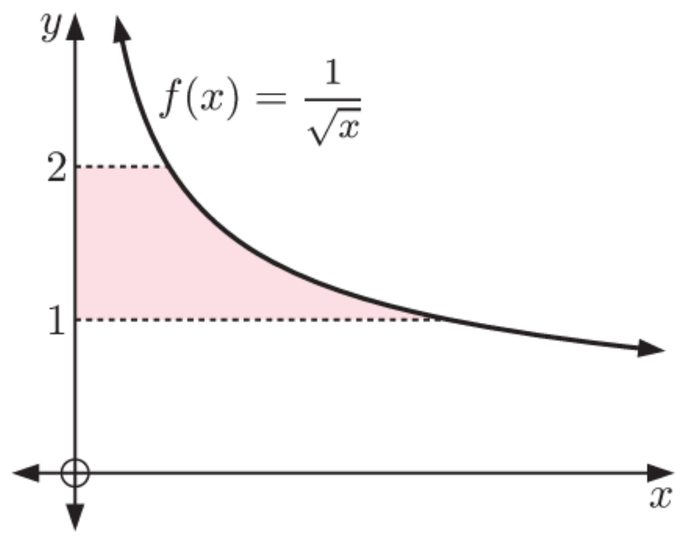


$$y = 2 - x^3$$
$$\therefore \quad x^3 = 2 - y$$
$$\therefore \quad x = \sqrt[3]{2-y}$$
$$\therefore \quad f^{-1}(y) = \sqrt[3]{2-y}$$
$$\therefore \quad \text{area} = -\int_3^{10} (2-y)^{\frac{1}{3}}\, dy$$
$$= -\left[-\tfrac{3}{4}(2-y)^{\frac{4}{3}}\right]_3^{10}$$
$$= -\left(-\tfrac{3}{4}(16) + \tfrac{3}{4}(1)\right)$$
$$= \tfrac{45}{4} \text{ units}^2$$

EXERCISE 23E

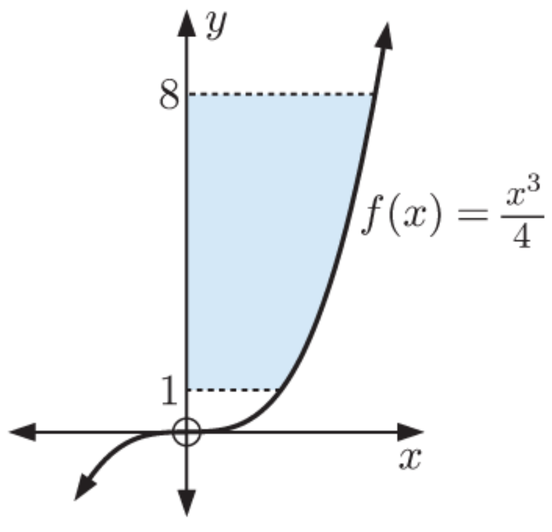
1 The graph of $f(x) = \dfrac{1}{\sqrt{x}}$ is shown alongside.

a Find $f^{-1}(y)$. **b** Hence find the shaded area.

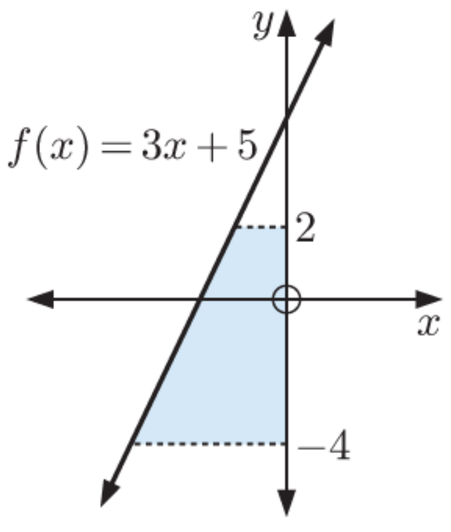


2 Find the shaded area:

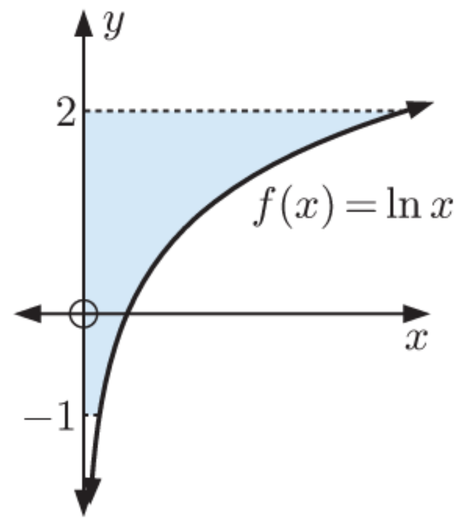
a



b



c



3 Find the area of the region bounded by:

a $x = y^2 + 1$, the y-axis, and the lines $y = 1$ and $y = 2$

b $x = \sqrt{y+5}$, the y-axis, and the lines $y = -1$ and $y = 4$.

4 Find the exact area of the region bounded by:

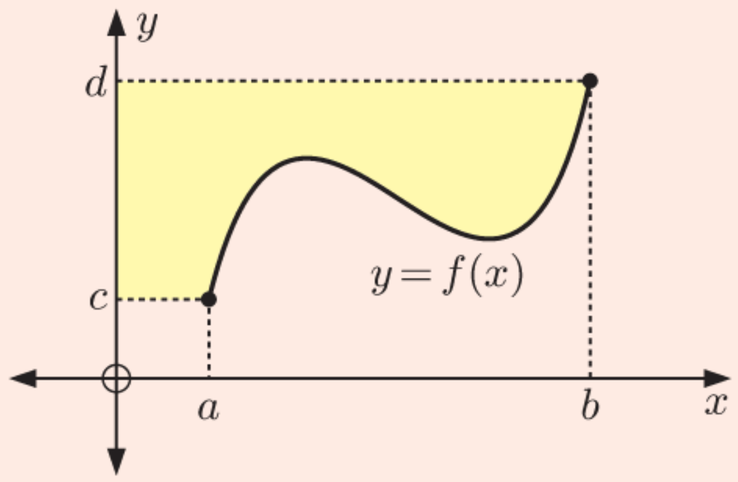
a $y = \sqrt{x}$, the y-axis, and the lines $y = 2$ and $y = 3$

b $y = \dfrac{2}{x}$, the y-axis, and the lines $y = -1$ and $y = -3$

c $y = \dfrac{x+1}{2-x}$, the y-axis, and the lines $y = 1$ and $y = 4$.

DISCUSSION

- Why did we require f to be invertible?
- How could you find the shaded area for this function?

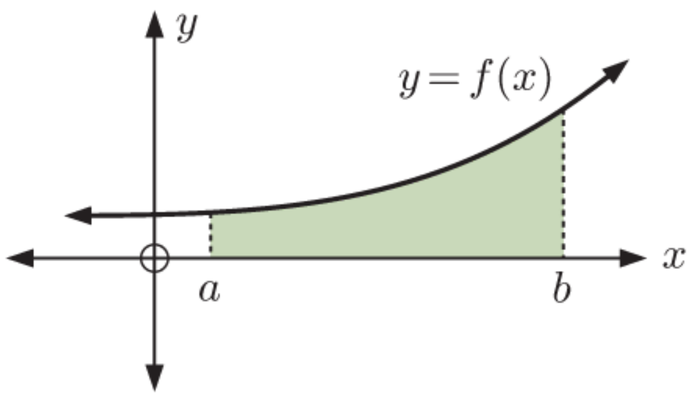


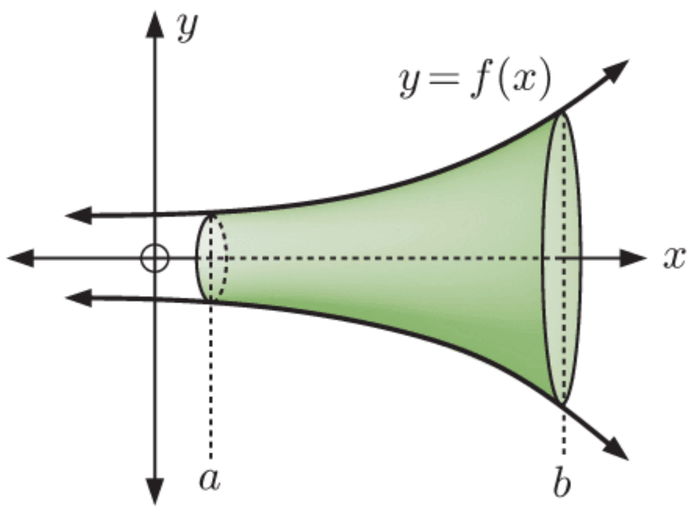
F SOLIDS OF REVOLUTION

SOLIDS OF REVOLUTION ABOUT THE x-AXIS

Consider the curve $y = f(x)$ for $a \leqslant x \leqslant b$.

If the shaded area below is revolved about the x-axis through $360°$ or 2π, a 3-dimensional solid will be formed. This solid is called a **solid of revolution**.

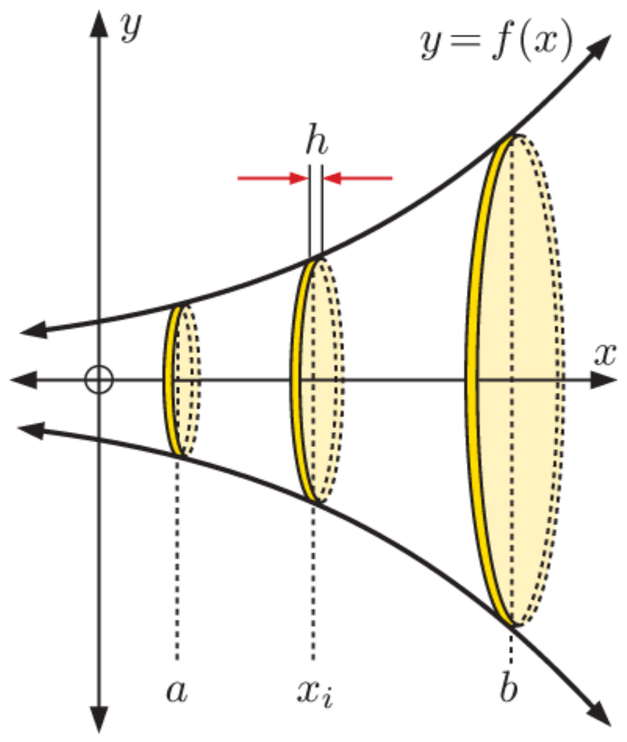






To find the volume of the solid, we can think of it as being made up of n thin cylindrical discs, where the radius of each disc is the value of the function at the left end $x = x_i$, and the height is h. The volume of the ith disc is $\pi[f(x_i)]^2 h$, so the total volume of the discs is

$$\sum_i \pi[f(x_i)]^2 h.$$



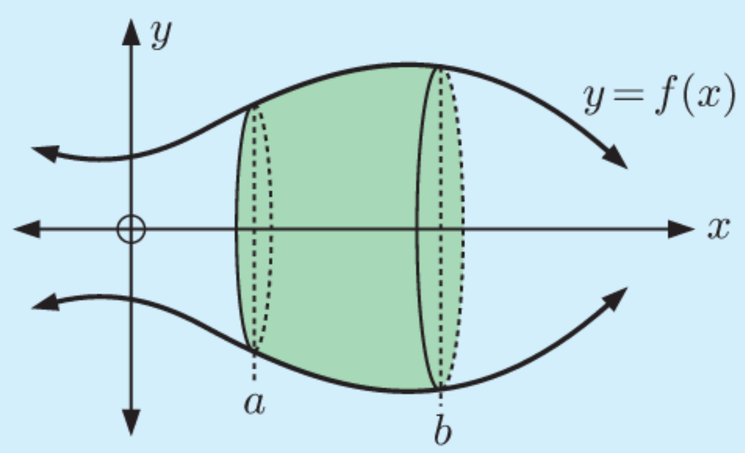
To find the volume *exactly*, we need infinitely many discs. We therefore take the limit as $h \to 0$.

$$\begin{aligned}\text{Volume of revolution} &= \lim_{h\to 0} \sum_i \pi[f(x_i)]^2 h \\ &= \int_a^b \pi[f(x)]^2 \, dx \\ &= \pi \int_a^b [f(x)]^2 \, dx\end{aligned}$$

When the region enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is revolved through 2π about the x-axis to generate a solid, the

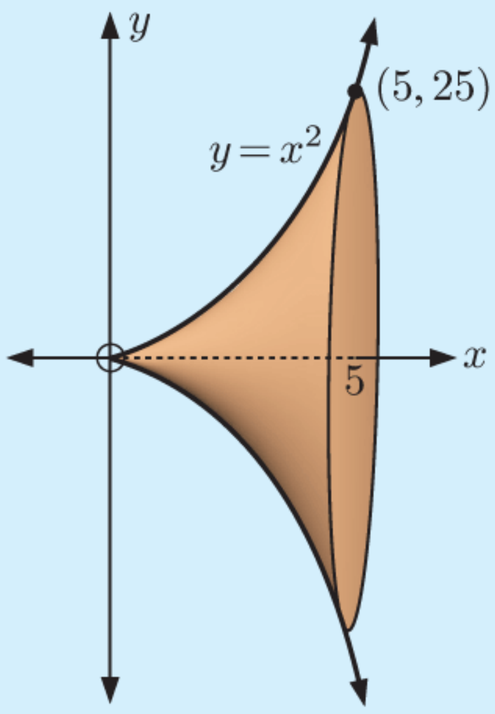
$$\textbf{Volume of revolution} = \pi \int_a^b [f(x)]^2 \, dx$$

$$\text{or} \quad \pi \int_a^b y^2 \, dx.$$



Example 11 **Self Tutor**

Find the volume of the solid formed when the graph of the function $y = x^2$ for $0 \leqslant x \leqslant 5$ is revolved through 2π about the x-axis.



$$\text{Volume of revolution} = \pi \int_a^b y^2 \, dx$$
$$= \pi \int_0^5 (x^2)^2 \, dx$$
$$= \pi \int_0^5 x^4 \, dx$$
$$= \pi \left[\frac{x^5}{5} \right]_0^5$$
$$= \pi(625 - 0)$$
$$= 625\pi \text{ units}^3$$



GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 23F.1

1 Find the volume of the solid formed when the following are revolved through 2π about the x-axis:

a $y = 2x$ for $0 \leqslant x \leqslant 3$

b $y = \sqrt{x}$ for $0 \leqslant x \leqslant 4$

c $y = x^3$ for $1 \leqslant x \leqslant 2$

d $y = x^{\frac{3}{2}}$ for $1 \leqslant x \leqslant 4$

e $y = x^2$ for $2 \leqslant x \leqslant 4$

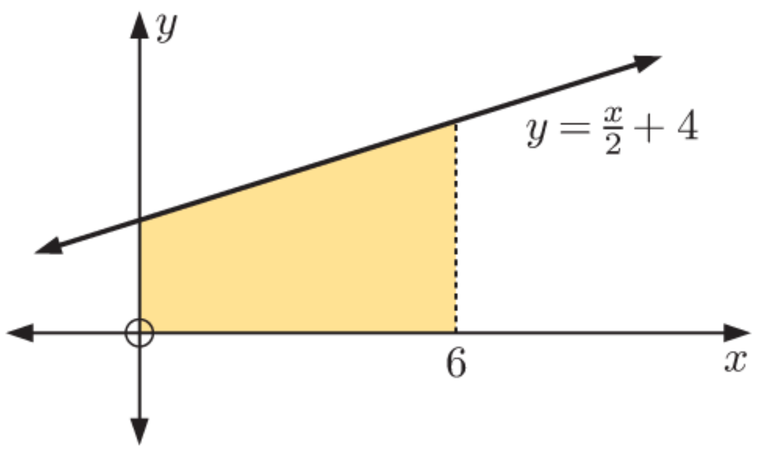
f $y = \sqrt{25 - x^2}$ for $0 \leqslant x \leqslant 5$

g $y = \dfrac{1}{x-1}$ for $2 \leqslant x \leqslant 3$

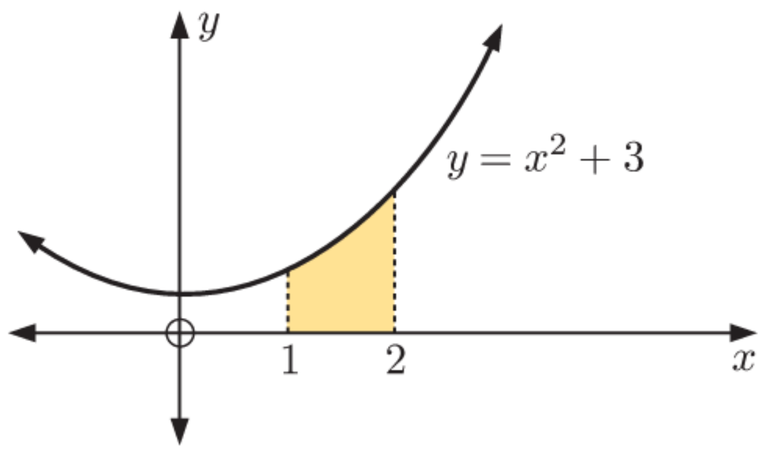
h $y = x + \dfrac{1}{x}$ for $1 \leqslant x \leqslant 3$

2 Find the volume of revolution when the shaded region is revolved through 2π about the x-axis.

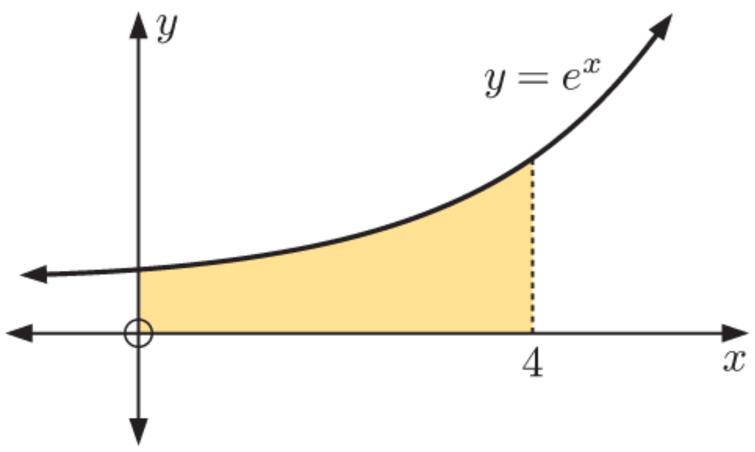
a



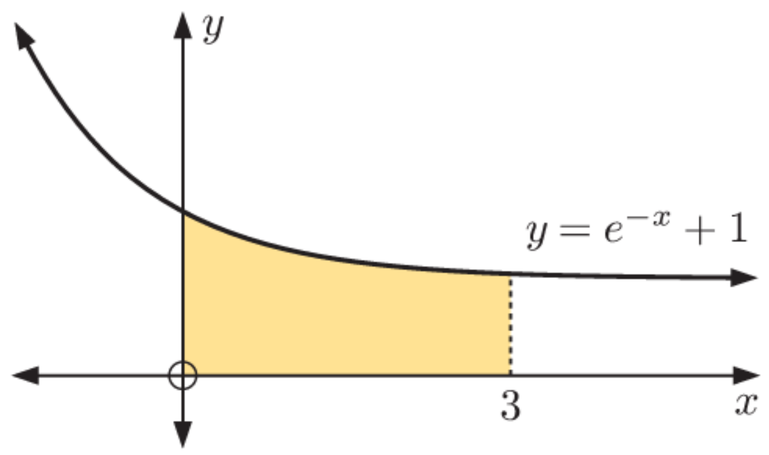
b



c



d

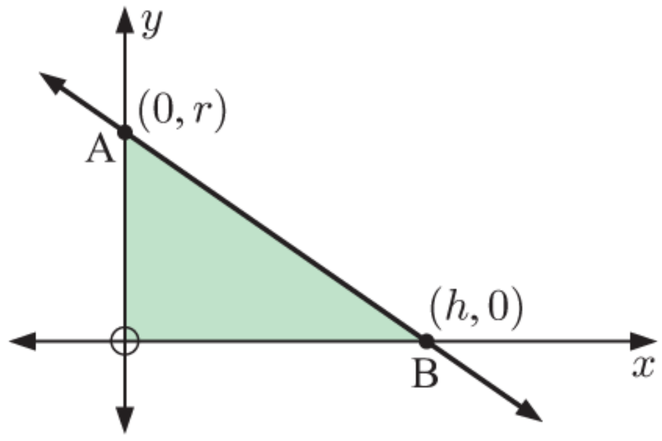


3 **a** Describe the solid of revolution when the shaded region is revolved about the x-axis.

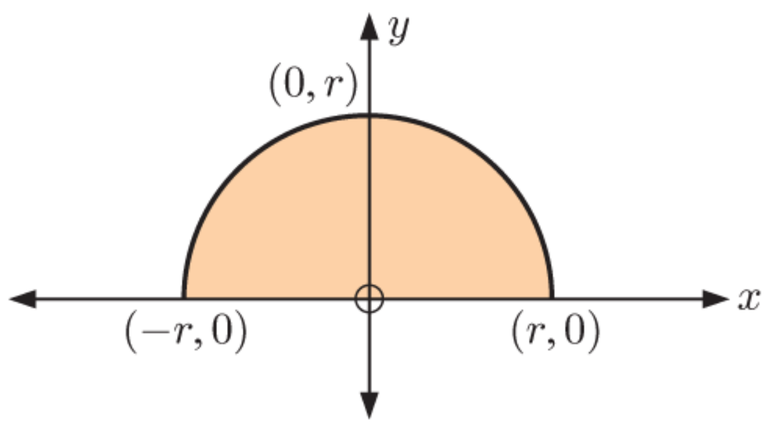
b Find the equation of the line (AB) in the form $y = mx + c$.

c Find a formula for the volume of the solid using

$$V = \pi \int_a^b y^2 \, dx.$$



4



A circle with centre $(0,\ 0)$ and radius r units has equation $x^2 + y^2 = r^2$.

a If the shaded region is revolved about the x-axis, what solid is formed?

b Show that the volume of revolution is $\frac{4}{3}\pi r^3$.

5 Find the volume of revolution when the following regions are revolved through 2π about the x-axis:

a $y = \sqrt{\sin x}$ for $0 \leqslant x \leqslant \pi$

b $y = \dfrac{1}{\cos x}$ for $0 \leqslant x \leqslant \frac{\pi}{3}$

6 Use technology to find, correct to 3 significant figures, the volume of the solid of revolution formed when these functions are rotated through 2π about the x-axis:

a $y = \ln x$ for $1 \leqslant x \leqslant 5$

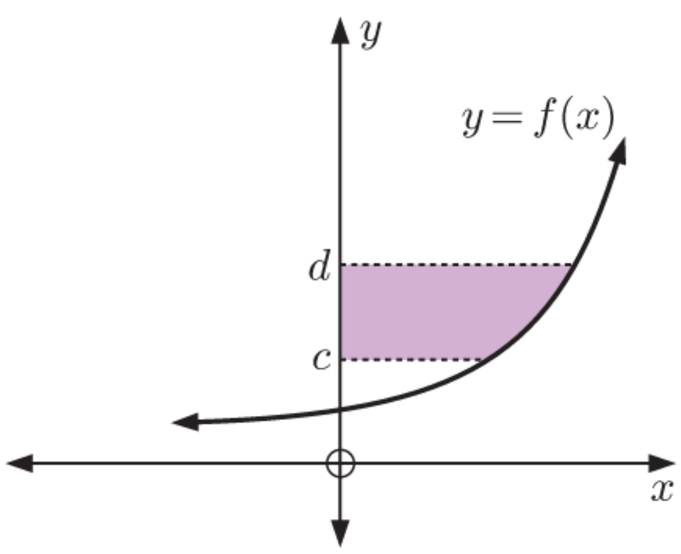
b $y = 4 \sin 2x$ for $0 \leqslant x \leqslant \frac{\pi}{4}$

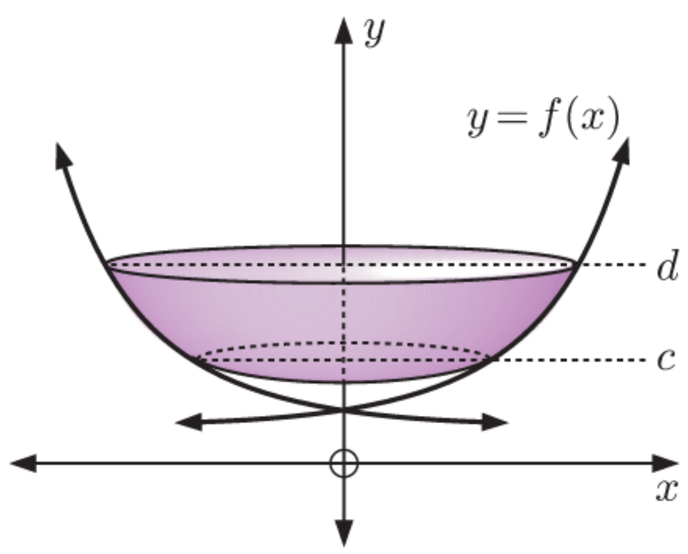
c $y = \dfrac{x^3}{x^2 + 1}$ for $1 \leqslant x \leqslant 3$

d $y = e^{\sin x}$ for $0 \leqslant x \leqslant 2$.

SOLIDS OF REVOLUTION ABOUT THE y-AXIS

If the function $f(x)$ is invertible, we can also define a solid of revolution about the y-axis.





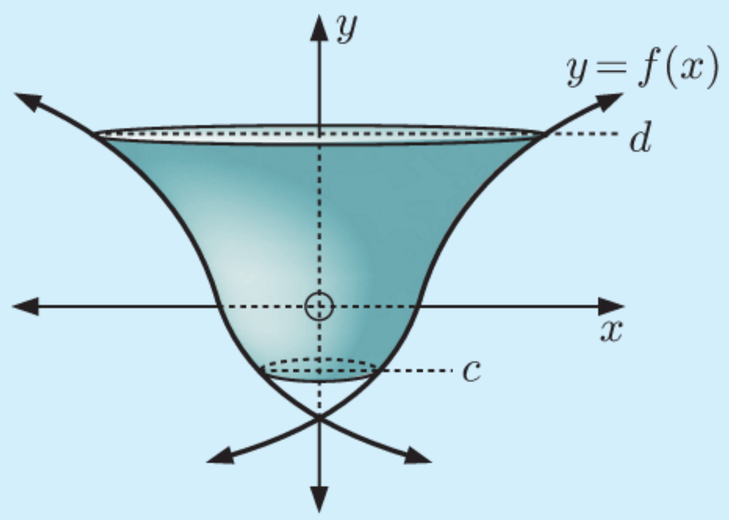
DEMO



When the region enclosed by the invertible function $y = f(x)$, the y-axis, and the horizontal lines $y = c$ and $y = d$ is revolved through 2π about the y-axis to generate a solid, the

$$\textbf{Volume of revolution} = \pi \int_c^d \left[f^{-1}(y)\right]^2 dy$$

$$\text{or} \quad \pi \int_c^d x^2 \, dy.$$



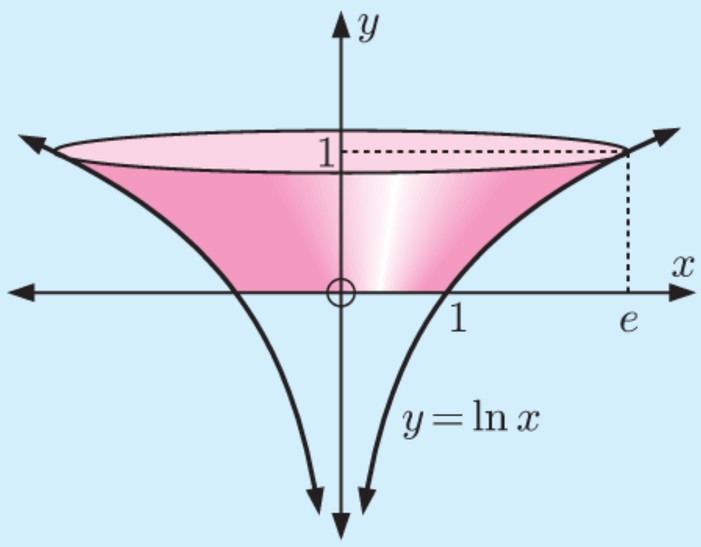
Example 12 **Self Tutor**

The graph of $y = \ln x$, $1 \leqslant x \leqslant e$ is revolved through 2π about the y-axis. Find the volume of revolution.

When $x = 1$, $y = 0$

When $x = e$, $y = 1$

$\therefore$ we rotate the function for $0 \leqslant y \leqslant 1$.



Volume of revolution

$$= \pi \int_0^1 x^2 \, dy$$

$$= \pi \int_0^1 (e^y)^2 \, dy$$

$$= \pi \int_0^1 e^{2y} \, dy$$

$$= \pi \left[\tfrac{1}{2}e^{2y}\right]_0^1$$

$$= \pi\left(\tfrac{1}{2}e^2 - \tfrac{1}{2}e^0\right)$$

$$= \tfrac{\pi}{2}(e^2 - 1) \text{ units}^3$$

EXERCISE 23F.2

1 Find the volume of the solid formed when the following are revolved through 2π about the y-axis:

a $y = x^2$, $x \geqslant 0$, between $y = 0$ and $y = 4$

b $y = \sqrt{x}$ between $y = 1$ and $y = 4$

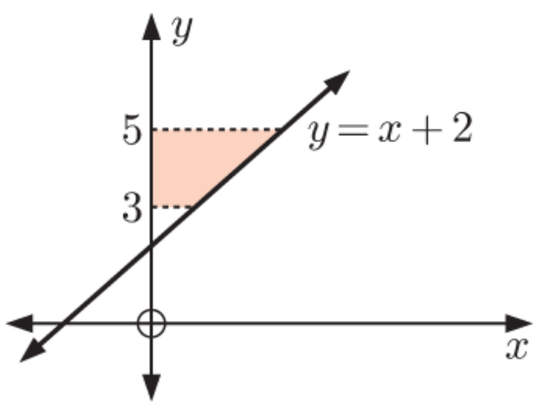
c $y = \ln x$ between $y = 0$ and $y = 2$

d $y = \sqrt{x - 2}$ between $x = 2$ and $x = 11$

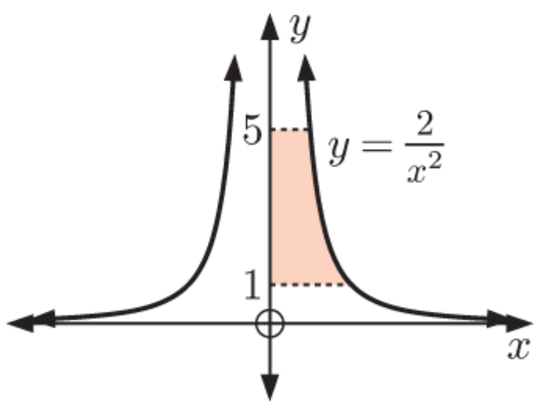
e $y = (x - 1)^3$ between $x = 1$ and $x = 3$.

2 Find the volume of revolution when the shaded region is revolved through 2π about the y-axis.

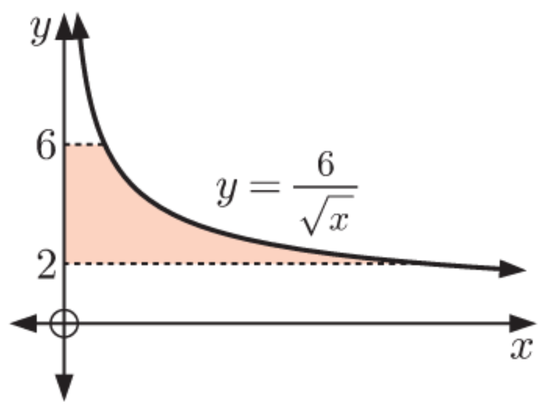
a



b



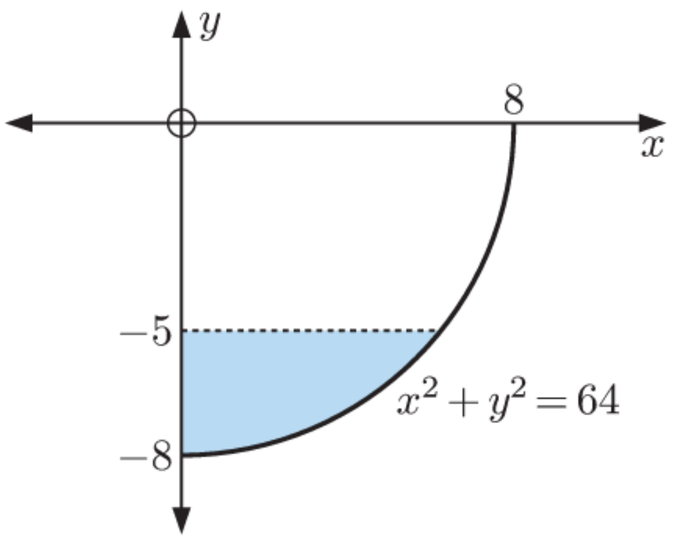
c



3 Find exactly the volume of the solid of revolution formed by revolving the relation $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, $x \geqslant 0$ through $360°$ about the y-axis.

4 A wooden bowl is made in the shape of a *paraboloid* by revolving the curve $y = \frac{1}{4}x^2$, $x \geqslant 0$, between $y = 0$ and $y = 4$ through 2π about the y-axis. Find the capacity of the bowl.

5

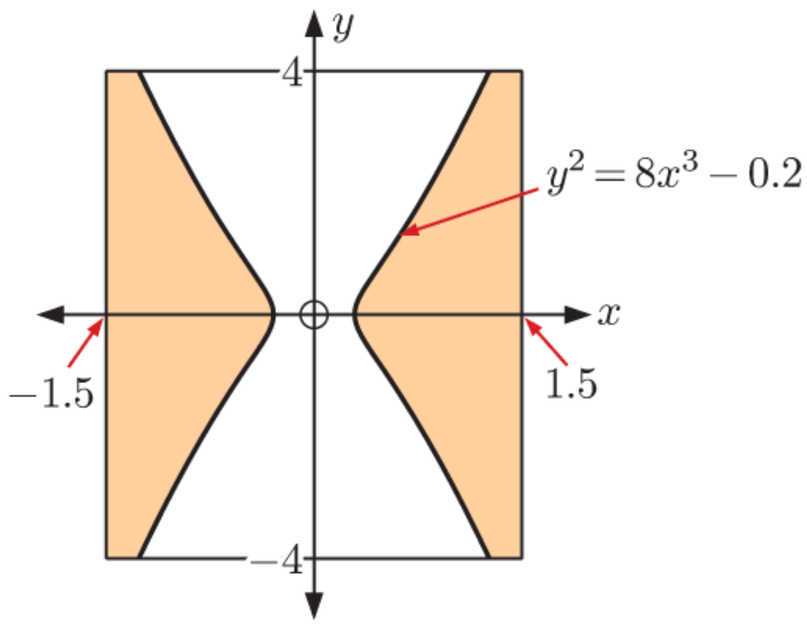


The shaded region is rotated through 360° about the y-axis.

a Find the volume of revolution.

b A hemispherical bowl of radius 8 cm contains water to a depth of 3 cm.
Find the volume of water in the bowl.

6 The outer casing of an hourglass is a cylinder with diameter 3 cm and height 8 cm. The inner casing which contains the sand is formed by rotating the curve $y^2 = 8x^3 - 0.2$, $-4 \leqslant y \leqslant 4$ about the y-axis.
Use technology to find the volume of air between the inner and outer casings.



G PROBLEM SOLVING BY INTEGRATION

When we studied differential calculus, we saw how to find the rate of change of a function by differentiation.

Since integration is the reverse process of differentiation, integration can be used to measure the change in a quantity from its rate of change.

When we introduced integration, we separated the area under a curve into rectangles. The area of each rectangle is the rate of change of the quantity times the subinterval width, which tells us the change in the quantity corresponding to that subinterval.

So, the area under a rate function for a particular interval tells us the overall change in the quantity over that interval.

Example 13 Self Tutor

The rate of power consumption by the city of Bristol can be modelled by the function

$$E(t) = 0.3 \sin\left(\frac{(t-10)\pi}{12}\right) + 0.1 \cos\left(\frac{(t-6)\pi}{6}\right) + 0.775 \text{ GW}$$

where t is the number of hours after midnight each day, $0 \leqslant t \leqslant 24$.

Find the following quantities and explain what they represent:

a $\int_0^{12} E(t)\,dt$ **b** $\int_{12}^{24} E(t)\,dt$ **c** $\int_0^{24} E(t)\,dt$



lpgphotos/Shutterstock.com

$$E(t) = 0.3\sin\left(\frac{(t-10)\pi}{12}\right) + 0.1\cos\left(\frac{(t-6)\pi}{6}\right) + 0.775 \text{ GW}$$

$$\therefore \int E(t)\,dt = 0.3\left(-\cos\left(\frac{(t-10)\pi}{12}\right)\right)\left(\frac{12}{\pi}\right) + 0.1\sin\left(\frac{(t-6)\pi}{6}\right)\left(\frac{6}{\pi}\right) + 0.775t + c$$

$$= -\frac{3.6}{\pi}\cos\left(\frac{(t-10)\pi}{12}\right) + \frac{0.6}{\pi}\sin\left(\frac{(t-6)\pi}{6}\right) + 0.775t + c$$

a $\int_0^{12} E(t)\,dt = \left(-\frac{3.6}{\pi}\cos\frac{\pi}{6} + \cancel{\frac{0.6}{\pi}\sin\pi} + 9.3\right) - \left(-\frac{3.6}{\pi}\cos\left(-\frac{5\pi}{6}\right) + \cancel{\frac{0.6}{\pi}\sin(-\pi)} + \cancel{0}\right)$

≈ 7.315 GWh

The morning power consumption of Bristol is about 7.32 GWh.

b $\int_{12}^{24} E(t)\,dt = \left(-\frac{3.6}{\pi}\cos\frac{7\pi}{6} + \cancel{\frac{0.6}{\pi}\sin 3\pi} + 18.6\right) - \left(-\frac{3.6}{\pi}\cos\frac{\pi}{6} + \cancel{\frac{0.6}{\pi}\sin\pi} + 9.3\right)$

≈ 11.285 GWh

The afternoon power consumption of Bristol is about 11.29 GWh.

c $\int_0^{24} E(t)\,dt = \int_0^{12} E(t)\,dt + \int_{12}^{24} E(t)\,dt \approx 18.6$ GWh

The total daily power consumption of Bristol is about 18.6 GWh.

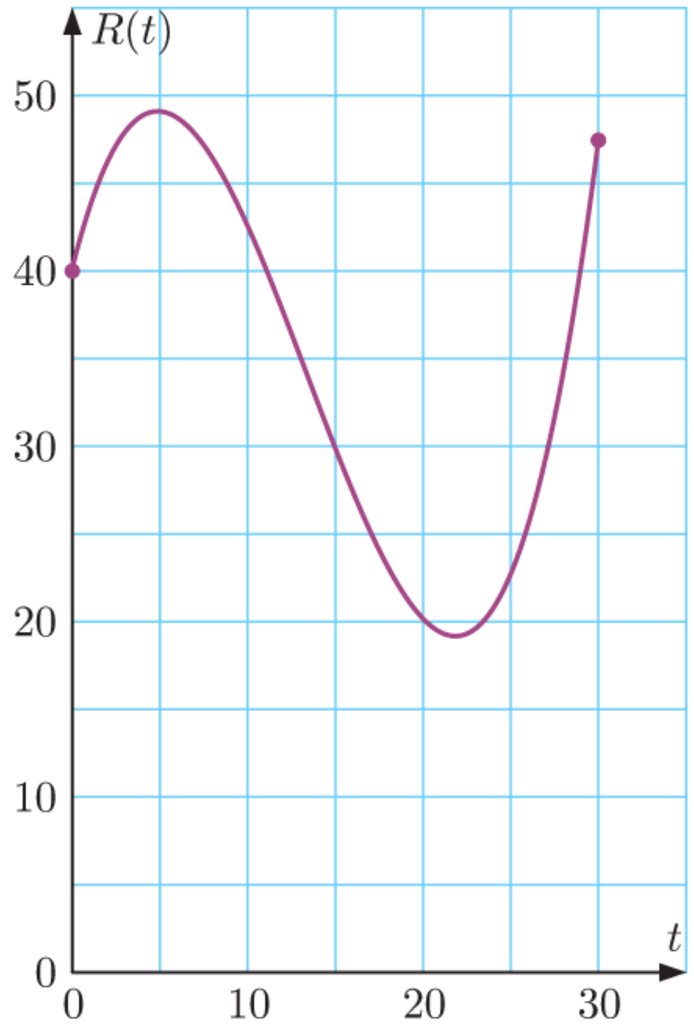
EXERCISE 23G

1 The rate of traffic flow past a pedestrian crossing between 8 am and 8:30 am is given by

$$R(t) = \frac{t^3}{80} - \frac{t^2}{2} + 4t + 40 \text{ cars per minute,}$$

where t is the number of minutes after 8 am, $0 \leqslant t \leqslant 30$.

The graph of $R(t)$ against t is shown alongside.



a Find the rate of traffic flow at 8:20 am.

b Use the graph to estimate the time at which the traffic flow was greatest.

c Copy the graph, and shade the region corresponding to $\int_{10}^{15} R(t)\,dt$. Explain what this region represents.

d How many cars passed the crossing between 8 am and 8:30 am?

PRINTABLE GRAPH

2 Evan is happily paddling until his kayak strikes a sharp rock.

Water begins to leak in at the rate $R_1(t) = 5 - 5e^{-0.2t}$ litres per minute, where t is the time in minutes.

Evan tries to bail the water out of the kayak, removing the water at the rate $R_2(t) = 6 - 6e^{-0.1t}$ litres per minute.

a After 2 minutes, at what rate is water:

i leaking into the kayak **ii** being bailed from the kayak?

b Is the amount of water in the kayak increasing or decreasing after 2 minutes? Explain your answer.

c Evaluate the following integrals, and interpret their meaning:

i $\int_0^3 R_1(t)\,dt$ **ii** $\int_2^5 R_2(t)\,dt$ **iii** $\int_0^8 [R_1(t) - R_2(t)]\,dt$

d How much water is in the kayak after 10 minutes?

3 Answer the **Opening Problem** on page **584**.

4 The rate of power consumption of the United Kingdom can be modelled by the function

$$E(t) = 13\sin\left(\tfrac{(t+3)\pi}{3}\right) + 70\cos\left(\tfrac{(t-1)\pi}{6}\right) + 196 \text{ TWh per month}$$

where t is the number of months after January 1st, $0 \leqslant t \leqslant 12$.

a Use technology to help sketch the function.

b Find the following quantities and explain what they represent:

i $\int_3^4 E(t)\,dt$ **ii** $\int_5^8 E(t)\,dt$ **iii** $\int_0^{12} E(t)\,dt$

ACTIVITY 2 BUFFON'S NEEDLE PROBLEM

The French naturalist and mathematician **Georges-Louis Leclerc, Comte de Buffon** was the first person to apply ideas in calculus to probability.

Click on the icon to obtain an Activity about Buffon's needle problem.



REVIEW SET 23A

1 Find the exact value of:

a $\int_{-2}^{0} (1 - 3x)\,dx$ **b** $\int_0^{\frac{1}{2}} \left(x - \sqrt{x}\right) dx$ **c** $\int_1^2 (x^2 + 1)^2\,dx$

2 Find the exact value of:

a $\int_{-5}^{-1} \sqrt{1 - 3x}\,dx$ **b** $\int_0^{\frac{\pi}{2}} \cos\tfrac{x}{2}\,dx$ **c** $\int_2^6 \frac{2}{x}\,dx$

3 Find $\frac{d}{dx}(e^{-2x}\sin x)$ and hence evaluate $\int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx$.

4 Evaluate correct to 6 significant figures:

a $\int_3^4 \frac{x}{\sqrt{2x+1}}\,dx$ **b** $\int_0^1 x^2 e^{x+1}\,dx$ **c** $\int_0^{\pi} \sin(x^2)\,dx$

5 **a** Find $\int 2x(x^2+1)^3\,dx$ using the substitution $u = x^2 + 1$.

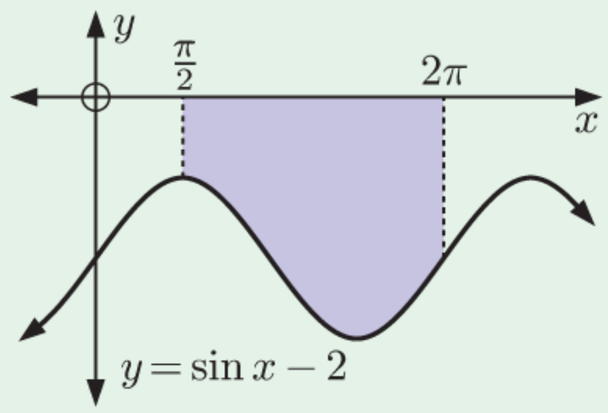
b *Hence* evaluate:

i $\int_0^1 2x(x^2+1)^3\,dx$ **ii** $\int_{-1}^2 -x(1+x^2)^3\,dx$

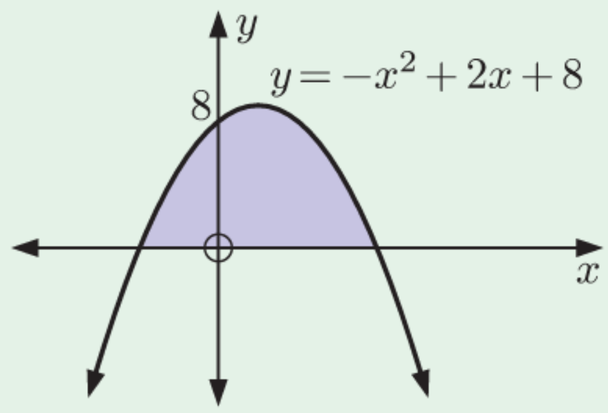
6 Find $\int_0^1 x^2 e^{1-x^3}\,dx$ using an appropriate substitution.

7 Find the shaded area:

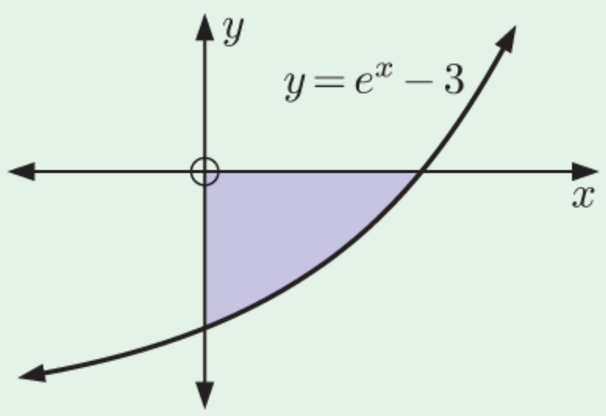
a



b



c



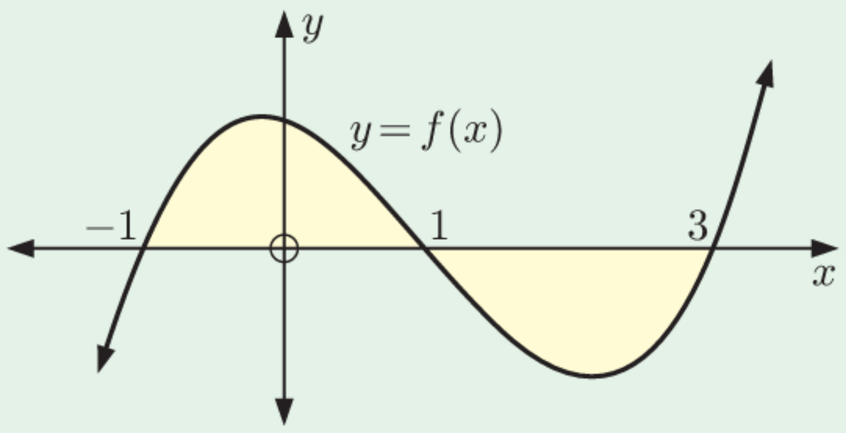
8 Find the area of the region bounded by:

a $y = x^2$, the x-axis, $x = 2$, and $x = 5$

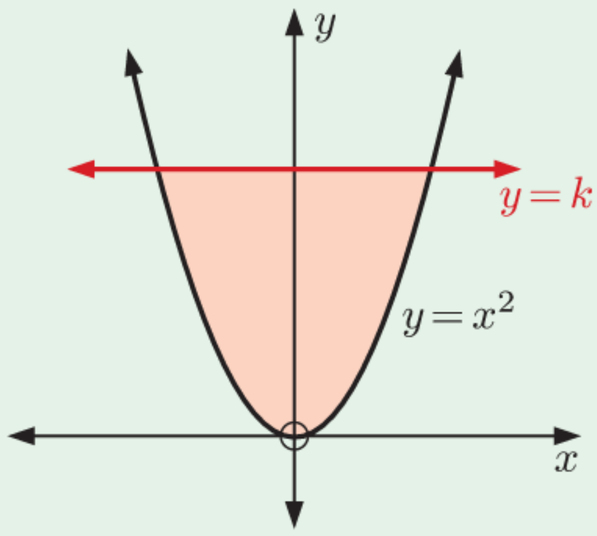
b $y = \sqrt{5 - x}$, the x-axis, $x = 1$, and $x = 2$

c $y = \sin\frac{x}{3}$, the x-axis, $x = \pi$, and $x = 2\pi$.

9 Does $\int_{-1}^{3} f(x)\,dx$ represent the area of the shaded region? Explain your answer.

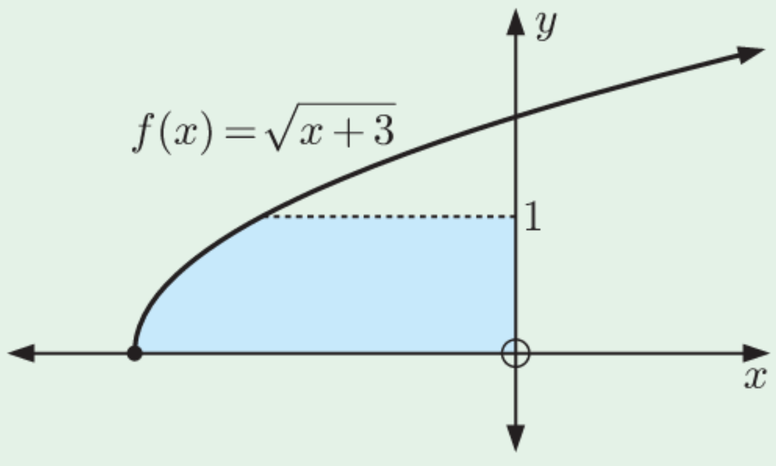


10 Determine k given that the enclosed region has area $5\frac{1}{3}$ units^2.

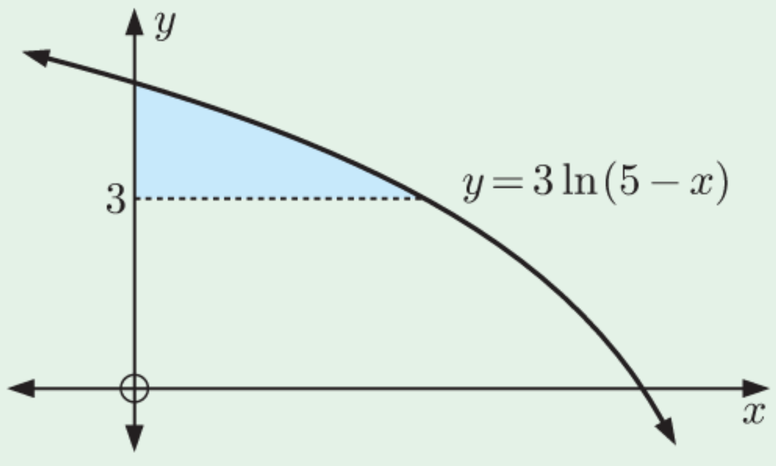


11 Find the shaded area:

a



b



12 Consider the function $f(x) = \sqrt{\frac{1}{x} - 1}$. Use technology to:

a sketch the graph of $y = f(x)$

b find the area bounded by $y = f(x)$ and the y-axis, between $y = 1$ and $y = \sqrt{3}$.

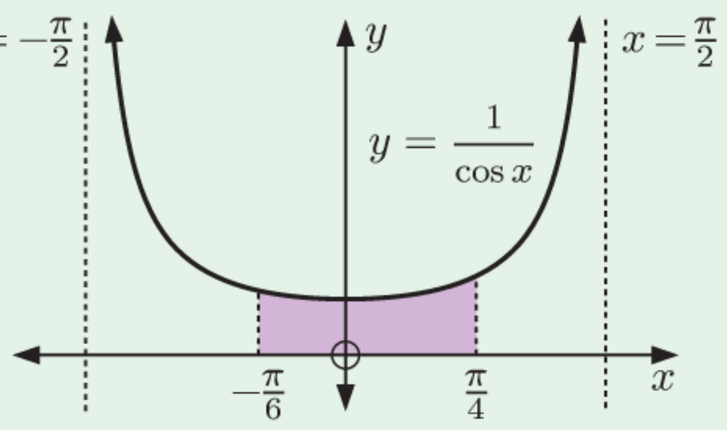
13 Find the volume of the solid of revolution formed when the following curves are revolved through $360°$ about the x-axis:

a $y = x$ between $x = 4$ and $x = 10$ **b** $y = x + 1$ between $x = 4$ and $x = 10$

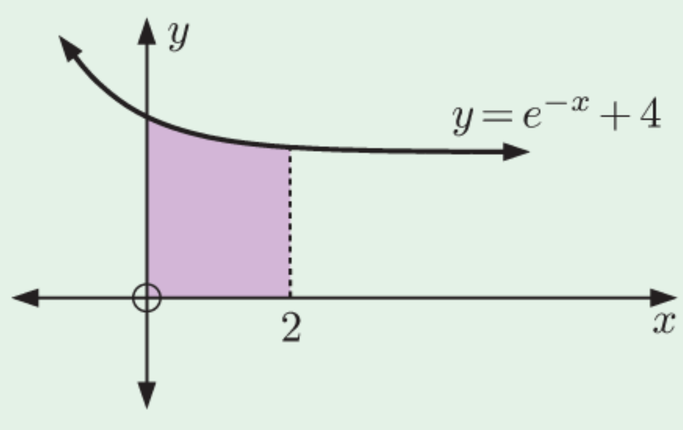
c $y = \sqrt{\sin x}$ for $0 \leqslant x \leqslant \pi$.

14 Find the volume of the solid of revolution formed when the shaded region is revolved through $360°$ about the x-axis:

a



b



15 Find the volume of the solid of revolution formed when the following are rotated through 2π about the y-axis:

a $y = \sqrt{x}$ between $y = 1$ and $y = 2$ **b** $y = \sqrt[3]{x^2}$ between $y = 2$ and $y = 3$.

16 Bettina is filling a watering can with water from a tap. The water enters at the rate $R_1(t) = 6.4$ litres per minute.

Bettina's watering can starts empty, and has capacity 16 litres. Unfortunately it has a hole in the bottom, so it leaks water at the rate $R_2(t) = 2.5 - 1.25e^{-0.2t}$ litres per minute, where t is the time in minutes.

a Evaluate the following integrals, and interpret their meaning:

i $\int_0^{\frac{1}{2}} R_2(t)\,dt$ **ii** $\int_0^1 [R_1(t) - R_2(t)]\,dt$

b How long will it take for the watering can to be full? Give your answer to the nearest second.

REVIEW SET 23B

1 Find the exact value of:

a $\int_2^3 \frac{1}{\sqrt{3x}}\,dx$ **b** $\int_1^4 (x - \frac{1}{2}x^2)\,dx$ **c** $\int_0^1 (x^2 + \frac{1}{3})^2\,dx$

2 Find the exact value of:

a $\int_2^3 \frac{1}{\sqrt{3x - 4}}\,dx$ **b** $\int_{2e}^{3e} \frac{4}{x + e}\,dx$ **c** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2\sin x + 1)\,dx$

3 Find the values of b such that $\int_0^b \cos x \, dx = \frac{1}{\sqrt{2}}$, $0 < b < \pi$.

4 Evaluate correct to 4 significant figures:

a $\int_{-2}^{0} 4e^{-x^2} \, dx$ **b** $\int_0^1 \frac{10x}{\sqrt{3x+1}} \, dx$ **c** $\int_0^{\frac{\pi}{3}} \cos^2\left(\frac{x}{2}\right) dx$

5 Find $\int_0^{\frac{\pi}{6}} \frac{\cos x}{(1+\sin x)^3} \, dx$ using an appropriate substitution.

6 Find the area of the region bounded by:

a $y = x^3 + 1$, the x-axis, $x = 1$, and $x = 3$

b $y = \frac{1}{x}$, the x-axis, $x = 3$, and $x = 9$

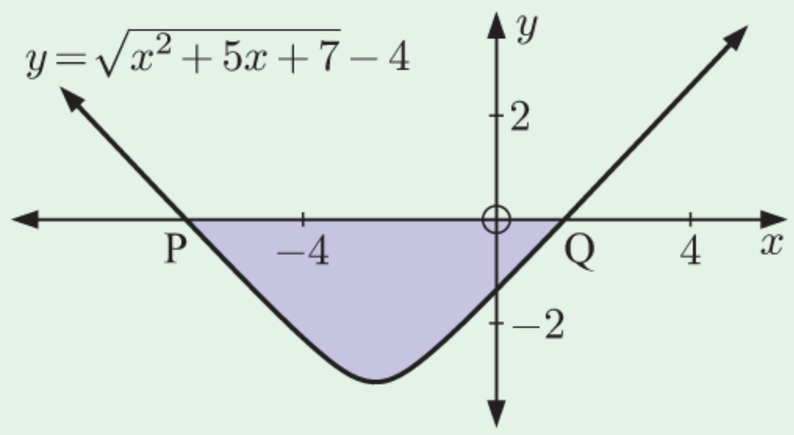
c $y = e^{2x}$, the x-axis, $x = 0$, and $x = 2$

d $y = -2 - 2\cos 3x$ and the x-axis from $x = -\frac{\pi}{3}$ to $x = \frac{\pi}{3}$.

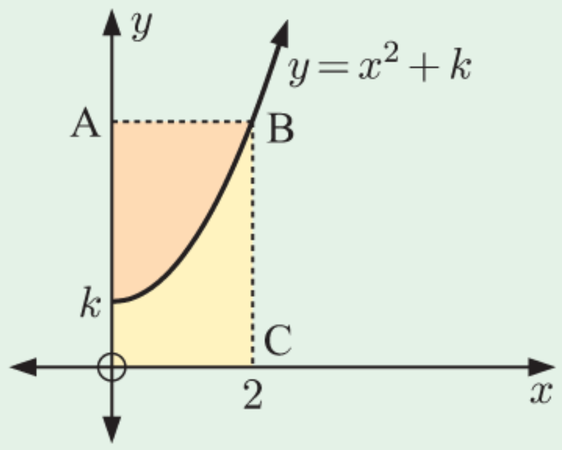
7 Use technology to find:

a the x-coordinates of P and Q

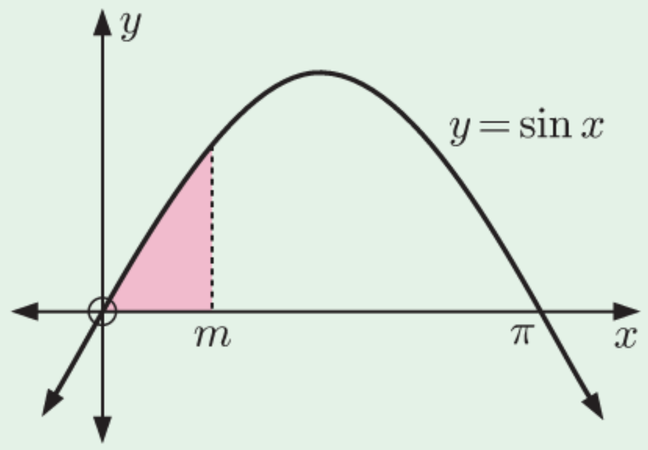
b the shaded area.



8 OABC is a rectangle and the two shaded regions are equal in area. Find k.



9 The shaded region has area $\frac{1}{2}$ unit^2. Find the value of m.



10 Find the area enclosed by:

a $y = \frac{1}{x}$, the y-axis, $y = 2$, and $y = 4$

b $y = x^3 + 2$, the y-axis, $y = 3$, and $y = 6$.

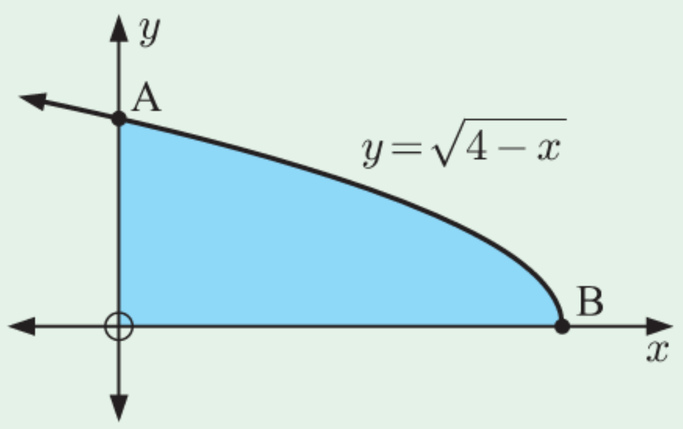
11 The graph of $y = \sqrt{4 - x}$ is shown alongside.

a Find the coordinates of A and B.

b Find the shaded area by considering it as the area between the curve and:

i the x-axis **ii** the y-axis.

Comment on your answers.



12 Find the volume of the solid of revolution formed when the following curves are rotated about the x-axis:

a $y = \dfrac{4}{x}$ between $x = 1$ and $x = 2$ **b** $y = \sqrt{9 - x^2}$, $0 \leqslant x \leqslant 3$.

13 Use technology to find the volume of the solid of revolution formed when the following curves are rotated about the x-axis:

a $y = \sin x$ between $x = 0$ and $x = \pi$

b $y = \ln(2x)$ between $x = 1.5$ and $x = 4.5$.

14 **a** Find the derivative of $\dfrac{\cos x}{\sin x}$.

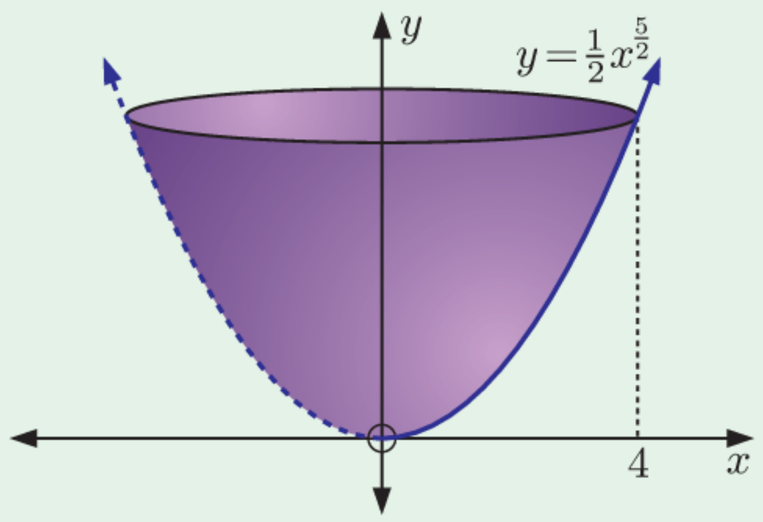
b Find the volume of revolution when $y = \dfrac{1}{\sin x}$ is rotated through $360°$ about the x-axis for $\frac{\pi}{4} \leqslant x \leqslant \frac{3\pi}{4}$.

15 The shape of a vase is found by revolving the curve $y = \frac{1}{2}x^{\frac{5}{2}}$, $0 \leqslant x \leqslant 4$ through $360°$ about the y-axis. The units are in centimetres.

a Find the height of the vase.

b Find the volume of the vase.

c The vase is filled with water to a height of 10 cm. Find the amount of water in the vase.



16 Over the course of a day, the rate of solar energy being transferred into Callum's solar panels is given by $E(t) = 2\sin\left(\frac{t-5}{5}\right) + \frac{1}{2}\sin\left(\frac{t-5}{4}\right)$ kW where t is the time in hours after midnight, $5 \leqslant t \leqslant 20$.

Find the following quantities and explain what they represent:

a $\displaystyle\int_5^{12} E(t)\,dt$ **b** $\displaystyle\int_{12}^{20} E(t)\,dt$ **c** $\displaystyle\int_5^{20} E(t)\,dt$

Chapter 24

Kinematics

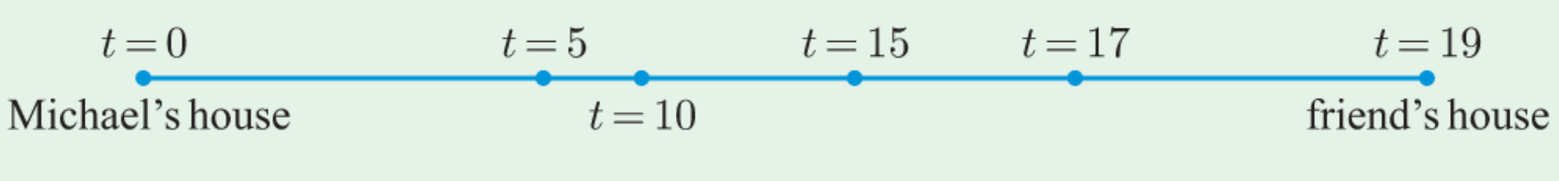
Contents:

A Displacement
B Velocity
C Acceleration
D Speed
E Velocity and acceleration in terms of displacement
F Motion with variable velocity
G Projectile motion



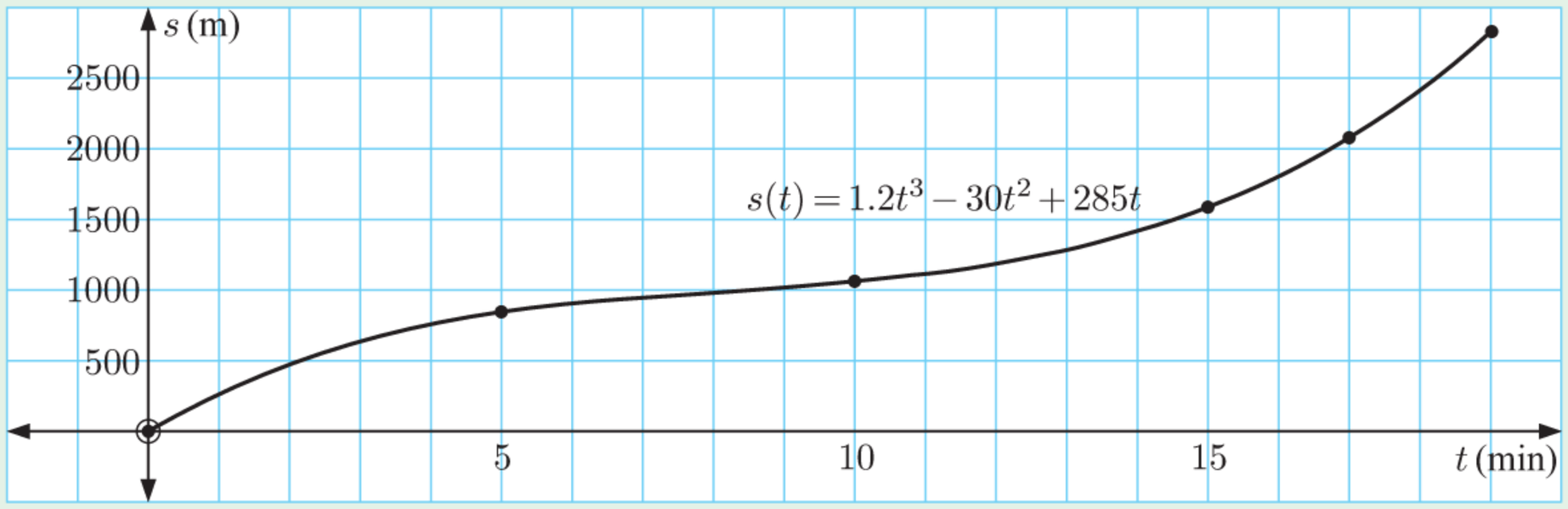
OPENING PROBLEM

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.





The distance Michael has travelled at various times is given by the function $s(t) = 1.2t^3 - 30t^2 + 285t$ metres for $0 \leqslant t \leqslant 19$ minutes.



Things to think about:

a At what point do you think the hill was steepest? How far had Michael travelled to this point?

b How can we find Michael's *average* speed between $t = 5$ and $t = 15$ minutes?

c How can we find Michael's speed at a particular instant? Can we write a *function* which will tell us Michael's speed at any time t?



Kinematics is the study of motion.

In previous years you should have studied **travel graphs** which plot distance travelled against time taken.

In this Chapter we consider kinematics more formally, taking into account *direction* as well as distance and speed. This will lead us to functions for **displacement**, **velocity**, and **acceleration**, and how they are linked by calculus.

THE LANGUAGE OF MOTION

Consider an object P which is in motion along a straight line.

- The **distance** it travels accumulates over time, irrespective of its direction of travel.
- The **speed** S of P is how fast it is travelling.
- The **displacement** s or x of P is its position relative to a fixed origin O.
- The **velocity** v of P is its rate of change of displacement.
- The **acceleration** a of P is its rate of change of velocity.

DISCUSSION

1 What is the difference between:

a distance travelled and displacement **b** speed and velocity?

2 Each of the following expressions refers to a zero of one of the variables time t, displacement s, velocity v, or acceleration a. Discuss which variable each expression refers to and what it means.

a initial conditions **b** at the origin

c stationary **d** reverses direction

e maximum or minimum displacement **f** constant velocity

g maximum or minimum velocity

3 Does a zero displacement correspond to:

a zero distance travelled **b** zero speed?

4 How should we interpret the sign of:

a displacement **b** velocity **c** acceleration?

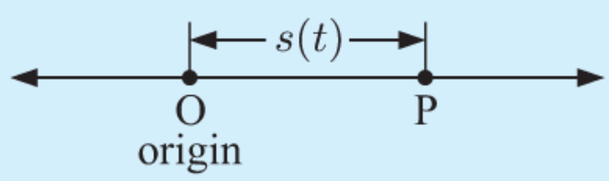
QUICK REFERENCE SUMMARY



A DISPLACEMENT

For an object P which is in motion along a straight line:

- The **displacement** s of P is its position relative to a fixed origin O.
- The **displacement function** $s(t)$ gives the displacement of the object at any time $t \geqslant 0$.
  - When $s(t) > 0$, P is to the right of the origin.
  - When $s(t) < 0$, P is to the left of the origin.



Example 1

Self Tutor

A lighthouse is situated on the top of a cliff. A stone is thrown from the top of the lighthouse. The stone falls past the base of the lighthouse, and lands after 6 seconds in the water below.



The displacement of the stone above ground level $s(t) = -5t^2 + 20t + 25$ m for $0 \leqslant t \leqslant 6$ s.

a Find the initial displacement of the stone.

b Use quadratic theory to graph $s(t)$.

c At what time and place does the stone change direction?

d For what time interval is the stone above ground level?

a $s(0) = 25$

$\therefore$ the initial displacement of the stone is 25 m.

b $s(t) = -5t^2 + 20t + 25$
$= -5(t^2 - 4t - 5)$
$= -5(t - 5)(t + 1)$

c The stone changes direction at the turning point of $s(t)$.

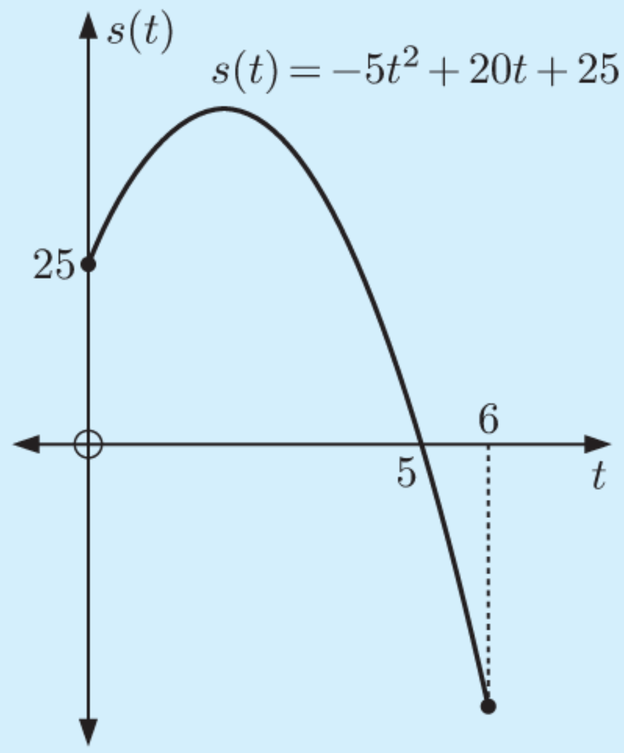
This occurs when $t = \dfrac{-20}{2(-5)} = 2$ s.

$s(2) = -5(2)^2 + 20(2) + 25 = 45$ m

The stone changes direction after 2 seconds, when it is 45 m above ground level.

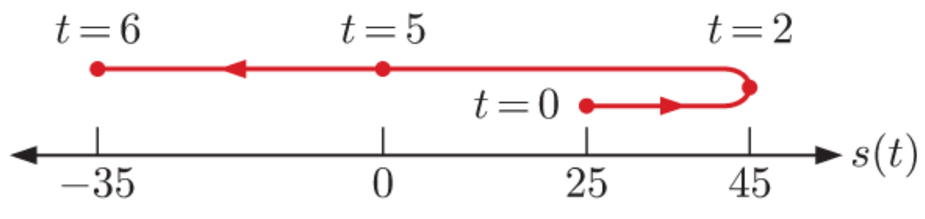
d The stone is above ground level whenever $s(t) > 0$.

This occurs for $0 \leqslant t < 5$ s.



A **motion diagram** is used to plot the position of an object on a single number line. It can be helpful in reminding us that we are looking at motion in one dimension only.

For the object in **Example 1**, the motion diagram is:



Click on the icon to explore motion diagrams for other displacement functions.



EXERCISE 24A

1 An object travels with displacement function $s(t) = 5 - t$ cm for $0 \leqslant t \leqslant 10$ s.

a Find the initial displacement of the object.

b Find the displacement of the object at time:

i $t = 3$ s **ii** $t = 10$ s.

c At what time does the object reach the origin?

d Does the object ever change direction? Explain your answer.

e Draw a motion diagram showing the information you have found.

2 An object travels with displacement function $s(t) = 10t^2 - 7t + 1$ m for $0 \leqslant t \leqslant 1$ s.

a Find the initial displacement of the object.

b Use quadratic theory to graph $s(t)$.

c At what time and place does the object change direction?

d For what time interval is the object to the right of the origin?

e Draw a motion diagram showing the information you have found.

3 A mass on a spring oscillates with displacement $x(t) = 8 \sin 8\pi t + 6$ cm where $0 \leqslant t \leqslant 0.25$ s.

a At what times is the displacement 6 cm?

b Plot the graph of $x(t)$ for $0 \leqslant t \leqslant 0.25$ s.

c Find where and when the mass changes direction.

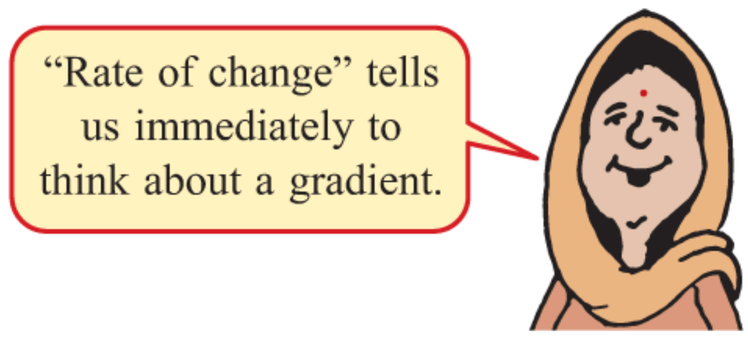
d Draw a motion diagram showing the information you have found.

B VELOCITY

The **velocity** of an object is its rate of change of displacement.

AVERAGE VELOCITY

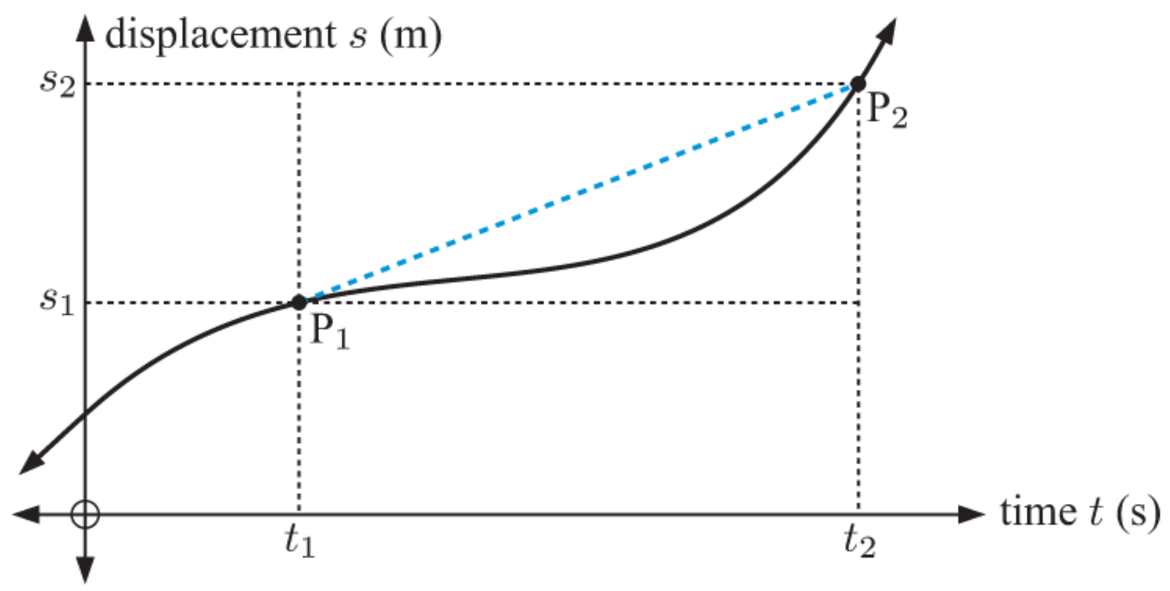
In many real-world scenarios, we only know certain data points for where an object was at a particular time. In these cases, we can only calculate the *average* velocity between data points.



The **average velocity** of an object moving in the time interval from $t = t_1$ to $t = t_2$ is given by

$$\textbf{average velocity} = \frac{\textbf{change in displacement}}{\textbf{change in time}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

On a graph of $s(t)$ against t, the average velocity is the gradient of the chord through the points $\mathrm{P}_1(t_1,\ s(t_1))$ and $\mathrm{P}_2(t_2,\ s(t_2))$.



INSTANTANEOUS VELOCITY

If we know the complete function $s(t)$, we can find the instantaneous velocity of the object at any time.

The **instantaneous velocity** or **velocity function** of the object at time t is $\boldsymbol{v(t) = s'(t)}$.
It is the gradient of the tangent to the function $s(t)$ at any given time.

If the displacement is represented by $x(t)$, it is common to denote the velocity function $v(t) = \dfrac{dx}{dt}$ by $\dot{x}(t)$. The dot over the x represents a single derivative.

Notice that:

- When $v(t) > 0$, the object is moving to the right.
- When $v(t) < 0$, the object is moving to the left.
- When $v(t) = 0$, the object is instantaneously at rest. A change in the sign of $v(t)$ at this time indicates that the object has changed direction.

When we take a derivative with respect to t, we calculate a rate per unit of time. So, for a displacement in metres and time in seconds, the units of velocity are $\mathrm{m\,s^{-1}}$.

Example 2 **Self Tutor**

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity for the time interval from $t = 2$ to $t = 5$ seconds

b the instantaneous velocity at $t = 2$ seconds.

a average velocity $= \dfrac{s(5) - s(2)}{5 - 2}$

$= \dfrac{(15 - 25) - (6 - 4)}{3}$

$= \dfrac{-10 - 2}{3}$

$= -4 \text{ m s}^{-1}$

b $s(t) = 3t - t^2$

$\therefore \ v(t) = s'(t) = 3 - 2t$

$\therefore \ v(2) = 3 - 2(2)$

$= -1 \text{ m s}^{-1}$

The particle is travelling at 1 m s^{-1} to the left.

EXERCISE 24B.1

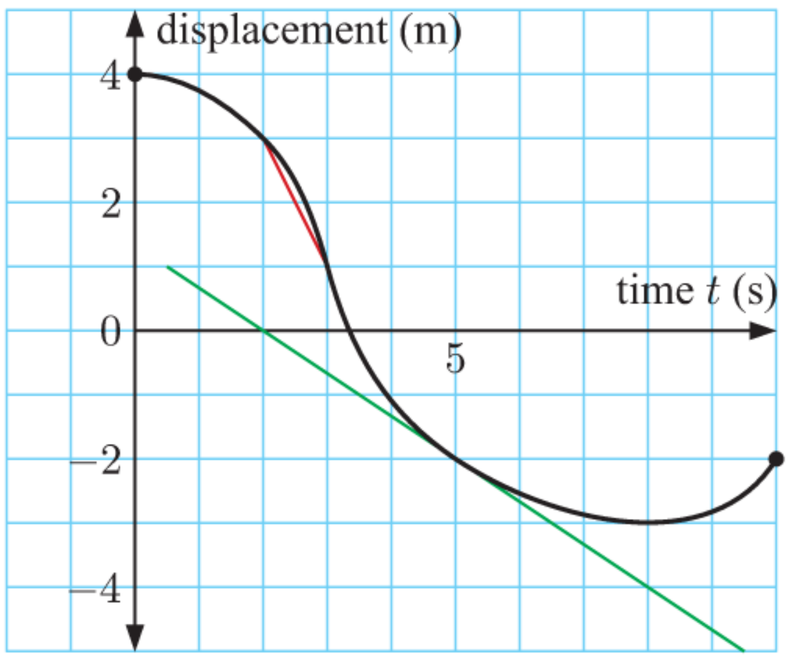
1 Consider the displacement-time graph alongside.

a Find the displacement of the object when:

i $t = 2$ seconds **ii** $t = 8$ seconds.

b Find the average velocity of the object from $t = 2$ to $t = 3$ seconds.

c Find the instantaneous velocity of the object when $t = 5$ seconds.



2 A particle moves in a straight line with displacement from O given by $s(t) = t^2 - 6t + 1$ metres at time t seconds, $t \geqslant 0$.

a Find the average velocity from $t = 1$ to $t = 3$ seconds.

b Find $v(t)$.

c Hence find the instantaneous velocity at:

i $t = 1$ second **ii** $t = 5$ seconds.

3 Consider the displacement-time graph alongside.

a Describe the initial position of the object.

b At what time is the object at the origin?

c In which direction is the object moving when $t = 5$ seconds?

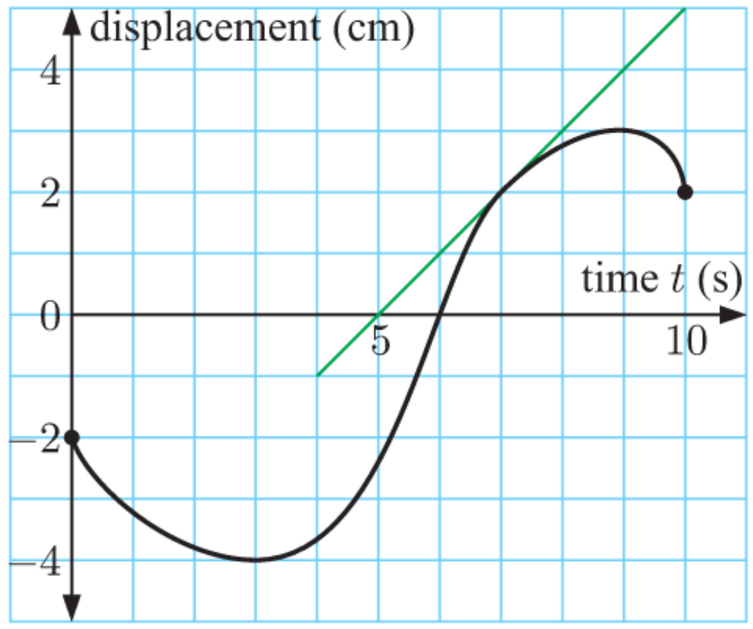
d At what times does the object change direction?

e Draw a sign diagram for:

i the displacement function $s(t)$

ii the velocity function $v(t)$.

f Find the instantaneous velocity of the object when $t = 7$ seconds.



4 An object moves in a straight line with displacement function $s(t) = 2\sqrt{t} + 3$ cm, where t is in seconds, $t \geqslant 0$.

a Find the average velocity from $t = 1$ to $t = 4$ seconds.

b Find the initial position of the object.

c Find $v(t)$.

d Hence find the instantaneous velocity at:

i $t = 4$ seconds **ii** $t = 16$ seconds.

5 An object moves in a straight line with position given by $x(t) = t^3 - 11t^2 + 24t$ m from O, where t is in seconds, $t \geqslant 0$.

a Find the velocity function $\dot{x}(t)$.

b Describe the initial conditions of the object.

c Draw a sign diagram for $x(t)$ and $\dot{x}(t)$.

d At what times is the object at O?

e At which times does the object reverse direction? Find the position of the object at each of these times.

f Describe in words the motion of the object.

g Draw a motion diagram for the object.

6 A shell is accidentally fired vertically from a mortar at ground level. Its height above the ground after t seconds is given by $s(t) = bt - 4.9t^2$ metres where b is constant.

a Show that the initial velocity of the shell is b m s^{-1} upwards.

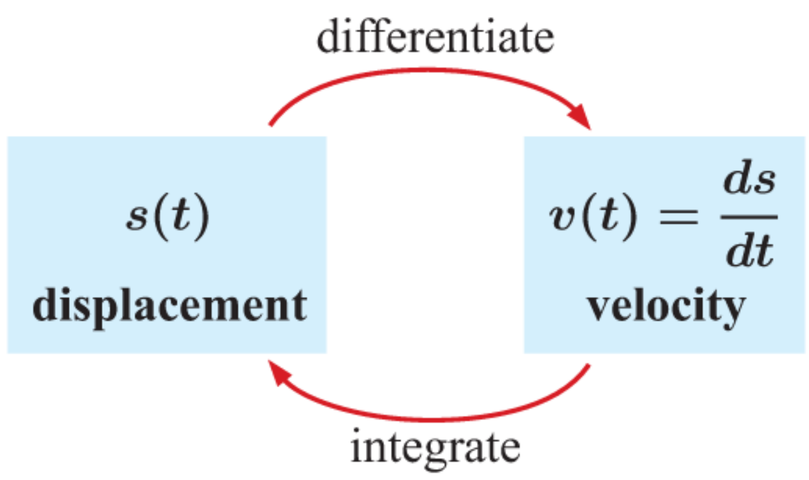
b The shell reaches its maximum height after 7.1 seconds.

i Find the initial velocity of the shell.

ii Find the maximum height reached by the shell.

FINDING DISPLACEMENT FROM VELOCITY

Since the velocity function is the derivative of the displacement function, we can use integration to reverse the process.



To find the constant of integration, we would need to know the displacement at a particular time. Most commonly, this would be the initial displacement $s(0)$.

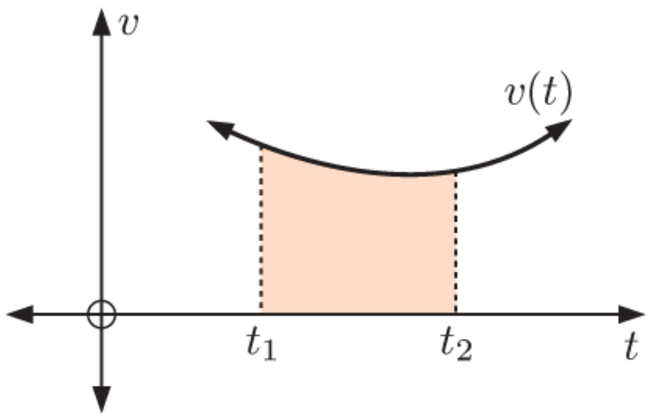
CHANGE IN DISPLACEMENT

We can determine the change in displacement in a time interval $t_1 \leqslant t \leqslant t_2$ using the integral:

$$\textbf{Change in displacement} = s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt$$

DISTANCE TRAVELLED

For a velocity-time function $v(t)$ where $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$, the distance travelled will simply be the change in displacement $\int_{t_1}^{t_2} v(t)\,dt$.





If we have a change of direction within the time interval then the velocity will change sign. To find the total distance travelled, we therefore need to add the components of area above and below the t-axis. Alternatively, we can integrate the absolute value of the velocity, which is the object's **speed**.

$$\textbf{Distance travelled} = \int_{t_1}^{t_2} |\,v(t)\,|\,dt$$

Example 3 — Self Tutor

The velocity-time graph for a train journey is illustrated in the graph alongside. Find the total distance travelled by the train.

Total distance travelled

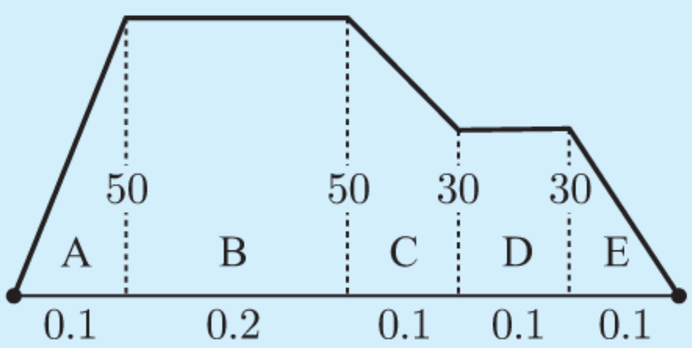
= total area under the graph

= area A + area B + area C + area D + area E

$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30 + \frac{1}{2}(0.1)30$

$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km



Example 4 — Self Tutor

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ m s^{-1}.

a Find the times when P reverses direction.

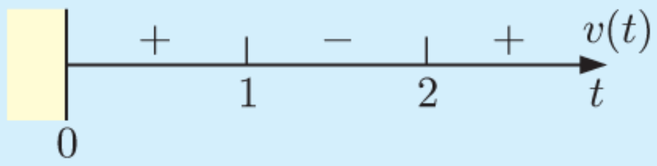
b Hence find the distance P travels in the first 4 seconds of motion.

Check your answer by evaluating $\int_0^4 |\,v(t)\,|\,dt$.

c Find the change in displacement of P in the first 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$
$= (t-1)(t-2)$

$\therefore$ the sign diagram of v is:



Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

b Now $s(t) = \int (t^2 - 3t + 2)\,dt$

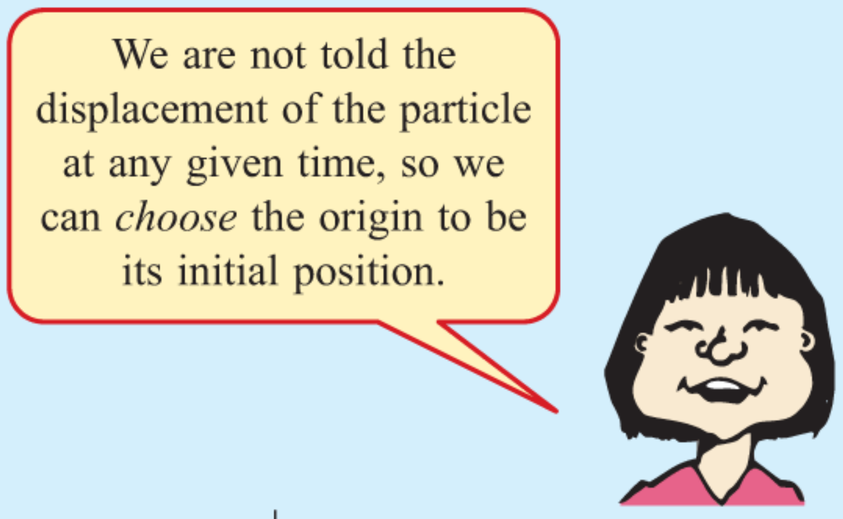
$= \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

We choose the initial displacement to be zero, so $s(0) = c = 0$.

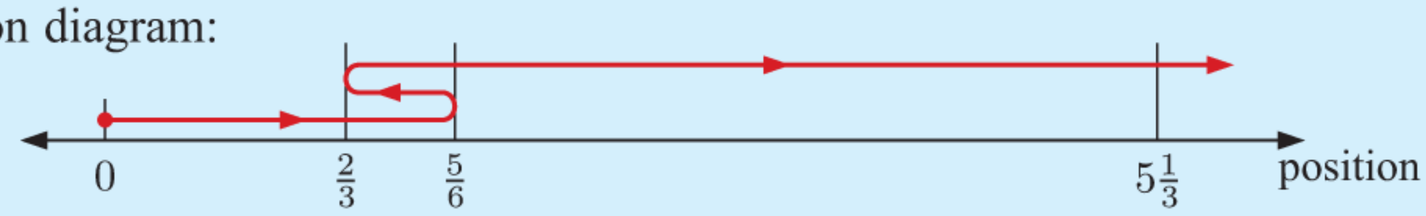
$\therefore \quad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = \frac{2}{3}$

$s(4) = \frac{64}{3} - 24 + 8 + c = 5\frac{1}{3}$

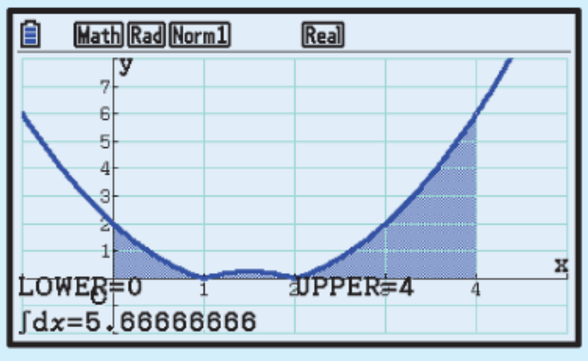


Motion diagram:

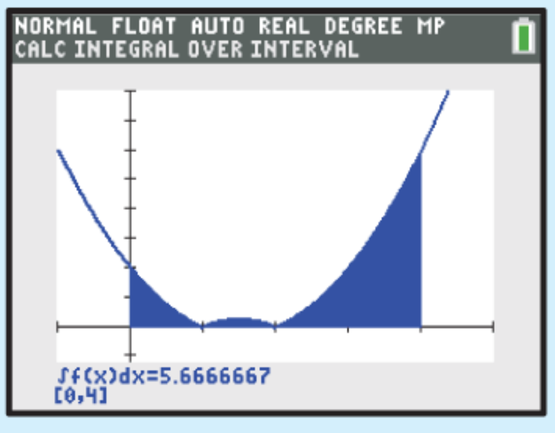


$\therefore$ total distance travelled $= \frac{5}{6} + (\frac{5}{6} - \frac{2}{3}) + (5\frac{1}{3} - \frac{2}{3}) = 5\frac{2}{3}$ m

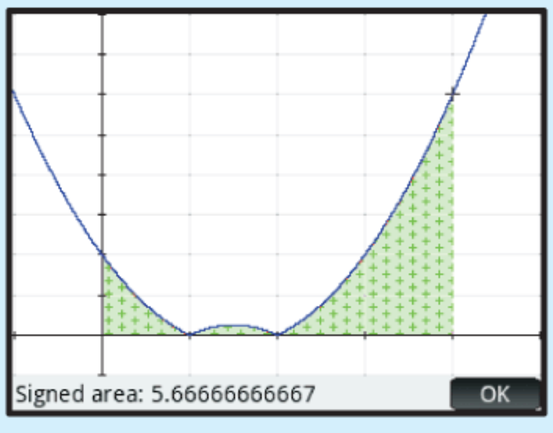
Casio fx-CG50



TI-84 Plus CE



HP Prime

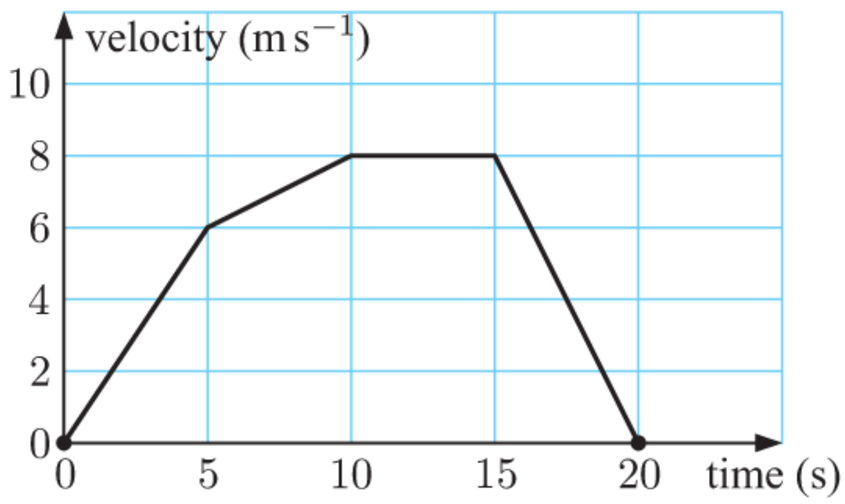


c Change in displacement = final position − original position
$= s(4) - s(0)$
$= 5\frac{1}{3}$ m

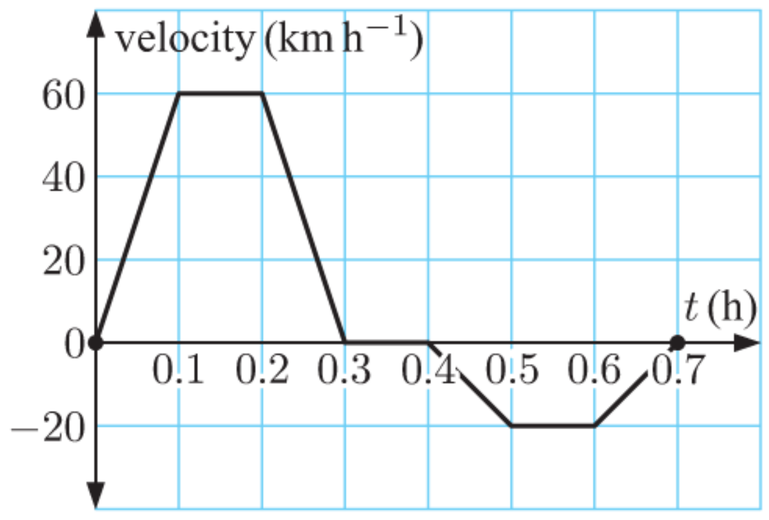
So, the displacement is $5\frac{1}{3}$ m to the right.

EXERCISE 24B.2

1 A runner has the velocity-time graph shown. Find the total distance travelled by the runner.



2



A car travels along a straight road with the velocity-time graph illustrated.

a What is the significance of the graph:

i above the t-axis

ii below the t-axis?

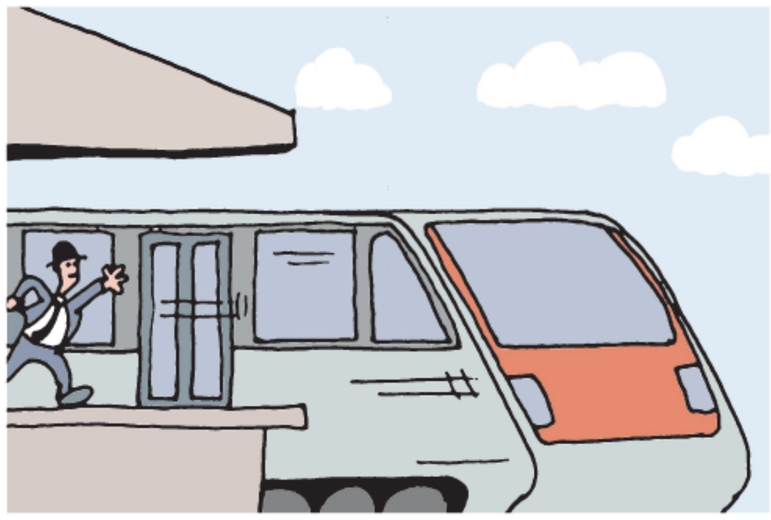
b Find the total *distance* travelled by the car.

c Find the final *displacement* of the car from its starting point.

3 After leaving a station, a train accelerates at a constant rate for 40 seconds until its speed reaches 15 m s^{-1}. The train then travels at this speed for 160 seconds. On its approach to the next station, the train slows down at a constant rate for 80 seconds until it is at rest.

a Draw a graph to show the train's motion.

b How far did the train travel between the stations?



4 A cyclist rides off from rest, accelerating at a constant rate for 3 minutes until she reaches 40 km h^{-1}. She then maintains a constant speed for 4 minutes until reaching a hill. She slows down at a constant rate over one minute to 20 km h^{-1}, then continues at this speed for 5 minutes. At the top of the hill she reduces her speed uniformly, and she is stationary 2 minutes later.

a Draw a graph to show the cyclist's motion.

b How far has the cyclist travelled?

5 A particle has velocity function $v(t) = 1 - 2t$ cm s^{-1} as it moves in a straight line.

a Find the time when the particle reverses direction.

b Hence find the total distance travelled in the first second of motion.

Check your answer by evaluating $\int_0^1 |\, v(t)\,|\, dt$.

c Find the change in displacement of the particle in the first second.

6 Particle P is initially at the origin O. It moves with the velocity function $\dot{x}(t) = t^2 - t - 2$ cm s^{-1}.

a Write a formula for the displacement function $x(t)$.

b Find the total distance travelled in the first 3 seconds of motion.

c Find the particle's displacement from its starting position after three seconds.

7 A ball is thrown from 1 m above ground level. Its velocity is $v(t) = 29.4 - 9.8t$ m s^{-1}.

a Find the displacement function $s(t)$. **b** Find the maximum height reached by the ball.

8 The velocity of a moving object is given by $v(t) = 32 + 4t$ m s^{-1}.

a If $s = 16$ m when $t = 0$ seconds, find the displacement function.

b Explain why the change in displacement of the object and its total distance travelled in the interval $0 \leqslant t \leqslant \tau$, can both be represented by the definite integral $\int_0^\tau (32 + 4t)\, dt$.

c Find the distance travelled by the object in the first 4 seconds.

9 An object has velocity function $v(t) = \cos 2t$ m s^{-1}.

a Show that the particle oscillates between two points, and find the distance between them.

b If $s(\frac{\pi}{4}) = 1$ m, determine $s(\frac{\pi}{3})$ exactly.

10 When a pendulum is released, its velocity along the arc of motion after t seconds is given by $v(t) = 20 + 5\sin 4t$ cm s^{-1}.

a Sketch the graph of $v(t)$ against t for $0 \leqslant t \leqslant 6$.

b Find the pendulum's velocity after 4.5 seconds.

c Find the total distance travelled by the tip of the pendulum in the first 2 seconds.

11 The velocity of a moving object is given by $\dot{x}(t) = -4 + \sqrt{t}$ m s^{-1}. Suppose the object is initially at the origin. Find:

a the displacement function **b** the time when the object changes direction

c the change in displacement after the first 30 seconds

d the total distance travelled in the first 30 seconds.

12 A stunt motorcyclist rides towards a ramp. His velocity after t seconds is given by $v(t) = 10\sqrt{t}$ m s^{-1}.



a Find the motorcyclist's velocity after:

i 1 second **ii** 2 seconds.

b Write a function $s(t)$ for the displacement of the motorcyclist after t seconds.

c Find $\int_0^2 v(t)\,dt$, and interpret your answer.

d In order to perform his stunt, the motorcyclist needs to be travelling at 20 m s^{-1} or faster when he reaches the ramp.

i How long will he take to reach a speed of 20 m s^{-1}?

ii The motorcyclist started his approach 55 m from the ramp. Has he given himself enough distance to reach the required speed?

13 Use technology and the formula distance travelled $= \int_{t_1}^{t_2} |v(t)|\,dt$ to find the total distance travelled in the following scenarios:

a a skydiver has velocity $v(t) = -54(1 - e^{-\frac{t}{6}})$ m s^{-1} for the first 15 seconds

b a mass on a spring has velocity $v(t) = e^{-t}\cos 16t$ cm s^{-1} for 10 seconds.

C ACCELERATION

The **acceleration** of an object is its rate of change of velocity.

If an object moves in a straight line with displacement function $s(t)$ and velocity function $v(t)$, then:

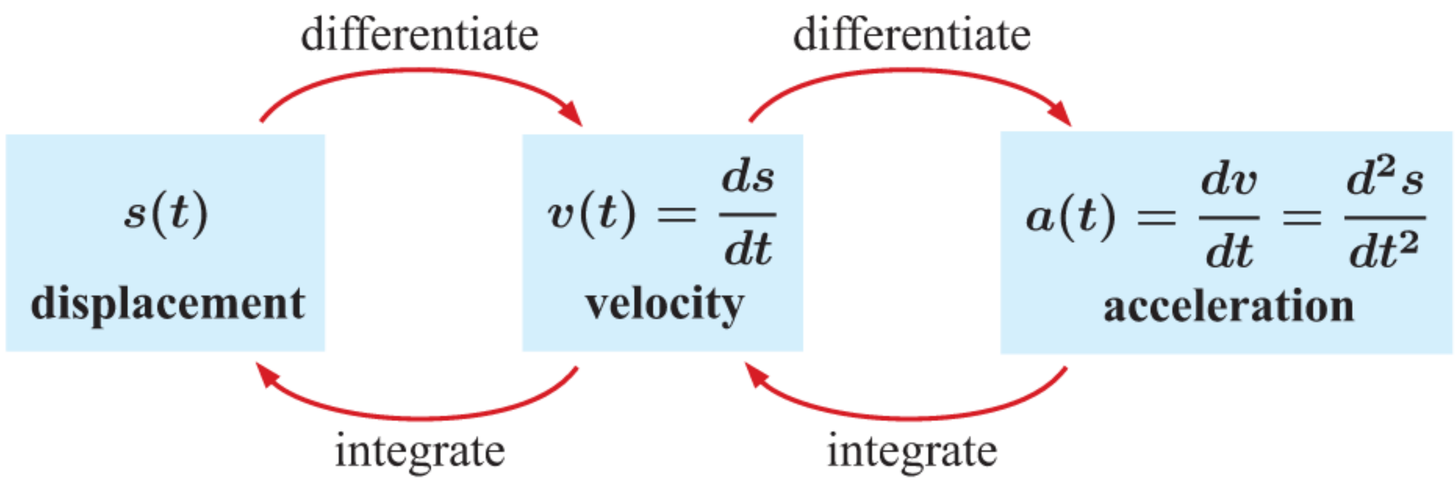
- The **average acceleration** for the time interval from $t = t_1$ to $t = t_2$ is given by

$$\textbf{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

- The **acceleration function** $a(t)$ of the object is given by $a(t) = v'(t) = s''(t)$.

For the displacement function $x(t)$, it is common to denote the acceleration function $a(t) = \dfrac{d^2x}{dt^2}$ by $\ddot{x}(t)$. The two dots over the x indicate a *second* derivative.

The displacement, velocity, and acceleration functions are therefore connected as follows:



To obtain acceleration from velocity, we must calculate a rate per unit of time. So, for displacement in metres and time in seconds, the units of acceleration are m s^{-2}.

Example 5 **Self Tutor**

A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 6t^2 + 9t - 1$ cm, where t is the time in seconds, $t \geqslant 0$.

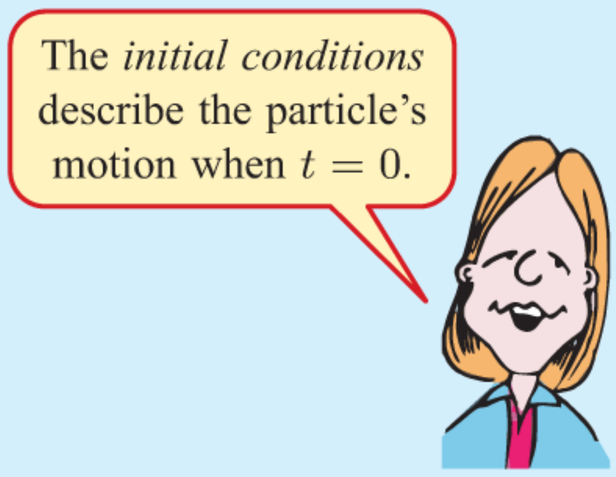
a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion at this instant.

c Find the position of the particle when it changes direction.

d Draw a motion diagram for the particle.

e Find the total distance travelled in the first 3 seconds.



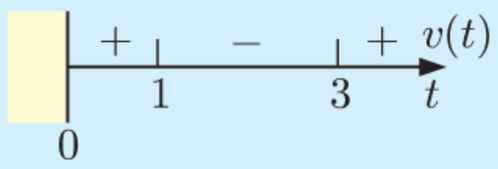
a $s(t) = t^3 - 6t^2 + 9t - 1$ cm

$\therefore\ v(t) = 3t^2 - 12t + 9 \qquad \{v(t) = s'(t)\}$

$= 3(t^2 - 4t + 3)$

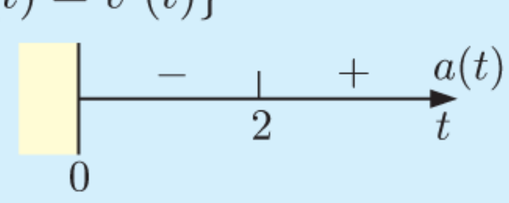
$= 3(t-1)(t-3)\ \text{cm s}^{-1}$

which has sign diagram:



and $a(t) = 6t - 12\ \text{cm s}^{-2} \qquad \{a(t) = v'(t)\}$

which has sign diagram:



b When $t = 0$, $s(0) = -1$ cm, $v(0) = 9\ \text{cm s}^{-1}$, and $a(0) = -12\ \text{cm s}^{-2}$

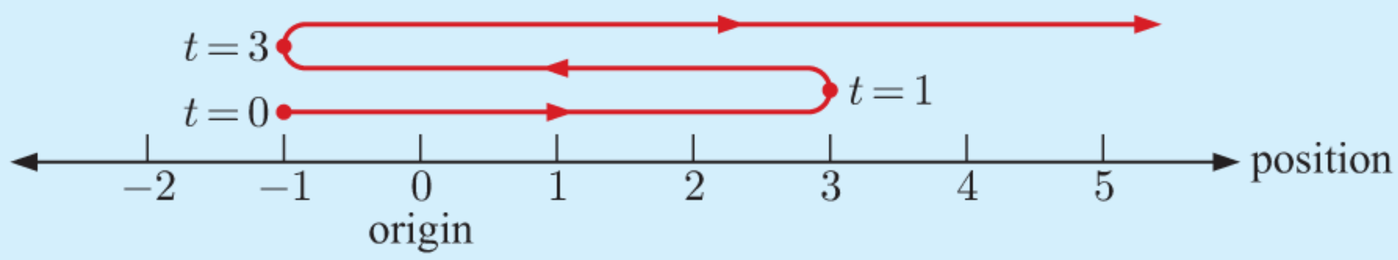
$\therefore$ the particle is 1 cm to the left of O, moving to the right with speed $9\ \text{cm s}^{-1}$, and has acceleration $-12\ \text{cm s}^{-2}$.

c Since $v(t)$ changes sign when $t = 1$ and $t = 3$, a change of direction occurs at these instants.

$s(1) = 1 - 6 + 9 - 1 = 3$, and $s(3) = 27 - 54 + 27 - 1 = -1$.

So, the particle changes direction when it is 3 cm to the right of O, and when it is 1 cm to the left of O.

d



e The total distance travelled $= 4 + 4 = 8$ cm.

EXERCISE 24C

1 A particle moves with velocity function $v(t) = 10t - t^2$ cm s^{-1}, $t \geqslant 0$. Find:

a the velocity of the particle when $t = 2$ seconds

b the average acceleration of the particle from $t = 1$ to $t = 3$ seconds

c the acceleration function $a(t)$

d the instantaneous acceleration of the particle when $t = 3$ seconds.

2 An object moves in a straight line with displacement function $x(t) = t^3 - t^2 - 5$ metres at time t seconds, $t \geqslant 0$.

a Find the object's displacement, velocity, and acceleration when $t = 2$ seconds.

b Find the time at which the object has zero acceleration.

3 A stone is fired from a catapult so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.

a Find the velocity and acceleration functions for the stone, and draw sign diagrams for each function.

b Find the initial position and velocity of the stone.

c Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.

d Find the maximum height reached by the stone.

e Find the time taken for the stone to hit the ground.

4 A particle P moves in a straight line with displacement function $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Discuss the velocity of P as $t \to \infty$.

d Discuss the acceleration of P as $t \to \infty$.

5 An object has displacement function $x(t) = t - \ln(2t + 1)$ cm, where t is in seconds, $t \geqslant 0$.

a Show that the object is initially at the origin.

b Find the velocity function $\dot{x}(t)$.

c Over what time interval is the object moving:

i to the right **ii** to the left?

d Find the acceleration function $\ddot{x}(t)$.

e Show that the object's acceleration is positive for all $t \geqslant 0$.

f Find the acceleration of the object after 2 seconds.

g Find the distance travelled by the object in the first 2 seconds.

6 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, where $t \geqslant 0$, t in seconds.

a State the initial velocity of the particle.

b Find the velocity of the particle after 3 seconds.

c How long will it take for the particle's velocity to reach 45 m s^{-1}?

d Discuss $v(t)$ as $t \to \infty$.

e Show that the particle's acceleration is always positive.

f Find the exact time at which the acceleration of the particle is 2 m s^{-2}.

g Draw the graph of $v(t)$ against t.

h Find the total distance travelled by the particle in the first 3 seconds of motion.

Example 6

Self Tutor

A train is initially at rest at a station. It begins to accelerate according to the function $a(t) = \frac{1}{2}e^{-\frac{t}{100}}$ m s^{-2}.



a Find the velocity function of the train, and sketch its graph.

b How long will it take for the train to reach the speed 40 m s^{-1}?

c How far will the train travel in the first 5 minutes?

a $a(t) = \frac{1}{2}e^{-\frac{t}{100}}$ m s^{-2}

$\therefore\ v(t) = \int \frac{1}{2}e^{-\frac{t}{100}}\,dt$

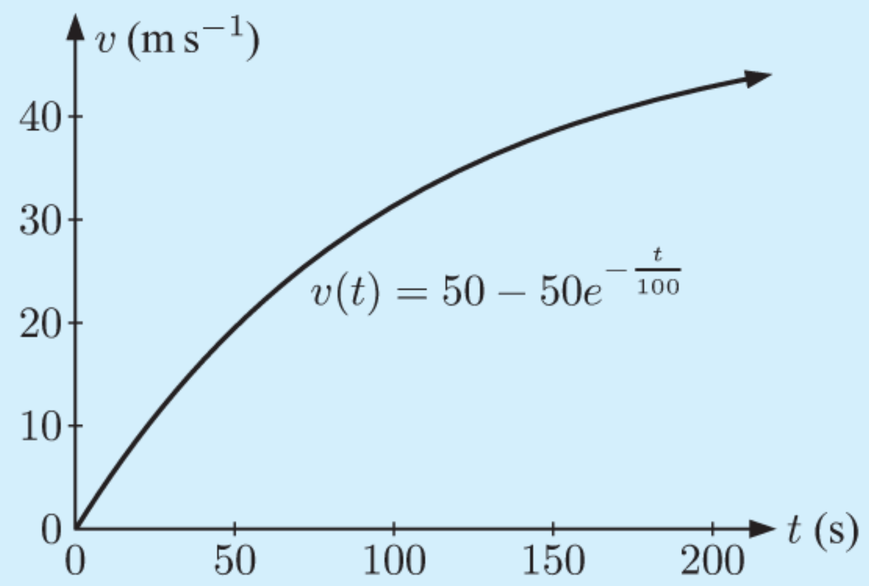
$= -50e^{-\frac{t}{100}} + c$ m s^{-1}

The train was initially stationary.

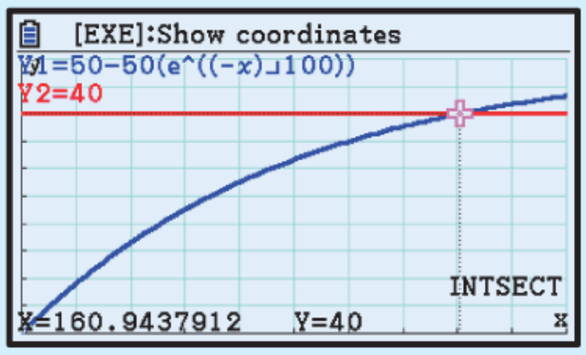
$\therefore\ -50e^0 + c = 0$

$\therefore\ c = 50$

$\therefore\ v(t) = 50 - 50e^{-\frac{t}{100}}$ m s^{-1}



b If $v = 40$ then $50 - 50e^{-\frac{t}{100}} = 40$



Using technology, it will take ≈ 161 seconds for the train to reach 40 m s^{-1}.

c The train does not change direction.

$\therefore$ distance travelled

$= \int_0^{300} (50 - 50e^{-\frac{t}{100}})\,dt$

$= \left[50t + 5000e^{-\frac{t}{100}}\right]_0^{300}$

$= 50(300) + 5000e^{-3} - 5000e^0$

$= 10\,000 + \frac{5000}{e^3}$

$\approx 10\,200$ m

7 A train moves along a straight track with acceleration $a(t) = \frac{t}{10} - 3$ m s^{-2}. The initial velocity of the train is 45 m s^{-1}.

a Determine the velocity function $v(t)$.

b Evaluate $\int_0^{60} v(t)\,dt$ and explain what this value represents.

8 An object has initial velocity 20 m s^{-1} as it moves in a straight line with acceleration function $\ddot{x}(t) = 4e^{-\frac{t}{20}}$ m s^{-2}.

a Show that as t increases, the object approaches a limiting velocity.

b Find the total distance travelled in the first 10 seconds of motion.

9 A particle is initially stationary at the origin. It accelerates according to the function $a(t) = \dfrac{2}{(t+1)^3}$ m s^{-2}.

a Find the velocity function $v(t)$ for the particle.

b Find the displacement function $s(t)$ for the particle.

c Describe the motion of the particle at the time $t = 2$ seconds.

D SPEED

We have seen that velocities have size (magnitude) and sign (direction). In contrast, speed simply measures *how fast* something is travelling, regardless of the direction of travel. Speed is a *scalar* quantity which has size but no sign. Speed cannot be negative.

The **speed** S of an object at any instant is the magnitude of the object's velocity.

$$S = |v|$$

For example, an object with velocity -5 m s^{-1} is moving to the left with speed 5 m s^{-1}.

To determine when the speed of an object P with displacement $s(t)$ is increasing or decreasing, we use a **sign test**.

- If the signs of $v(t)$ and $a(t)$ are the same (both positive or both negative), then the speed of P is increasing.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the speed of P is decreasing.

DISCUSSION

Discuss why the sign test for speed works by considering each case below:

- The object is travelling to the right $(v > 0)$ and the acceleration $a > 0$.
- The object is travelling to the right $(v > 0)$ and the acceleration $a < 0$.
- The object is travelling to the left $(v < 0)$ and the acceleration $a > 0$.
- The object is travelling to the left $(v < 0)$ and the acceleration $a < 0$.

Example 7 **Self Tutor**

A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion of the particle at this instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the position of the particle when it changes direction.

e Draw a motion diagram for the particle.

f For what time interval is the particle's speed increasing?

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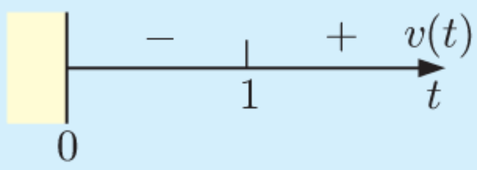
a $s(t) = t^3 - 3t + 1$ cm

$\therefore\ v(t) = 3t^2 - 3$ $\{v(t) = s'(t)\}$

$= 3(t^2 - 1)$

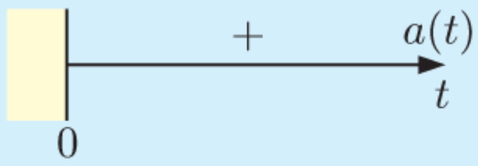
$= 3(t+1)(t-1)$ cm s^{-1}

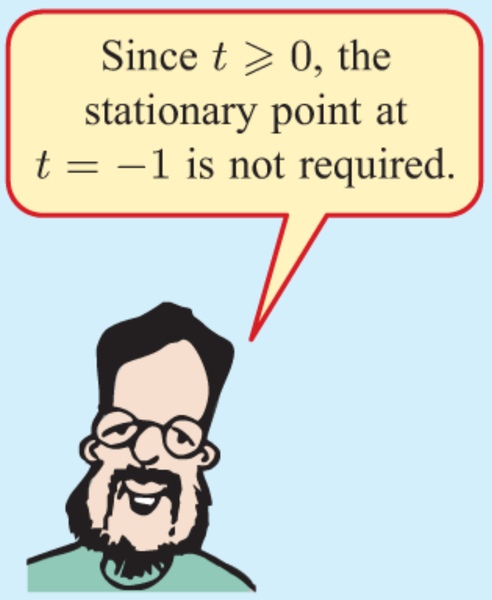
which has sign diagram:



and $a(t) = 6t$ cm s^{-2} $\{a(t) = v'(t)\}$

which has sign diagram:





b When $t = 0$, $s(0) = 1$ cm

$v(0) = -3$ cm s^{-1}

$a(0) = 0$ cm s^{-2}

$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.

c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm

$v(2) = 12 - 3 = 9$ cm s^{-1}

$a(2) = 12$ cm s^{-2}

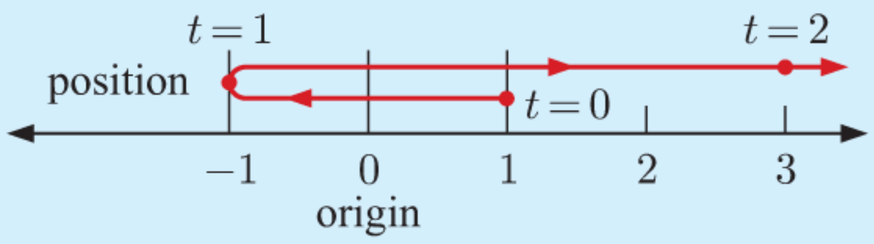
$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 cm s^{-1}.

Since a and v have the same sign, the speed of the particle is increasing.

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.

$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

e



f Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

EXERCISE 24D

1 An object moves in a straight line with position from O given by $s(t) = t^2 - 6t + 7$ m, where t is in seconds, $t \geqslant 0$.

a Find expressions for the object's velocity and acceleration, and show sign diagrams for each of them.

b Find the speed of the object after:

i 1 second **ii** 4 seconds.

c Find the initial conditions and hence describe the motion at this instant.

d Find the position of the object when it changes direction.

e Draw a motion diagram for the object.

f During which time interval is the object's speed decreasing?

2 When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres, where t is the time in seconds.

a From what distance above the ground was the ball released?

b Find $s'(t)$ and state what it represents.

c Find the maximum height reached by the ball.

d Find the ball's speed:

i when released **ii** at $t = 2$ s **iii** at $t = 5$ s.

3 A particle moves in a straight line with displacement function $x(t) = 12t - 2t^3 - 1$ cm where t is in seconds, $t \geqslant 0$.

a Find velocity and acceleration functions for the particle's motion.

b Find the initial conditions, and interpret their meaning.

c Find the times and positions when the particle reverses direction.

d At what times is the particle's:

i speed increasing **ii** velocity increasing?

e Draw a motion diagram for the particle.

4 A particle has displacement function $s(t) = 4 - \sqrt{t+1}$ m, where t is in seconds, $t \geqslant 0$.

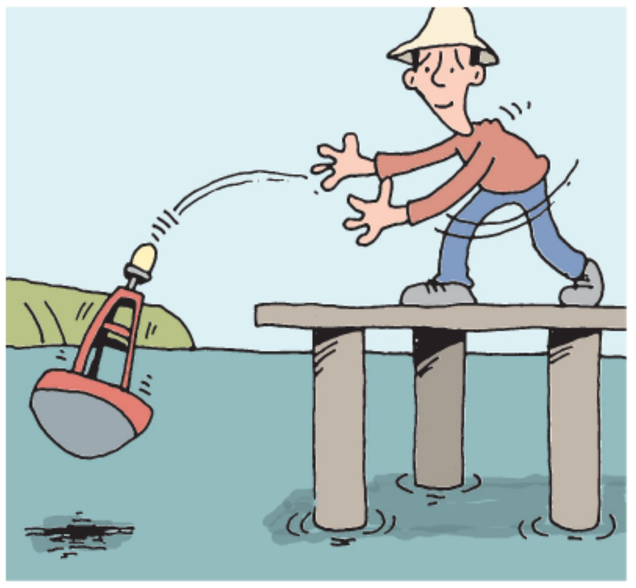
a Find the velocity and acceleration functions for the particle's motion, and draw sign diagrams for each of them.

b Find the initial conditions and interpret their meaning.

c Describe the motion of the particle after 3 seconds.

d Describe what is happening to the speed of the particle.

5 A flotation device is thrown from a jetty into the water below. It takes k seconds for the device to reach the water. The height of the device above sea level is given by $s(t) = -4.9t^2 + 4.9t + 8$ m, where t is in seconds, $0 \leqslant t \leqslant k$ seconds.



a Find the value of k.

b Draw sign diagrams for the device's velocity and acceleration functions.

c Was the speed of the device increasing or decreasing after:

i 0.2 seconds **ii** 1 second?

6 A particle P moves along the x-axis with position given by $x(t) = 1 - 2\cos t$ cm, $t \geqslant 0$ seconds.

a State the initial position, velocity, and acceleration of P.

b Describe the motion of P when $t = \frac{\pi}{4}$ seconds.

c Find the times when the particle reverses direction on $0 < t < 2\pi$, and find the position of the particle at these instants.

d When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

7 A dog paces back and forth along a fence, guarding its owner's property. The dog's horizontal displacement relative to its kennel is given by $x(t) = 8\sin\frac{t}{2}$ m, $t \geqslant 0$ seconds.

a Is the dog to the left or the right of its kennel after: **i** 3 seconds **ii** 7 seconds?

b Find the velocity function $\dot{x}(t)$.

c Is the dog moving to the left or the right after: **i** 4 seconds **ii** 10 seconds?

d Find the acceleration function $\ddot{x}(t)$.

e Is the dog's speed increasing or decreasing after 2 seconds?

f Show that the dog's speed is maximised when it is moving past its kennel.

8 The velocity of an object after t seconds is given by $v(t) = 25te^{-2t}$ cm s^{-1}, $t \geqslant 0$.

a Use technology to help sketch the velocity function.

b Show that the object's acceleration at time t is given by $a(t) = 25(1-2t)e^{-2t}$ cm s^{-2}, $t \geqslant 0$.

c When is the velocity of the object increasing?

d When is the speed of the object decreasing?

9 A lion is chasing a zebra. The zebra notices the lion when the lion is 40 m away. From this time, the lion approaches with speed $v_1(t) = 15e^{-0.1t}$ m s^{-1}, and the zebra runs away with speed $v_2(t) = 20 - 20e^{-0.1t}$ m s^{-1}.

a Find the speed of each animal after 1 second.

b Show that the lion's speed decreases over time, whereas the zebra's speed increases over time.

c Find $\int_0^3 v_1(t)\,dt$, and interpret your answer.

d Find $\int_0^3 [v_1(t) - v_2(t)]\,dt$, and interpret your answer.

e Explain why the lion will be closest to the zebra when $v_1(t) = v_2(t)$.

f Use technology to find the time at which $v_1(t) = v_2(t)$.

g Did the lion catch the zebra? If not, how close did the lion get to the zebra?

E VELOCITY AND ACCELERATION IN TERMS OF DISPLACEMENT

We have seen how displacement, velocity, and acceleration of an object can be expressed as functions of time t. We can differentiate and integrate with respect to t to move between these quantities.

However, in some situations it is more appropriate to express the velocity or acceleration in terms of the displacement s.

For example, we could say that:

- the velocity of the water in a river is twice its displacement from the coast, in which case $v = -2s$
- the acceleration of an object in water is half its displacement from the surface. In this case $a = -\frac{1}{2}s$.

DISCUSSION

Discuss the motion of an object which has:

- velocity $v = s$, and initial displacement:
  - $s = 2$
  - $s = -2$
  - $s = 0$
- velocity $v = 5 - s$, and initial displacement:
  - $s = 5$
  - $s = 0$
  - $s = 8$

If we are given an object's velocity in terms of s, we can write its acceleration in terms of s using the formula

$$a = v\frac{dv}{ds}$$

Proof:

$$\begin{aligned} a &= \frac{dv}{dt} \\ &= \frac{dv}{ds}\frac{ds}{dt} && \{\text{chain rule}\} \\ &= \frac{dv}{ds}v && \left\{v = \frac{ds}{dt}\right\} \end{aligned}$$

Example 8 — Self Tutor

An object moves with velocity $v = 2s^2$ m s^{-1}, where s is its displacement. Write the acceleration of the object in terms of s.

$$\begin{aligned} a &= v\frac{dv}{ds} \\ &= 2s^2 \times 4s \\ &= 8s^3 \text{ m s}^{-2} \end{aligned}$$

EXERCISE 24E

1 An object moves with velocity $v = 4s^3$ m s^{-1}, where s is the displacement. Write the acceleration of the object in terms of s.

2 A particle with displacement s m moves with velocity $v = -3s^2$ m s^{-1}.

a Find the acceleration of the particle in terms of s.

b Find the velocity and acceleration of the particle when it is 4 m to the right of the origin.

3 An object with displacement s m moves with velocity $v = \frac{6}{s}$ m s^{-1}.

a Find the acceleration of the object in terms of s.

b Find the acceleration of the object when its velocity is 4 m s^{-1}.

4 When a ball is dropped from the top of a building, its velocity is proportional to the square root of its displacement from the top of the building.
Show that the acceleration of the ball is constant.

5 A particle with displacement s m moves with velocity $v = 3s - s^2$ m s^{-1}.

a Write the acceleration of the particle in terms of s.

b The particle is initially 5 m to the right of the origin.

i Find the initial velocity and acceleration of the particle.

ii Describe the displacement, velocity, and acceleration of the particle as time progresses.

DISCUSSION

From the formula $a = v\dfrac{dv}{ds}$, Lewis concludes that when an object's velocity is zero, its acceleration must also be zero.

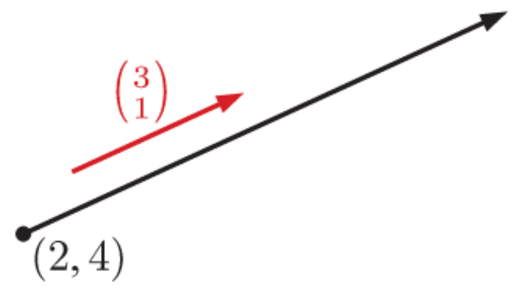
However, Lewis observes that when a projectile reaches its maximum height, its velocity is zero, but its acceleration is $g \approx -9.8$ m s^{-2}.

Can you identify the error in Lewis' thinking?

F MOTION WITH VARIABLE VELOCITY

In **Chapter 10** we used vectors to study the motion of an object with constant velocity.

For example, if an object moves from $(2,\ 4)$ with velocity vector $\begin{pmatrix}3\\1\end{pmatrix}$, its motion is given in vector form by $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix}t$, $t \geqslant 0$, or in parametric form by $x(t) = 2 + 3t$, $y(t) = 4 + t$, $t \geqslant 0$.



In this Section we will consider the movement of an object whose velocity varies over time.

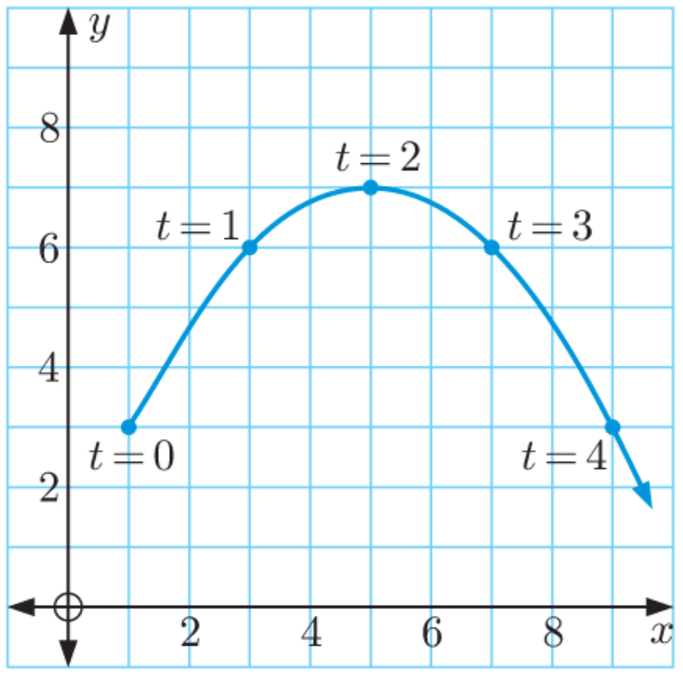
For example, suppose an object P moves with parametric equations $x(t) = 2t + 1$, $y(t) = -t^2 + 4t + 3$.

We can plot the path of P by finding its position at times $t = 0$, 1, 2, 3, and 4.

t	$x(t)$	$y(t)$
0	1	3
1	3	6
2	5	7
3	7	6
4	9	3



Click on the path plotter icon to observe the motion of P.



THE VELOCITY VECTOR

To find the velocity vector of an object moving with parametric equations $x(t)$, $y(t)$, $t \geqslant 0$, we differentiate each of the components with respect to t.

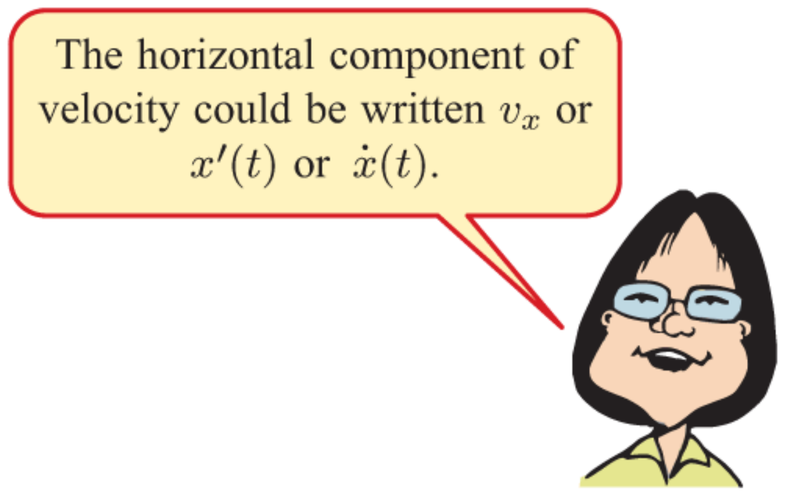
For the moving object $\mathrm{P}(x(t), y(t))$, the velocity vector $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$.

In component form, we sometimes write $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$, so $v_x = x'(t)$ and $v_y = y'(t)$.

For example, for the object P with parametric equations $x(t) = 2t + 1$, $y(t) = -t^2 + 4t + 3$, we find $v_x = x'(t) = 2$ and $v_y = y'(t) = -2t + 4$.

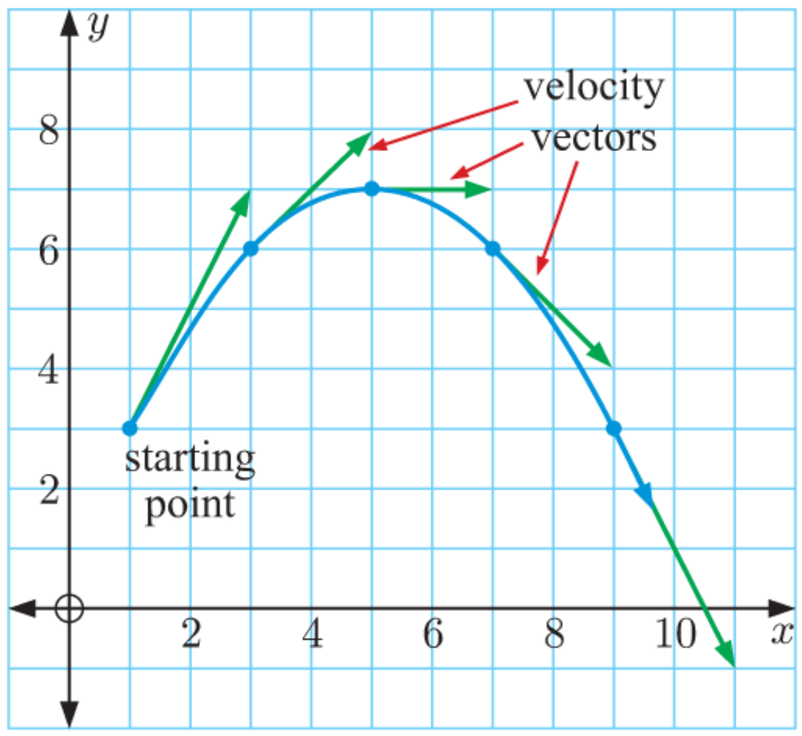
So, the velocity vector $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2 \\ -2t + 4 \end{pmatrix}$.

Notice that the x-component of the velocity is constant, but the y-component is variable.



We can use the velocity vector to find the instantaneous velocity at any time $t \geqslant 0$.

t	$x'(t)$	$y'(t)$
0	2	4
1	2	2
2	2	0
3	2	−2
4	2	−4



SPEED

As with constant velocity problems, the speed of an object at any time is given by the length of the velocity vector.

$$\text{speed} = |\mathbf{v}| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

In the diagram above, we can see that the speed of the object varies because the length of the velocity vector varies.

Example 9 **Self Tutor**

The position of a squirrel running through a field is given by $x(t) = 0.5t^2 + t$, $y(t) = 3 - t$, where $0 \leqslant t \leqslant 10$ is the time in seconds, and the distance units are metres.

a Find the initial position of the squirrel.

b Find the velocity vector of the squirrel.

c Find the speed of the squirrel after:

i 2 seconds **ii** 5 seconds.

d At what time is the squirrel running:

i parallel to $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ **ii** perpendicular to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$?

a $x(0) = 0$ and $y(0) = 3$

$\therefore$ the squirrel is initially at $(0, 3)$.

b $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} t+1 \\ -1 \end{pmatrix}$

The velocity vector can be used to find the speed and direction of the squirrel at any time.



c **i** When $t = 2$, $\mathbf{v} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\therefore$ speed $= \sqrt{3^2 + (-1)^2}$
$\approx 3.16 \text{ m s}^{-1}$

ii When $t = 5$, $\mathbf{v} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

$\therefore$ speed $= \sqrt{6^2 + (-1)^2}$
$\approx 6.08 \text{ m s}^{-1}$

d **i** The squirrel is running parallel to $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ when

$\begin{pmatrix} t+1 \\ -1 \end{pmatrix} = k\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ for some $k \in \mathbb{R}$

$\therefore \ t + 1 = -5k$ and $-1 = k$

$\therefore \ t + 1 = 5$

$\therefore \ t = 4$ s

ii The squirrel is running perpendicular to $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ when $\begin{pmatrix} t+1 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 0$

$\therefore \ 3t + 3 - 4 = 0$

$\therefore \ 3t = 1$

$\therefore \ t = \frac{1}{3}$ s

EXERCISE 24F

In the following questions, distances are in metres and time is in seconds unless otherwise specified.

1 A particle P moves with position equations $x(t) = 2t + 1$, $y(t) = -t^2 + t - 2$, $t \geqslant 0$.

a Find the initial position of P.

b Illustrate the motion of P for $t = 0$, 1, 2, and 3.

c Find the velocity vector of P.

d Find the speed of P: **i** initially **ii** after 2 seconds.

e At what time is P moving: **i** parallel to $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ **ii** perpendicular to $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$?

2 A truck moves through a construction site with position equations $x(t) = 6\sqrt{t+1} + 5$, $y(t) = 4t - 10$, $t \geqslant 0$.

a Find the initial position of the truck.

b Find the velocity vector of the truck.

c Find the initial speed of the truck.

d Describe what happens to the speed of the truck over time.

e At what time is the truck travelling at 4.5 m s^{-1}?

3 A cat is slowly approaching her toy. After t seconds, the cat has position $x(t) = 4e^{-t} - 10$, $y(t) = 3e^{-t} + 4$, $0 \leqslant t < \tau$. The distance units are metres, and τ is the time when the cat is within 10 cm of her toy at $(-10, 4)$, and can reach it with her paw.



a Find the initial position of the cat.

b Find the velocity vector of the cat.

c Discuss whether the cat is moving:

i in a straight line

ii with constant velocity.

d Discuss what happens to the position and speed of the cat over time.

e Use technology to find the value of τ.

4 Particle A is initially at S, and moves with position equations $x_A(t) = t + 2$, $y_A(t) = \frac{1}{2}t^2 - 2t - 6$, $t \geqslant 0$.

a Find the coordinates of S.

b Find the velocity vector of particle A.

c Find the speed of particle A after 1 second.

d Particle B is released from S, k seconds after particle A was released, and moves with the same position equations as particle A.

i Explain why the position of particle B at time t is given by $x_B(t) = (t - k) + 2$, $y_B(t) = \frac{1}{2}(t-k)^2 - 2(t-k) - 6$, $t \geqslant k$.

ii Given that particle B crosses the x-axis when $t = 10$, find the value of k.

iii Find the distance between the particles when $t = 6$.

iv At what time(s) are the particles moving perpendicular to each other?

5 A particle is initially at $(2, 5)$. It moves with velocity vector $\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2t - 2 \\ 3t^2 - 3 \end{pmatrix}$, $t \geqslant 0$.

a Integrate each component of the velocity vector, and hence determine the equations for the particle's motion.

b Find the position of the particle after 2 seconds.

c Find the speed of the particle after 3 seconds.

6 A speedboat initially at $(0, 0)$ moves through the water with velocity vector $\mathbf{v} = \begin{pmatrix} 6 + 2\sin t \\ 8 - 2\sin t \end{pmatrix}$, $t \geqslant 0$.

a Find the initial speed of the boat.

b If the boat has speed $S \text{ m s}^{-1}$, show that $S^2 = 100 - 8\sin t + 8\sin^2 t$.

c Find the highest and lowest speeds achieved by the boat.

d Find the position equations of the boat.

e How far is the boat from its starting position after 10 seconds?

Example 10 **Self Tutor**

An object P moves with equations $x(t) = 3\cos 2t$, $y(t) = 3\sin 2t$, $t \geqslant 0$ seconds.

a Find the initial position of P.

b Find the position of P after $\frac{\pi}{2}$ seconds.

c Show that the equations satisfy $x^2 + y^2 = 9$. Hence describe the path traced out by P.

d Find the velocity vector of P.

e At what speed is P travelling?

a $x(0) = 3$, $y(0) = 0$

$\therefore$ P is initially at $(3, 0)$.

b $x\left(\frac{\pi}{2}\right) = 3\cos\pi = -3$, $y\left(\frac{\pi}{2}\right) = 3\sin\pi = 0$

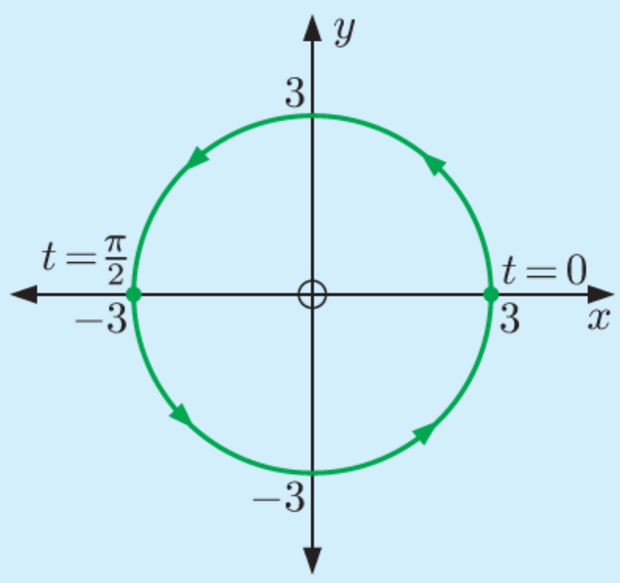
$\therefore$ after $\frac{\pi}{2}$ seconds, P is at $(-3, 0)$.

c $x^2 + y^2 = (3\cos 2t)^2 + (3\sin 2t)^2$

$= 9(\cos^2 2t + \sin^2 2t)$

$= 9 \qquad \{\cos^2\theta + \sin^2\theta = 1\}$

$\therefore$ P moves in a circle centred at $(0, 0)$ with radius 3 units.

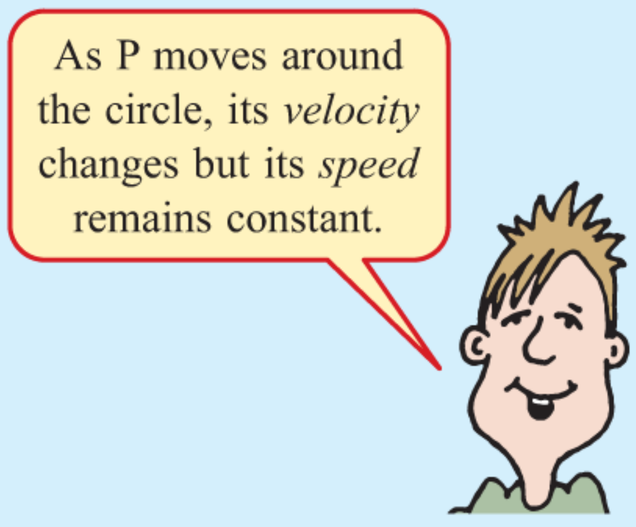


d $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -6\sin 2t \\ 6\cos 2t \end{pmatrix}$

e speed $= |\mathbf{v}| = \sqrt{(-6\sin 2t)^2 + (6\cos 2t)^2}$

$= 6\sqrt{\sin^2 2t + \cos^2 2t}$

$= 6$ units per second



7 An object P moves with position equations $x(t) = 2\cos t$, $y(t) = 2\sin t$, $t \geqslant 0$ seconds.

a Find the initial position of P.

b Find the position of P after $\frac{\pi}{2}$ seconds.

c Show that the equations satisfy $x^2 + y^2 = 4$. Hence describe the path traced out by P.

d Find the velocity vector of P.

e At what speed is P travelling?

8 A point P moves according to the parametric equations $x(t) = 5\cos 2t$, $y(t) = 5\sin 2t$, $t \geqslant 0$ seconds.

a Find the position of P:

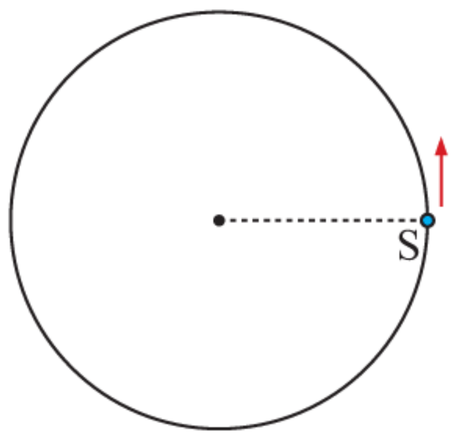
i initially **ii** after $\frac{\pi}{3}$ seconds.

b Find the position and velocity vector of P after $\frac{3\pi}{4}$ seconds.

c Describe the path traced out by P. Sketch the path, including the information found in **a** and **b**.

d At what speed is P travelling?

9



A model train starts at point S, and moves anticlockwise around a circular track. Its position is given by the parametric equations $x(t) = 30\cos\frac{t}{6}$, $y(t) = 30\sin\frac{t}{6}$, where x and y are in centimetres, and $t \geqslant 0$ is in seconds.

a Find the radius of the track.

b How long does the train take to complete a lap of the track?

c Find the velocity vector of the train.

d Find the speed at which the train is travelling.

e A second train departs from S, 4 seconds after the first, and moves anticlockwise around the track at the same speed as the first train.

i Find the position equations for the second train at time $t \geqslant 4$.

ii Find the position of each train when $t = 12$.

iii Find the straight-line distance between the trains at any time.

10 Kristen is running laps around a lake. Her position is described by the parametric equations $x(t) = 50\cos\left(\frac{\pi t}{40}\right)$, $y(t) = 60\sin\left(\frac{\pi t}{40}\right)$, where x and y are in metres and $t \geqslant 0$ is in seconds.

a Find Kristen's starting position.

b Show that $\dfrac{x^2}{50^2} + \dfrac{y^2}{60^2} = 1$.

c Hence sketch the path Kristen runs along, and describe this shape.

d How long does Kristen take to complete one lap of the lake?

e Find Kristen's speed in m s^{-1}, correct to 2 decimal places.

ACTIVITY 1 ORBITS

In physics, an **orbit** is the curved path followed by an object under the gravitational influence of a much larger nearby object.

For example, we talk about the orbits of planets around a star, and moons around a planet.

Following Kepler's laws of planetary motion, most orbits can be well approximated by ellipses, and sometimes even circles.

What to do:

1 Consider the motion of P given by $x(t) = R\cos\omega t$, $y(t) = R\sin\omega t$ where $t \geqslant 0$, $R > 0$, $\omega > 0$.

a Describe the orbit of P.

b Why do you think we specify:

i $R > 0$ **ii** $\omega > 0$?

2 What motion would be represented by:

a $x(t) = R\cos(-\omega t)$, $y(t) = R\sin(-\omega t)$, $t \geqslant 0$, $R > 0$, $\omega > 0$

b $x(t) = R\cos(\omega t + \theta)$, $y(t) = R\sin(\omega t + \theta)$, $t \geqslant 0$, $R > 0$, $\omega > 0$

c $x(t) = A\cos(\omega t + \theta)$, $y(t) = B\sin(\omega t + \theta)$, $t \geqslant 0$, $A > 0$, $B > 0$, $\omega > 0$?



Click on the icon to experiment with these cases and check your answers.

3 Research the orbits of the planets in our Solar System, including:

- the number of Earth solar days in the planet's orbital year
- how close each orbit is to being circular
- how close the planets' orbits are to lying in the same plane
- at what stage of their orbits the planets were on a particular date.

4 Use your research to create mathematical models for the orbits of the planets. Discuss the assumptions you have made in your model.

ACTIVITY 2 BÉZIER CURVES

A **Bézier curve** is a mathematically defined curve which can be constructed between any two points in the Cartesian plane. Click on the icon to obtain this Activity.



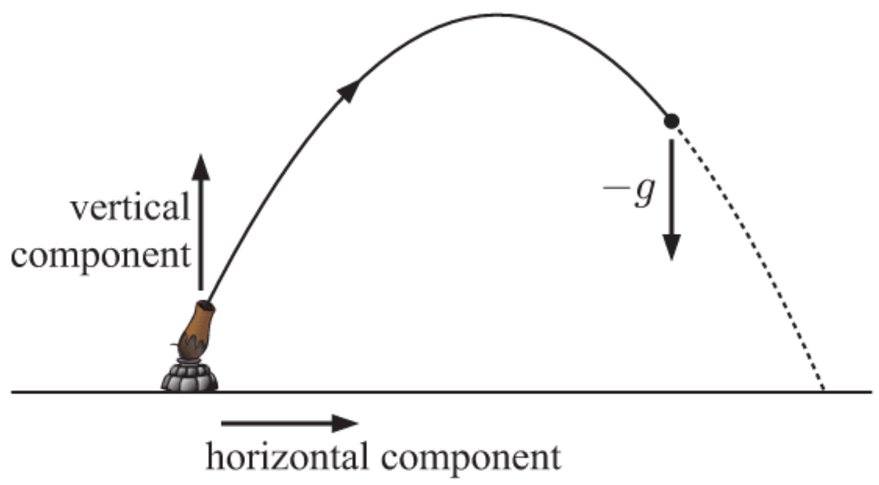
G PROJECTILE MOTION

A **projectile** is an object which is released into the air, and then moves under the force of gravity. Examples of projectiles include a ball thrown into the air, and a bullet fired from a gun.

When considering problems involving projectile motion, we assume there is no air resistance. So, once the object has been released, the *only* force acting on the object is gravity.

When an object is acted upon by gravity, it accelerates vertically downwards at $g \approx 9.8 \text{ m s}^{-2}$. So:

- the horizontal component of the projectile motion has zero acceleration
- the vertical component of the projectile motion has constant acceleration which is $-g$.



INVESTIGATION PROJECTILE MOTION

Suppose an object is released from the position (x_0, y_0) with initial velocity $\begin{pmatrix} a \\ b \end{pmatrix}$.

Under projectile motion, the object moves with constant acceleration $\mathbf{a} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$, where $g \approx 9.8 \text{ m s}^{-2}$.

What to do:

1 By integrating each component of the acceleration and using the initial velocity conditions, find the velocity vector of the object at time t.

2 By integrating each component of the velocity vector and using the initial position, find the position of the object at time t.

From the **Investigation**, you should have found that:

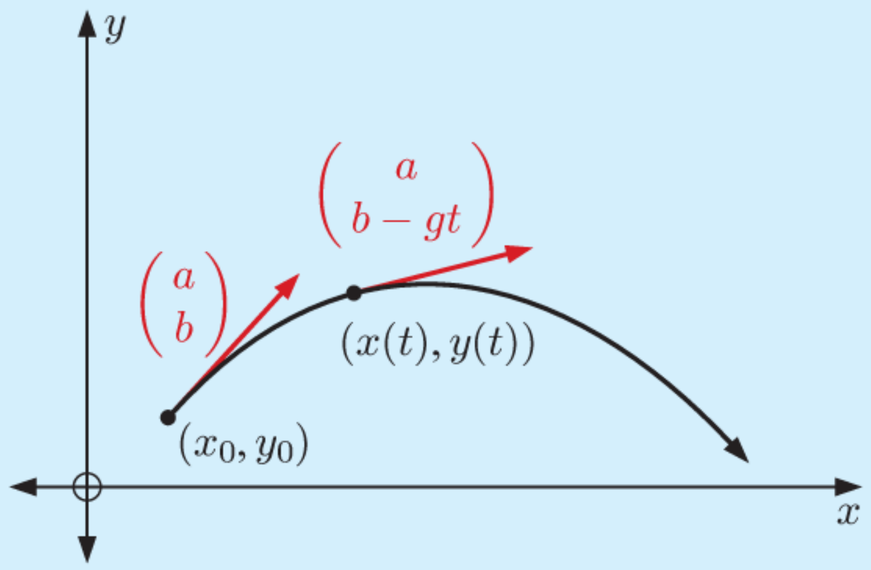
An object moving with projectile motion from (x_0, y_0) with initial velocity $\begin{pmatrix} a \\ b \end{pmatrix}$ has:

- position equations

$$x(t) = x_0 + at$$
$$y(t) = y_0 + bt - \tfrac{1}{2}gt^2, \quad t \geqslant 0$$

- velocity vector

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} a \\ b - gt \end{pmatrix}, \quad t \geqslant 0.$$



Example 11

Self Tutor

The equations $x(t) = 2 + 25t$, $y(t) = 7 + 28t - 4.9t^2$, $t \geqslant 0$ seconds, represent the flight of an arrow fired by an archer sitting in a tree. $y = 0$ is ground level. The distance units are metres.

a How high in the tree was the archer when he fired the arrow?

b What was the total time of flight?

c How far did the arrow fly horizontally?

d What was the maximum height reached by the arrow?

e Find the velocity vector for the arrow.

f What was the arrow's speed: **i** initially **ii** after 4 seconds?

g At what angle was the arrow fired?

h At what angle did the arrow strike the horizontal ground?

---

a $y(0) = 7$ $\therefore$ the archer was 7 m up the tree when he fired the arrow.

b The arrow strikes the ground when $y(t) = 0$

$$\therefore \ 7 + 28t - 4.9t^2 = 0$$
$$\therefore \ 4.9t^2 - 28t - 7 = 0$$
$$\therefore \ t = \frac{28 \pm \sqrt{28^2 - 4(4.9)(-7)}}{9.8}$$
$$\therefore \ t \approx 5.954 \quad \{\text{as } t \geqslant 0\}$$

The total flight time ≈ 5.95 seconds.

c $x(0) = 2$, and $x(5.954) \approx 2 + 25(5.954) \approx 151$ m

$\therefore$ the arrow flew $\approx 151 - 2 \approx 149$ m horizontally.

d The maximum height was reached when $y'(t) = 0$,

$$\therefore \ 28 - 9.8t = 0$$
$$\therefore \ t \approx 2.857$$

Now $y(2.857) \approx 47.0$

So, the maximum height reached was approximately 47.0 m.

e $\mathbf{v} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 25 \\ 28 - 9.8t \end{pmatrix}$

f **i** When $t = 0$, $\mathbf{v} = \begin{pmatrix} 25 \\ 28 \end{pmatrix}$

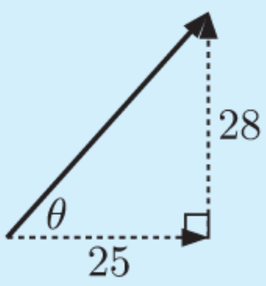
$\therefore$ speed $= \sqrt{25^2 + 28^2}$
$\approx 37.5 \text{ m s}^{-1}$

ii When $t = 4$, $\mathbf{v} = \begin{pmatrix} 25 \\ -11.2 \end{pmatrix}$

$\therefore$ speed $= \sqrt{25^2 + (-11.2)^2}$
$\approx 27.4 \text{ m s}^{-1}$

g When $t = 0$, $\mathbf{v} = \begin{pmatrix} 25 \\ 28 \end{pmatrix}$

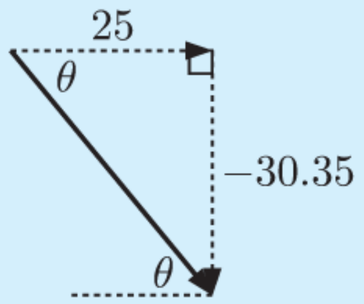
$\tan\theta = \frac{28}{25}$
$= 1.12$
$\therefore \quad \theta \approx 48.2°$



The arrow was fired at $\approx 48.2°$ above the horizontal.

h When $t \approx 5.954$, $\mathbf{v} \approx \begin{pmatrix} 25 \\ -30.35 \end{pmatrix}$

$\tan\theta \approx \frac{30.35}{25}$
$\therefore \quad \theta \approx 50.5°$



The arrow met the ground at an angle of $\approx 50.5°$.

EXERCISE 24G

In the following questions, distances are in metres and time is in seconds unless otherwise specified.

1 The equations of motion for the tip of a javelin are $x(t) = -0.6 + 20.2t$, $y(t) = 1.8 + 22.2t - 4.9t^2$, $t \geqslant 0$. $y = 0$ is ground level.

a At what position was the tip of the javelin when the javelin was released?

b Find the initial velocity and initial speed of the javelin's tip.

c How far from the line at $x = 0$ was the tip of the javelin when it was released?

d What was the total time of flight?

e Find the maximum height reached by the javelin's tip.

f Find the horizontal distance travelled by the javelin.

g At what angle was the javelin released?

h At what angle did the javelin strike the ground?

2 When a stone is thrown, its equations of motion are $x(t) = 3 + 12.7t$, $y(t) = 1.3 + 13.2t - 4.9t^2$, $t \geqslant 0$.

a What was the position of the stone: **i** initially **ii** after 2 seconds?

b Find the time taken for the stone to hit the horizontal ground at $y = 0$.

c Find the horizontal distance the stone travelled.

d What was the maximum height reached by the stone?

e Find the speed of the stone after 2 seconds.

f Find the angle of release when the stone is thrown.

g Find the angle at which the stone strikes the ground.

3 A golf ball is hit across a level field. It moves with velocity $\mathbf{v} = \begin{pmatrix} 60 \\ 18 - gt \end{pmatrix} \text{ m s}^{-1}$, where $g \approx 9.8$.

a Find the initial speed and angle of trajectory of the golf ball.

b At what time will the golf ball reach its maximum height?

c Find the maximum height the golf ball will reach.

d Do you think that in reality, the golf ball would reach this height? Explain your answer.

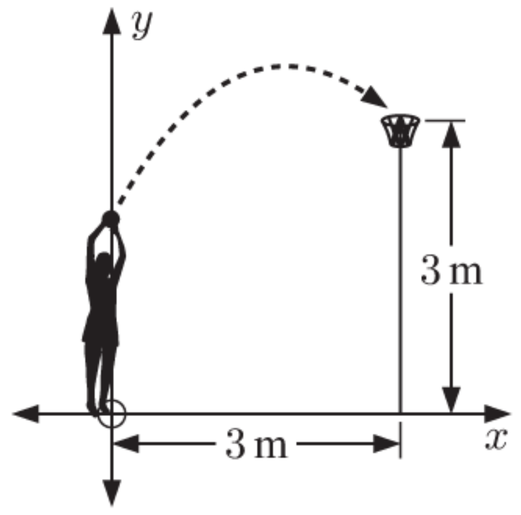
4 A rock is hurled from a castle battlement 30 m above the field below. Its velocity is given by $\mathbf{v} = \begin{pmatrix} 5 \\ 5 - gt \end{pmatrix}$ m s^{-1}.

a Find the speed of the rock: **i** initially **ii** after 0.5 seconds.

b For how long will the rock remain in flight?

c How far from the castle will the rock land?

5 A netballer is shooting at a goal which is 3 m away and 3 m high. She releases the ball 2 m above ground level.



a Given the initial velocity vector $\begin{pmatrix} a \\ b \end{pmatrix}$, find the parametric equations for the path of the ball.

b The netballer shoots at an angle of 60° to the horizontal. Find b in terms of a.

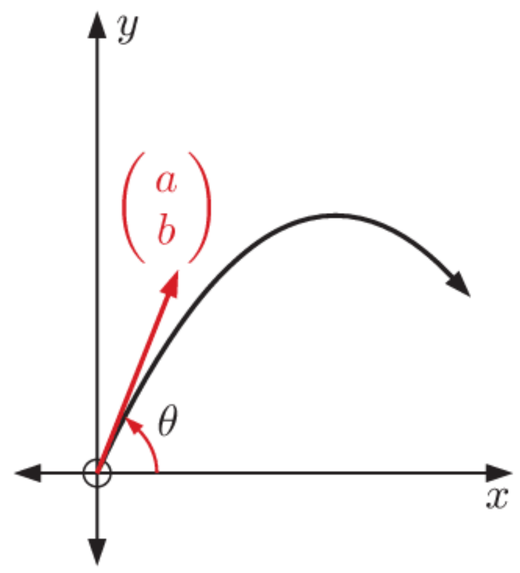
c Find a such that the ball passes through the goal.

d Find the speed at which the netballer should shoot the ball.

e Find the maximum height reached by the ball.

f At what speed will the ball pass through the goal?

6 A projectile is fired across level ground from the origin, with initial velocity $\mathbf{v}_0 = \begin{pmatrix} a \\ b \end{pmatrix}$.



a Write an expression for:

i the time when the maximum height is reached

ii the maximum height reached

iii the flight time until the projectile returns to the ground

iv the horizontal distance the projectile will travel.

b Let D be the distance of the projectile from the origin at time t.

i Write an expression for D in terms of a and b. **ii** Find $\frac{d}{dt}(D^2)$.

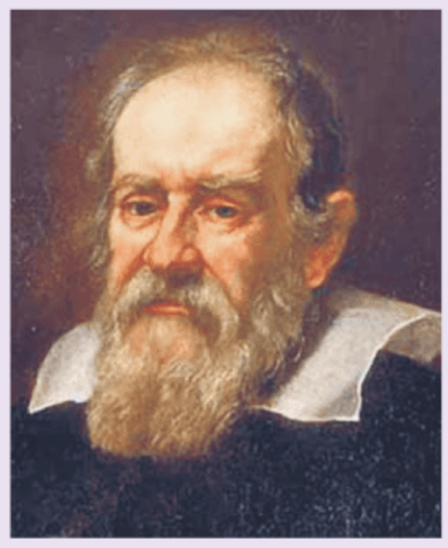
iii Suppose the projectile is fired with angle θ to the horizontal. Show that if $\tan\theta > 2\sqrt{2}$, the distance of the projectile from the origin will reach a local maximum value while in flight, whereas if $0 \leqslant \tan\theta \leqslant 2\sqrt{2}$, the projectile will continually move further from the origin while in flight.

HISTORICAL NOTE — PROJECTILE MOTION

The theory of projectile motion was developed in Europe in the 14th century, driven by the desire to improve guns and cannons. At that time, scientists were still using Aristotle's theory which suggested that forces gave rise to momentum. This would mean that as soon as you stopped pushing something (even an object on wheels) it would stop moving.

It was **Galileo Galilei** (1564 - 1642) who first suggested that in the absence of resistance, a projectile would move in a quadratic curve.

Within his research he conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.



Galileo

REVIEW SET 24A

1 An object travels with displacement function $s(t) = 12 - 2t$ m for $0 \leqslant t \leqslant 10$ s.

a Find the initial displacement of the object.

b Find the displacement of the object at time: **i** $t = 1$ s **ii** $t = 3$ s.

c At what time does the object reach the origin?

d Does the object ever change direction? Explain your answer.

e Draw a motion diagram showing the information you have found.

2 A particle P moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity and acceleration functions, and draw a sign diagram for each.

b Find the initial conditions.

c Describe the motion of the particle at time $t = 2$ seconds.

d Find the times and positions where the particle changes direction.

e Draw a diagram to illustrate the motion of P.

f Determine the time intervals when the particle's speed is increasing.

3 A particle moves in a straight line with velocity $\dot{x}(t) = t^2 - 6t + 8$ m s^{-1}, $t \geqslant 0$ seconds.

a Draw a sign diagram for $\dot{x}(t)$.

b Describe what happens to the particle in the first 5 seconds of motion.

c After 5 seconds, how far is the particle from its original position?

d Find the total distance travelled in the first 5 seconds of motion.

4 When a kayaker stops paddling, the velocity of the kayak in the following 6 seconds is given by $v(t) = 2.75 - t + 0.5t^{1.2}$ m s^{-1}, where t is the time in seconds.



a Find the velocity of the kayak:

i when the kayaker stops paddling

ii after 3 seconds.

b Show that the kayak's speed is decreasing during the 6 second period.

c Find $\int_0^2 v(t)\,dt$ and interpret your answer.

5 A particle P moves in a straight line with position relative to O given by $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find velocity and acceleration functions for P's motion.

b Describe the motion of P at $t = 3$ seconds.

c For what values of t is the particle's speed increasing?

6 A spotlight moves back and forth across a stage at a concert. Its position after t seconds is given by $x(t) = 3 + 2\sin \pi t$ m.

a Find the initial position, velocity, and acceleration of the spotlight.

b Find the times when the spotlight changes direction during $0 \leqslant t \leqslant 5$ seconds.

c Find the total distance travelled by the spotlight in the first 5 seconds.

7 A particle has velocity function $v(t) = 2\cos 4t$ m s^{-1}.

a Show that the particle oscillates between two points, and find the distance between them.

b Given that $s\left(\frac{\pi}{12}\right) = 6$ m, determine $s\left(\frac{\pi}{6}\right)$.

c Find the total distance travelled by the particle in the first π seconds.

8 When an aeroplane lands, its velocity is 65 m s^{-1}. Its acceleration t seconds after landing is given by $a(t) = -2$ m s^{-2}.

a Find the velocity $v(t)$ of the plane t seconds after landing.

b Find the displacement $s(t)$ of the plane t seconds after landing.

c **i** How long will it take the plane to reduce its speed to 3 m s^{-1}?

ii How far would the plane have travelled along the runway at this time?

9 A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity at time t seconds thereafter is given by $v(t) = \dfrac{100}{(t+2)^2}$ m s^{-1}.

a Find the initial velocity of the boat, and its velocity after 3 seconds.

b Discuss $v(t)$ as $t \to \infty$.

c Sketch the graph of $v(t)$ against t.

d Find $\displaystyle\int_0^2 v(t)\,dt$, and interpret your answer.

e From when the engine is turned off, how long will it take for the boat to travel 30 metres?

10 An object with displacement s m moves with velocity $v = s^2$ m s^{-1}.

a Write the acceleration of the object in terms of s.

b The object is initially 2 m to the left of the origin. Find its initial velocity and acceleration.

11 An object P moves with position equations $x(t) = t - 2$, $y(t) = -t^2 + 3t - 5$, $t \geqslant 0$ seconds. The distance units are metres.

a Find the initial position of P.

b Illustrate the motion of P for $t = 0$, 1, 2, and 3.

c Find the velocity vector of P.

d Find the speed of P: **i** initially **ii** after 4 seconds.

12 A seal at the zoo swims around the edge of a circular pool. Its position is given by $x(t) = 10\cos\frac{t}{2}$, $y(t) = 10\sin\frac{t}{2}$, where x and y are in metres, and $t \geqslant 0$ is in seconds.

a Find the seal's position:

i initially **ii** after π seconds.

b Is the seal swimming clockwise or anticlockwise?

c How long does it take the seal to complete a lap of the pool?

d Find the velocity vector of the seal.

e At what speed is the seal travelling?



13 A competitor in the shot put releases the shot from 1.5 m above ground level. The velocity of the shot after t seconds is $\mathbf{v} = \begin{pmatrix} 10 \\ 9 - gt \end{pmatrix}$ m s^{-1}, where $g \approx 9.8$.

a Find the speed at which the shot was released.

b Find the angle to the horizontal at which the shot was released.

c Find the maximum height reached by the shot.

d For how long was the shot in the air?

e How far did the shot travel horizontally before hitting the ground?

REVIEW SET 24B

1 An object moves in a straight line with displacement function $s(t) = t^2 + 4t + 1$ m, where t is in seconds, $t \geqslant 0$.

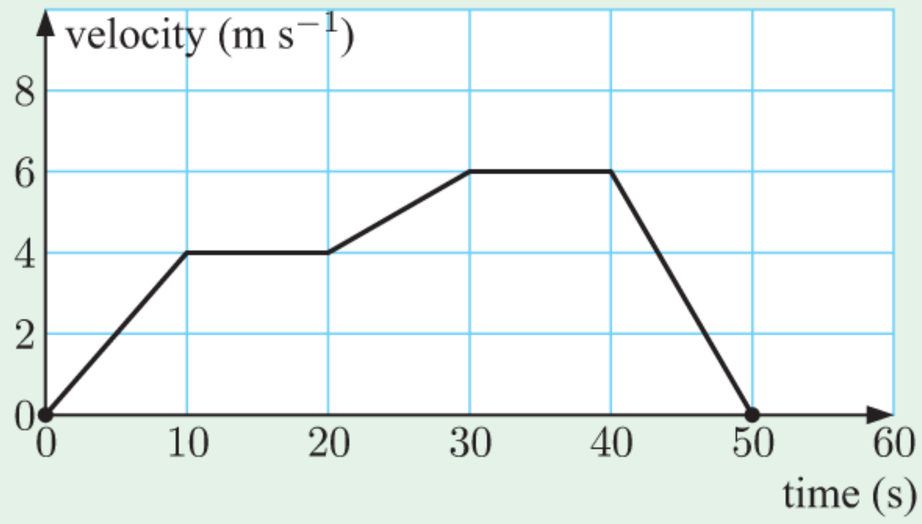
a Find the initial position of the object.

b Find the average velocity from $t = 1$ to $t = 3$ seconds.

c Find $v(t)$.

d Find the instantaneous velocity at $t = 1$ second.

2 A jogger has the velocity-time graph shown. Find the total distance travelled by the jogger.



3 At time $t = 0$ a particle passes through the origin with velocity 27 cm s^{-1}. Its acceleration t seconds later is $6t - 30$ cm s^{-2}.

a Write an expression for the particle's velocity.

b Calculate the displacement from the origin after 6 seconds.

4 A particle moves along the x-axis with position relative to origin O given by $x(t) = 3t - t\sqrt{t}$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find the velocity function $\dot{x}(t)$ and the acceleration function $\ddot{x}(t)$, and draw a sign diagram for each.

b Find the initial position and velocity of the particle and hence describe the motion at that instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the time and position when the particle reverses direction.

e Determine the time interval when the particle's speed is decreasing.

f Draw a motion diagram for the particle.

g Find the distance travelled by the particle in the first 6 seconds of motion.

5 The velocity of a human cannonball is given by $v(t) = 4.8t^2 - 0.8t^3$ m s^{-1}, $0 \leqslant t \leqslant 6$.

a Find the acceleration of the human cannonball after:

i 1 second **ii** 2 seconds **iii** 4 seconds **iv** 5 seconds.

b Find $\int_0^3 v(t)\,dt$ and interpret your answer.

c How long does it take for the human cannonball to travel 30 m?

6 A particle P moves in a straight line with position given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m, $t \geqslant 0$ seconds.

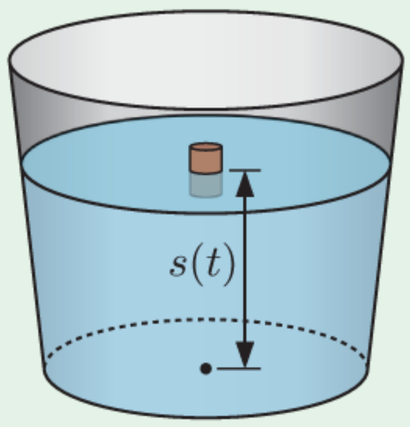
a Find the velocity and acceleration functions.

b Find the initial position, velocity, and acceleration of P.

c Sketch the graph of the velocity function.

d Find the exact time when the velocity is -44 m s^{-1}.

7 A cork bobs up and down in a bucket of water. The distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos \pi t$ cm, $t \geqslant 0$ seconds.



a Find the cork's velocity at times $t = 0$, $\frac{1}{2}$, 1, $1\frac{1}{2}$, and 2 seconds.

b Find the time intervals when the cork is falling.

8 A skier is travelling down a hill. Her velocity after t seconds is given by $v(t) = \dfrac{\left(t^{1.1} + 3t\right)^{1.5}}{10}$ m s^{-1}.

a Find the velocity of the skier after 4 seconds.

b Write an expression for the acceleration of the skier after t seconds.

c Find the acceleration of the skier after 2 seconds.

d Use technology to find the total distance travelled by the skier in the first 10 seconds.

9 A feather is falling with velocity function $\dot{x}(t) = -\frac{1}{24}t^3 - \frac{1}{12}t$ m s^{-1}. The feather is initially 2 metres above the ground.

a Find the displacement function $x(t)$.

b Find the time taken for the feather to reach the ground.

10 Tyson and Maurice are competing in a 100 m sprint.

Tyson runs with speed $v_1(t) = 10(1 - e^{-1.25t})$ m s^{-1}.

Maurice runs with speed $v_2(t) = 10.5(1 - e^{-t})$ m s^{-1}, for $t \geqslant 0$ seconds.

a Who is travelling faster after 2 seconds?

b Find $\int_0^5 v_1(t)\,dt$, and interpret your answer.

c Find displacement functions $s_1(t)$ and $s_2(t)$ for each sprinter.

d Who is winning the race after 3 seconds?

e Show that Tyson completes the 100 m in approximately 10.8 seconds.

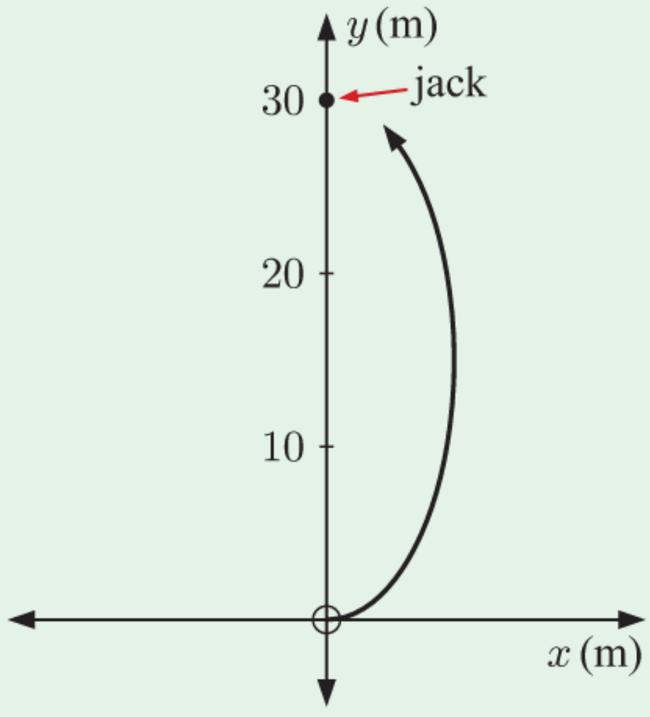
f Who will win the race?

11 The velocity of an object is inversely proportional to the square root of its displacement. Show that the acceleration of the object is always negative.

12 **a** Show that $\dfrac{d}{dt}\left(\dfrac{\sin\frac{t}{5}}{e^{\frac{t}{5}}}\right) = \dfrac{\cos\frac{t}{5} - \sin\frac{t}{5}}{5e^{\frac{t}{5}}}$.

b In a game of lawn bowls, the jack is located at $(0, 30)$. Jess bowled her ball from the origin with velocity $\mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$ where $v_x = \frac{6}{5}e^{-\frac{t}{5}}\left(\cos\frac{t}{5} - \sin\frac{t}{5}\right)$ m s^{-1}

and $v_y = 5e^{-\frac{t}{8}}$ m s^{-1}.



Her ball came to rest after 10 seconds, when it hit another ball.

i Find the initial speed of the ball.

ii Find the position equations of the ball.

iii How far was the ball from the jack when it came to rest?

iv Find the speed of Jess' ball the instant before it hit the other ball.

Chapter 25

Differential equations

Contents:

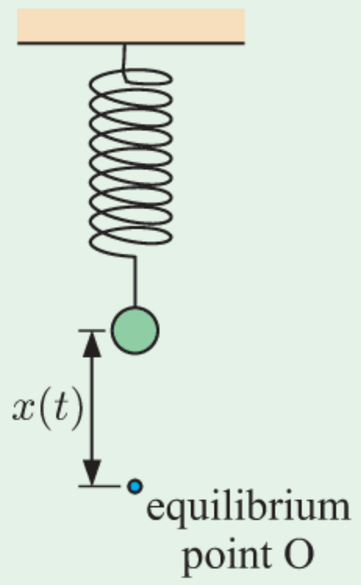
A Differential equations
B Solutions of differential equations
C Differential equations of the form $\frac{dy}{dx} = f(x)$
D Separable differential equations
E Slope fields
F Euler's method for numerical integration



OPENING PROBLEM

Consider an object attached to a spring, hanging at rest from an equilibrium position O.

Suppose we lift the object and release it from rest. We let $x(t)$ be the displacement of the object above the spring's equilibrium point O, at time t.



Things to think about:

a What will happen when the object is released?

b **Hooke's law** says that the force F exerted by a spring is proportional to the distance it has been compressed or stretched, and in the direction of the equilibrium point.

Can you explain why Hooke's law gives us $F = -kx$?

c **Newton's second law of motion** states that when a force F acts on an object with mass m, resulting in an acceleration a, the variables are related by $F = ma$.

Can you use Newton's second law of motion to explain why $m\dfrac{d^2x}{dt^2} = -kx$?

d The solution to $m\dfrac{d^2x}{dt^2} = -kx$ can be written as a cosine function. Can you explain why this is reasonable:

i using the physical context **ii** using calculus?

In the **Opening Problem**, the equation $m\dfrac{d^2x}{dt^2} = -kx$ describes the relationship between the function $x(t)$ and its second derivative. This is an example of a **differential equation**.

A DIFFERENTIAL EQUATIONS

A **differential equation** is an equation involving a *derivative* of a function.

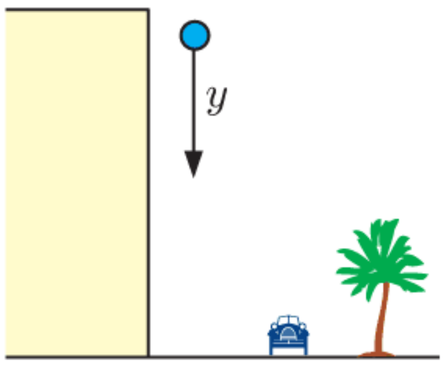
Suppose y is a function of x, so $y = y(x)$. Examples of differential equations for this function are:

- $\dfrac{dy}{dx} = \dfrac{x^2}{y}$
- $\dfrac{dy}{dx} = -0.075y^3$
- $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 4y = 0$

Differential equations not only arise in pure mathematics, but are also used to model and solve problems in applied mathematics, physics, engineering, economics, and other subjects.

For example:

A falling object



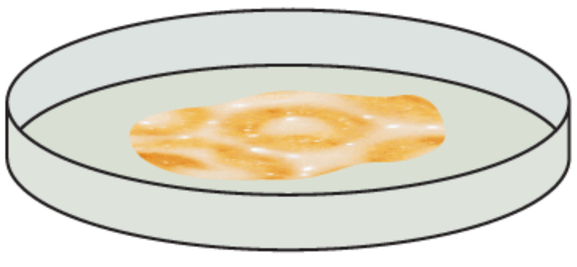
$$\frac{d^2y}{dt^2} = 9.8$$

A parachutist



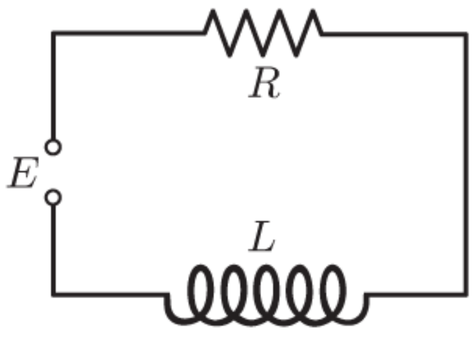
$$m\frac{dv}{dt} = mg - kv^2$$

Constrained population growth



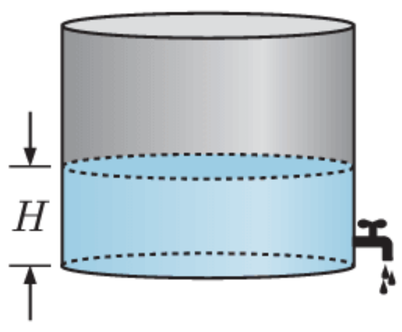
$$\frac{dP}{dt} = aP - bP^2$$

Current in an RL Circuit



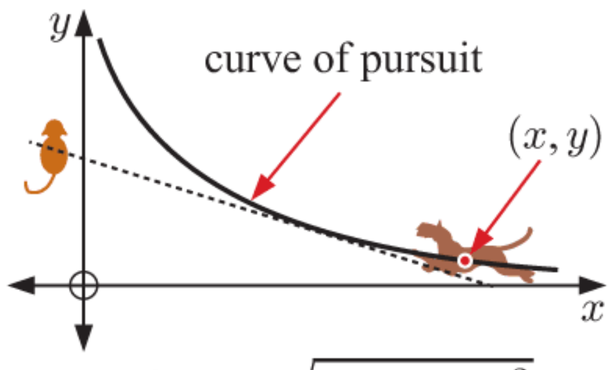
$$L\frac{dI}{dt} + RI = E$$

Water leaking from a tank



$$\frac{dH}{dt} = -a\sqrt{H}$$

A dog pursuing a cat



$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

CONSTRUCTING DIFFERENTIAL EQUATIONS

When modelling a physical system, we often use relevant physical laws to establish differential equations. Each term in the differential equation has a physical meaning.

For example, in the **Opening Problem**, we used Newton's second law and Hooke's law to establish a differential equation which can be written in the form

$$m\frac{d^2x}{dt^2} = -kx$$

resultant force (Newton's second law) — spring's "bounce" (Hooke's law)

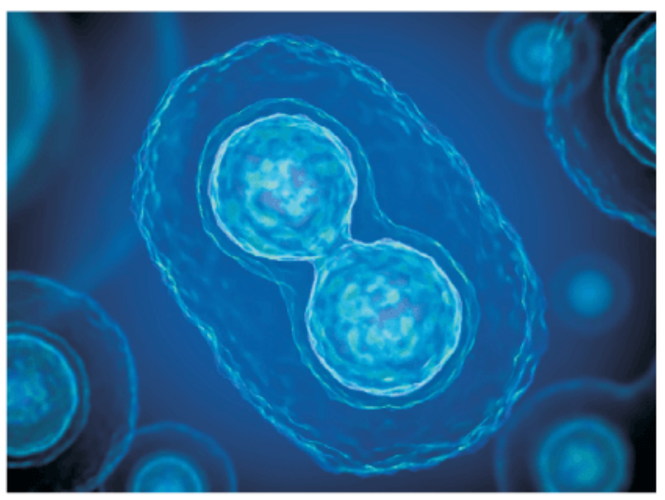
To improve our model, we might take into account the air resistance acting on the object, which is proportional to the object's velocity $\frac{dx}{dt}$. A more realistic model is therefore:

$$m\frac{d^2x}{dt^2} = -r\frac{dx}{dt} - kx$$

resultant force — air resistance — spring's "bounce"

EXERCISE 25A

1 Given sufficient resources, bacteria reproduce by "splitting" themselves at a constant rate. This means that the population P increases at a rate which is proportional to P. Write this as a differential equation.



2 Algae is a kind of plant that grows in water. If $G(t)$ is the amount of algae present in a pond at time t, the rate of algae growth is proportional to $\sqrt{G}$. Write this as a differential equation.

3 When a hot cup of coffee cools in a room with temperature τ, the rate at which its temperature T changes is proportional to the temperature difference with the surrounding room. Write this as a differential equation.

4 The equation of motion for a parachutist falling with downwards velocity v, is $m\frac{dv}{dt} = mg - kv^2$, $k > 0$. The term $m\frac{dv}{dt}$ is the resultant force as predicted by Newton's second law.

a Identify the term which models: **i** acceleration due to gravity **ii** air resistance.

b Explain the term which has a negative coefficient.

5 Newton's law of universal gravitation states that the force of attraction between two objects with masses m_1 and m_2 a distance r apart is given by $F = \dfrac{Gm_1m_2}{r^2}$ where $G \approx 6.7743 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

Consider a small satellite with mass m_S which is at altitude x above the Earth's surface.

The Earth has mass $m_E \approx 5.9722 \times 10^{24}$ kg and radius $r_E \approx 6.378 \times 10^6$ m.

a Use the law of universal gravitation and Newton's second law to show that the acceleration of the satellite due to gravity is $\ddot{x} = -\dfrac{Gm_E}{(x + r_E)^2}$.

b *Hence* estimate g, the acceleration due to gravity for an object at low altitude.

B SOLUTIONS OF DIFFERENTIAL EQUATIONS

A **solution** of a differential equation is a function y in terms of x, which satisfies the differential equation.

For example:

- a solution to the differential equation $\dfrac{dy}{dx} = 2xy$ is $y = e^{x^2}$
- a solution to the differential equation $\dfrac{dy}{dx} = y^2$ is $y = -\dfrac{1}{x}$.

Example 1 **Self Tutor**

Verify that $y = e^{x^2}$ is a solution to the differential equation $\dfrac{dy}{dx} = 2xy$.

If $y = e^{x^2}$, then $\dfrac{dy}{dx} = 2x(e^{x^2})$

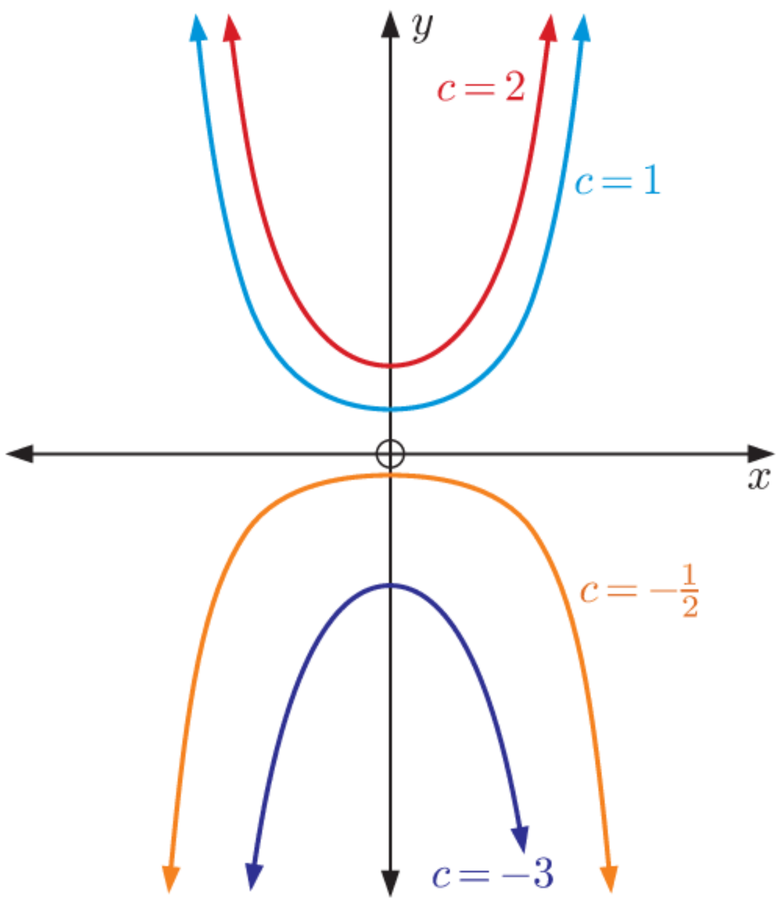
$\therefore \dfrac{dy}{dx} = 2xy$ as required.

THE GENERAL SOLUTION

In addition to the solution $y = e^{x^2}$, the differential equation $\dfrac{dy}{dx} = 2xy$ has infinitely many other solutions, including $y = 2e^{x^2}$, $y = 3e^{x^2}$, and $y = -7e^{x^2}$.

All of these solutions have the form $y = ce^{x^2}$ where c is a constant, so we say that $y = ce^{x^2}$ is the **general solution** to the differential equation.

If we are given **initial conditions** for the problem, such as a value of y or $\dfrac{dy}{dx}$ for a specific value of x, then we can evaluate c. This gives us a **particular solution** to the problem, which is one particular curve from the family of curves described by the general solution.



Example 2 **Self Tutor**

A cup of water is placed in a refrigerator. The temperature $T°C$ of the water decreases according to $\frac{dT}{dt} + \frac{1}{4}T = 1$, where t is the time in minutes.

a Show that $T = ce^{-\frac{t}{4}} + 4$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 6$, 26, and 56.

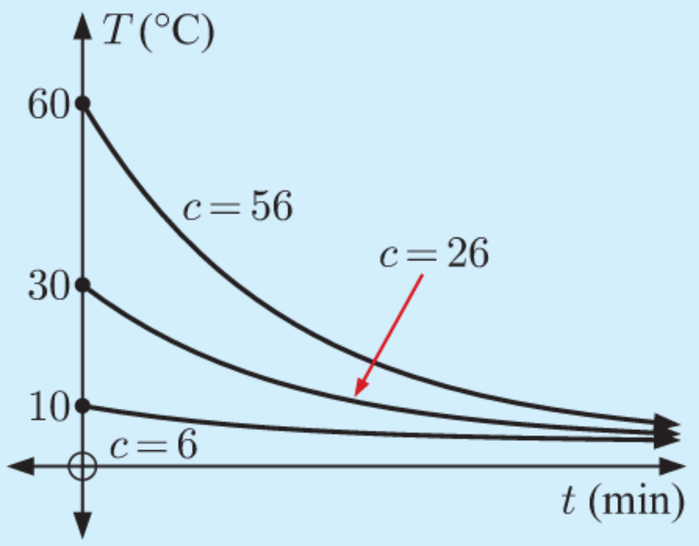
c Given that the initial water temperature was 80°C, find the particular solution.

a If $T = ce^{-\frac{t}{4}} + 4$ then $\frac{dT}{dt} = -\frac{c}{4}e^{-\frac{t}{4}}$.

$$\therefore \quad \frac{dT}{dt} + \frac{1}{4}T = -\frac{c}{4}e^{-\frac{t}{4}} + \frac{1}{4}\left(ce^{-\frac{t}{4}} + 4\right)$$
$$= -\frac{c}{4}e^{-\frac{t}{4}} + \frac{c}{4}e^{-\frac{t}{4}} + 1$$
$$= 1$$

$\therefore$ the differential equation is satisfied for any constant c.

b



c $T = ce^{-\frac{t}{4}} + 4$ is a general solution to the differential equation.

When $t = 0$, $T = 80$, so $80 = ce^{-\frac{0}{4}} + 4$

$$\therefore \quad c = 76$$

$\therefore$ the particular solution is $T = 76e^{-\frac{t}{4}} + 4$.

EXERCISE 25B

1 Verify that:

a $y = x^4$ is a solution to $\frac{dy}{dx} = 4x^3$

b $y = 5e^{2x}$ is a solution to $\frac{dy}{dx} = 2y$

c $y = \sqrt{x^2 + 1}$ is a solution to $\frac{dy}{dx} = \frac{x}{y}$

d $y = -\frac{1}{x}$ is a solution to $\frac{dy}{dx} = y^2$

e $y = 3e^{\frac{x^2}{2} + x}$ is a solution to $\frac{dy}{dx} - y = xy$.

2 Match each differential equation with the correct solution:

a $\frac{dy}{dx} = \frac{1}{2y}$ **b** $\frac{dy}{dx} = -\frac{2y}{x}$ **c** $\frac{dy}{dx} = \frac{2x}{3y^2}$

A $y = \sqrt[3]{x^2 + 1}$ **B** $y = \sqrt{x + 3}$ **C** $y = \frac{1}{x^2}$

3 Verify that for any constant c:

a $y = x^3 + c$ is a solution to $\frac{dy}{dx} = 3x^2$

b $y = ce^{-x}$ is a solution to $\frac{dy}{dx} = -y$

c $y = -\frac{2}{x^2 + c}$ is a solution to $\frac{dy}{dx} = xy^2$.

4 The population of guinea pigs P grows according to the model $\frac{dP}{dt} - \frac{P}{6} = 0$, where t is the time in months.

a Show that $P = ce^{\frac{t}{6}}$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = -2,\ 0,\ 2,\ 6,\ 10$.

c Explain why c must be positive for the model to be realistic.

d Given that there were initially 30 guinea pigs in the field, find the particular solution.

5 Consider the algae growth model $\frac{dG}{dt} = k\sqrt{G},\ k > 0$.



a Show that $G(t) = \left(\frac{kt}{2} + \sqrt{a}\right)^2$ is a solution for any constant a.

b Explain the significance of a by calculating $G(0)$.

c Show that $G(t)$ is a quadratic function whose vertex occurs for some $t < 0$.

d Sketch $G(t)$ for $t > 0$.

6 Consider the differential equation $\frac{dy}{dx} = 4x$.

a Show that $y = 2x^2 + c$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 0,\ \pm1,\ \pm2$.

c Find the particular solution which passes through $(1,\ \frac{1}{2})$.

d Find the equation of the tangent to the particular solution at $(1,\ \frac{1}{2})$.

7 Consider the differential equation $\frac{dy}{dx} = 2x - y$.

a Show that $y = 2x - 2 + ce^{-x}$ is a solution to the differential equation for any constant c.

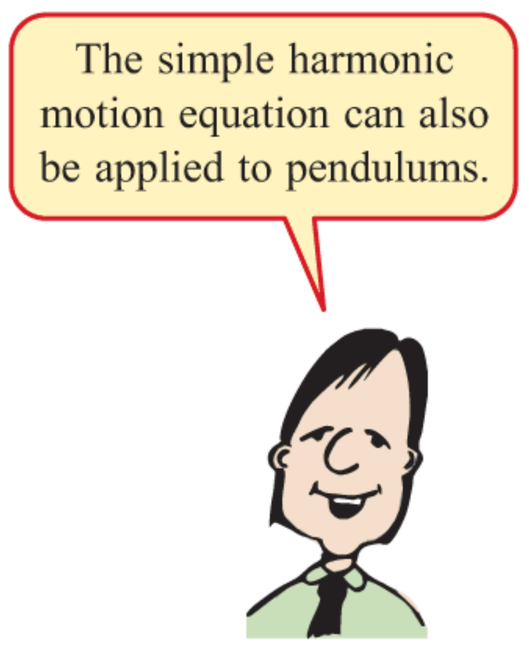
b Sketch the solution curves for $c = 0,\ \pm1,\ \pm2$.

c Find the particular solution which passes through $(0,\ 1)$.

d Find the equation of the tangent to the particular solution at $(0,\ 1)$.

8 For the frictionless mass on a spring in the **Opening Problem**, we can write the differential equation as $\frac{d^2x}{dt^2} + \omega^2 x = 0$ where $\omega^2 = \frac{k}{m}$.

This is a particularly important equation called the **simple harmonic motion** (SHM) equation.



a Verify that each of the following is a solution to the SHM equation:

i $x = \sin \omega t$ **ii** $x = \sin(\omega t + b)$

iii $x = \cos(\omega t + b)$

b Prove that $x = e^{\lambda t}$ is a solution to the SHM equation provided λ is purely imaginary. Find λ in this case.

c Prove that if $x_1(t)$ and $x_2(t)$ are both solutions to the SHM equation then $x = Ax_1(t) + Bx_2(t)$ is also a solution.

C DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dy}{dx} = f(x)$

For many differential equations, we can find analytical solutions.

The first form we consider are differential equations of the form $\frac{dy}{dx} = f(x)$. If $f(x)$ is *integrable*, which means we can write a formula for its indefinite integral, then we simply integrate with respect to x.

Example 3 Self Tutor

Find the general solution to:

a $\frac{dy}{dx} = e^{2x}$ b $\frac{dy}{dx} = \frac{2x}{x^2+1}$

a $\frac{dy}{dx} = e^{2x}$

$\therefore\ y = \int e^{2x}\,dx$

$\therefore\ y = \frac{1}{2}e^{2x} + c$

b $\frac{dy}{dx} = \frac{2x}{x^2+1}$

$\therefore\ y = \int \frac{2x}{x^2+1}\,dx$

$\therefore\ y = \ln|\,x^2+1\,| + c$

Example 4 Self Tutor

Find the particular solution to $\frac{dy}{dx} + 1 = \cos x$ given $y(0) = \frac{1}{2}$.

$\frac{dy}{dx} + 1 = \cos x$

$\therefore\ \frac{dy}{dx} = \cos x - 1$

$\therefore\ y = \int (\cos x - 1)\,dx$

$\therefore\ y = \sin x - x + c$

Now $y(0) = \frac{1}{2}$, so $\sin 0 - 0 + c = \frac{1}{2}$

$\therefore\ c = \frac{1}{2}$

So, the solution is $y = \sin x - x + \frac{1}{2}$.

EXERCISE 25C

1 Find the general solution to:

a $\frac{dy}{dx} = 4x^3$ b $\frac{dy}{dx} = x^2 + 6x$ c $\frac{dy}{dx} = e^{3x} + 4$

d $\frac{dy}{dx} = \cos x + \sin 2x$ e $\frac{dy}{dx} = \frac{1}{x+4}$ f $\frac{dy}{dx} + \frac{2}{x} = \sqrt{x}$

2 Find the general solution to:

a $\frac{dM}{dt} = \frac{3t^2}{t^3 - 4}$ **b** $\frac{dy}{dt} = \frac{t}{\sqrt{25 - t^2}}$ **c** $f'(t) = te^{-t^2+1} + 2$

3 Find the particular solution to:

a $\frac{dy}{dx} = 3x - 2$ given $y(0) = 5$ **b** $\frac{dy}{dx} = e^{3x} + 1$ given $y(0) = 0$

c $\frac{dy}{dx} = \frac{1}{x}$ given $y(2) = \ln 12$

4 Find the particular solution to:

a $\frac{dy}{dt} = 2e^{2t} - e^{-t}$ given that $y = 2.5$ when $t = \ln 2$

b $\frac{dM}{d\alpha} = \cos 2\alpha - 3\sin\alpha$ given that $M = 2$ when $\alpha = \frac{\pi}{2}$.

5 A function $f(x)$ has gradient function $f'(x) = 2x - 5$, and the point $(2, -18)$ lies on the curve. Find the value of $f(-2)$.

6 **a** Find the particular solution to the differential equation $\frac{dy}{dx} = e^x - e^{-x}$ given $y(0) = 1$.

b Sketch the graph of the particular solution.

c Find the equation of the tangent to the solution curve at the point where $x = \ln 2$.

7 A curve has gradient function $f'(x) = ax + bx^{-2}$ where a and b are constants. It passes through $(-1, -2)$ and has a turning point at $(1, 0)$. Find the function $f(x)$.

8 Find the particular solution to:

a $f''(x) = 6x - 4$ given $f'(1) = 3$ and $f(2) = 7$

b $\frac{d^2y}{dx^2} = \sin 2x$ given $y(0) = 0$ and $y(\frac{\pi}{2}) = 2\pi$.

Example 5 Self Tutor

The marginal cost of producing x urns per week is given by

$\frac{dC}{dx} = 2.15 - 0.02x + 0.000\,36x^2$ pounds per urn provided $0 \leqslant x \leqslant 120$.

The initial costs before production starts are £185.

Find the total cost of producing 100 urns per day.

The marginal cost is $\frac{dC}{dx} = 2.15 - 0.02x + 0.000\,36x^2$ pounds per urn

$\therefore\ C(x) = \int (2.15 - 0.02x + 0.000\,36x^2)\,dx$

$= 2.15x - 0.01x^2 + 0.000\,12x^3 + c$ pounds

But $C(0) = 185$ pounds, so $c = 185$

$\therefore\ C(x) = 2.15x - 0.01x^2 + 0.000\,12x^3 + 185$ pounds

$\therefore\ C(100) = 2.15(100) - 0.01(100)^2 + 0.000\,12(100)^3 + 185$ pounds

$= 420$ pounds

$\therefore$ the total cost is £420.

9 The marginal cost per day of producing x gadgets is $C'(x) = 3.15 + 0.004x$ pounds per gadget. Find the total daily production cost for 800 gadgets given that the fixed costs before production commences are £450 per day.

10 The marginal profit for producing x dinner plates per week is given by $P'(x) = 15 - 0.03x$ pounds per plate. If no plates are made then a loss of £650 each week occurs.

a Find the profit function $P(x)$.

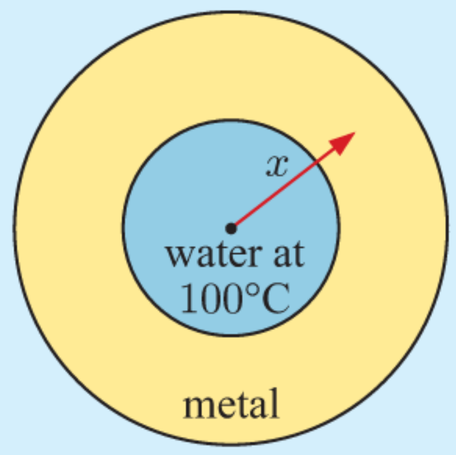
b What is the maximum profit, and when does it occur?

c What production levels enable a profit to be made?

Example 6 **Self Tutor**

A metal water pipe has an annulus cross-section as shown. The outer radius is 4 cm and the inner radius is 2 cm. Within the pipe, the water temperature is maintained at 100°C. Within the metal, the temperature drops off from inside to outside according to $\frac{dT}{dx} = -\frac{10}{x}$, where x is the distance from the central axis and $2 \leqslant x \leqslant 4$.

Find the temperature of the outer surface of the pipe.



tube cross-section

Since $\frac{dT}{dx} = -\frac{10}{x}$, $\quad T = \int -\frac{10}{x}\,dx$

$\therefore\ T = -10\ln x + c, \quad x > 0$

But when $x = 2$, $\quad T = 100$

$\therefore\ 100 = -10\ln 2 + c$

$\therefore\ c = 100 + 10\ln 2$

Thus $\quad T = -10\ln x + 100 + 10\ln 2$

$\therefore\ T = 100 + 10\ln\left(\frac{2}{x}\right)$

When $x = 4$, $\quad T = 100 + 10\ln\left(\frac{1}{2}\right) \approx 93.1$

The outer surface temperature is approximately 93.1°C.

11 An insulation tube has inner radius 0.02 m and outer radius 0.04 m. Fluid flowing through the tube maintains the temperature on the inner wall at 600°C.

Heat is lost through each metre of tube length according to *Fourier's law* $\frac{dT}{dr} = -\frac{q}{2\pi kr}$ where:

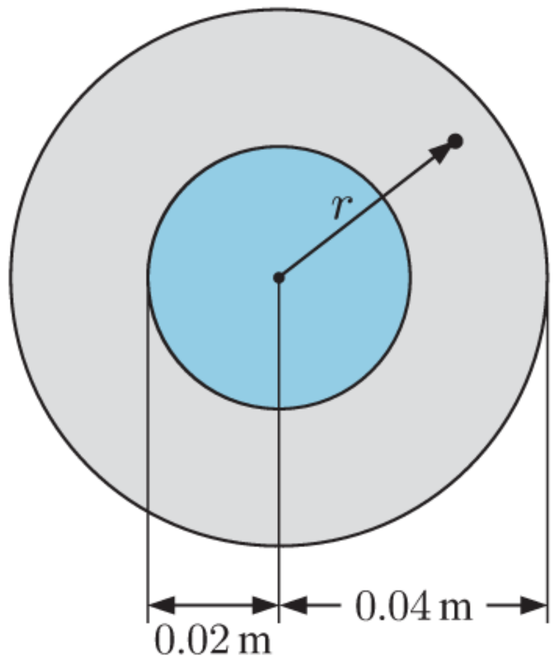
$q = 680\ \text{W m}^{-1}$ is the heat transfer rate per metre of length

$k = 0.2\ \text{W m}^{-1}\,^\circ\text{C}$ is the thermal conductivity constant

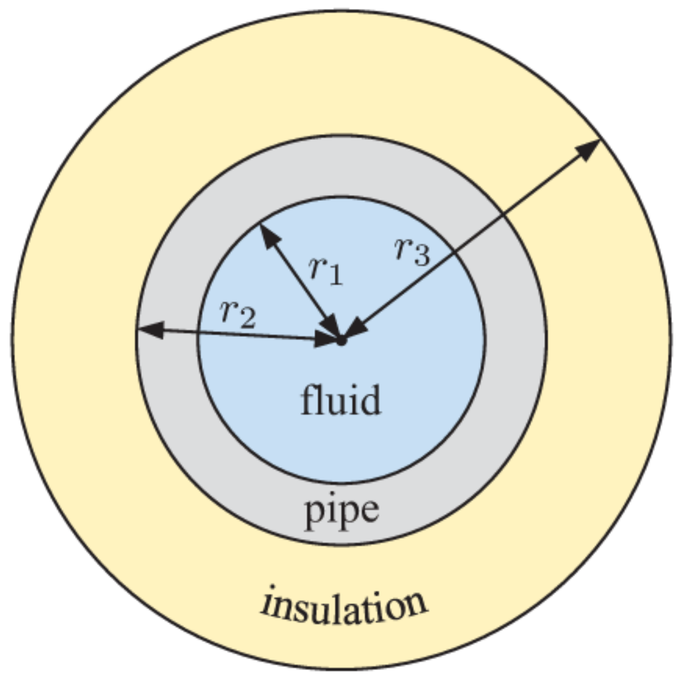
r is the radius from the centre of the tube

T is the temperature in °C.

Calculate the external temperature of the tube.



12 Fluid flows through a stainless steel pipe with thermal conductivity $k = 19\ \text{W m}^{-1}\ ^\circ\text{C}$.
The pipe has inner radius $r_1 = 0.14$ m and outer radius $r_2 = 0.20$ m.
The inner wall temperature is maintained at 400°C.
The stainless steel pipe needs to be insulated with urethane foam with $k = 0.018\ \text{W m}^{-1}\ ^\circ\text{C}$ so that the temperature on the outside of the foam is not more than 50°C.
The heat loss $q = 60\ \text{W m}^{-1}$ is constant throughout the pipe and the insulation. Use Fourier's law $\dfrac{dT}{dr} = -\dfrac{q}{2\pi kr}$ to find:

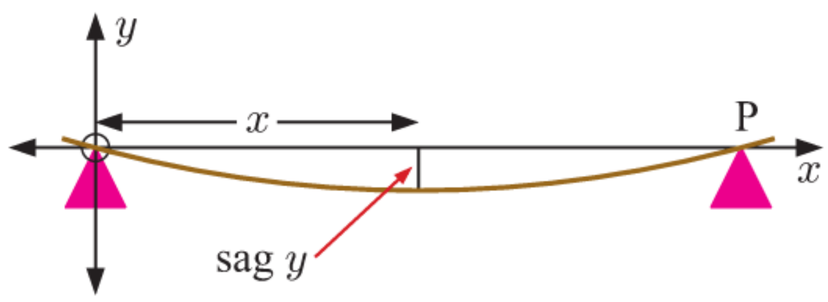


a the temperature of the outer surface of the pipe

b the outer radius r_3 of the insulation

c the thickness of insulation needed.

13 A wooden plank is supported only at its ends O and P, which are 4 m apart. The plank sags under its own weight by y m at the distance x m from O.



The differential equation $\dfrac{d^2y}{dx^2} = 0.01\left(2x - \dfrac{x^2}{2}\right)$ relates the variables x and y, for $0 \leqslant x \leqslant 4$.

a Find the function $y(x)$ which measures the sag from the horizontal at any point along the plank.

b Find the maximum sag from the horizontal. Does the position of the point of maximum sag seem reasonable?

c Find the sag 1 m away from P.

d Find the angle the plank makes with the horizontal at the point 1 m from P.

D SEPARABLE DIFFERENTIAL EQUATIONS

Differential equations of the form $\dfrac{dy}{dx} = f(x)\,g(y)$ are called **separable differential equations**.

Examples of separable differential equations include:

- $\dfrac{dy}{dx} = xy^2 \qquad \{f(x) = x,\ \ g(y) = y^2\}$
- $\dfrac{dy}{dx} = y^2 + 3 \quad \{f(x) = 1,\ \ g(y) = y^2 + 3\}$
- $\dfrac{dy}{dx} = \dfrac{x+2}{y^3} \qquad \{f(x) = x + 2,\ \ g(y) = \dfrac{1}{y^3}\}$



Differential equations of the forms $\dfrac{dy}{dx} = f(x)$ and $\dfrac{dy}{dx} = g(y)$ are special cases of separable differential equations.

These equations are called separable differential equations because we can *separate* the variables so that everything involving y is on one side, and everything involving x is on the other side.

For the separable differential equation $\dfrac{dy}{dx} = f(x)\,g(y)$,

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Integrating both sides of this equation with respect to x, $\displaystyle\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$

$\therefore$ by the chain rule, $\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx.$

The variables are now separated, so we can now find the two integrals separately.

Example 7 Self Tutor

Solve the following differential equations:

a $\dfrac{dy}{dx} = xy$ **b** $\dfrac{dy}{dx} = \dfrac{1}{y^2}$

a

$$\frac{dy}{dx} = xy$$
$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = x$$
$$\therefore \quad \int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$
$$\therefore \quad \int \frac{1}{y}\,dy = \int x\,dx$$
$$\therefore \quad \ln|y| = \tfrac{1}{2}x^2 + c$$
$$\therefore \quad y = \pm e^{\frac{1}{2}x^2 + c}$$
$$\therefore \quad y = Ae^{\frac{1}{2}x^2} \qquad \{A = \pm e^c\}$$

b

$$\frac{dy}{dx} = \frac{1}{y^2}$$
$$\therefore \quad y^2\frac{dy}{dx} = 1$$
$$\therefore \quad \int y^2\frac{dy}{dx}\,dx = \int 1\,dx$$
$$\therefore \quad \int y^2\,dy = \int 1\,dx$$
$$\therefore \quad \frac{y^3}{3} = x + c$$
$$\therefore \quad y^3 = 3x + c$$
$$\therefore \quad y = \sqrt[3]{3x + c}$$

EXERCISE 25D

1 Solve the following separable differential equations:

a $\dfrac{dy}{dx} = \dfrac{x}{y^2}$ **b** $\dfrac{dy}{dx} = \dfrac{2x}{e^y}$ **c** $\dfrac{dy}{dx} = 3xy$

d $\dfrac{dy}{dx} = 2x\sqrt{y}$ **e** $\dfrac{dy}{dx} = y\sin x$ **f** $\dfrac{dy}{dx} = -x\sqrt{y+1}$

g $\dfrac{dy}{dx} = \dfrac{y}{x}$ **h** $\dfrac{dy}{dx} = 3x^2e^y$ **i** $\dfrac{dy}{dx} = \dfrac{y+2}{x-1}$

2 Solve:

a $\dfrac{dy}{dx} = y$ **b** $\dfrac{dy}{dx} = \dfrac{1}{y}$ **c** $\dfrac{dy}{dt} = y - 4$

d $\dfrac{dP}{dt} = 3\sqrt{P}$ **e** $\dfrac{dQ}{dt} = 2Q + 3$ **f** $\dfrac{dQ}{dt} = \dfrac{1}{2Q+3}$

3 Solve the differential equation $\dfrac{dy}{dx} = y^2$. Discuss the values of x for which the solution is defined.

Example 8 Self Tutor

Solve the differential equation $\frac{dy}{dx} = e^{2y}\cos 3x$ given that $y(0) = 0$.

$$\frac{dy}{dx} = e^{2y}\cos 3x$$

$$\therefore \quad \frac{1}{e^{2y}}\frac{dy}{dx} = \cos 3x$$

$$\therefore \quad \int \frac{1}{e^{2y}}\frac{dy}{dx}\,dx = \int \cos 3x\,dx$$

$$\therefore \quad \int e^{-2y}\,dy = \int \cos 3x\,dx$$

$$\therefore \quad -\tfrac{1}{2}e^{-2y} = \tfrac{1}{3}\sin 3x + c$$

$$\therefore \quad e^{-2y} = -\tfrac{2}{3}\sin 3x + c$$

$$\therefore \quad -2y = \ln\left(-\tfrac{2}{3}\sin 3x + c\right)$$

$$\therefore \quad y = -\tfrac{1}{2}\ln\left(-\tfrac{2}{3}\sin 3x + c\right)$$

But $y(0) = 0$, so $0 = -\frac{1}{2}\ln c$

$$\therefore \quad \ln c = 0$$

$$\therefore \quad c = 1$$

The particular solution to the differential equation is $y = -\frac{1}{2}\ln\left(-\frac{2}{3}\sin 3x + 1\right)$.

4 Find the particular solution to:

a $\frac{dy}{dx} = \frac{3x}{y^2}$ given that $y(0) = 1$

b $\frac{dy}{dx} = \frac{\sqrt{y}}{3}$ given that $y(44) = 9$

c $\frac{dy}{dx} = y + yx^2$ given that $y(0) = 1$

d $\frac{dy}{dx} = \frac{3x}{\cos y}$ given that $y(1) = 0$

e $\frac{dy}{dx} = \frac{6\cos 2x}{\sqrt{y}}$ given that $y(0) = 3$

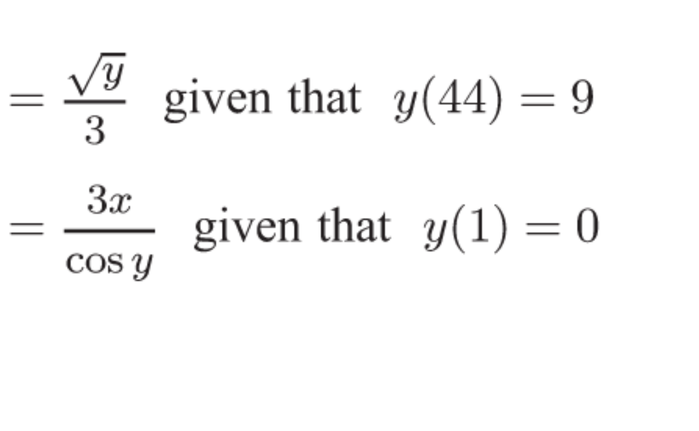
5 Find the particular solution to $\frac{dy}{dx} = 2\sqrt{y}$ given that $y(0) = 9$. State the values of x for which the solution is defined.

Example 9 Self Tutor

A salmon farm has an initial population of 80. The population grows according to the differential equation $\frac{dP}{dt} = \frac{1}{5}P$, where t is the time in years.

a Write an expression for P in terms of t.

b Find the salmon population after 10 years.



a
$$\frac{dP}{dt} = \frac{1}{5}P$$
$$\therefore \quad \frac{1}{P}\frac{dP}{dt} = \frac{1}{5}$$
$$\therefore \quad \int \frac{1}{P}\frac{dP}{dt}\,dt = \int \frac{1}{5}\,dt$$
$$\therefore \quad \int \frac{1}{P}\,dP = \int \frac{1}{5}\,dt$$
$$\therefore \quad \ln|P| = \frac{1}{5}t + c$$
$$\therefore \quad P = e^{\frac{t}{5}+c} \qquad \{\text{since } P \geqslant 0\}$$
$$\therefore \quad P = Ae^{\frac{t}{5}} \qquad \{A = e^c\}$$
When $t = 0$, $P = 80$
$$\therefore \quad 80 = Ae^0 \quad \text{and so} \quad A = 80$$
$$\therefore \quad P = 80e^{\frac{t}{5}}$$

b When $t = 10$, $P = 80e^2$
$$\therefore \quad P \approx 591$$
After 10 years, there are approximately 591 salmon.

$\frac{dP}{dt} = kP$ for some constant k indicates an **exponential relationship** between P and t.



6 A population of rabbits in a field grows according to the differential equation $\frac{dP}{dt} = \frac{1}{2}P$, where t is the time in months. The initial population was 40 rabbits.

a Write P in terms of t.

b Find the rabbit population after 6 months.

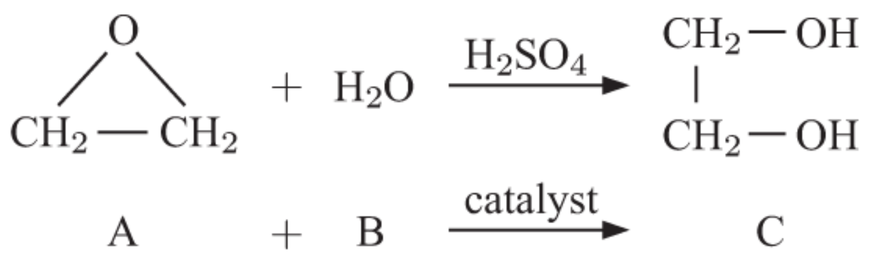
7 When a transistor radio is switched off, the current I (in milliamps) falls away according to the differential equation $\frac{dI}{dt} = -0.4I$ where t is the time in milliseconds. At the instant the radio is switched off, the current is 350 milliamps.

a Write I in terms of t.

b Find the current after 5 milliseconds.

c How long will it take for the current to fall to 20 milliamps?

8 Ethylene oxide reacts with water in the presence of the catalyst sulphuric acid to form ethylene glycol. Since water is present in excess, the rate of change in concentration of ethylene oxide (A) will be given by $\frac{dC_A}{dt} = -kC_A$ where the reaction rate constant $k = 0.31$ min^{-1}.



The initial concentration of ethylene oxide is 1 mol L^{-1}. Find the time required for 80% of the ethylene oxide to be used up.

9 In the "inversion" of raw sugar, the rate of change in the weight w kg of raw sugar is directly proportional to the weight w. After 10 hours, 80% of the sugar has been "inverted". What percentage of raw sugar remains after 30 hours?

10 A bloom of toxic blue-green algae grows in proportion to the square root of its existing size. When it is first noticed, the bloom covers 16 m^2. Its area doubles in the next 3 days. Find a function $A(t)$ for the area of the bloom after t days.

11 Water evaporates from a lake at a rate proportional to the volume of water remaining.
Suppose V is the total amount of water evaporated after t days, and V_0 is the initial volume of water in the lake.

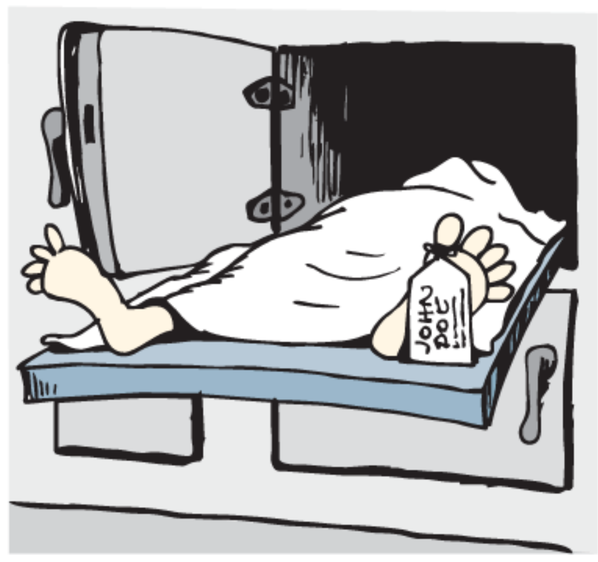
a Explain why $\dfrac{dV}{dt} = k(V_0 - V)$.

b If 50% of the water evaporates in 20 days, find the percentage of the original water remaining after 50 days without rain.

12 *Newton's law of cooling* states that the rate at which an object changes temperature is proportional to the difference between its temperature T and that of the surrounding medium T_m.

So, $\dfrac{dT}{dt} \propto (T - T_m)$.

Use Newton's law of cooling to solve these problems:

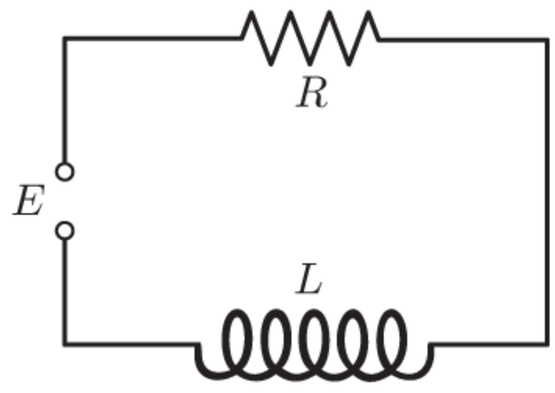


a The temperature inside a refrigerator is maintained at 5°C. An object at 100°C is placed in the refrigerator to cool. After 1 minute its temperature drops to 80°C. How long will it take for the temperature to drop to 10°C?

b At 6 am the temperature of a corpse was 13°C. By 9 am it had fallen to 9°C. Given that the temperature of a living body is 37°C and the temperature of the surroundings is constant at 5°C, estimate the time of death.

13 In an RL-circuit, the current I amps changes according to the differential equation $L\dfrac{dI}{dt} + RI = E$ where

L is the inductance in henrys,
R is the resistance in ohms,
E is the voltage drop in volts, and
t is the time in seconds.



Suppose $L = 0.3$, $R = 10$, and $E = 20$.

a Find a general solution for $I(t)$.

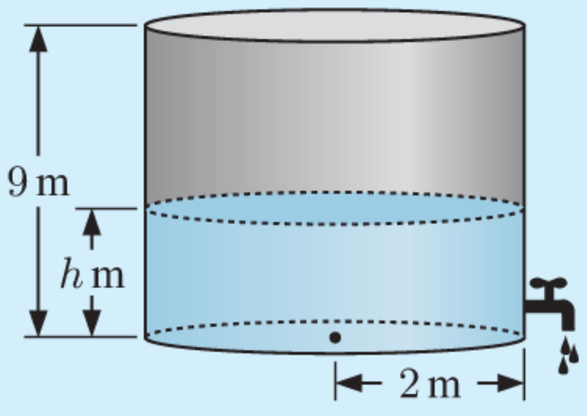
b Find a particular solution for $I(t)$ if $I(0) = 0$ amps.

c By considering I as $t \to \infty$, find the limiting current.

d Find the time required for the current to reach 99% of its limiting value.

Example 10 **Self Tutor**

Water flows out of a tap at the bottom of a large cylindrical tank with base radius 2 m. The rate at which the water flows is proportional to the square root of the depth of the water remaining in the tank. Initially the tank is full to a depth of 9 m. After 15 minutes the water is 4 m deep. How long will it take for the tank to empty?



We are given that $\dfrac{dV}{dt} \propto \sqrt{h}$ where h is the depth of the water.

$\therefore \dfrac{dV}{dt} = k\sqrt{h}$ where k is a constant.

$$\therefore \quad \frac{dV}{dh}\frac{dh}{dt} = k\sqrt{h} \qquad \{\text{chain rule}\}$$

$$\therefore \quad 4\pi\frac{dh}{dt} = k\sqrt{h} \qquad \{V = \pi r^2 h = 4\pi h \quad \therefore \quad \frac{dV}{dh} = 4\pi\}$$

$$\therefore \quad \frac{4\pi}{\sqrt{h}}\frac{dh}{dt} = k$$

$$\therefore \quad \int 4\pi h^{-\frac{1}{2}}\frac{dh}{dt}\,dt = \int k\,dt$$

$$\therefore \quad 4\pi\int h^{-\frac{1}{2}}\,dh = \int k\,dt$$

$$\therefore \quad 4\pi\frac{h^{\frac{1}{2}}}{\frac{1}{2}} = kt + c$$

$$\therefore \quad 8\pi\sqrt{h} = kt + c$$

Now when $t = 0$, $h = 9$, so $8\pi\sqrt{9} = c$

$$\therefore \quad c = 24\pi$$

$$\therefore \quad 8\pi\sqrt{h} = kt + 24\pi$$

Also, when $t = 15$, $h = 4$, so $8\pi\sqrt{4} = 15k + 24\pi$

$$\therefore \quad 15k = -8\pi$$

$$\therefore \quad k = -\tfrac{8\pi}{15}$$

So, the equation connecting the depth of the water and the time t is $8\pi\sqrt{h} = -\frac{8\pi}{15}t + 24\pi$.

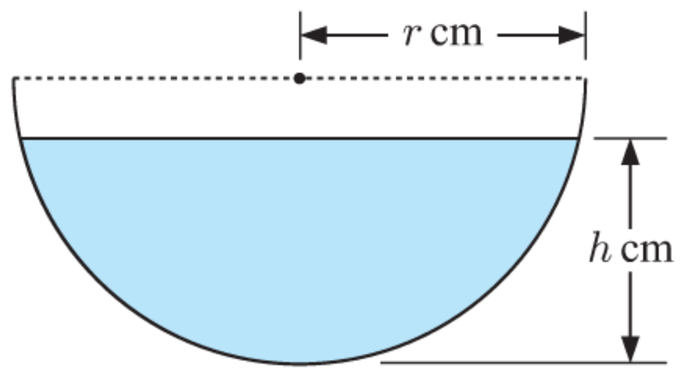
The tank is empty when $h = 0$. This occurs when $\frac{8\pi}{15}t = 24\pi$

$$\therefore \quad t = 45$$

The tank empties in 45 minutes.

14 Water flows out of a tap at the bottom of a cylindrical tank of height 4 m and radius 2 m. The tank is initially full, and the water escapes at a rate proportional to the square root of the depth of the water remaining. After 2 hours, the water is 1 m deep. How long will it take for the tank to empty?

15 Water evaporates from a hemispherical bowl with radius r cm at the rate $\frac{dV}{dt} = -r^2$, where t is the time in hours.



The volume of water of depth h, in a hemispherical bowl of radius r, is given by $V = \frac{1}{3}\pi h^2(3r - h)$.

a Use $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$ to write a differential equation connecting h and t, given that r is a constant.

b Suppose the bowl's radius is 10 cm and that initially the bowl is full of water.

i Show that $t = \frac{\pi}{300}(h^3 - 30h^2 + 2000)$.

ii Find the time taken for the depth of the water to fall to 5 cm.

iii How long will it take for the bowl to empty?

16 Since water and oil are *immiscible*, oil spilt in water will form a cylindrical patch on the surface of the water. The patch spreads at a rate proportional to the thickness of the patch, which is the height of the cylinder.

a Suppose V cm^3 of oil is spilt on a still surface. Show that the radius r of the spill after t seconds is given by $r = \sqrt[3]{\dfrac{3kV}{\pi}t + c}$ cm where c, k are constants.

b Suppose 1 litre of oil is spilt on a still lake. The initial radius of the spill is 20 cm, and after 2 seconds it has increased to 50 cm. How long will it take for the spill radius to reach 5 m?

E SLOPE FIELDS

The differential equation $\dfrac{dy}{dx} = 2x + y$ may be interpreted geometrically as a statement about the tangents to any solution curve of the equation.

The gradient of the tangent at point (x, y) on the solution curve can be found by calculating the value of $2x + y$ at that point.

For example, at the point $(1, -1)$, $\dfrac{dy}{dx} = 2(1) + (-1) = 1$. This tells us that the tangent to the solution curve at $(1, -1)$ has gradient 1.

The table alongside summarises the values of $\dfrac{dy}{dx} = 2x + y$ for the integer grid points $x, y \in [-2, 2]$.

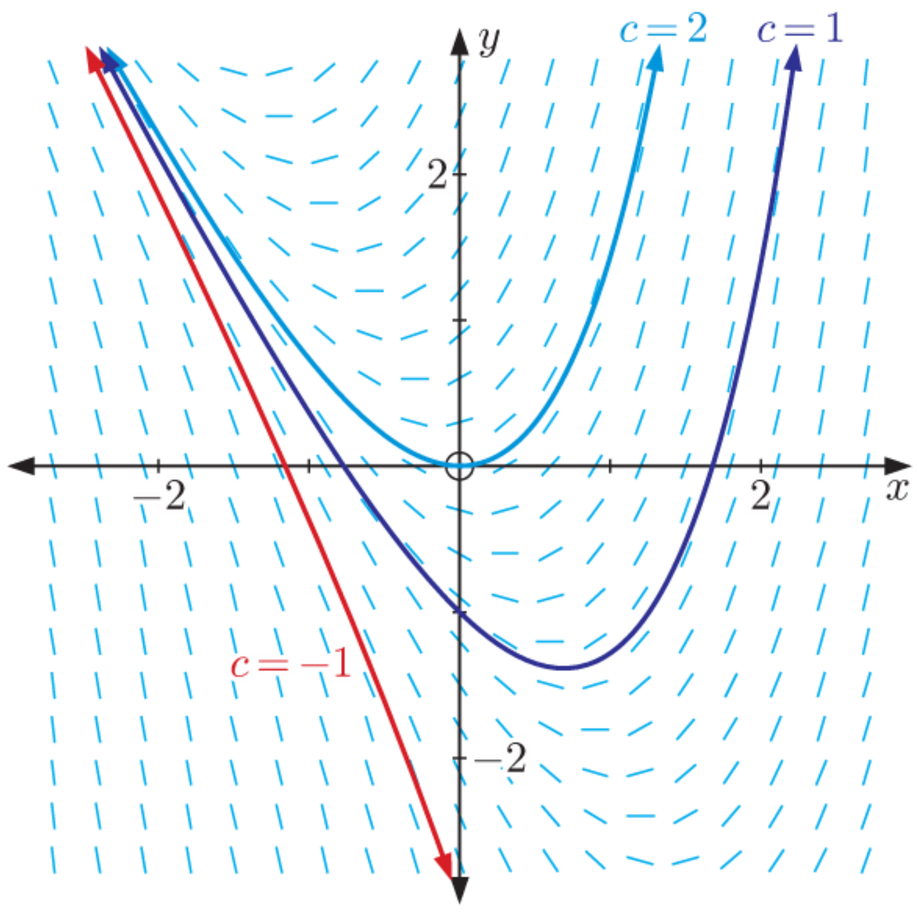
y \ x	-2	-1	0	1	2
-2	-6	-4	-2	0	2
-1	-5	-3	-1	1	3
0	-4	-2	0	2	4
1	-3	-1	1	3	5
2	-2	0	2	4	6

By representing the gradients at many different grid points with short line segments, we obtain a **slope field** of the tangents to the solution curves.

The slope field for $\dfrac{dy}{dx} = 2x + y$ is shown alongside. Three of the possible solution curves are drawn.

Each curve is tangential to any segment it touches, and remains parallel to any segment it passes near. This controls the curve's path through the slope field.

In this case the family of solution curves has the form $y = ce^x - 2x - 2$. Each solution curve corresponds to a particular value of c.

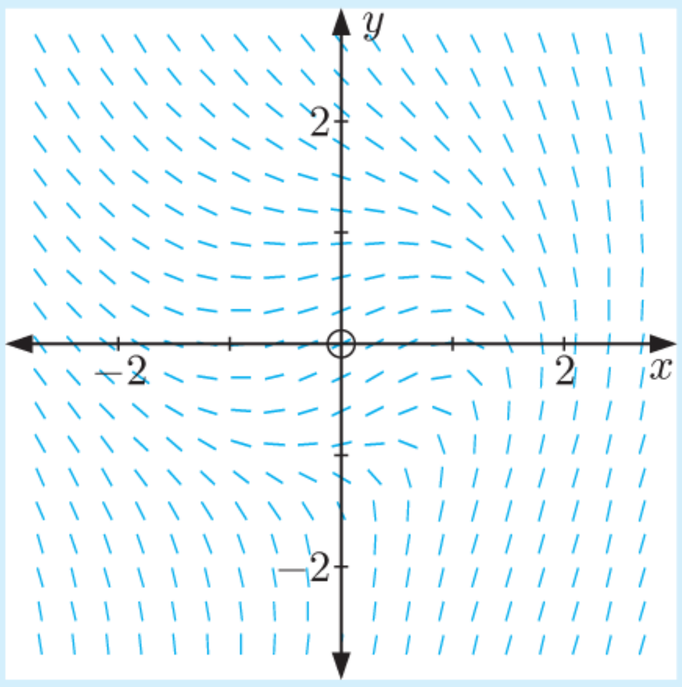


Example 11 **Self Tutor**

The slope field for $\dfrac{dy}{dx} = \dfrac{1-x^2-y^2}{y-x+2}$ is shown.

a Find the gradient of the tangent to the solution curve at $(1, 1)$.

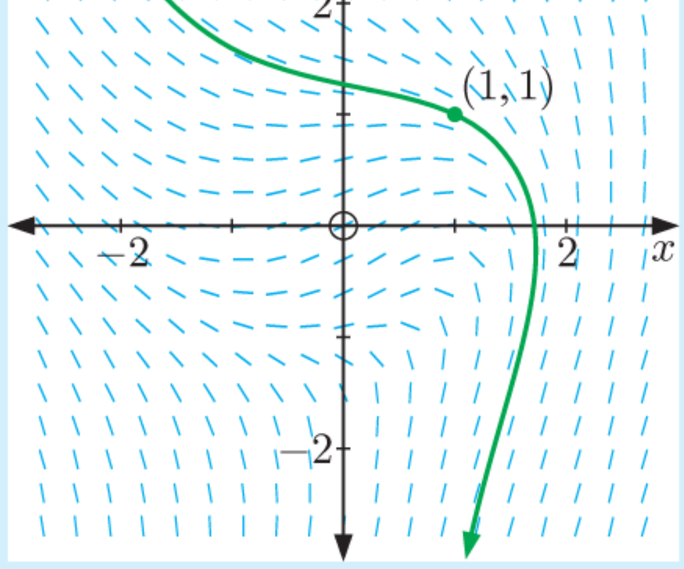
b Sketch the solution curve which passes through $(1, 1)$.



a At $(1, 1)$, $\dfrac{dy}{dx} = \dfrac{1-1^2-1^2}{1-1+2} = -\dfrac{1}{2}$

$\therefore$ the gradient of the tangent to the solution curve at $(1, 1)$ is $-\frac{1}{2}$.

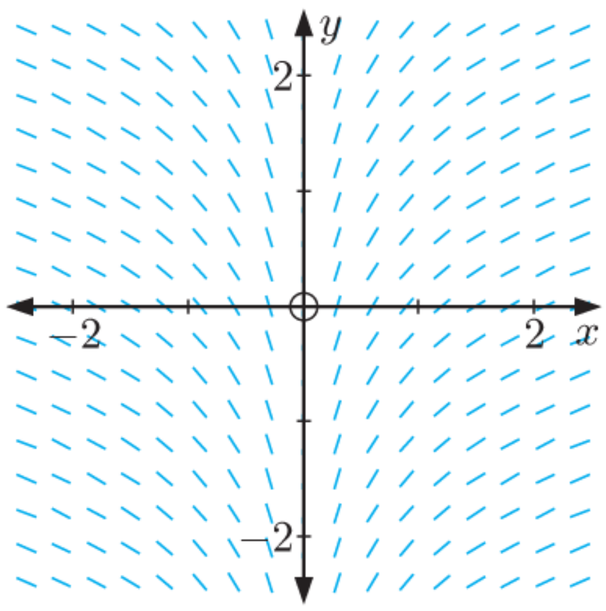
b



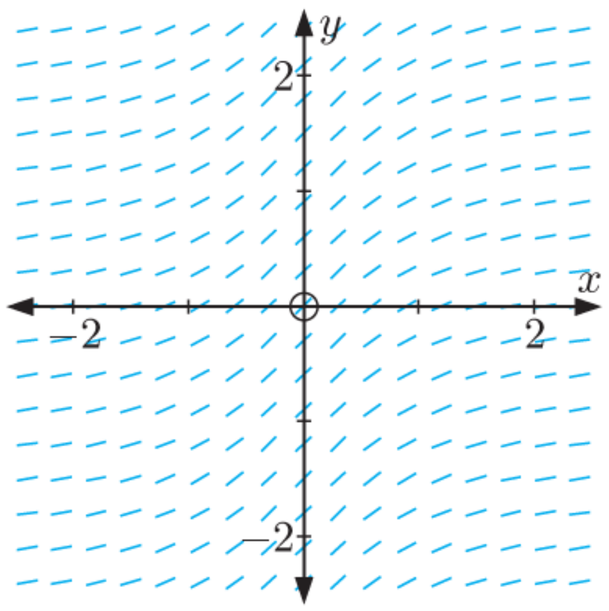
EXERCISE 25E

1 Use the slope field to graph the solution curve passing through $(1, 1)$.

a



b



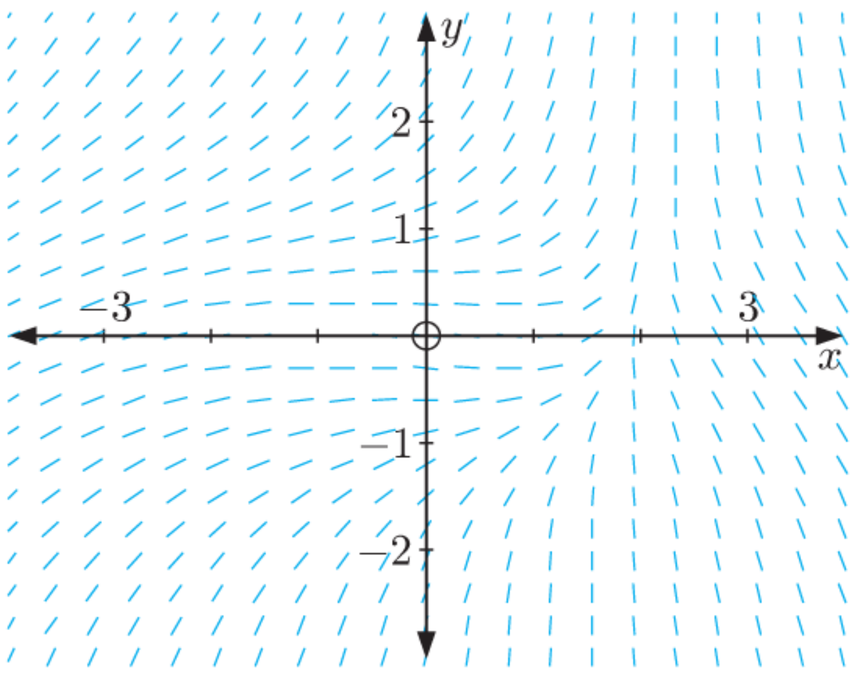
PRINTABLE SLOPE FIELDS



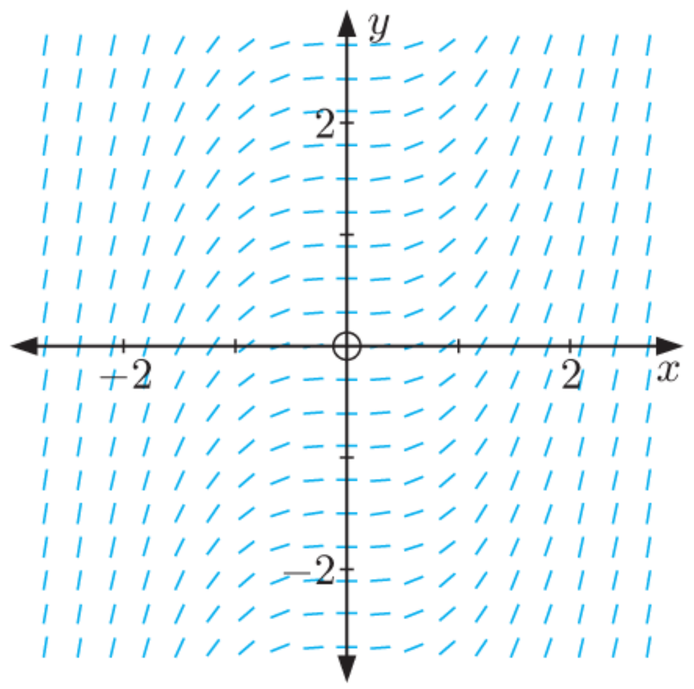
2 The slope field for the differential equation $\dfrac{dy}{dx} = \dfrac{-1+x^2+4y^2}{y-5x+10}$ is shown alongside.

a Find the gradient of the tangent to the solution curve at the origin.

b Sketch the particular solution passing through the origin.



3 The slope field for $\frac{dy}{dx} = x^2$ is shown.



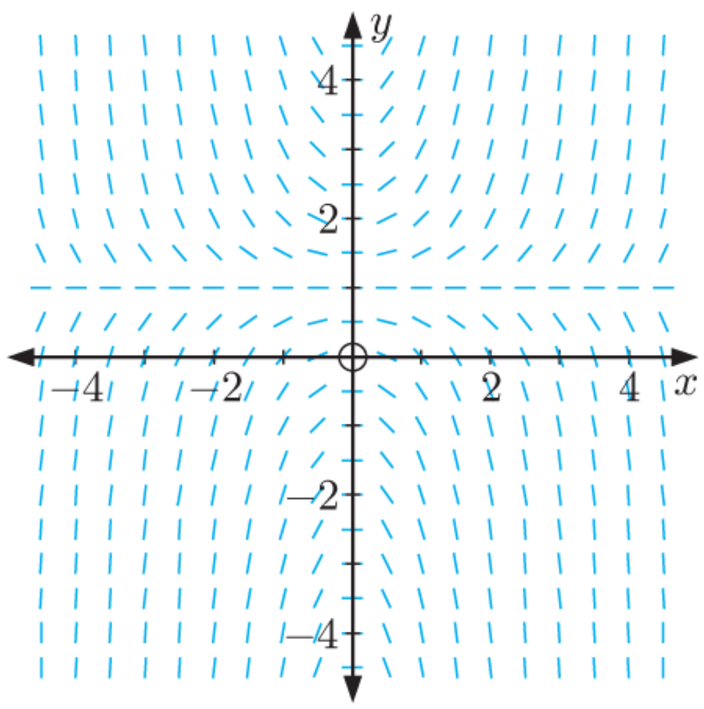
a Find the general solution to the differential equation.

b Find the particular solution for:

i $c = 1$ **ii** $c = -2$

Sketch these solutions on the slope field.

4



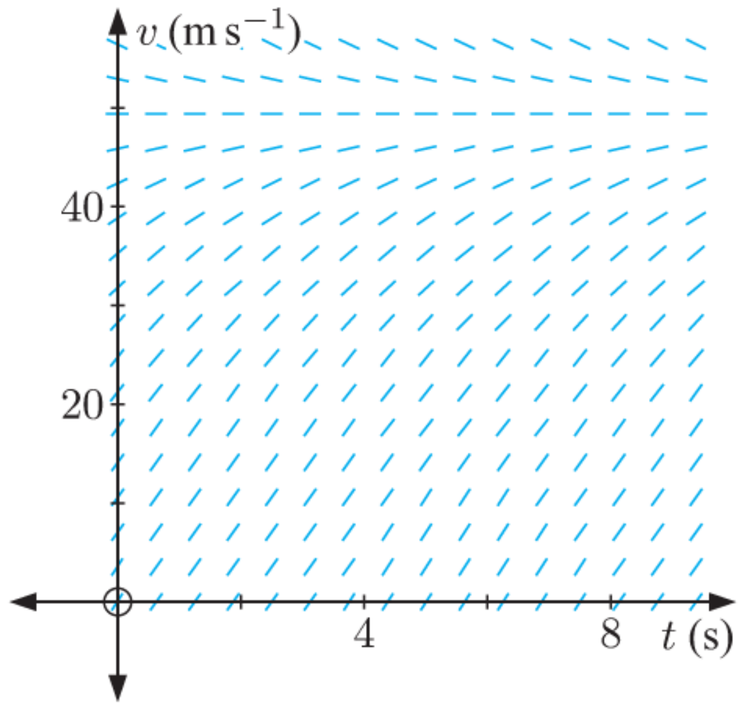
The slope field for the differential equation $\frac{dy}{dx} = x(y-1)$ is shown alongside.

a Sketch the solution curve which passes through $(0, 2)$.

b Find the equation of the solution curve drawn in **a**.

5 The velocity v m s^{-1} of a skydiver t seconds after she jumps from a helicopter satisfies the differential equation $60\frac{dv}{dt} = 60g - \frac{3g}{125}v^2$, where $g \approx 9.8$ m s^{-2} is the acceleration due to gravity.

The slope field for this differential equation is shown alongside.



a Given that the skydiver initially had velocity 0 m s^{-1}, sketch the particular solution curve.

b Describe what happens to the velocity of the skydiver as she falls.

Example 12 **Self Tutor**

Consider the differential equation $\frac{dy}{dx} = x - y + 1$, which has general solution $y = ce^{-x} + x$.

a Find the particular solution which passes through $(0, -1)$.

b Construct the slope field for the differential equation using the integer grid points for $x, y \in [-2, 2]$. Include the particular solution curve from **a**.

a The particular solution passes through $(0, -1)$ $\quad \therefore \ -1 = ce^{-0} + 0$

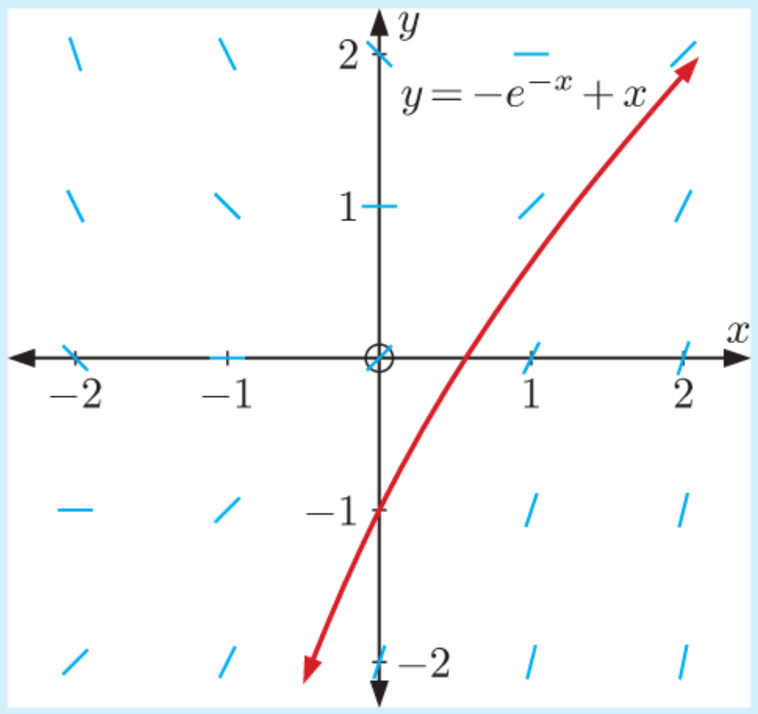
$\therefore \ c = -1$

So, the particular solution is $y = -e^{-x} + x$.



b We calculate $\frac{dy}{dx}$ for each grid point:

y \ x	-2	-1	0	1	2
-2	1	2	3	4	5
-1	0	1	2	3	4
0	-1	0	1	2	3
1	-2	-1	0	1	2
2	-3	-2	-1	0	1



6 Consider the differential equation $\frac{dy}{dx} = 1 - x$, which has general solution $y = x - \frac{1}{2}x^2 + c$.

a Find the particular solution which passes through $(-1, 0)$.

b Construct the slope field for the differential equation using the integer grid points for $x, y \in [-2, 2]$. Include the particular solution curve from **a**.

7 Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$.

a Find the particular solution which passes through $(1, -1)$.

b Construct the slope field for the differential equation using the integer grid points for $x, y \in [-2, 2]$. Include the particular solution curve from **a**.

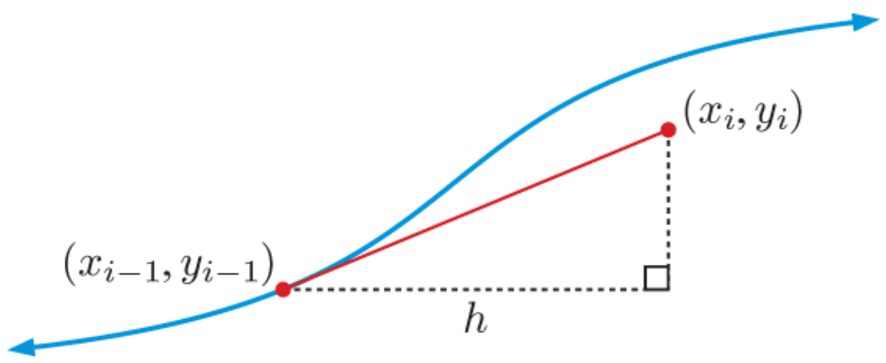
F EULER'S METHOD FOR NUMERICAL INTEGRATION

In the book *Institutionum calculi integralis*, **Leonhard Euler** (1707 - 1783) described a method which allows us to numerically approximate the solution curve to a differential equation where $\frac{dy}{dx} = f(x, y)$ with particular solution (x_0, y_0).

The particular solution gives us initial values for x_0 and y_0. At each iteration of Euler's method, we generate another point for our solution curve.

We choose a step size h and let $x_i = x_{i-1} + h$ be the x-coordinate of the ith point.

Since $\frac{dy}{dx}$ gives us the gradient of the tangent to the curve, we follow the tangent to find the next value of y.



Now $\frac{y_i - y_{i-1}}{h} = f(x_{i-1}, y_{i-1})$

$\therefore \quad y_i - y_{i-1} = h\, f(x_{i-1}, y_{i-1})$

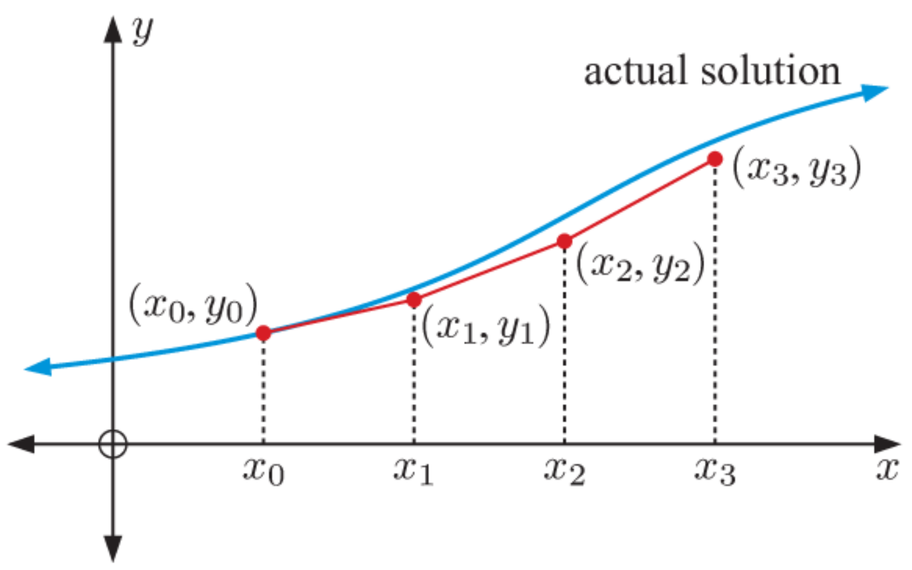
$\therefore \quad y_i = y_{i-1} + h\, f(x_{i-1}, y_{i-1})$

The iterative procedure

$$\begin{cases} x_i = x_{i-1} + h \\ y_i = y_{i-1} + h\,f(x_{i-1},\, y_{i-1}) \end{cases}$$

creates a polygonal approximation to the solution curve.

We correct our direction at each iteration by recalculating the gradient of the tangent to the curve.



Suppose we apply Euler's method for n iterations to reach $x = b$.

If we let $n \to \infty$ so we use infinitely many steps, Euler's method gives $y(b) = y(x_0) + \int_{x_0}^{b} \frac{dy}{dx}\,dx$, which is a statement of the Fundamental Theorem of Calculus.

Example 13 **Self Tutor**

Consider the differential equation $\frac{dy}{dx} = e^x + 1$ with $y(0) = 1$.

a Estimate $y(0.5)$ by applying Euler's method with:

i $h = 0.25$ for two steps ii $h = 0.1$ for five steps.

b Find $y(0.5)$ exactly using the Fundamental Theorem of Calculus.

$y(0) = 1$ gives us $x_0 = 0$ and $y_0 = 1$.

a i

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	1	2	0.25	1.5
2	0.25	1.5	2.2840	0.5	2.0710

$\therefore \quad y(0.5) \approx 2.0710$

ii

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	1	2	0.1	1.2
2	0.1	1.2	2.1052	0.2	1.4105
3	0.2	1.4105	2.2214	0.3	1.6327
4	0.3	1.6327	2.3499	0.4	1.8676
5	0.4	1.8676	2.4918	0.5	2.1168

$\therefore \quad y(0.5) \approx 2.1168$

b $y(0.5) = y(0) + \int_0^{0.5} \frac{dy}{dx}\,dx$ {Fundamental Theorem of Calculus}

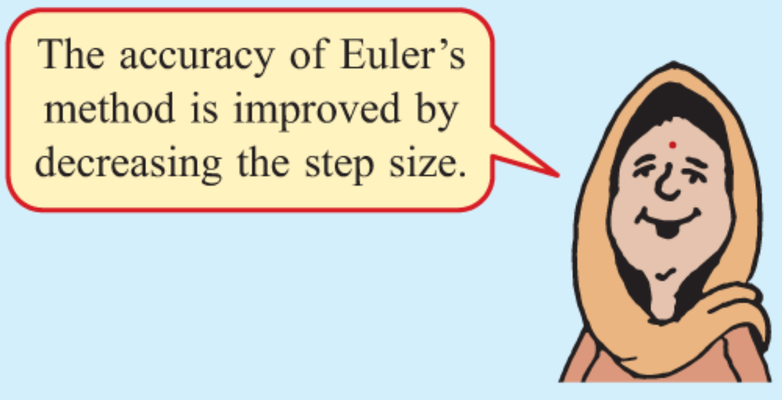
$= 1 + \int_0^{0.5} (e^x + 1)\,dx$

$= 1 + \left[e^x + x\right]_0^{0.5}$

$= 1 + (e^{0.5} + 0.5) - (e^0 + 0)$

$= e^{0.5} + 0.5$

≈ 2.1487



The table of values for Euler's method can be conveniently generated using a spreadsheet. Click on the icon to obtain a spreadsheet set up for the Example on the previous page.

EULER'S METHOD SPREADSHEET

Notice how the formula for $\frac{dy}{dx}$ is used in column D.

You can also apply Euler's method using online software or your graphics calculator.

EULER'S METHOD

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 25F

1 Consider the differential equation $\frac{dy}{dx} = xy - 1$ with initial point $(0, 3)$.

a Copy and complete this table for Euler's method with step size $h = 0.1$.

b Plot the solution curve for $0 \leqslant x \leqslant 0.5$.

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	3	-1	0.1	2.9
2	0.1	2.9			
3					
4					
5					

2 Consider the differential equation $\frac{dy}{dx} = 3e^{2x} - 1$ with $y(0) = 2$.

a Estimate $y(0.5)$ by applying Euler's method with:

i $h = 0.25$ for two steps **ii** $h = 0.1$ for five steps.

b Find $y(0.5)$ exactly using the Fundamental Theorem of Calculus. Comment on your results.

3 Use Euler's method with step size 0.2 to estimate $y(1)$ given $\frac{dy}{dx} = 1 + 2x - 3y$, $y(0) = 1$.

4 Consider the differential equation $\frac{dy}{dx} = -\cos x$ with $y(0) = 0$.

a Estimate $y(0.5)$ using Euler's method with step size 0.1.

b Use technology to apply Euler's method with step size 0.001 for 500 steps.

c Find the exact value of $y(0.5)$. Compare it with your numerical results.

5 Consider the differential equation $\frac{dy}{dx} = x \cos y$ with initial point $(0, 0)$.

Use Euler's method to estimate the value of y when $x = 2$, using:

a $h = 1$ **b** $h = 0.1$ **c** $h = 0.01$

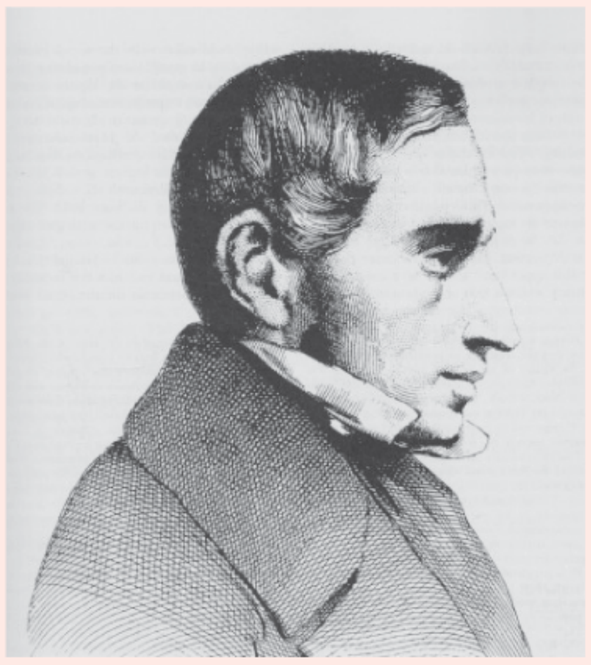
DISCUSSION

- What features of Euler's method are most useful?
- As the number of iterations n increases, what happens to the accuracy of the solution curve?
- In the absence of an analytical solution, how do we know how accurate a numerical approximation is?
- Suppose we keep halving the step size until consecutive numerical approximations obtained are within an acceptable error of each other. Will this guarantee that the numerical solution is close to the actual solution?

ACTIVITY 1 LOGISTIC GROWTH

The **logistic growth** or **constrained population growth** model describes populations which grow exponentially at first, but then approach a limiting value due to restricted land or food.

The logistic model was first given its name in 1845 by the Belgian mathematician **Pierre François Verhulst**, in his study of population growth.



Pierre François Verhulst

Verhulst described the logistic model using the differential equation

$$\frac{dN}{dt} = \underbrace{rN}_{\text{This factor controls the exponential nature of the growth when } N \text{ is small.}} \underbrace{\left(1 - \frac{N}{K}\right)}_{\text{This factor ensures that the growth approaches 0 as } N \text{ approaches } K.}$$

The r/K **selection theory** in ecology derives its name from the symbols in Verhulst's logistic model. The theory hypothesises that species are driven towards either *r-selection* or *K-selection*.

r-selected species such as insects and small rodents have high growth rates and high birth rates, but individuals have a relatively low chance of surviving to adulthood.

K-selected species such as humans, elephants, and whales operate in dense groups close to their limiting population. They typically produce fewer offspring, but their offspring have a high chance of survival. They are characterised by their ability to compete for limited resources.

Verhulst's differential equation is separable, but its solution is non-trivial. You can click on the icon to see how it is done.

ANALYTIC SOLUTION

If N_0 is the initial population, the solution is $N(t) = \dfrac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}$ which is equivalent to the logistic model we studied in **Chapter 1**.

What to do:

1 A lily in a pond initially has area 20 cm^2. Its area A grows according to the model $\dfrac{dA}{dt} = 0.1A\left(1 - \dfrac{A}{500}\right)$, where t is the time in days.

SPREADSHEET

a By comparison with the analytic solution above:

i write A as a function of t **ii** state the limiting size of the lily.

b Use a spreadsheet to apply Euler's method with step size 1 for the given scenario. Estimate the area of the lily each day for 40 days, and plot the analytic solution and your numerical approximation on the same set of axes.

c Discuss the accuracy of your approximation in **b**. Reduce the step size in Euler's method until you are satisfied with its accuracy.

2 In a small country town, rumours spread very fast. At 8 am on Monday, a rumour begins with 2 people. The number of people N who have heard the rumour grows according to the model $\dfrac{dN}{dt} = 0.8N\left(1 - \dfrac{N}{600}\right)$, where t is the time in hours after 8 am.

a Change your spreadsheet from **1** to apply the differential equation for this scenario.
Apply Euler's method with step size 1 for the first 10 hours.
Reduce the step size until you are satisfied that your model will be accurate.

b By comparison with the analytic solution for Verhulst's differential equation, write N in terms of t.

c Plot the analytic solution on the same set of axes as the numerical approximation you were satisfied with. Comment on your results.

INVESTIGATION IMPROVING EULER'S METHOD

We have seen that Euler's method can be quite inaccurate, depending on the function we are integrating.

Click on the icon to see how Euler's method can be improved.



ACTIVITY 2 THE SPRUCE BUDWORM

In this online Activity we consider the population dynamics of the spruce budworm. To help analyse the behaviour of the population, we consider **equilibrium points** at which the population is stable, and how they change depending on the reproduction rate of the budworm.



THEORY OF KNOWLEDGE

Sir Isaac Newton developed his *Laws of Motion* in the mid 1600s in England.

Since acceleration is the second derivative of position, Newton's second law $F = ma$ gives us a differential equation. Combined with other physics, such as the Law of Gravitation, we can write equations which describe the future position of the planets. For example, by solving a differential equation, Newton showed that a single planet could move in an elliptical path around the sun, in agreement with astronomical observations.

In the seventeenth and eighteenth centuries, the study of differential equations became a central part of classical mathematics. **Leonhard Euler** (1707 - 1783), **Pierre-Simon Laplace** (1749 - 1827), and **Jean-Baptiste Fourier** (1768 - 1830) all made significant contributions to the techniques for solving differential equations.

By the middle of the twentieth century, mathematicians had realised that some differential equations do not have simple analytical solutions. Furthermore, while solutions may remain finite, they can be chaotically unstable so that even with the computer power available today, long term numerical simulation is essentially impossible. An example of this is long term weather forecasting.



Jean-Baptiste Fourier

Edward Lorenz (1917 - 2008) was one of the first people to recognise this problem, and the *Lorenz equations* are one of the simplest systems which show this behaviour.

Mathematicians are still developing *Chaos theory* to better understand complex physical processes such as turbulent flow and improve long range forecasting.

1 What are the relative advantages and disadvantages of a numerical solution compared with a traditional analytic solution?

2 At what point does the use of numerical methods become closer to scientific experimentation than logic and deduction?

3 What does Chaos theory tell us about the limitations of our knowledge of the world?

REVIEW SET 25A

1 The rate at which the mass M of a Schwarzchild black hole decreases over time is inversely proportional to M^2. Write this as a differential equation.

2 Verify that:

a $y = 3e^{-\frac{1}{x}}$ is a solution to $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

b $y = 2 + ce^{-x^3}$ is a solution to $\dfrac{dy}{dx} + 3x^2y = 6x^2$ for any constant c.

3 Solve the following differential equations:

a $\dfrac{dy}{dx} = \cos 2x - \sin x$

b $\dfrac{dy}{dx} = 3 - e^{-2x}$

c $\dfrac{dy}{dx} = \dfrac{1}{2x+1}, \quad y(0) = 2$

d $\dfrac{dy}{dt} - t = te^{t^2}, \quad y(1) = 2e$

4 The rate at which spectators entered a stadium before a football match is given by

$S'(t) = \dfrac{4000e^{-0.05t}}{(1+4e^{-0.05t})^2}$ spectators per minute, where t is the time in minutes after noon.

There were 4000 spectators in the stadium at noon.



a At what rate were the spectators entering the stadium at:

i noon **ii** 12:30 pm?

b Find the derivative of $\dfrac{1}{1+4e^{-0.05t}}$.

c Hence find $S(t)$, the number of spectators in the stadium t minutes after noon.

d Find the number of spectators in the stadium at 1:40 pm when the match started.

5 Solve:

a $\dfrac{dy}{dx} = 5x^2y$

b $\dfrac{dy}{dx} = 2xy^2 - y^2$

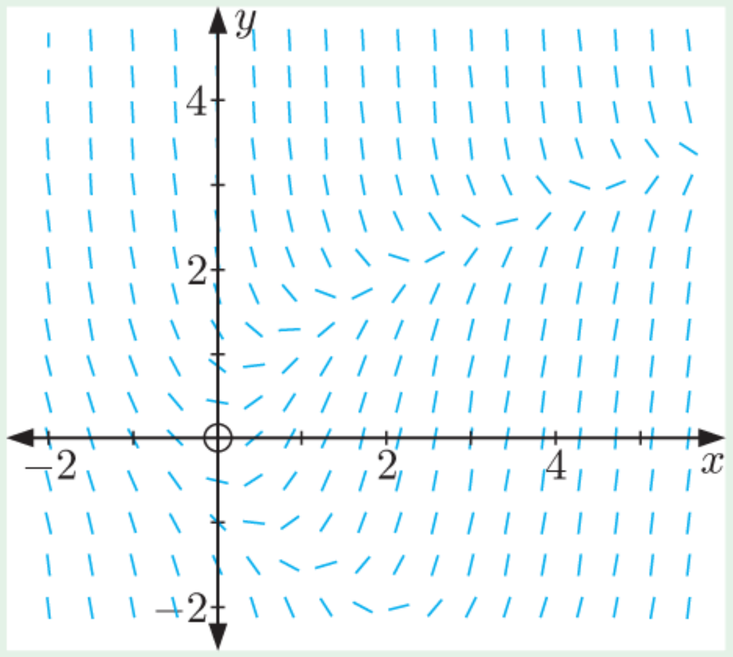
6 A radioactive substance decays at a rate proportional to the mass M remaining at time t. Suppose the initial mass is M_0.

a Construct a differential equation involving M, t, and a constant k to model this situation.

b If the substance is reduced to $\frac{4}{5}$ of its original mass in 30 days, calculate the time required for the substance to decay to half its original mass.

7 Solve the differential equation $\dfrac{dy}{dx} = -\dfrac{3}{e^y}$ given that $y = 1$ when $x = 0$.

8 The slope field for $\dfrac{dy}{dx} = 2x - y^2$ is shown alongside.



a Find the gradient of the tangent to the solution curve at $(1, -1)$.

b Sketch the solution curves through:

i $(0, 0)$ **ii** $(2, 3)$

PRINTABLE SLOPE FIELD



9 Consider the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

a Show that the solution to the differential equation is $x^2 + y^2 = c$ for any constant c

b Using integer grid points for $x, y \in [-3, 3]$, draw the slope field for the differential equation. Illustrate the particular solutions corresponding to $c = 1$ and $c = 6\frac{1}{4}$.

10 A curve passes through the point $(1, 2)$ and satisfies the differential equation $\dfrac{dy}{dx} = x - 2y$.

Use Euler's method with step size 0.1 to estimate the value of y when $x = 1.6$.

REVIEW SET 25B

1 Let $D(t)$ be the distance a falling object has travelled after t seconds. The rate at which D increases is proportional to $\sqrt{D}$. Write this as a differential equation.

2 Verify that $y = 3\sin x - 2\cos x$ is a solution to $3\dfrac{dy}{dx} - 2y = 13\cos x$.

3 Consider the differential equation $\dfrac{dy}{dx} = y - 1$.

a Show that $y = ce^x + 1$ is a solution to the differential equation for any constant c.

b Sketch the solution curves for $c = 0, \pm1, \pm2$.

c Find the particular solution which passes through $(0, 4)$.

d Find the equation of the tangent to the particular solution at $(0, 4)$.

4 Solve the following differential equations:

a $\dfrac{dy}{dx} = \dfrac{e^x}{e^x - 2}$

b $\dfrac{dy}{dx} = \frac{1}{2}\cos\left(\frac{\pi}{3} - 2x\right), \quad y\left(\frac{\pi}{2}\right) = 0$

5 Solve:

a $\dfrac{dy}{dx} = 2y^4$ **b** $(t^2+1)\dfrac{dP}{dt} = Pt$

6 Find the particular solution to:

a $\dfrac{dy}{dx} = \sqrt{y}$ given that $y(0) = 4$ **b** $\dfrac{dy}{dx} = y\cos x$ given that $y\left(\frac{\pi}{2}\right) = \dfrac{1}{e^2}$.

7 A water tank of height 1 m has a square base 2 m $\times$ 2 m. When a tap at its base is opened, the water flows out at a rate proportional to the square root of the depth of the water at any given time. Suppose the depth of the water is h m, and V is the volume of water remaining in the tank after t minutes.

a Write down a differential equation involving $\dfrac{dV}{dt}$ and h to model this situation.

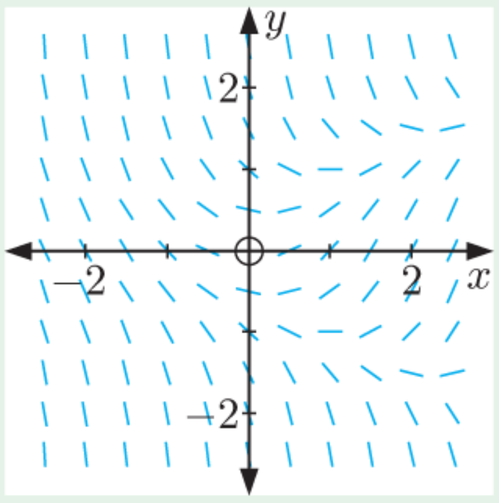
b Explain why $V = 4h$ m^3 at time t, then use the chain rule to write down a differential equation involving $\dfrac{dh}{dt}$ and h only.

c The tank is initially full. When the tap is opened, the water level drops by 19 cm in 2 minutes. Find the time it takes for the tank to empty.

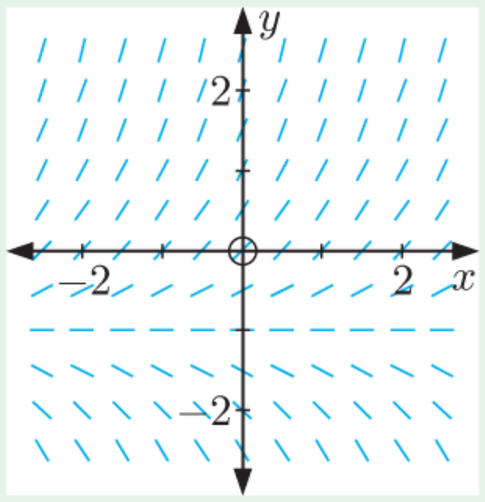
8 Match the slope fields to the differential equations:

a $\dfrac{dy}{dx} = y+1$ **b** $\dfrac{dy}{dx} = x-y$ **c** $\dfrac{dy}{dx} = x-y^2$

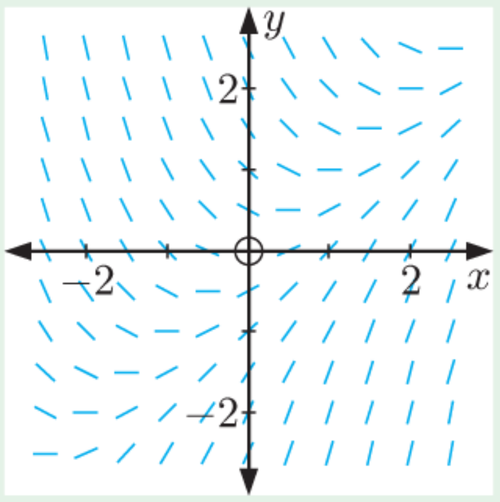
A



B

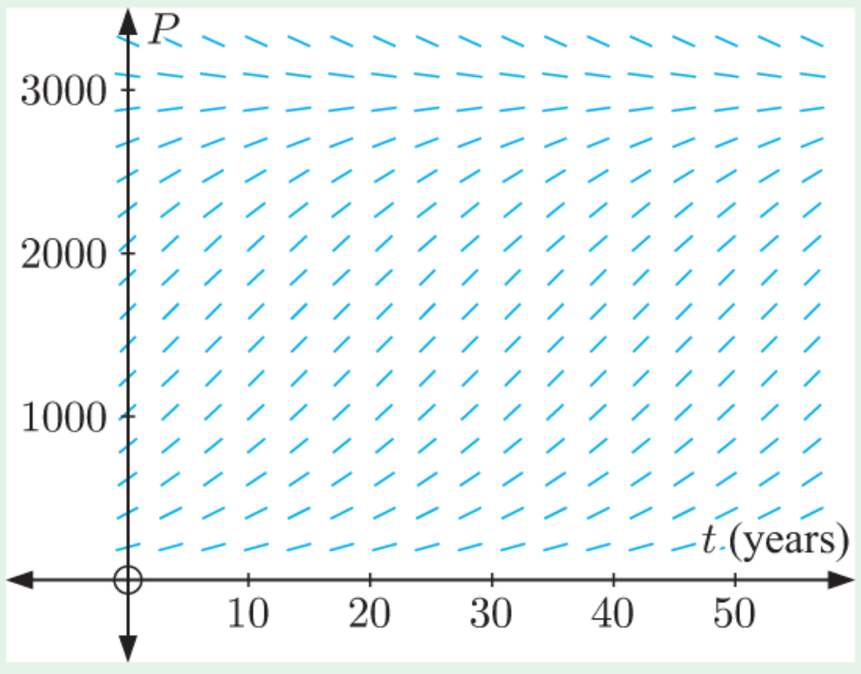


C



9 The population P of rodents on an island is expected to grow according to the model $\dfrac{dP}{dt} = 0.1P\left(1 - \dfrac{P}{3000}\right)$, where t is the time in years from now.

The slope field for this differential equation is shown alongside.



a Given that the population is currently 500, sketch the particular solution curve.

b Describe what happens to the population over time.

PRINTABLE SLOPE FIELD



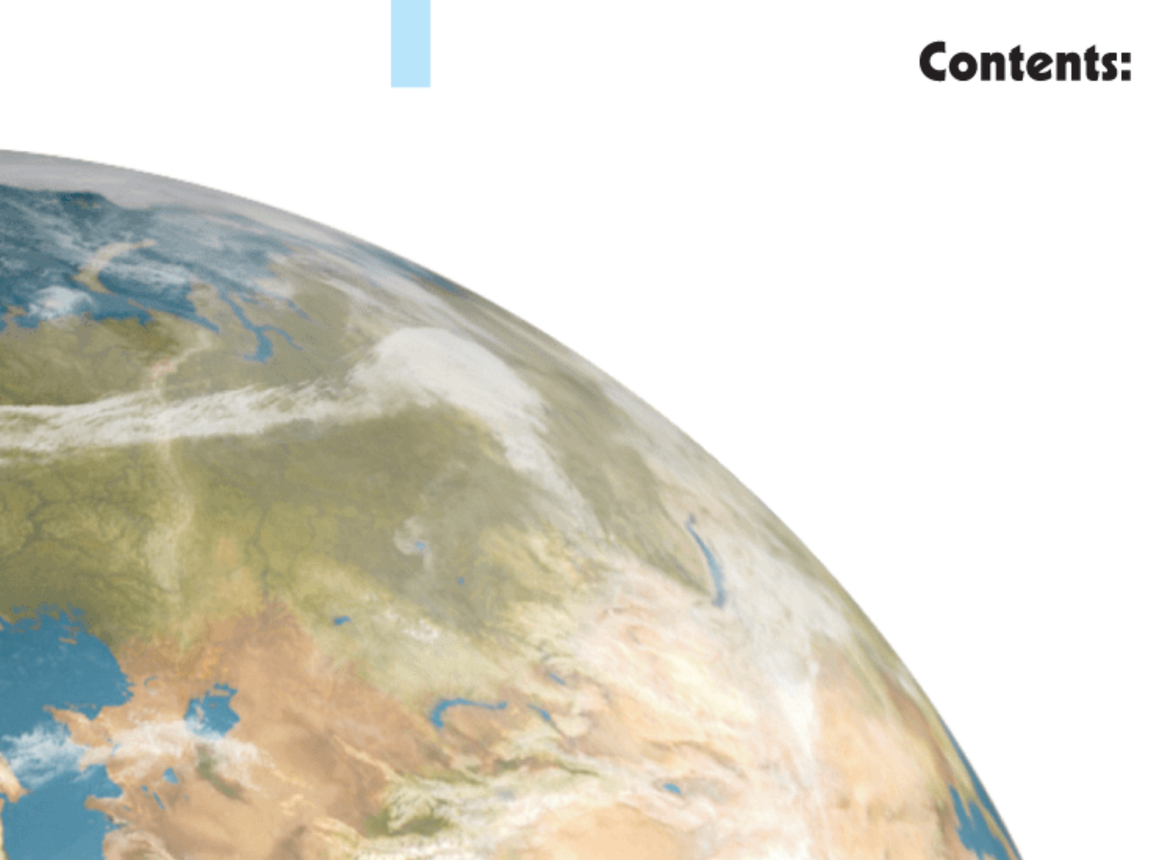
10 Use Euler's method with step size 0.1 to estimate $y(0.5)$ given $\dfrac{dy}{dx} = \sin(x+y)$, $y(0) = 0.5$.

Chapter 26

Coupled differential equations

Contents:

A Phase portraits
B Coupled linear differential equations
C Second order differential equations
D Euler's method for coupled equations



OPENING PROBLEM

In 1910, the American mathematician **Alfred J. Lotka** (1880 - 1949) formulated a model to describe the rates of chemical reactions. It was extended to problems in mathematical biology in the 1920s by the Soviet mathematician **Andrey Kolmogorov** (1903 - 1987) and Italian mathematician **Vito Volterra** (1860 - 1940).

The **Lotka-Volterra predator-prey model**, as it is now known, is used to model the populations of predators (y), and prey (x) as they change over time t.

Suppose a population of y orcas feed on a population of x seals. Clearly, these two populations interact, and so the changes in their population are affected by *both* species. We therefore have *two* differential equations to solve:

$$\begin{cases} \dfrac{dx}{dt} = 30x - xy \\ \dfrac{dy}{dt} = 0.015xy - 30y \end{cases}$$



Things to think about:

a What constraints are there on the variables x and y?

b Suppose there are no predators, so $y = 0$.

i Find $\dfrac{dy}{dt}$ and interpret this value.

ii Find $\dfrac{dx}{dt}$. Explain what will happen to the population of seals.

c Suppose there are no prey, so $x = 0$.

i Find $\dfrac{dx}{dt}$ and interpret this value.

ii Find $\dfrac{dy}{dt}$. Explain what will happen to the population of orcas and why.

d If $x = 0$ and $y = 0$ simultaneously, then the populations are both extinct and clearly $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ must both be zero.

Can you find another pair of values (x, y) such that $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are both zero? What does this *mean* for the two populations?

e How can we find solutions to these differential equations which will tell us what happens to the populations over time?

Just as we can have simultaneous equations, we can also have simultaneous *differential* equations.

> Differential equations which need to be solved simultaneously are said to be **coupled**.

For example, the x and y components of the velocity of an object may be given as

$$\begin{cases} \dfrac{dx}{dt} = f_1(x, y) \\ \dfrac{dy}{dt} = f_2(x, y) \end{cases} \quad \text{or} \quad \begin{cases} \dot{x} = f_1(x, y) \\ \dot{y} = f_2(x, y). \end{cases}$$

In this case:

- The velocity vector of the object is $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ or $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.
- The solution to the differential equations will be the parametric equations $x(t)$ and $y(t)$ for the object's position at time t.
- We need an initial point (x_0, y_0) to fully specify $x(t)$ and $y(t)$.

EQUILIBRIUM POINTS

If we can find a point (x, y) such that $\frac{dx}{dt} = \frac{dy}{dt} = 0$, then (x, y) is a point at which the system will stay in *equilibrium* forever.

An **equilibrium point** is a point at which $\frac{dx}{dt} = \frac{dy}{dt} = 0$ simultaneously.

A PHASE PORTRAITS

For the coupled system of differential equations $\begin{cases} \frac{dx}{dt} = f_1(x, y) \\ \frac{dy}{dt} = f_2(x, y) \end{cases}$ the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are uniquely defined by x and y independent of time.

For any point (x, y) we can therefore calculate a **trajectory vector** $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$.

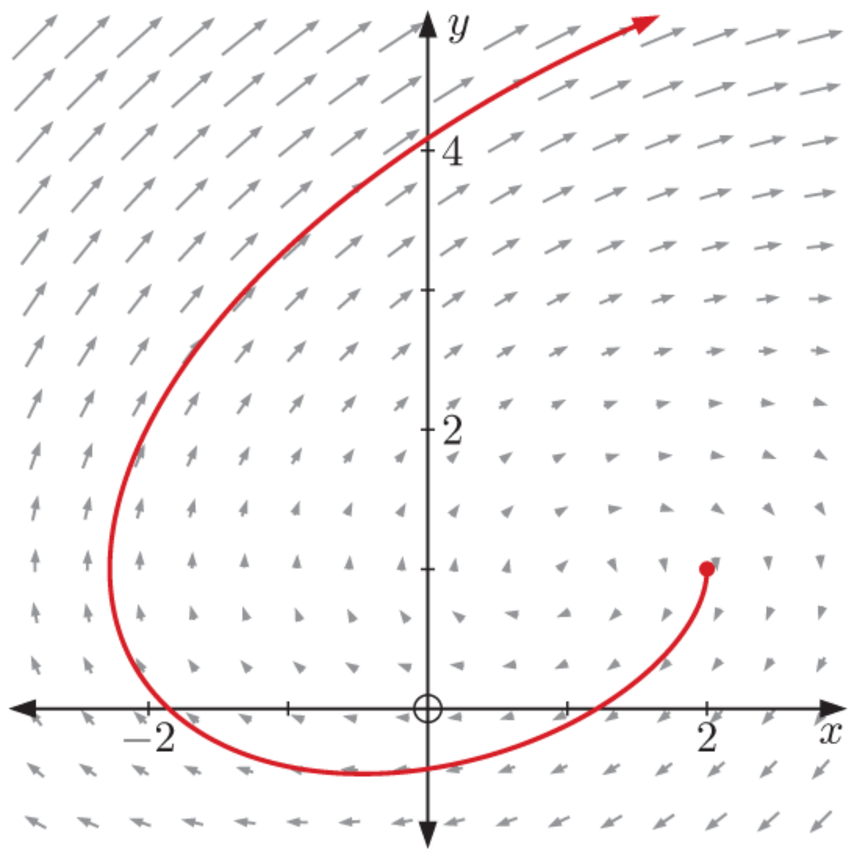
Calculating the trajectory vector at each point on a grid allows us to generate a diagram similar to a slope field. However, we use vectors instead of line segments, so the magnitude and direction are also indicated at each point. We call such a diagram a **phase portrait**.

We can solve the system of differential equations by following the vector arrows from any starting point. This generates the **trajectory** or **solution curve** from the given point.

For example, the phase portrait for the coupled system of differential equations

$$\begin{cases} \frac{dx}{dt} = 2y - 2 \\ \frac{dy}{dt} = y - x \end{cases}$$

is shown alongside, along with the trajectory from $(2, 1)$.

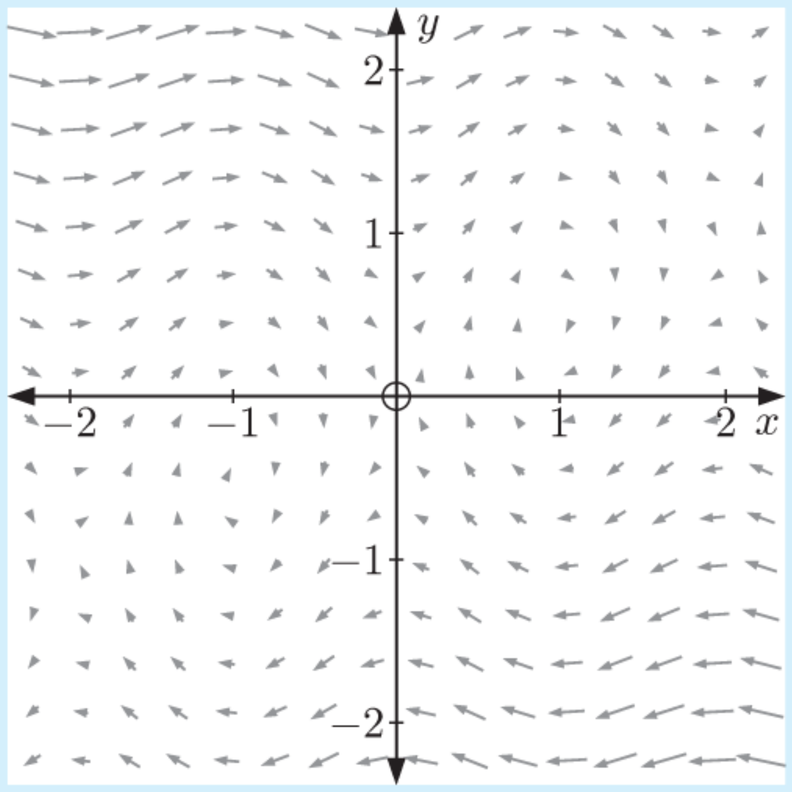


Example 1

Self Tutor

The phase portrait for the coupled system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = y - \frac{1}{2}x \\ \dfrac{dy}{dt} = \sin \pi x \end{cases}$$ is shown alongside.



a Calculate $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix}$ at the point:

i $(1, -1)$ **ii** $\left(\frac{1}{2}, 1\right)$

Explain how these vectors are observed on the phase portrait.

b Illustrate the solution curves with the following initial points on the phase portrait:

i $(2, 0)$ **ii** $\left(-\frac{3}{2}, \frac{1}{2}\right)$ **iii** $\left(2, -\frac{3}{2}\right)$

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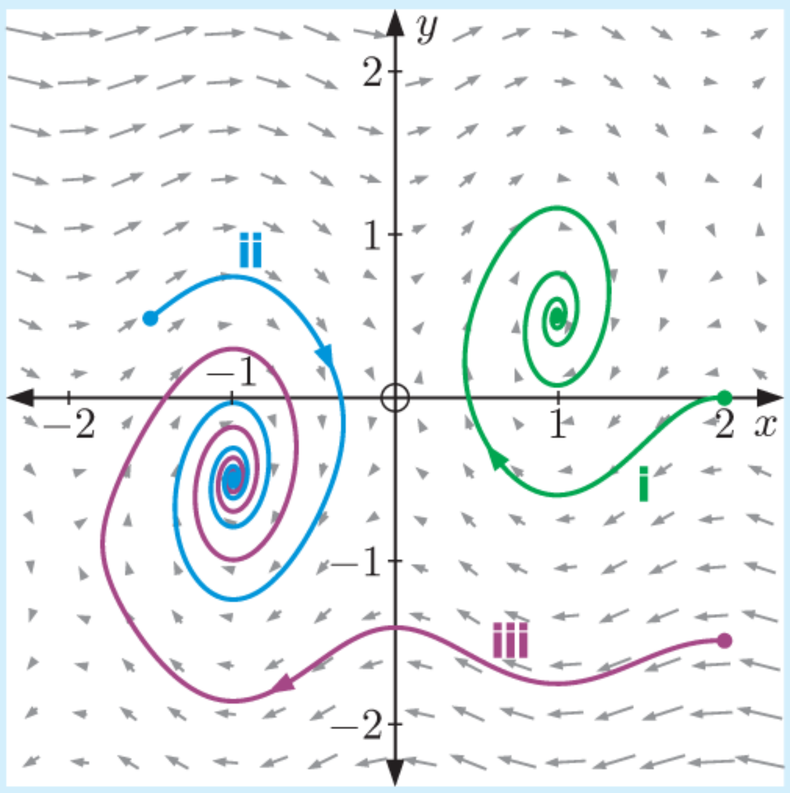
a **i** At $(1, -1)$, $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -1 - \frac{1}{2} \\ \sin \pi \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ 0 \end{pmatrix}$

On the phase portrait, the vector at $(1, -1)$ is horizontal and pointing to the left.

ii At $\left(\frac{1}{2}, 1\right)$, $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}\left(\frac{1}{2}\right) \\ \sin \frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$

On the phase portrait, the vector at $\left(\frac{1}{2}, 1\right)$ has the direction $\begin{pmatrix} \frac{3}{4} \\ 1 \end{pmatrix}$.

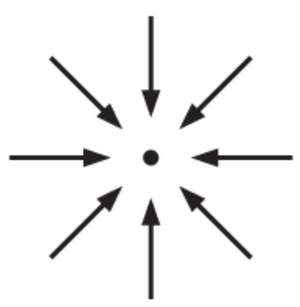
b



EQUILIBRIUM POINTS

We can describe an equilibrium point according to the behaviour of the phase portrait around it.

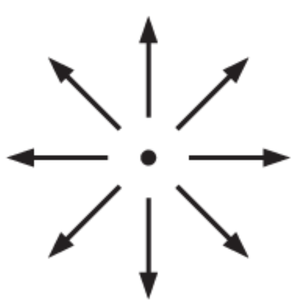
stable fixed



saddle point



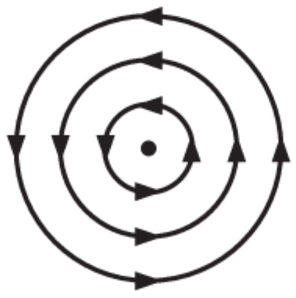
unstable fixed



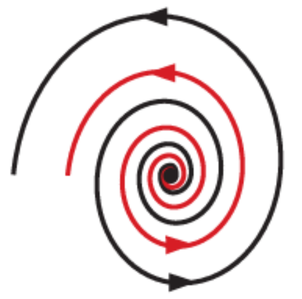
stable spiral



centre



unstable spiral



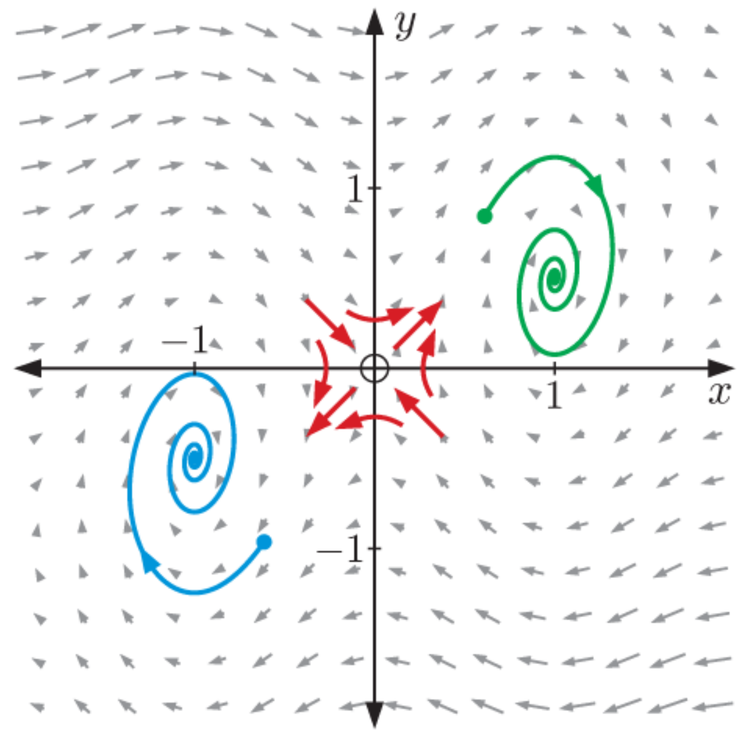
For example, for the coupled system in **Example 1**:

$$\frac{dy}{dt} = 0 \text{ whenever } \pi x = k\pi, \ k \in \mathbb{Z}$$
$$\therefore \ x = k, \ k \in \mathbb{Z}$$

$$\frac{dx}{dt} = 0 \text{ when } y - \tfrac{1}{2}x = 0$$

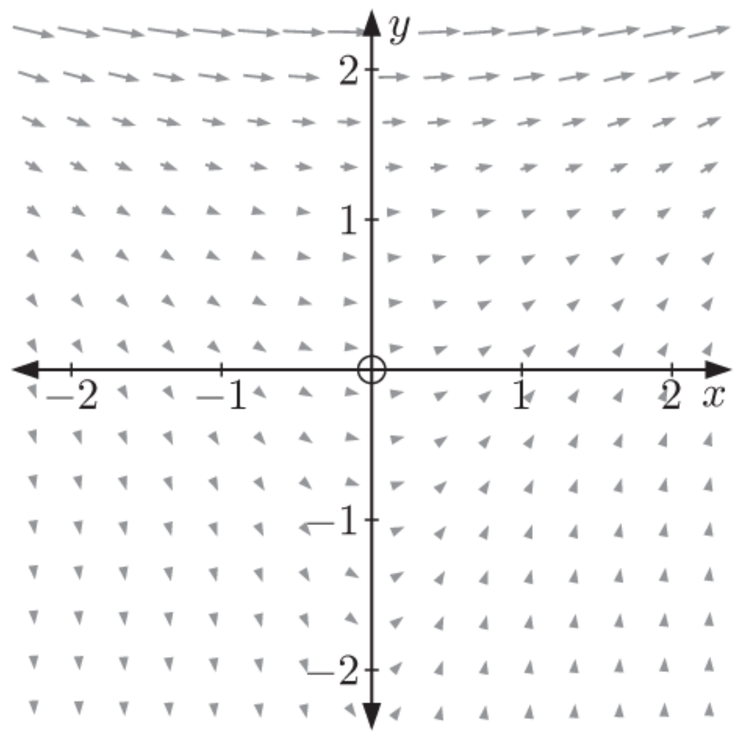
$\therefore$ for an equilibrium point, $x = k$, $y = \frac{1}{2}k$, $k \in \mathbb{Z}$.

On the phase portrait we observe two types of equilibrium point. There is a saddle point at $(0, 0)$ and there are stable spirals at $\left(-1, -\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$.



EXERCISE 26A

1 The phase portrait for the coupled system of differential equations $\begin{cases} \frac{dx}{dt} = e^y \\ \frac{dy}{dt} = x \end{cases}$ is shown alongside.



a Calculate $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ at the point:

i $(1, 0)$ **ii** $(-1, 1)$

Explain how these vectors are observed on the phase portrait.

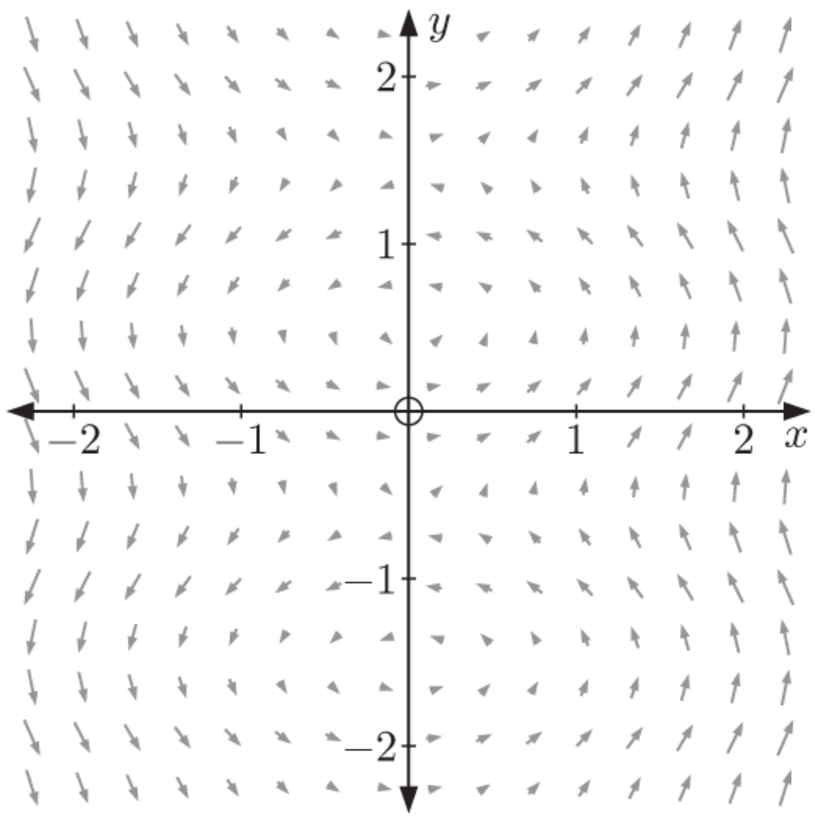
b Illustrate the solution curves with the following initial points on the phase portrait:

i $\left(-2, \frac{3}{2}\right)$ **ii** $(-2, 1)$ **iii** $\left(-2, \frac{1}{2}\right)$



2 The phase portrait for the coupled system

$$\begin{cases} \dot{x} = \cos \pi y \\ \dot{y} = x \end{cases}$$ is shown alongside.



a Show that the following points are equilibrium points. Describe the equilibrium point in each case:

i $\left(0, \frac{1}{2}\right)$ **ii** $\left(0, \frac{3}{2}\right)$

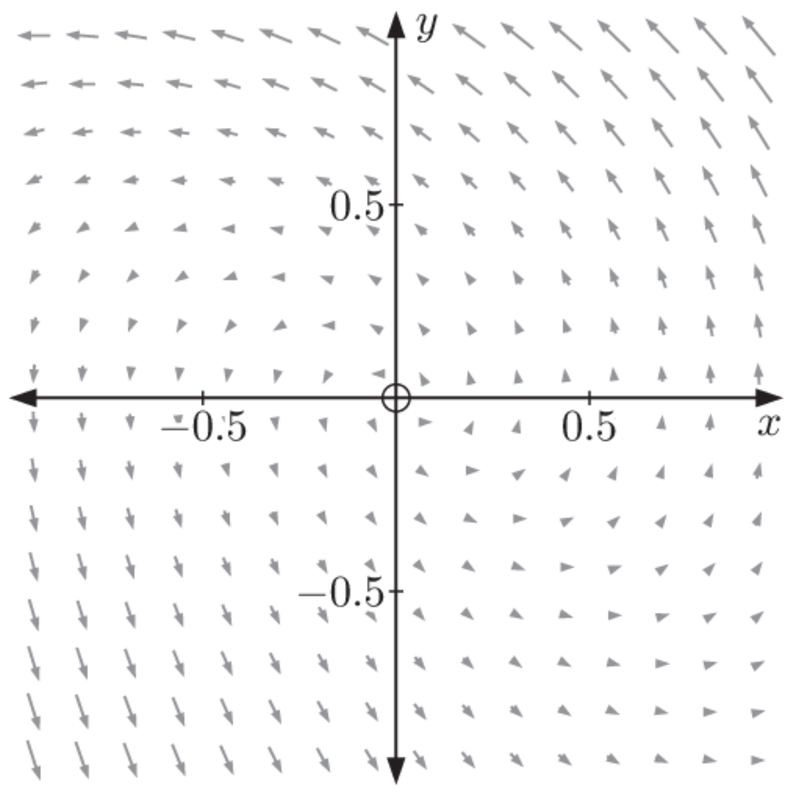
b Illustrate the solution curve starting at:

i $(-1, 2)$ **ii** $(1, -2)$

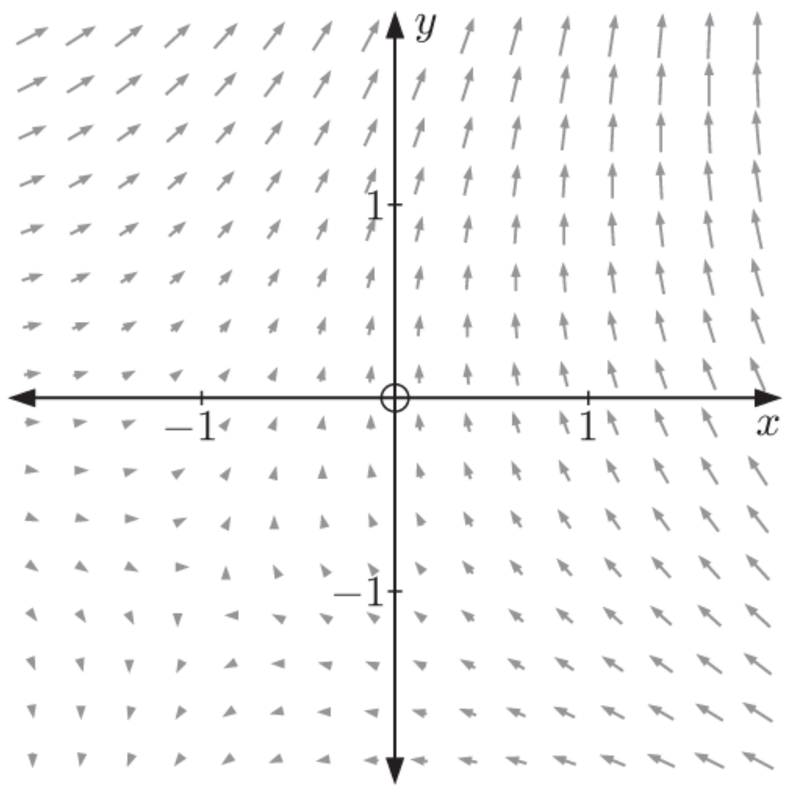
iii $(0, 0)$ **iv** $(0, -2)$

3 For each of the following phase portraits, locate and describe the equilibrium point.

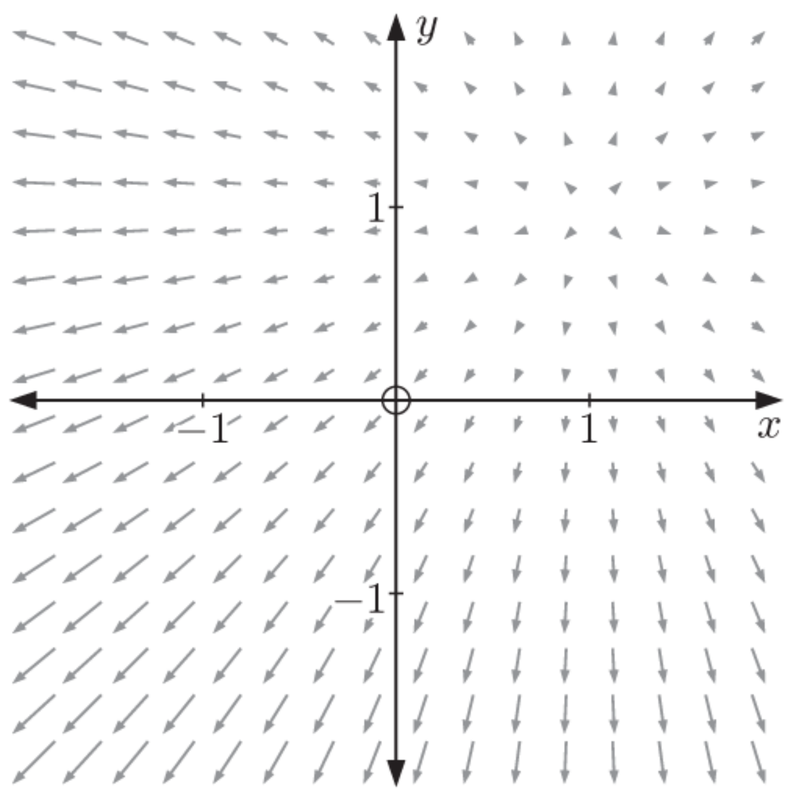
a



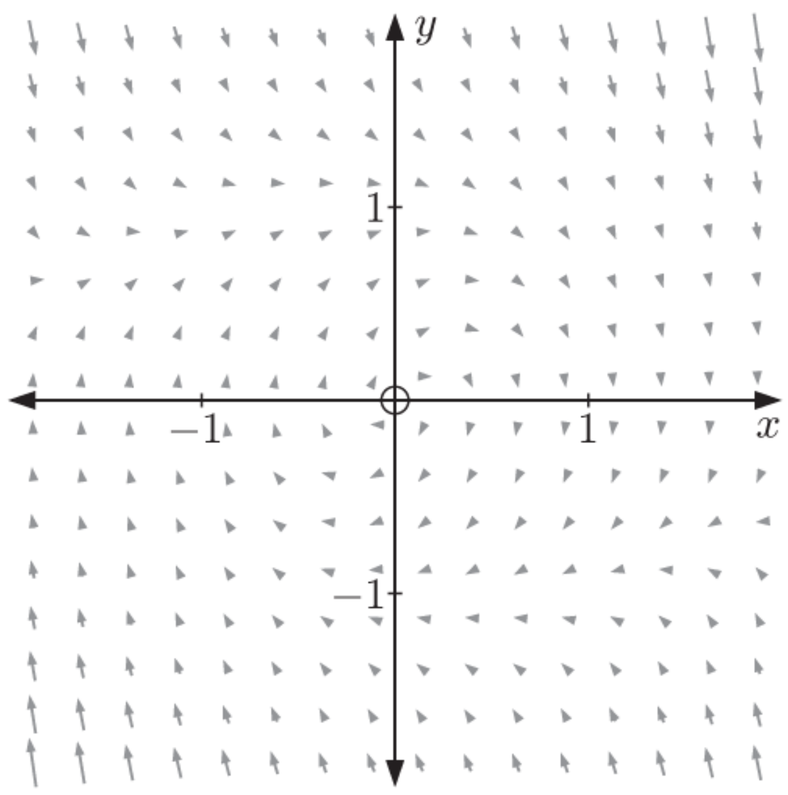
b

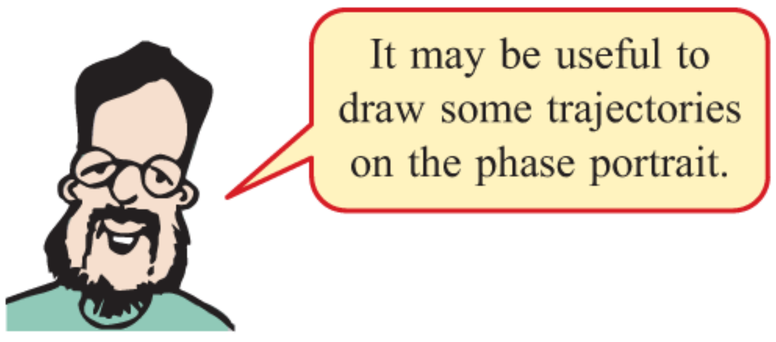


c



d



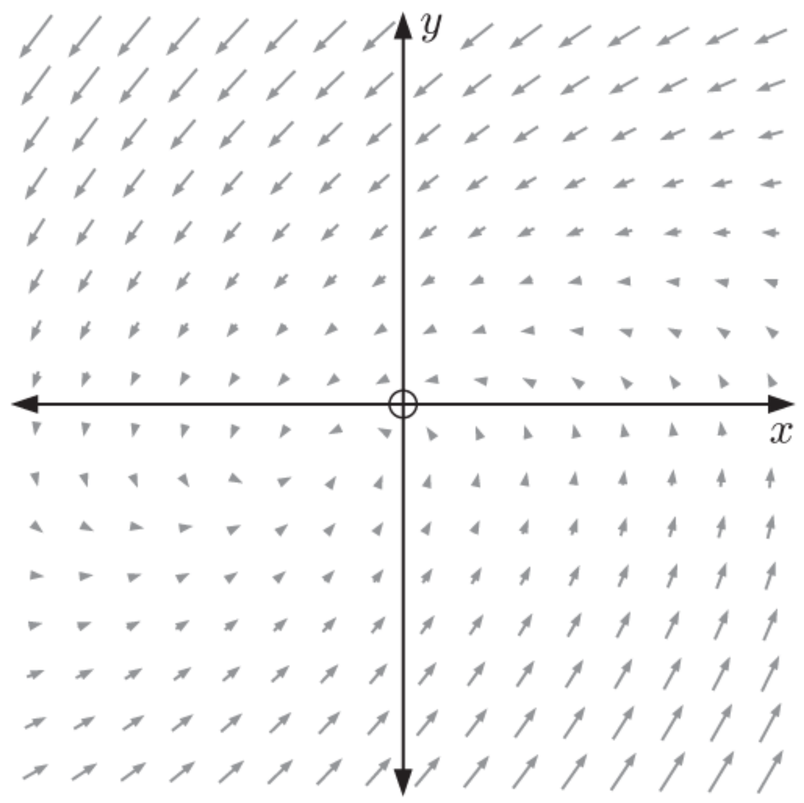


PRINTABLE DIAGRAMS



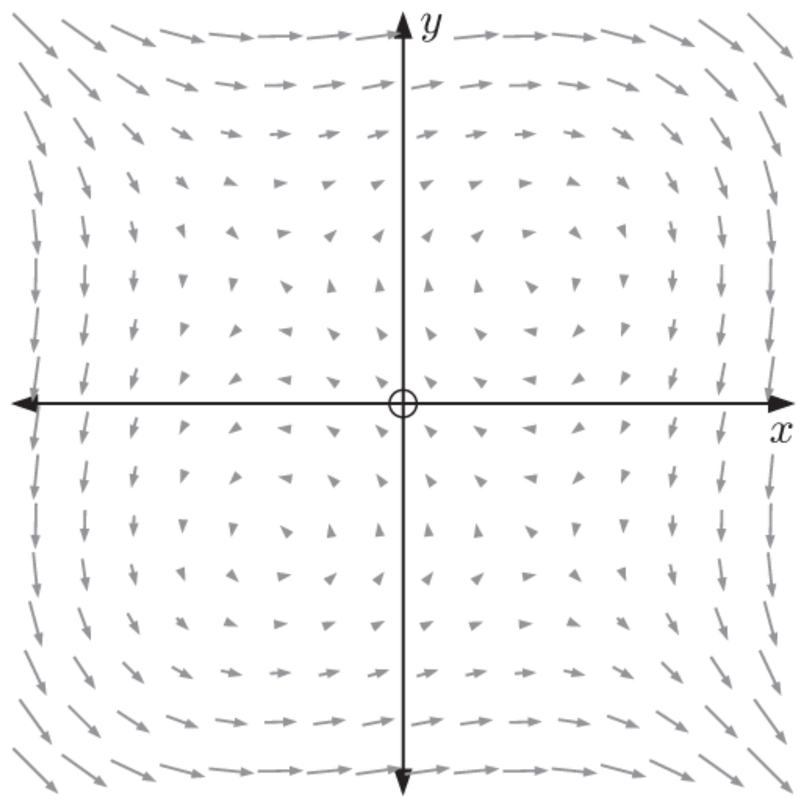
4 The diagram alongside shows the phase portrait for the system $\begin{cases} \dfrac{dx}{dt} = -2y - \frac{1}{2} \\ \dfrac{dy}{dt} = x - 2y. \end{cases}$

Locate and describe the equilibrium point.



5 The diagram alongside shows the phase portrait for the system $\begin{cases} \dfrac{dx}{dt} = y^2 - 1 \\ \dfrac{dy}{dt} = 1 - x^2. \end{cases}$

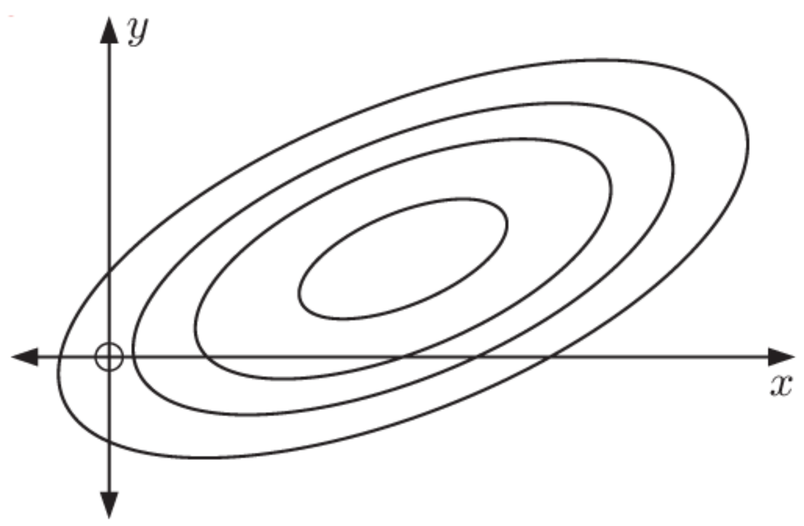
Locate and describe the equilibrium points.



6 The diagram alongside shows solution curves for the coupled system $\begin{cases} \dot{x} = x - 3y \\ \dot{y} = x - y - 1. \end{cases}$

a Locate and describe the equilibrium point.

b Do the solution curves rotate clockwise or anticlockwise? Explain your answer.



ACTIVITY 1 POPULATION MODELS

For a single population N, we have studied the important models:

- **exponential growth** $\dfrac{dN}{dt} = aN$, $a > 0$ if resources are unlimited
- **logistic growth** $\dfrac{dN}{dt} = aN - bN^2$, $a, b > 0$ where the second term accounts for *self-competition* as members of the population compete for limited resources.

In this Activity we will use phase portraits to study some population models where two species interact.

What to do:

1 In the **Opening Problem** we saw a **Lotka-Volterra predator-prey** model of the form:

$$\begin{cases} \dfrac{dx}{dt} = \overbrace{ax}^{\text{exponential growth}} \overbrace{- bxy}^{\text{loss due to predators}} \\ \dfrac{dy}{dt} = \underbrace{cxy}_{\text{growth with available prey}} \underbrace{- dy}_{\text{natural death}} \end{cases}$$

where x is the prey,
y is the predator,
and $a, b, c, d > 0$.

a Each term in the differential equations above has been labelled with its meaning.

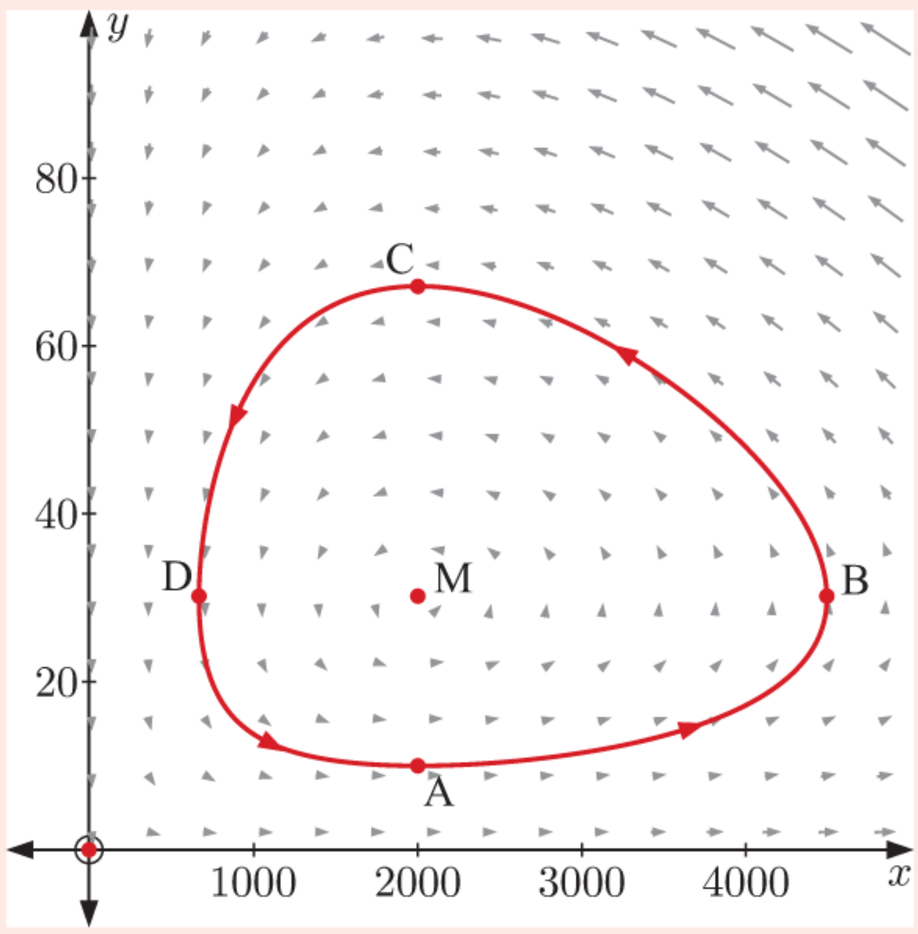
i Why are the terms for the predator different from the terms for the prey?

ii Consider only the differential equation for $\dfrac{dy}{dt}$ and assume x is constant. Is there now an "exponential growth" term for y?

b The model in the **Opening Problem** was

$$\begin{cases} \dfrac{dx}{dt} = 30x - xy \\ \dfrac{dy}{dt} = 0.015xy - 30y. \end{cases}$$

The phase portrait for this model is shown alongside.



i Use the differential equations to explain why $(0, 0)$ is an equilibrium point. Describe this equilibrium point using the phase portrait.

ii Inside the marked trajectory is an equilibrium point. Locate and describe this equilibrium point, and explain its significance in the context of the model.

iii Consider the trajectory ABCD marked. At point A, there are very few orcas and a healthy population of seals. With few predators, the seals multiply rapidly, eventually allowing the number of orcas to increase.
Describe what is happening to the populations at each of the points B, C, and D.

c Click on the icon to obtain software which allows you to generate phase portraits for yourself.
Experiment with the parameters a, b, c, and d.
Discuss how these parameters effect the Lotka-Volterra predator-prey model.



2 A more sophisticated Kolmogorov predator-prey model is

$$\begin{cases} \dfrac{dx}{dt} = ax - bx^2 - cxy \\ \dfrac{dy}{dt} = dxy - ey - fy^2. \end{cases}$$

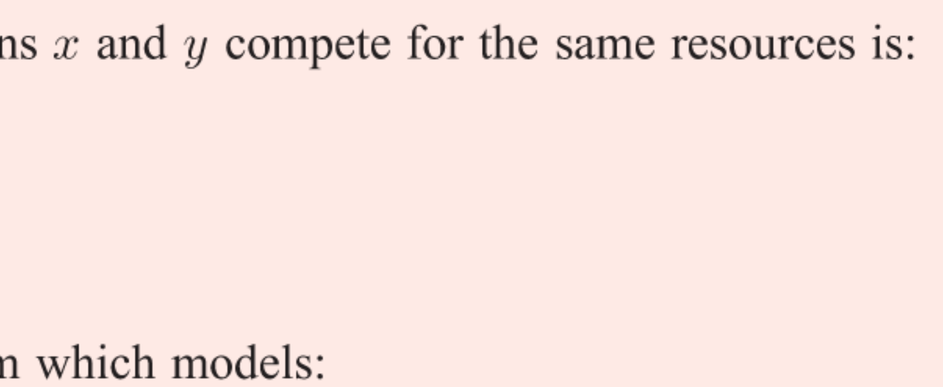
a Compare the differential equations with those from the Lotka-Volterra model. Explain the new terms which have been added by comparison with logistic growth.

b The phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} = 5x - 0.0003x^2 - 0.03xy \\ \dfrac{dy}{dt} = 0.0006xy - y - 0.003y^2 \end{cases}$$

is shown alongside, including the trajectory from $(2000, 10)$.

Locate and describe the equilibrium point which is *not* $(0, 0)$. Explain what it means in the context of the model, using the given trajectory as an example.



3 A **competing species** model in which populations x and y compete for the same resources is:

$$\begin{cases} \dfrac{dx}{dt} = ax - bx^2 - cxy \\ \dfrac{dy}{dt} = dy - ey^2 - fxy. \end{cases}$$

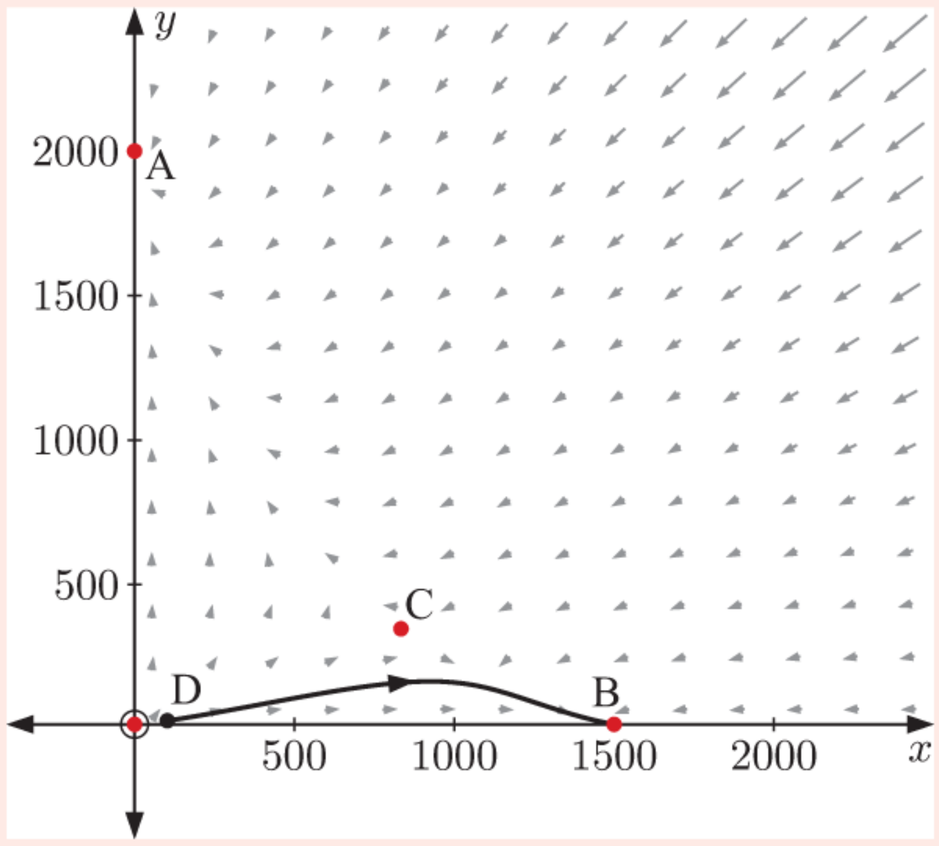
a In each differential equation, identify a term which models:

i exponential growth **ii** self-competition **iii** inter-species competition.

b The phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} = 1.5x - 0.001x^2 - 0.002xy \\ \dfrac{dy}{dt} = 2y - 0.001y^2 - 0.002xy \end{cases}$$

is shown alongside.



i Locate each of the *four* equilibrium points O, A, B, and C, and describe what each equilibrium point means for the two species.

ii What *type* of equilibrium point is C?

iii Discuss the trajectory which starts at D.

iv Discuss whether it is possible for the two species to coexist in the long term.

B COUPLED LINEAR DIFFERENTIAL EQUATIONS

A special form of coupled differential equations is $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$

which can be written in the matrix form $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

Letting $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix}$, we can write the system concisely as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the **matrix of coefficients**. We assume that $\mathbf{A}$ is invertible.

If a system can be written in this form, we say we have **coupled linear differential equations**.

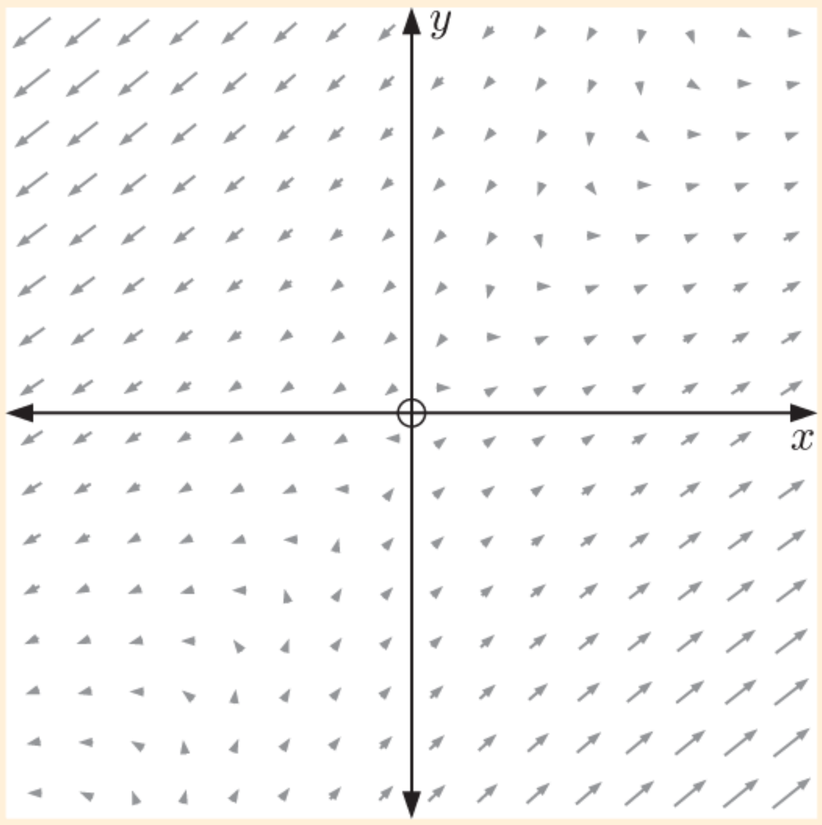
INVESTIGATION 1 COUPLED LINEAR DIFFERENTIAL EQUATIONS

In this Investigation we look for properties of a system of coupled linear differential equations of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

What to do:

1 State the equilibrium point for *any* system of coupled linear differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

2 The system $\begin{cases} \dfrac{dx}{dt} = 3x - 2y \\ \dfrac{dy}{dt} = 2x - 2y \end{cases}$ has the phase portrait alongside.



a Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

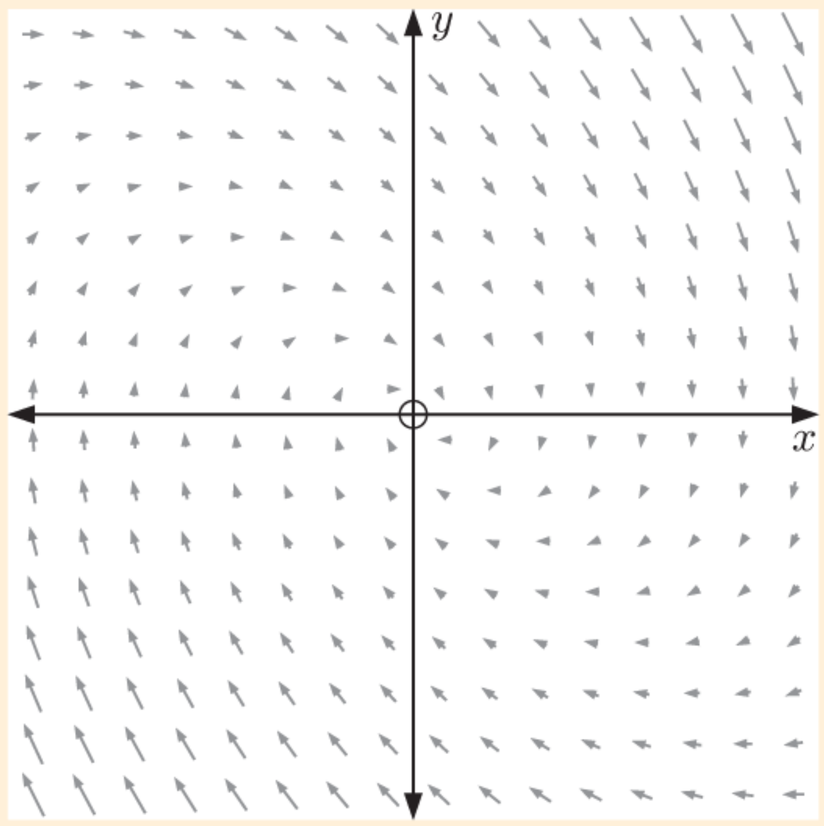
b Describe the equilibrium point of the system.

c Show that the matrix $\mathbf{A}$ has eigenvalues -1, 2 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

d Are there any trajectories which are straight lines? How do these relate to the eigenvectors of $\mathbf{A}$?

e Do any trajectories ever cross the eigenvectors of $\mathbf{A}$?

3 The system $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x - y \end{cases}$ has the phase portrait alongside.



a Write the system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

b Describe the equilibrium point of the system.

c Show that the matrix $\mathbf{A}$ has complex eigenvalues.

d Are there any trajectories which are straight lines?

From the **Investigation**, you should have found that:

For the system of coupled linear differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$:

- $(0, 0)$ is the only equilibrium point.
- If $\mathbf{A}$ has real eigenvectors, then these will appear as straight line trajectories on the phase portrait.

GENERAL SOLUTION TO THE SYSTEM

We have already seen that the one variable exponential growth equation $\dot{x} = kx$ has the solution $x(t) = x_0 e^{kt}$. This provides a clue that for $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, we should try a solution of the form $\mathbf{x} = \mathbf{v}e^{kt}$.

Suppose $\mathbf{x} = \mathbf{v}e^{kt} = \begin{pmatrix} v_1 e^{kt} \\ v_2 e^{kt} \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\therefore \quad \dot{x} = kv_1 e^{kt} = kx \quad$ and $\quad \dot{y} = kv_2 e^{kt} = ky$

$\therefore \quad \dot{\mathbf{x}} = \begin{pmatrix} kx \\ ky \end{pmatrix} = k\,\mathbf{x}$

So, $\mathbf{x} = \mathbf{v}e^{kt}$ satisfies $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ provided $k\mathbf{x} = \mathbf{A}\mathbf{x}$.

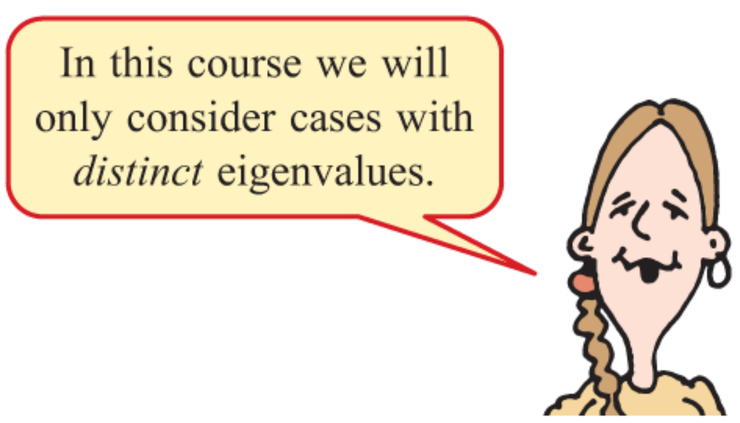
This condition is satisfied if k is an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is a corresponding eigenvector.

This observation gives *some* solutions to the system. In particular, since e^{kt} is a scalar, if a particle begins on a real eigenvector of the matrix $\mathbf{A}$, then it will always remain at a scalar multiple of that eigenvector. This explains why real eigenvectors produce straight line trajectories on a phase portrait.

Other solutions for the system are found as linear combinations of the eigenvector solutions as follows:

Suppose $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where the matrix of coefficients $\mathbf{A}$ has eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

Since $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are scalars for all t,
$\mathbf{y} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.



$$\begin{aligned}
\text{Now}\quad \mathbf{Ay} &= \mathbf{A}(Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2) \\
&= Ae^{\lambda_1 t}\mathbf{Av}_1 + Be^{\lambda_2 t}\mathbf{Av}_2 \\
&= Ae^{\lambda_1 t}\lambda_1\mathbf{v}_1 + Be^{\lambda_2 t}\lambda_2\mathbf{v}_2 \qquad \{\text{definition of eigenvalues and eigenvectors}\} \\
&= \begin{pmatrix} A\lambda_1 e^{\lambda_1 t}v_{11} + B\lambda_2 e^{\lambda_2 t}v_{21} \\ A\lambda_1 e^{\lambda_1 t}v_{12} + B\lambda_2 e^{\lambda_2 t}v_{22} \end{pmatrix} \qquad \left\{\text{letting } \mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix} \text{ and } \mathbf{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}\right\} \\
&= \dot{\mathbf{y}}
\end{aligned}$$

so $\mathbf{y}$ satisfies the system of coupled differential equations.

For the system of coupled linear differential equations $\dot{\mathbf{x}} = \mathbf{Ax}$ where $\mathbf{A}$ has distinct eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$, the **general solution** is $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$.

INVESTIGATION 2 — TRAJECTORIES

In this Investigation we use the general solution to the system $\dot{\mathbf{x}} = \mathbf{Ax}$ to identify the behaviour of the trajectories near the equilibrium point $(0, 0)$, and the nature of the eigenvalues of $\mathbf{A}$ needed to produce each type of equilibrium point.

What to do:

1 For a given input matrix $\mathbf{A}$, this online software will generate the phase portrait for $\dot{\mathbf{x}} = \mathbf{Ax}$ as well as the eigenvalues of $\mathbf{A}$.

PHASE PORTRAITS

Use the software to complete the following table:

PRINTABLE TABLE

Matrix $\mathbf{A}$	*Eigenvalues*	*Phase portrait*	*Type of equilibrium point*
$\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$			
$\begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix}$			
$\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$			

2 Consider carefully the data in your table. Describe the nature of the eigenvalues needed to produce an equilibrium point which is:

a stable fixed **b** a saddle point **c** unstable fixed

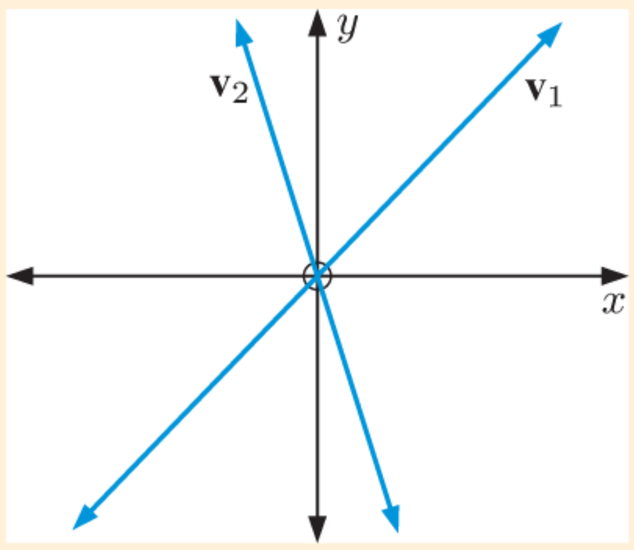
d a stable spiral **e** a centre **f** an unstable spiral.

Check your answers by considering other matrices **A** of your own choosing.

3 Suppose λ_1, λ_2 are real.

The general solution is $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$.

On the phase portrait, $\mathbf{v}_1$ and $\mathbf{v}_2$ give straight line trajectories.



a Consider an eigenvector $\mathbf{v}$ with corresponding eigenvalue λ. Explain why the trajectories along $\mathbf{v}$ move:

i *away* from O if $\lambda > 0$ **ii** *towards* O if $\lambda < 0$.

b Suppose $\lambda_1 > \lambda_2 > 0$.

i Describe the equilibrium point.

ii Which of $e^{\lambda_1 t}$ or $e^{\lambda_2 t}$ will be greater for large t? How is this observed on the phase portrait?

c Suppose $\lambda_1 > 0 > \lambda_2$.

i Describe the equilibrium point.

ii Determine the behaviour of the system as $t \to \infty$. How is this observed on the phase portrait?

d Suppose $0 > \lambda_1 > \lambda_2$.

i Describe the equilibrium point.

ii Which of $e^{\lambda_1 t}$ or $e^{\lambda_2 t}$ tends to 0 faster for large t? How is this observed on the phase portrait?

4 Suppose λ_1, λ_2 are complex.

We let $\lambda_1, \lambda_2 = \alpha \pm i\theta$ where α and $\theta \neq 0$ are real.

$\therefore$ the general solution is $\mathbf{x} = Ae^{(\alpha+i\theta)t}\mathbf{v}_1 + Be^{(\alpha-i\theta)t}\mathbf{v}_2$

$= e^{\alpha t}\left[Ae^{i\theta t}\mathbf{v}_1 + Be^{-i\theta t}\mathbf{v}_2\right]$

where we know that $e^{i\theta}$ and $e^{-i\theta}$ result in rotations of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Explain why:

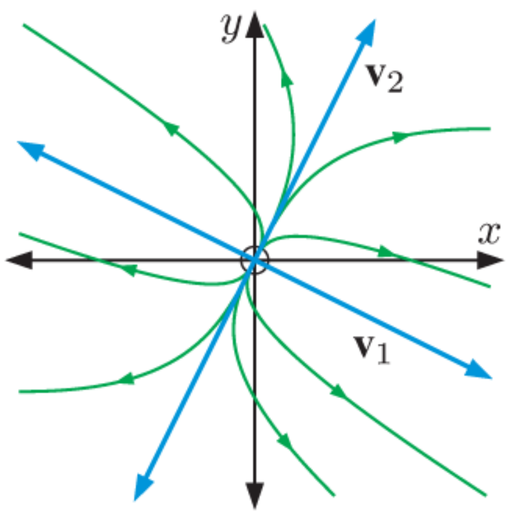
a if $\alpha < 0$, the trajectory spirals inwards

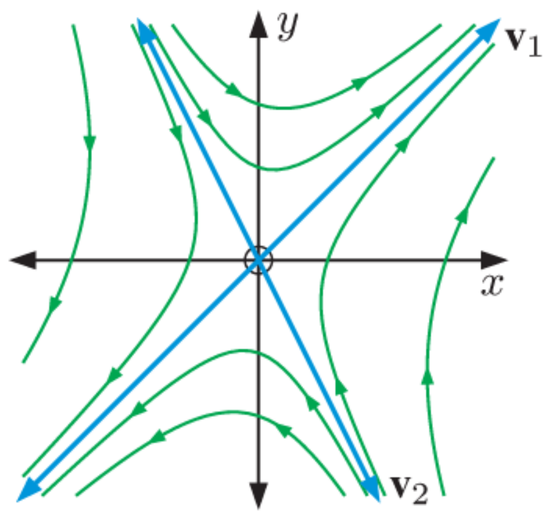
b if $\alpha = 0$, the trajectory is circular

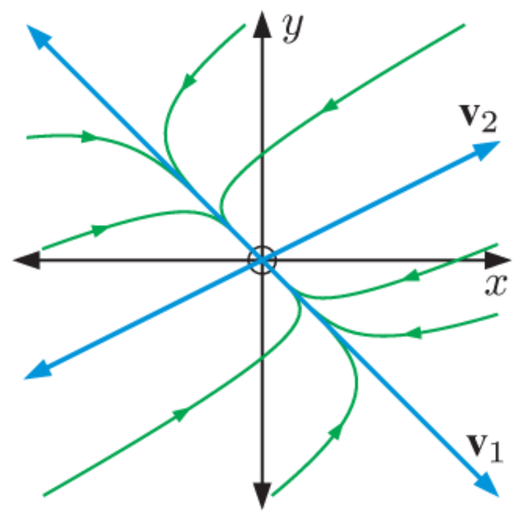
c if $\alpha > 0$, the trajectory spirals outwards.

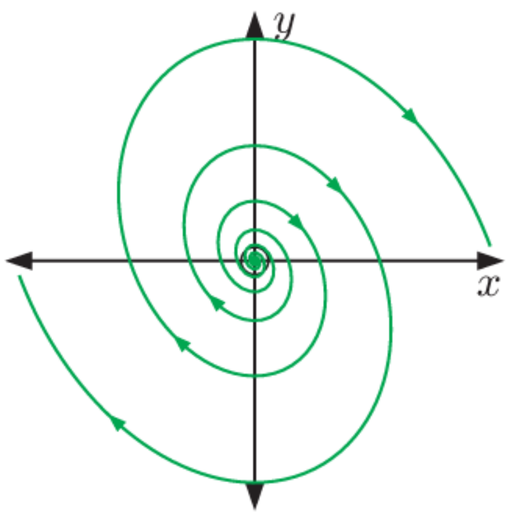
From the **Investigation**, you should have concluded:

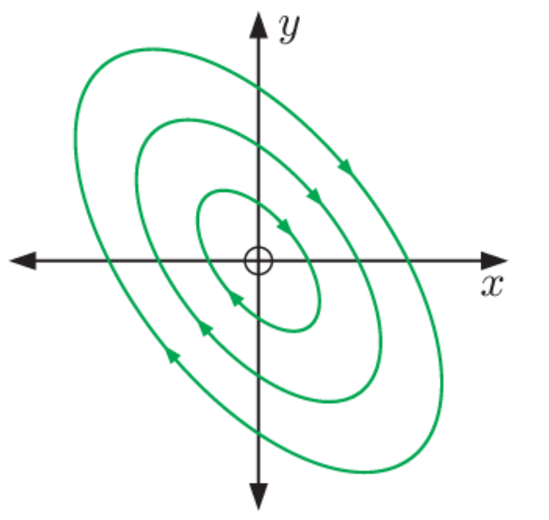
For the system of coupled linear differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the behaviour of the system around the equilibrium point $(0, 0)$ is determined by the eigenvalues of $\mathbf{A}$.

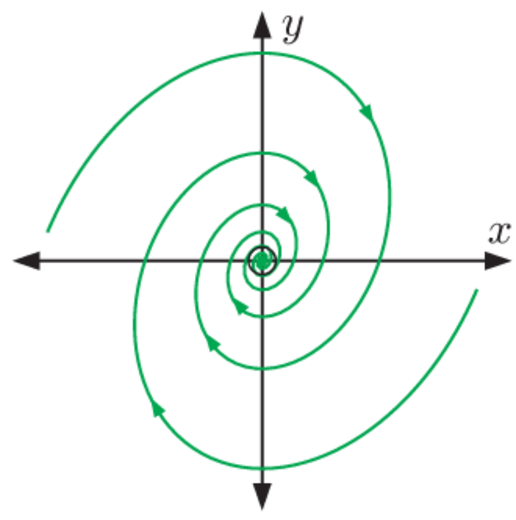
<table>
<tr>
<td>real and positive
 

unstable fixed
If $\lambda_1 > \lambda_2 > 0$, trajectories become parallel to $\mathbf{v}_1$ as $t \to \infty$.</td>
<td>real and opposite in sign
 

saddle point
If $\lambda_1 > 0 > \lambda_2$, trajectories approach $k\mathbf{v}_1$, $k \in \mathbb{R}$ as $t \to \infty$.</td>
<td>real and negative
 

stable fixed
If $0 > \lambda_1 > \lambda_2$, trajectories approach O along $k\mathbf{v}_1$, $k \in \mathbb{R}$.</td>
</tr>
<tr>
<td>complex with positive real part
 

unstable spiral</td>
<td>purely imaginary
 

centre</td>
<td>complex with negative real part
 

stable spiral</td>
</tr>
</table>

As per the table above, in this course we will only consider cases in which the eigenvalues are distinct.

- For both real and complex eigenvalues, you should be able to sketch a phase portrait and trajectories, and describe the behaviour of the system.
- If the eigenvalues are real, you should be able to calculate the corresponding eigenvectors, write down the general solution to the system, and find the particular solution for a given starting point.

Example 2 **Self Tutor**

The system $\begin{cases} \dfrac{dx}{dt} = -8x - 5y \\ \dfrac{dy}{dt} = 10x + 7y \end{cases}$ can be written in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$\mathbf{A} = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix}$ has eigenvalues $2, -3$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively.

a Given the initial point $(1, 2)$, find:

i $\dot{\mathbf{x}}$ when $t = 0$ **ii** the particular solution to the system.

b Sketch the phase portrait, including the particular solution.

c Discuss the behaviour of the system in the long term.

---

a **i** When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\therefore \quad \dot{\mathbf{x}} = \begin{pmatrix} -8 & -5 \\ 10 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -18 \\ 24 \end{pmatrix}$$

ii Using the given eigenvalues and eigenvectors, a general solution to the system is

$$\mathbf{x} = Ae^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + Be^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

When $t = 0$, we know $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\therefore \quad A \begin{pmatrix} 1 \\ -2 \end{pmatrix} + B \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

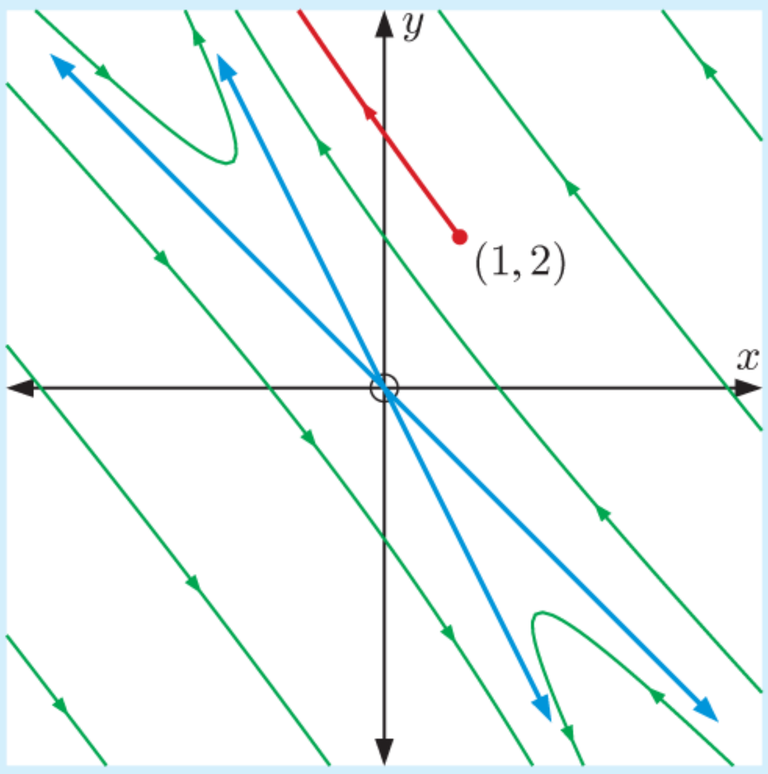
$$\therefore \quad \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

The particular solution is $\mathbf{x} = -3e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} - 4e^{-3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

b The equilibrium point at O is a saddle point.



c As $t \to \infty$, $e^{-3t} \to 0$

$$\therefore \quad \mathbf{x} \to -3e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\therefore$ the line $k \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ will be an asymptote as $t \to \infty$.

Example 3 **Self Tutor**

For each system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, use the given information to sketch the phase portrait and the trajectory from the initial point $(1, 2)$.

a $\dot{\mathbf{x}} = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\mathbf{x}$, where $\begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}$ has eigenvalues $1, 4$ and corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

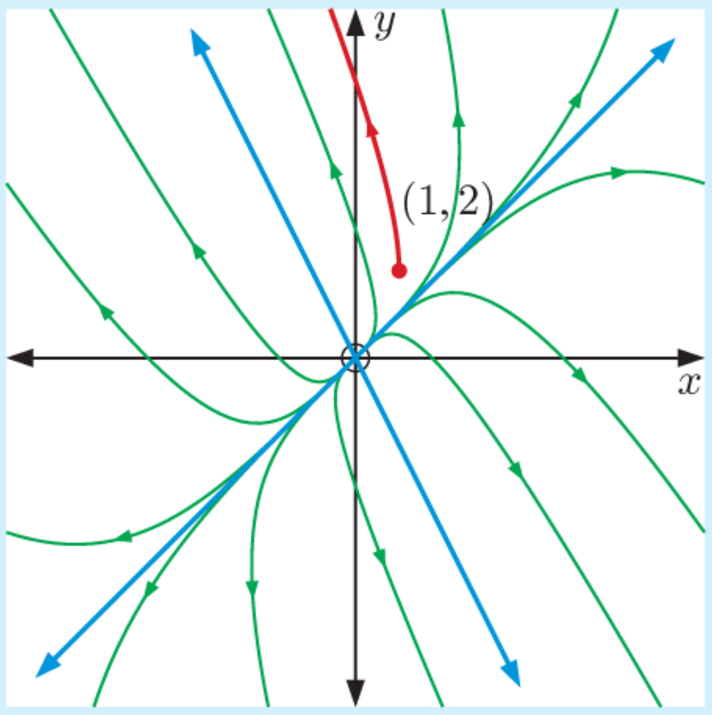
b $\dot{\mathbf{x}} = \begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix}\mathbf{x}$, where $\begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix}$ has eigenvalues $-3 \pm 2i$.

a The equilibrium point at O is an unstable fixed point.

When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore$ the initial trajectory is

$$\dot{\mathbf{x}} = \begin{pmatrix} 2 & -1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$



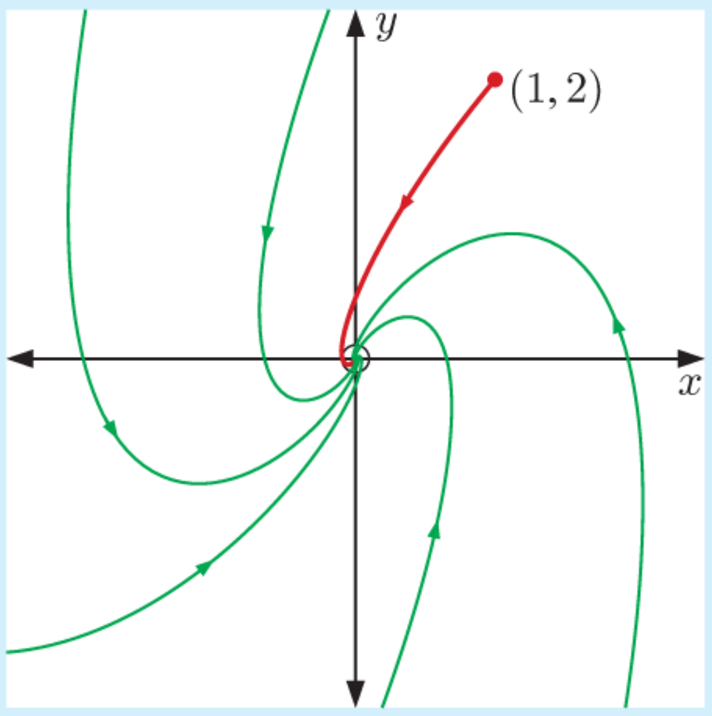
b The equilibrium point at O is a stable spiral.

When $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\therefore$ the initial trajectory is

$$\dot{\mathbf{x}} = \begin{pmatrix} -1 & -2 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ -6 \end{pmatrix}.$$

So the rotation is anticlockwise.



EXERCISE 26B

1 The system $\begin{cases} \dfrac{dx}{dt} = -2y \\ \dfrac{dy}{dt} = -x + y \end{cases}$ can be written in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$ has eigenvalues $-1, 2$ with corresponding eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively.

a Find the general solution to the system.

b Given the initial point $(3, 3)$, find:

i $\dot{\mathbf{x}}$ when $t = 0$ **ii** the particular solution to the system.

c Describe the equilibrium point at $(0, 0)$.

d Sketch the phase portrait, including the particular solution.

e Discuss the behaviour of the system in the long term.



2 The system $\begin{cases} \dot{x} = -x - 5y \\ \dot{y} = 2x + y \end{cases}$ can be written in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} -1 & -5 \\ 2 & 1 \end{pmatrix}$ has eigenvalues $\pm 3i$.

a Explain why the equilibrium point of the system is a centre.

b **i** Find the trajectory at the point $(1, 0)$.

ii Hence determine whether the trajectories rotate clockwise or anticlockwise.

c Sketch the phase portrait for the system.

3 For each system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, use the given information to sketch the phase portrait and the trajectory from the initial point $(2, 1)$.

a $\dot{\mathbf{x}} = \begin{pmatrix} -3 & 0 \\ -1 & -2 \end{pmatrix}\mathbf{x}$, where $\begin{pmatrix} -3 & 0 \\ -1 & -2 \end{pmatrix}$ has eigenvalues $-3, -2$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

b $\dot{\mathbf{x}} = \begin{pmatrix} 2 & -6 \\ 1 & 0 \end{pmatrix}\mathbf{x}$, where $\begin{pmatrix} 2 & -6 \\ 1 & 0 \end{pmatrix}$ has eigenvalues $1 \pm i\sqrt{5}$.

4 Consider the system $\begin{cases} \dfrac{dx}{dt} = -5x + 7y \\ \dfrac{dy}{dt} = -2x + y \end{cases}$ with initial point $(0, 3)$.

a Find the eigenvalues of $\begin{pmatrix} -5 & 7 \\ -2 & 1 \end{pmatrix}$.

b Hence describe the equilibrium point of the system.

c Find the trajectory when $t = 0$, and explain what this tells you about the phase portrait.

d Sketch the phase portrait including the trajectory from the initial point $(0, 3)$.

5 Consider the system $\begin{cases} \dot{x} = -3x + 4y \\ \dot{y} = x - 3y. \end{cases}$

a Find the eigenvalues and corresponding eigenvectors of $\begin{pmatrix} -3 & 4 \\ 1 & -3 \end{pmatrix}$.

b Describe the equilibrium point of the system.

c Given the initial point $(-2, 3)$, find:

i $\dot{\mathbf{x}}$ when $t = 0$ **ii** the particular solution to the system.

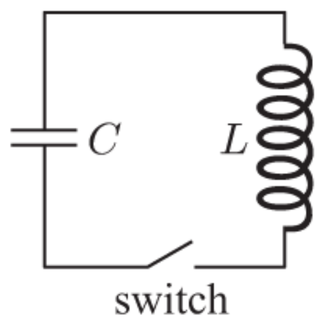
d Find the position of the particular solution when $t = 1$. Round your answer to 3 decimal places.

e Sketch the phase portrait, including the particular solution.

f Discuss the behaviour of the system in the long term.

6 An inductance-capacitance (LC) circuit is a simple electrical circuit used for producing radio signals.

A simple LC circuit may be described as a combination of two interacting elements:



- A time-varying current I flows through a coil, inducing a voltage V against the flow and proportional to the rate of change of current.
- The current is produced by discharge from a capacitor, and is proportional to the rate of change of voltage across the capacitor.

a Explain why:

i $V = -L\dfrac{dI}{dt}$ for some "inductance" L **ii** $I = C\dfrac{dV}{dt}$ for some "capacitance" C.

b A particular LC circuit has inductance $L = 4 \times 10^{-2}$ H and capacitance $C = 10^{-5}$ F.

i Write a system of coupled linear differential equations for the circuit in the form

$$\begin{pmatrix} \dfrac{dI}{dt} \\ \dfrac{dV}{dt} \end{pmatrix} = \mathbf{A}\begin{pmatrix} I \\ V \end{pmatrix}.$$

ii Find the eigenvalues of $\mathbf{A}$.

c At time zero, the capacitor is fully "charged" with voltage 10^{-4} V across it. The switch is turned on so current will be able to flow.

i Sketch a phase portrait for the system, including the trajectory from the initial conditions.

ii Describe what happens over time.

ACTIVITY 2 EQUILIBRIUM REACTIONS

Coupled systems of linear differential equations can be used to solve problems involving equilibrium reactions in chemistry.

Click on the icon to obtain this Activity.



C SECOND ORDER DIFFERENTIAL EQUATIONS

The **order** of a differential equation is the highest order derivative present.

For example:

- $\dfrac{dx}{dt} = kx(t)(1 - x(t))$ involves the first derivative $\dfrac{dx}{dt}$, so this is a first order differential equation.
- $\dfrac{d^2x}{dt^2} + w^2x(t) = 0$ involves the second derivative $\dfrac{d^2x}{dt^2}$, so this is a second order differential equation.

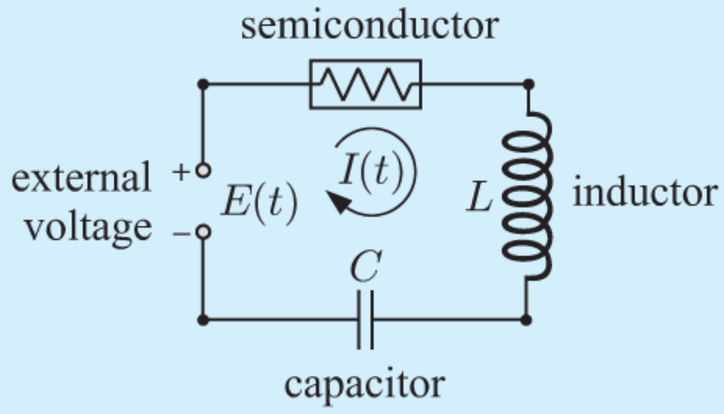
In this Section we see how second order differential equations can be transformed into a system of coupled first order differential equations. This allows us to use the phase portrait methods already studied.

Differential equations of the form $a\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0$ can be written as a system of coupled first order linear differential equations, allowing us to use matrix methods.

Example 4 **Self Tutor**

A Van der Pol oscillator is an electrical circuit involving a triode. It evolves in time according to the second order differential equation $\dfrac{d^2I}{dt^2} - \mu(1 - I^2)\dfrac{dI}{dt} + I = 0$ where $0 < \mu < 4$ is a parameter. The equation was first studied in the 1920s by **Balthasar van der Pol** (1889 - 1959).

Write this as a coupled system of first order differential equations.



Let $y = \dfrac{dI}{dt}$

$\therefore \dfrac{dy}{dt} = \dfrac{d^2I}{dt^2}$

$\therefore$ the system is $\begin{cases} \dfrac{dI}{dt} = y \\ \dfrac{dy}{dt} = \mu(1 - I^2)y - I. \end{cases}$

EXERCISE 26C

1 The displacement x of a mass on a spring changes according to the second order differential equation $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$.

a Write this as a coupled system of first order differential equations.

b Write this system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

2 A more realistic model for a mass on a spring includes *damping* due to air resistance. The displacement x can be modelled using $\dfrac{d^2x}{dt^2} + k\dfrac{dx}{dt} + \omega^2 x = 0$ where $k \geqslant 0$ is the level of damping.

a Write this as a coupled system of first order differential equations.

b Write this system in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

c Show that the eigenvalues of $\mathbf{A}$ are $\dfrac{-k \pm \sqrt{k^2 - 4\omega^2}}{2}$.

d If $k = 0$, the spring is undamped. State the nature of the equilibrium point and its significance in this case.

e Suppose $0 < k < 2\omega$. State the nature of the equilibrium point and its significance in this case.

3 When an object with mass m falls under gravity, its displacement x has equation of motion $m\ddot{x} = -mg - k(\dot{x})^2$ where $g \approx 9.8$ is the acceleration due to gravity alone and k is a positive constant which accounts for air resistance.
Use the substitution $v = \dot{x}$ to write this as a coupled system of differential equations.

4 The temperature x of a drink t seconds after ice cubes are added to it, satisfies the differential equation $50\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 0$.

a Use the substitution $y = \dfrac{dx}{dt}$ to write this as a coupled system of first order differential equations.

b The equation for $\dfrac{dy}{dt}$ is separable and independent of x. Solve this equation for $y(t)$.

c Hence find a general solution for $x(t)$.

d The temperature of the drink was initially 25°C. After 60 seconds it has dropped to 19°C.

i Find the particular solution for $x(t)$.

ii Discuss the temperature of the drink in the long term.

ACTIVITY 3 THE SIMPLE RIGID PENDULUM

The behaviour of a simple rigid pendulum is also modelled by a second order differential equation.

Click on the icon to obtain this Activity.



D EULER'S METHOD FOR COUPLED EQUATIONS

Euler's method can also be used to find a numerical approximation to the solution of the coupled equations

$$\begin{cases} \dfrac{dx}{dt} = f_1(x, y, t) \\ \dfrac{dy}{dt} = f_2(x, y, t). \end{cases}$$

We choose initial conditions x_0 and y_0 and an initial time t_0.

We choose a small time step $h > 0$, then just as before we calculate:

$$\begin{aligned} t_1 &= t_0 + h \\ x_1 &= x_0 + hf_1(x_0, y_0, t_0) \\ y_1 &= y_0 + hf_2(x_0, y_0, t_0). \end{aligned}$$

This generates the point (x_1, y_1) for our solution curve, corresponding to time $t = t_1$.

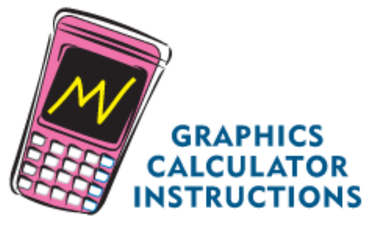
We continue this process to generate a sequence of coordinates (x_i, y_i) for each time t_i, using:

$$\begin{aligned} t_i &= t_{i-1} + h \\ x_i &= x_{i-1} + hf_1(x_{i-1}, y_{i-1}, t_{i-1}) \\ y_i &= y_{i-1} + hf_2(x_{i-1}, y_{i-1}, t_{i-1}). \end{aligned}$$

The values can be calculated using a spreadsheet, online software, or your graphics calculator.





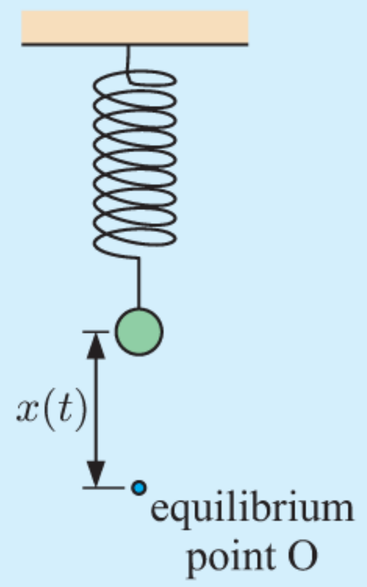


You should be aware that:

- The solution curve is only an approximation, and we have already observed how Euler's method can gather error over time.
- If the system has the form $\begin{cases} \frac{dx}{dt} = f_1(x, y) \\ \frac{dy}{dt} = f_2(x, y) \end{cases}$ where the derivatives are independent of time, then the system is **steady**.
  In these cases we can also generate a phase portrait to study the system.
  For an unsteady system, the phase portrait would change over time.
- Depending on the situation, it may be appropriate to view the solution as separate functions $x(t)$ and $y(t)$, or in a graph of y against x.

Example 5 **Self Tutor**

The damped simple harmonic motion of a mass on a spring is given by $\frac{d^2x}{dt^2} + 0.2\frac{dx}{dt} + 4.01x = 0$ where x is the displacement at time t.
The mass has initial displacement 5 cm, and is released with velocity $0.5\ \text{cm s}^{-1}$ downwards.



a Let $y = \frac{dx}{dt}$ be the velocity function. Hence describe the system using coupled differential equations.

b Apply Euler's method by hand with $h = 0.01$ to generate (x_1, y_1).

c Use technology to apply Euler's method with $h = 0.01$ for the first 5 seconds of motion. Plot the set of points $\{(t_i, x_i)\}$ on the same set of axes as the analytic solution $x(t) = 5e^{-0.1t}\cos 2t$. Discuss your answer.

a Let $\frac{dx}{dt} = y$

$\therefore \frac{d^2x}{dt^2} = \frac{dy}{dt}$

$\therefore \frac{dy}{dt} + 0.2y + 4.01x = 0$

$\therefore$ the system is $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -4.01x - 0.2y \end{cases}$

b At time $t_0 = 0$,

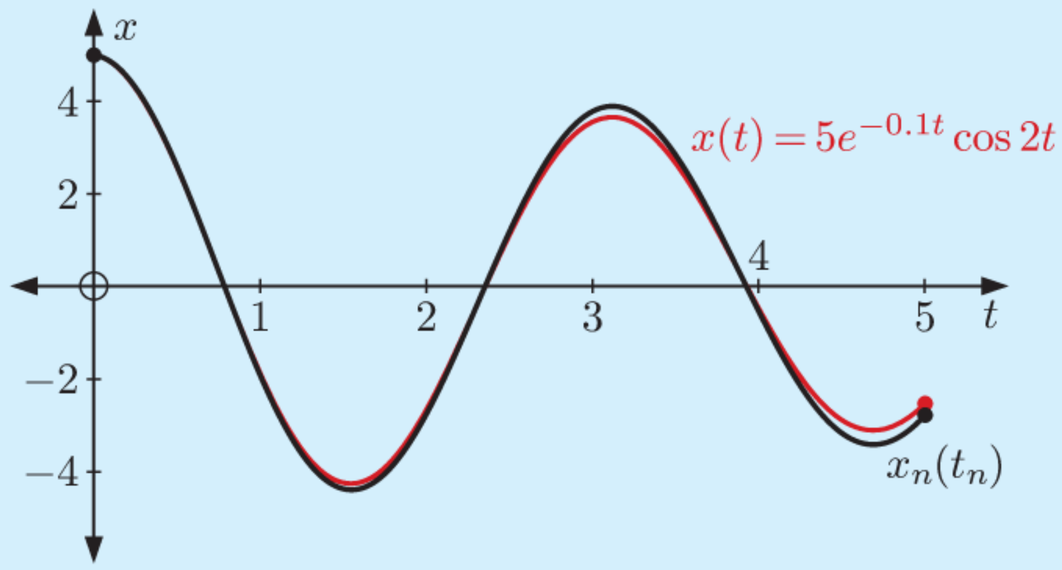
$x_0 = 5, \quad y_0 = -0.5$

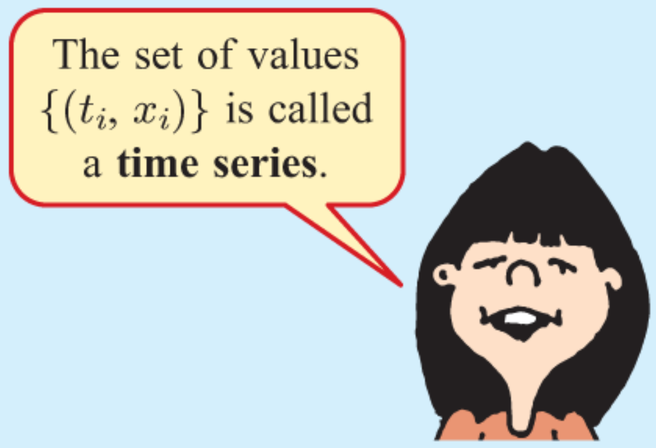
$\frac{dx}{dt} = y_0 = -0.5$

$\frac{dy}{dt} = -4.01x_0 - 0.2y_0 = -19.95$

$\therefore x_1 = 5 + 0.01(-0.5)$
$= 4.995$
$y_1 = -0.5 + 0.01(-19.95)$
$= -0.6995$

c





With $h = 0.01$, the points (t_i, x_i) approximate a smooth curve similar to the analytic solution $x(t) = 5e^{-0.1t}\cos 2t$. Euler's method is less accurate at times when the curve changes gradient faster.

You should be aware that with an inappropriate choice of h, Euler's method can fail spectacularly.



For example, click on the icon to see how Euler's method is applied to the Example above. Try applying Euler's method with these values of h: 0.005, 0.01, 0.02, 0.05, 0.1 .

EXERCISE 26D

1 Consider the differential equation $\dfrac{d^2x}{dt^2} + 9x = 0$. Initially, $x = 2$ and $y = \dfrac{dx}{dt} = 0$.

a Show that $x(t) = 2\cos 3t$ is a particular solution to the differential equation which also satisfies the initial conditions.

b Describe the system using coupled differential equations.

c If Euler's method is to be applied with step size h, write down expressions for t_i, x_i, and y_i in terms of t_{i-1}, x_{i-1}, and y_{i-1}.

d Apply Euler's method by hand with $h = 0.01$ to generate (x_1, y_1).

e Use technology to apply Euler's method with $h = 0.01$ for $0 \leqslant t \leqslant 2$.
Plot the set of points $\{(t_i, x_i)\}$ on the same set of axes as the analytic solution $x(t) = 2\cos 3t$. Discuss your answer.

2 Consider the Lotka-Volterra predator-prey model $\begin{cases} \dfrac{dx}{dt} = 2x - xy \\ \dfrac{dy}{dt} = xy - y \end{cases}$

with initial values $x(0) = 2$, $y(0) = 1$.

a If Euler's method is to be applied with step size h, write down expressions for t_i, x_i, and y_i in terms of t_{i-1}, x_{i-1}, and y_{i-1}.

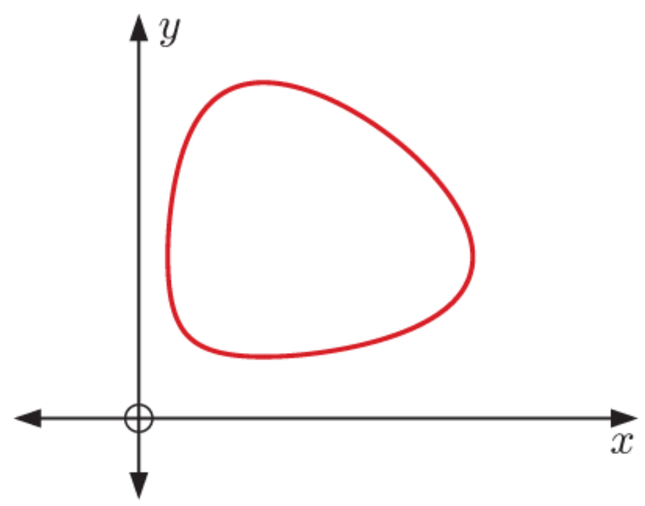
b Apply Euler's method with $h = 0.05$ to generate values of t_i, x_i, y_i for $0 \leqslant t \leqslant 10$.
Hence:

i Plot the time series $\{(t_i, x_i)\}$ and $\{(t_i, y_i)\}$ on the same set of axes.

ii Plot your set of data points $\{(x_i, y_i)\}$ as a trajectory on a phase portrait. Clearly indicate the direction of the trajectory.

iii The exact solution to this system is the relation $x - \ln x + y - 2\ln y = 3 - \ln 2$ which is graphed alongside.

Compare your trajectory with the exact solution. Discuss how your numerical solution could be misleading in the long term.



$x - \ln x + y - 2\ln y = 3 - \ln 2$

3 Consider the Van der Pol oscillator $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 1.5(1 - x^2)y - x \end{cases}$

with initial values $x(0) = 1,\ y(0) = 0$.

Apply Euler's method with $h = 0.1$ to generate values of $t_i,\ x_i,\ y_i$ for $0 \leqslant t \leqslant 20$. Hence:

a Plot the time series $\{(t_i,\ x_i)\}$ and $\{(t_i,\ y_i)\}$ on the same set of axes.

b Plot your set of data points $\{(x_i,\ y_i)\}$ as a trajectory on a phase portrait. Clearly indicate the direction of the trajectory.

4 When an anchor is released into the ocean, it falls through the water with acceleration $\dfrac{d^2x}{dt^2} = 9.81 - 0.375\left(\dfrac{dx}{dt}\right)^2 \text{ m s}^{-2}$.

a Letting $y = \dfrac{dx}{dt}$ be the velocity of the anchor, describe the anchor's fall using a system of first order differential equations.

b Use Euler's method with a suitable step size h to find how far the anchor falls in the first 5 seconds correct to 1 decimal place.

c Use Euler's method to establish the terminal velocity of the anchor correct to 4 decimal places.

d At terminal velocity the anchor has no acceleration. Calculate the terminal velocity using the original differential equation. Hence check your answer to **c**.

5 Juan is a conservationist monitoring populations of the common hermit crab (x) and the yellow-footed hermit crab (y) in a particular bay on the Queenstown coast. The two species of hermit crab compete for the same food source. Using data he has collected, Juan believes the crab populations can be modelled using the system

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{600}(1200 - x - 2y) \\ \dfrac{dy}{dt} = \dfrac{y}{600}(1500 - \frac{3}{2}x - y) \end{cases}$$ where t is the time in years.



a State the constraints on x and y.

b Find all *four* equilibrium points for the populations.

c Juan has carefully counted the current populations. They are $x_0 = 175$ and $y_0 = 25$.

i Use Euler's method with step size $\frac{1}{4}$ to estimate the populations after 2 years.

ii Use Euler's method to predict the long-term behaviour of the populations.

d In three other bays along the coast, Juan counts the populations of crabs displayed alongside.

Use Euler's method to generate trajectories for the populations in each bay.

Plot the trajectories for all *four* bays on the same phase portrait.

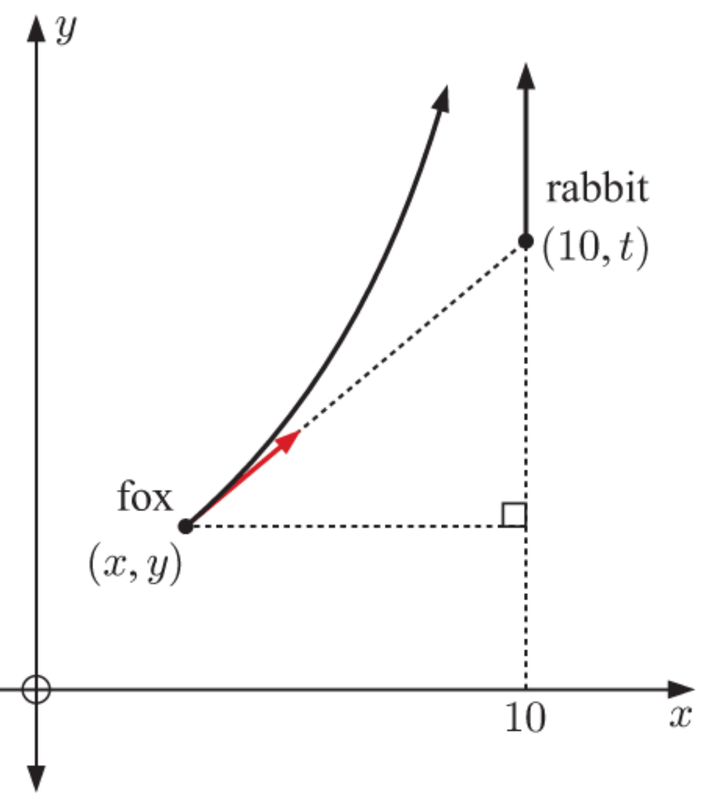
Bay	x_0	y_0
1	250	20
2	1000	300
3	1500	200

e Discuss whether it is possible for the two species to coexist in the long term in any one bay.

6 A **radiodrome** is a curve formed by the *pursuit* of one point which chases another point moving in a straight line. It was first described by the French mathematician **Pierre Bouguer** in 1732.

Suppose a rabbit is initially at $(10, 0)$, and runs with velocity vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

A fox pursues the rabbit, and is always running directly towards the rabbit. The rabbit is therefore always on a tangent to the fox's path.



a Explain why the position of the rabbit at time t is $(10, t)$.

b The fox runs at a constant speed F units per second. Explain why the position (x, y) of the fox satisfies

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{10 - x}{\sqrt{(10-x)^2 + (t-y)^2}} F \\ \dfrac{dy}{dt} = \dfrac{t - y}{\sqrt{(10-x)^2 + (t-y)^2}} F. \end{cases}$$

c Suppose the fox is initially at $(0, 0)$, and runs at the same speed as the rabbit, so $F = 1$.

i Use Euler's method with $h = 0.1$ to find the position of the fox each second for the first 20 seconds.

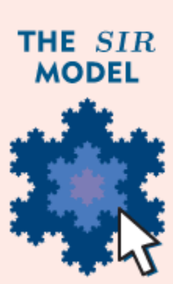
ii Does the fox catch the rabbit? If not, what happens to the distance between the fox and the rabbit as time progresses?

d Now suppose the fox runs twice as fast as the rabbit, so $F = 2$.

i Use Euler's method with $h = 0.05$ to find the position of the fox each second for the first 10 seconds.

ii Does the fox catch the rabbit? If so, at what time do you think this occurs?

ACTIVITY 4 THE SIR MODEL FOR INFECTIOUS DISEASE



The SIR model is used to study the spread of an infectious disease in a population. It considers changes in the number of susceptible, infectious, and recovered individuals over time.

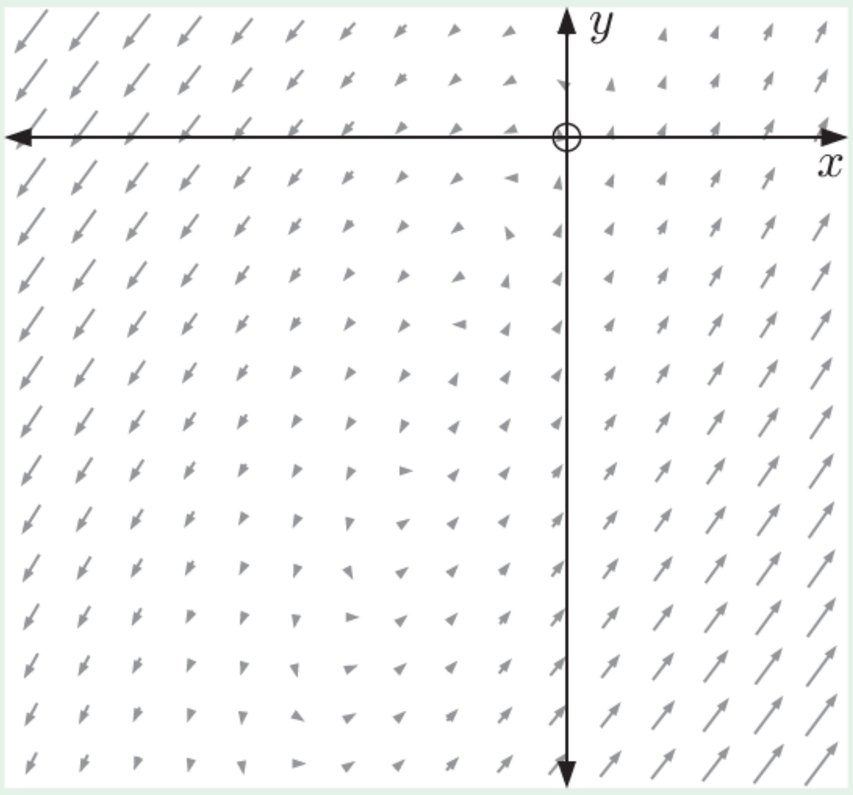
It therefore requires the solution of *three* simultaneous differential equations.

In this Activity we consider a number of useful models including:

- a short-term, non-fatal disease such as a cold
- a short-term, highly dangerous disease such as *Ebola*
- the long-term behaviour of a disease in a population
- vaccine models.

REVIEW SET 26A

1



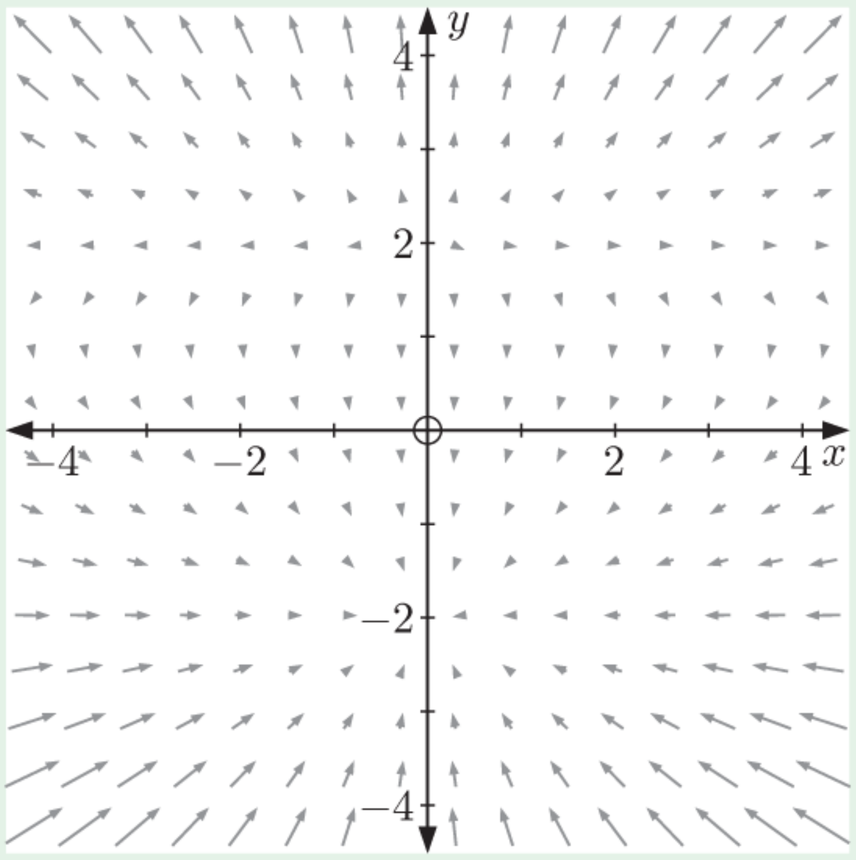
The phase portrait for the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = 3x - y + 1 \end{cases}$$ is shown alongside.

Locate and describe the equilibrium point.

2 The phase portrait for the coupled system

$$\begin{cases} \dot{x} = xy - x \\ \dot{y} = y^2 - 4 \end{cases}$$ is shown alongside.



a Calculate $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ at the point $(-1, 3)$.

b Locate and describe the equilibrium points.

c Illustrate the solution curve starting at $(4, 1)$.

3 Consider the system $\begin{cases} \dot{x} = 4x - 3y \\ \dot{y} = 5x - 3y. \end{cases}$

a Find the eigenvalues of $\begin{pmatrix} 4 & -3 \\ 5 & -3 \end{pmatrix}$.

b Hence explain why the equilibrium point is an unstable spiral.

c Initially, $x = -1$ and $y = 0$.

i Find the initial trajectory vector.

ii Do the spirals rotate clockwise or anticlockwise?

d Sketch the phase portrait including the trajectory from the initial point $(-1, 0)$.

4 Taking into account production and stock levels, the price P of an item to be sold is modelled by $\dfrac{d^2P}{dt^2} + 8\dfrac{dP}{dt} + 30P = 200$.

By letting $S = \dfrac{dP}{dt}$, write this as a system of coupled first order differential equations.

5 The system $\begin{cases} \dot{x} = x + 3y \\ \dot{y} = 3x + y \end{cases}$ can be written in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ has eigenvalues $-2,\ 4$ with corresponding eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

a Describe the equilibrium point.

b Find the general solution to the system.

c Given that $x = 3,\ y = -3$ when $t = 0$, find the particular solution.

d Discuss the behaviour of the particular solution in the long term.

e Sketch the phase portrait for the system, including the particular solution.

6 The number of zebras (x) and lions (y) in a region can be modelled by the differential equations

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{50}(30 - 0.5y - 0.003x) \\ \dfrac{dy}{dt} = \dfrac{y}{50}(-2 + 0.02x - 0.9y) \end{cases}$$ where t is the time in years.

a Find the equilibrium points. Describe the populations in each case.

b Suppose that initially there are 800 zebras and 35 lions.

i Use Euler's method with step length 0.1 to estimate the population of each species after 1 year.

ii Use further iterations of Euler's method to predict the populations of each species in the long term.

iii Estimate the lowest population of lions, and the time when it occurs.

iv Sketch the trajectory for the populations.

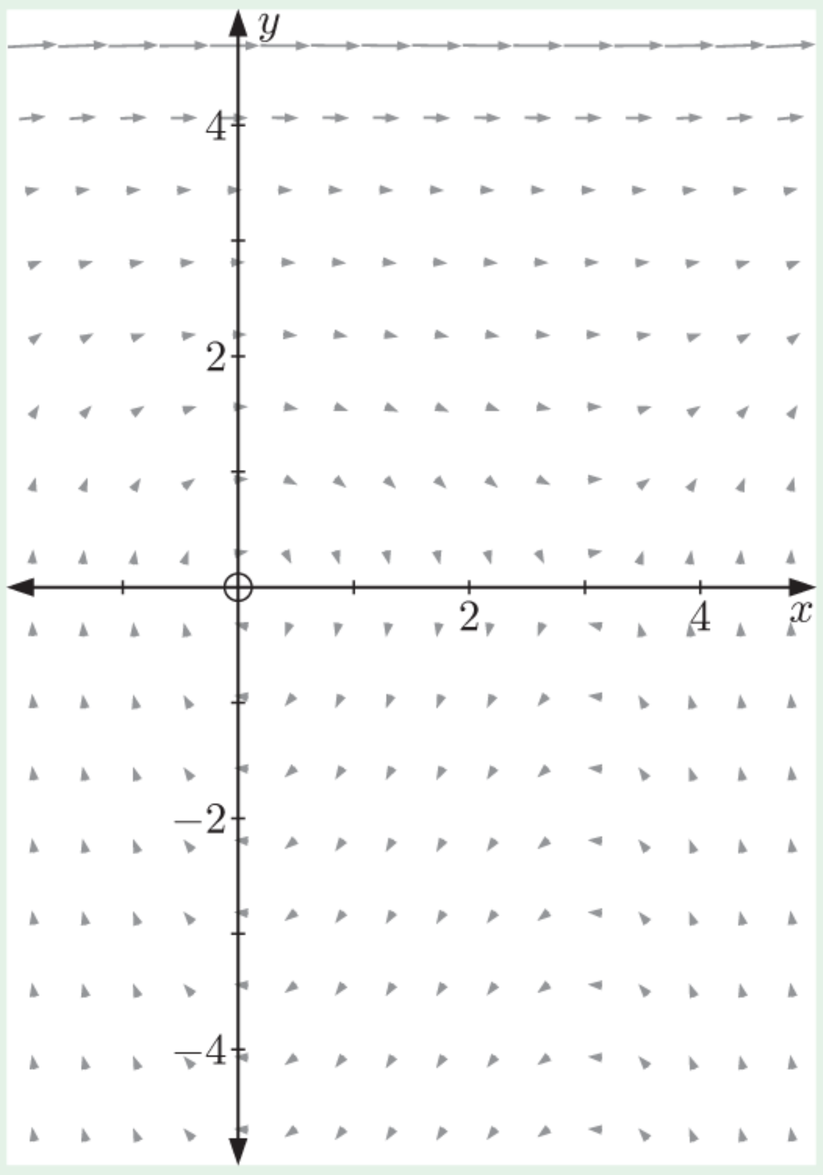
REVIEW SET 26B

1 The diagram alongside shows the phase portrait for the system $\begin{cases} \dot{x} = e^y - 1 \\ \dot{y} = x^2 - 3x. \end{cases}$

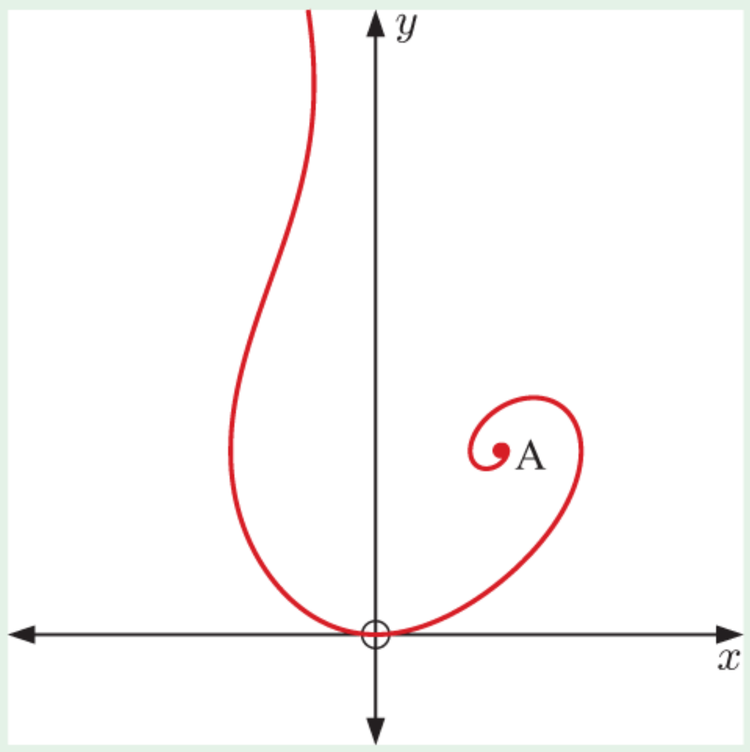
a Show that the following are equilibrium points, and describe the equilibrium point in each case.

i $(0,\ 0)$ **ii** $(3,\ 0)$

b Illustrate the solution curve starting at $(-1,\ -2)$.



2 The solution curve shown for the coupled system $\begin{cases} \dfrac{dx}{dt} = \cos \pi y \\ \dfrac{dy}{dt} = 3x - 2y \end{cases}$ passes through the origin.



a Find the coordinates of the equilibrium point A.

b Does the solution curve rotate inwards or outwards? Explain your answer.

3 The system $\begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = x + 6y \end{cases}$ can be written in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where

$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$ has eigenvalues $3, 5$ with corresponding eigenvectors $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ respectively.

a Find the general solution to the system.

b Given the initial point $(-3, 5)$, find:

i $\dot{\mathbf{x}}$ when $t = 0$ **ii** the particular solution to the system.

c Describe the equilibrium point.

d Sketch the phase portrait, including the particular solution.

e Discuss the behaviour of the system in the long term.

4 An undamped spring is pushed from its equilibrium position with initial velocity 6 cm s^{-1}. Its motion is determined by the differential equation $\dfrac{d^2x}{dt^2} + 4x = 0$.

a Show that $x(t) = 3 \sin 2t$ is a particular solution to the differential equation which also satisfies the initial conditions.

b Letting $y = \dfrac{dx}{dt}$, describe the system using coupled differential equations.

c If Euler's method is to be applied with step size h, write down expressions for t_i, x_i, and y_i in terms of t_{i-1}, x_{i-1}, and y_{i-1}.

d Apply Euler's method by hand with $h = 0.01$ to generate (x_1, y_1).

e Use technology to apply Euler's method with $h = 0.01$ for $0 \leqslant t \leqslant 2$. Plot the set of points $\{(t_i, x_i)\}$ on the same set of axes as the analytic solution $x(t) = 3 \sin 2t$. Discuss the accuracy of your numerical approximation.

5 In an RLC series circuit, the current I is governed by the differential equation $\dfrac{d^2I}{dt^2} + 4\dfrac{dI}{dt} + 16I = 0$.

a Letting $J = \dfrac{dI}{dt}$, describe the system using coupled linear differential equations.

b Write the system in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

c Find the eigenvalues of **A**, and hence describe the equilibrium point of the system.

d Initially, $I = 4$ mA and $\dfrac{dI}{dt} = 5$ mA m s^{-1}.

i Find the initial trajectory vector.

ii Use Euler's method with step size $h = 0.05$ to estimate the first time at which the current is zero.

iii Sketch the trajectory of the system on a phase portrait.

iv Describe what happens to the current in the long term.

6 In **Chapter 24** on Kinematics, we saw that $\begin{cases} x(t) = R\cos(\omega t + \phi) \\ y(t) = R\sin(\omega t + \phi) \end{cases}$ describes a circular orbit around the origin.

a Find $\dot{x}$ and $\dot{y}$.

b Hence write a system of coupled differential equations for the orbit in the matrix form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$.

c Find the eigenvalues of **A** and describe the significance of this result.

d Calculate $\dot{\mathbf{x}}$ when $t = 0$.

e **i** Explain how the direction of rotation is determined by ω.

ii How do we observe this in the matrix **A**?

f Consider the general system of coupled linear differential equations $\dot{\mathbf{x}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{x}$.

i Show that the eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are $\dfrac{a+d}{2} \pm \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.

ii For the eigenvalues to be complex, what can we say about the sign of bc?

iii Hence describe the features of the matrix **A** which will tell us the direction of rotation of a spiral or orbit.

Chapter 27

Discrete random variables

Contents:

A Random variables
B Discrete probability distributions
C Expectation
D Variance and standard deviation
E Properties of $aX + b$
F The binomial distribution
G Using technology to find binomial probabilities
H The mean and standard deviation of a binomial distribution
I The Poisson distribution



OPENING PROBLEM

In a sideshow game, players have a 50% chance of winning a prize worth \$2, \$5, \$10, or \$20. The probabilities of winning these prizes are given in the table below.

Prize value	\$0	\$2	\$5	\$10	\$20
Probability	0.5	0.35	0.1	0.04	0.01



Things to think about:

a What is the sum of the probabilities in the table? Why must this be the answer?

b What is the most likely outcome from playing the game?

c What is the *average* result from playing the game?

d What is a *fair* price for playing the game?

Many variables in the world around us depend on chance events. Examples of such variables are:

- the number of players in your football team who will score a goal in the next match
- the time it will take you to travel to school tomorrow
- the sum of the values when three dice are rolled.

Because of the element of chance in these variables, we cannot predict the exact value they will take when next measured. However, we can often determine the *possible values* the variable can take, and assign to each possible value the **probability** of it occurring.

In this Chapter we use probability to model the random variation or **distribution** of numerical variables.

A RANDOM VARIABLES

A **random variable** uses numbers to describe the possible outcomes which could result from a random experiment.

A random variable is often represented by a capital letter such as X.

Random variables can be either **discrete** or **continuous**.

A **discrete random variable** X has a set of distinct possible values.

For example, X could be:

- the number of wickets a bowler takes in an innings of cricket, so $X \in \{0, 1, 2, ..., 10\}$.
- the number of defective light bulbs in a purchase order of 50, so $X \in \{0, 1, 2, ..., 50\}$.

To determine the value of a discrete random variable, we need to **count**.

A **continuous random variable** X can take any value within some interval on the number line.

For example, X could be:

- the heights of men, which lie in the interval $50 \text{ cm} < X < 250 \text{ cm}$
- the volume of water in a tank, which could lie in the interval $0 \text{ m}^3 < X < 100 \text{ m}^3$.

To determine the value of a continuous random variable, we need to **measure**.

DISCRETE RANDOM VARIABLES

In this Chapter, we will focus on how discrete random variables and their distributions arise.

Example 1 **Self Tutor**

A supermarket has three checkouts A, B, and C. A government inspector checks the weighing scales for accuracy at each checkout. The random variable X is the number of accurate weighing scales at the supermarket.

a List the possible outcomes and the corresponding values of X.

b What value(s) of X correspond to there being:

i one accurate scale **ii** at least one accurate scale?

a Possible outcomes:

A	B	C	X
✗	✗	✗	0
✓	✗	✗	1
✗	✓	✗	1
✗	✗	✓	1
✗	✓	✓	2
✓	✗	✓	2
✓	✓	✗	2
✓	✓	✓	3

b **i** $X = 1$

ii $X = 1$, 2, or 3

EXERCISE 27A

1 Classify each random variable as continuous or discrete:

a the quantity of fat in a sausage **b** the mark out of 50 for a geography test

c the weight of a Year 12 student **d** the volume of water in a cup of coffee

e the number of trout in a lake **f** the number of hairs on a cat

g the length of a horse's mane **h** the height of a skyscraper.

2 For each scenario:

i Identify the random variable being considered.

ii State whether the variable is continuous or discrete.

iii Give possible values for the random variable.

a To measure the rainfall over a 24-hour period in Singapore, water is collected in a rain gauge.

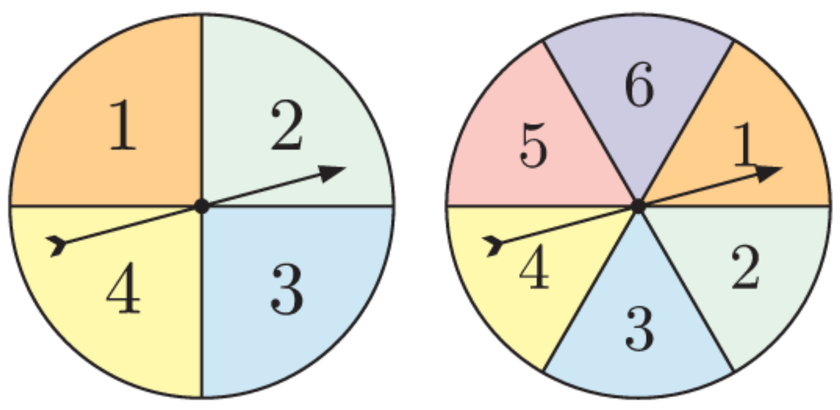
b To investigate the stopping distance for a tyre with a new tread pattern, a braking experiment is carried out.

c To check the reliability of a new type of light switch, switches are repeatedly turned off and on until they fail.

3 Suppose the spinners alongside are spun, and X is the sum of the numbers.

a Explain why X is a discrete random variable.

b State the possible values of X.



4 In the finals series of a baseball championship, the first team to win 4 games wins the championship. Let X represent the number of games played in the finals series.

a State the possible values of X.

b What value(s) of X correspond to the series lasting:

i exactly 5 games **ii** at least 6 games?

5 A supermarket has four checkouts A, B, C, and D. Management checks the weighing devices at each checkout. The random variable X is the number of weighing devices which are accurate.

a What values can X have?

b List the possible outcomes and the corresponding values of X.

c What value(s) of X correspond to:

i exactly two devices being accurate **ii** at least two devices being accurate?

6 Suppose three coins are tossed simultaneously. Let X be the number of heads that result.

a State the possible values of X.

b List the possible outcomes and the corresponding values of X.

c Are the possible values of X equally likely to occur? Explain your answer.

B DISCRETE PROBABILITY DISTRIBUTIONS

For any random variable, there is a corresponding **probability distribution** which describes the probability that the variable will take a particular value.

The probability that the variable X takes value x is denoted $\mathrm{P}(X = x)$.

If X is a random variable with possible values $\{x_1, x_2, x_3, ..., x_n\}$ and corresponding probabilities $\{p_1, p_2, p_3, ..., p_n\}$ such that $\mathrm{P}(X = x_i) = p_i$, $i = 1, ..., n$, then:

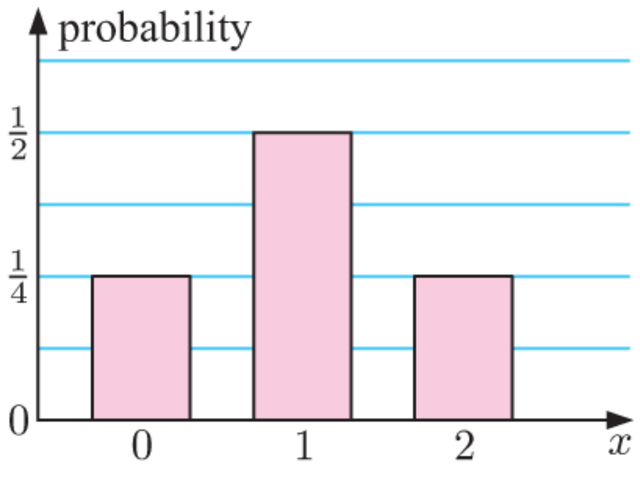
- $0 \leqslant p_i \leqslant 1$ for all $i = 1, ..., n$
- $\sum_{i=1}^{n} p_i = p_1 + p_2 + p_3 + ... + p_n = 1$
- $\{p_1, ..., p_n\}$ describes the **probability distribution** of X.

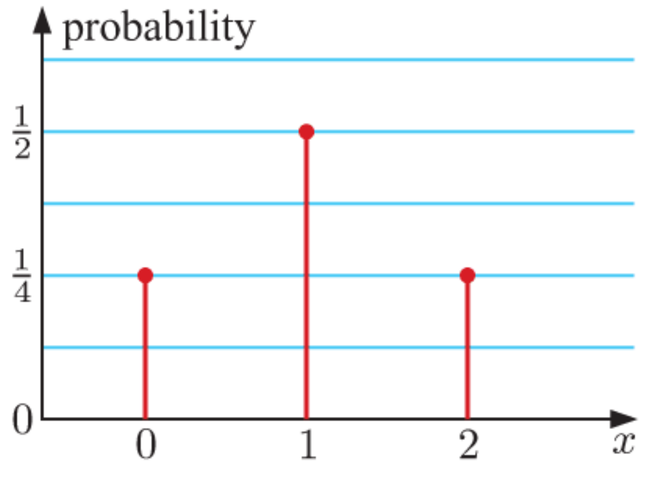
For example, suppose X is the number of heads obtained when 2 coins are tossed. The possible values for X are $\{0, 1, 2\}$ with corresponding probabilities $\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\}$. We see that $0 \leqslant p_i \leqslant 1$ for each value of i, and that the probabilities add up to 1.



We can display this probability distribution in a **table** or a **graph**.

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$





UNIFORM DISCRETE RANDOM VARIABLES

If the possible values $x_1, x_2, ..., x_n$ of a discrete random variable X all have the same probability $\frac{1}{n}$ of occurring, then X is a **uniform discrete random variable**.

An example of a uniform discrete random variable is the result X when a die is rolled. The possible values of X are 1, 2, 3, 4, 5, and 6, and each value has probability $\frac{1}{6}$ of occurring.



By contrast, if two dice are rolled, the sum of the resulting numbers Y is *not* a uniform discrete random variable.

THE MODE AND MEDIAN

The **mode** of a discrete probability distribution is the most frequently occurring value of the variable. This is the data value x_i whose probability p_i is the highest.

The **median** of the distribution corresponds to the 50th percentile. If the possible values $\{x_1, x_2, ..., x_n\}$ are listed in ascending order, the median is the value x_j when the cumulative sum $p_1 + p_2 + ... + p_j$ reaches 0.5.

Example 2 **Self Tutor**

A magazine store recorded the number of magazines purchased by its customers in one week. 23% purchased one magazine, 38% purchased two, 21% purchased three, 13% purchased four, and 5% purchased five. Let X be the number of magazines sold to a randomly selected customer.

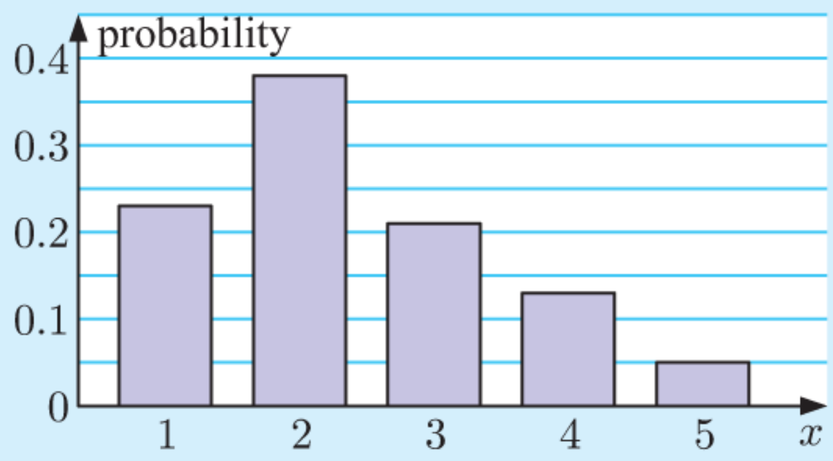
a State the possible values of X.
b Construct a probability table for X.
c Graph the probability distribution.
d Find the mode and median of X.

a $X = 1, 2, 3, 4,$ or 5

b

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.23	0.38	0.21	0.13	0.05

c



d Customers are most likely to buy 2 magazines, so this is the mode of X.
We now find the median:

$$p_1 = 0.23$$
$$p_1 + p_2 = 0.23 + 0.38 = 0.61$$

Since $p_1 + p_2 \geqslant 0.5$, the median is 2 magazines.

We can also describe the probability distribution of a discrete random variable using a **probability mass function** $P(x) = \mathrm{P}(X = x)$. The domain of the probability mass function is the set of possible values of the variable, and the range is the set of values in the probability distribution.

Example 3 **Self Tutor**

Show that $P(x) = \dfrac{x^2 + 1}{34}$, $x = 1, 2, 3, 4$ is a valid probability mass function.

$P(1) = \frac{2}{34}$, $P(2) = \frac{5}{34}$, $P(3) = \frac{10}{34}$, $P(4) = \frac{17}{34}$

All of these values obey $0 \leqslant P(x_i) \leqslant 1$, and $\sum_{i=1}^{n} P(x_i) = \frac{2}{34} + \frac{5}{34} + \frac{10}{34} + \frac{17}{34} = 1$

$\therefore$ $P(x)$ is a valid probability mass function.

EXERCISE 27B

1 **a** State whether each of the following is a valid probability distribution:

i

x	1	2	3	4
$\mathrm{P}(X = x)$	0.2	0.4	0.15	0.25

ii

x	0	1	2	3
$\mathrm{P}(X = x)$	0.2	0.3	0.4	0.2

iii

x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

iv

x	2	3	4	5
$\mathrm{P}(X = x)$	0.3	0.4	0.5	-0.2

b For which of the probability distributions in **a** is X a uniform random variable?

2 Find k in each of these probability distributions:

a

x	0	1	2
$\mathrm{P}(X = x)$	0.3	k	0.5

b

x	0	1	2	3
$\mathrm{P}(X = x)$	k	$2k$	$3k$	k

3 Consider the probability distribution alongside.

x	0	1	2	3
$\mathrm{P}(X = x)$	0.1	0.25	0.45	a

a Find the value of a.

b Is X a uniform discrete random variable? Explain your answer.

c State the mode of the distribution.

d Find $\mathrm{P}(X \geqslant 2)$.

4 The probability distribution for Jason scoring X home runs in each game during his baseball career is given in the following table:

x	0	1	2	3	4	5
$P(x)$	a	0.3333	0.1088	0.0084	0.0007	0.0000

a State the value of $P(2)$.

b Find the value of a. Explain what this number means.

c Find the value of $P(1) + P(2) + P(3) + P(4) + P(5)$. Explain what this means.

d Draw a graph of $P(x)$ against x.

e Find the mode and median of the distribution.

5 A policeman inspected the safety of tyres on cars passing through a checkpoint. The number of tyres X which needed replacing on each car followed the probability distribution below.

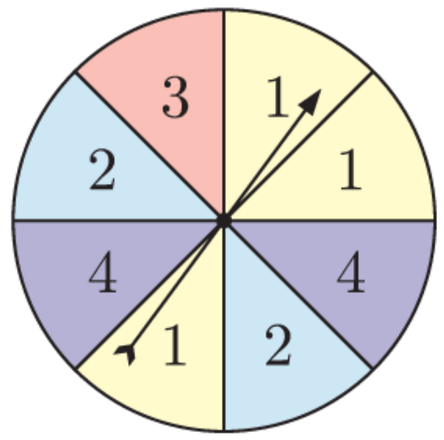
x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.68	0.2	0.06	k	0.02

a Find the value of k.

b Find the mode of the distribution.

c Find $\mathrm{P}(X > 1)$, and interpret this value.

6 Let X be the result when the spinner alongside is spun.



a Display the probability distribution of X in a table.

b Graph the probability distribution.

c Find the mode and median of the distribution.

d Find $\mathrm{P}(X \leqslant 3)$.

7 100 people were surveyed about the number of bedrooms in their house. 24 people had one bedroom, 35 people had two bedrooms, 27 people had three bedrooms, and 14 people had four bedrooms. Let X be the number of bedrooms a randomly selected person has in their house.

a State the possible values of X.

b Construct a probability table for X.

c Find the mode and median of the distribution.

8 A group of 25 basketballers took shots from the free throw line until they scored a goal. 12 of the players only needed one shot, 7 players took two shots, 2 players took three shots, and the rest took four shots. Let X be the number of shots a randomly selected player needs to score a goal.

a State the possible values of X.

b Construct a probability table for X.

c Find the mode and median of the distribution.

9 Show that the following are valid probability mass functions:

a $P(x) = \dfrac{x+1}{10}$ for $x = 0, 1, 2, 3$

b $P(x) = \dfrac{6}{11x}$ for $x = 1, 2, 3$.

10 Find k for the following probability mass functions:

a $P(x) = k(x+2)$ for $x = 1, 2, 3$

b $P(x) = \dfrac{k}{x+1}$ for $x = 0, 1, 2, 3$.

11 A discrete random variable X has the probability mass function $P(x) = \dfrac{4x - x^2}{a}$ for $x = 0, 1, 2, 3$.

a Find the value of a.

b Find $\mathrm{P}(X = 1)$.

c Find the mode of the distribution.

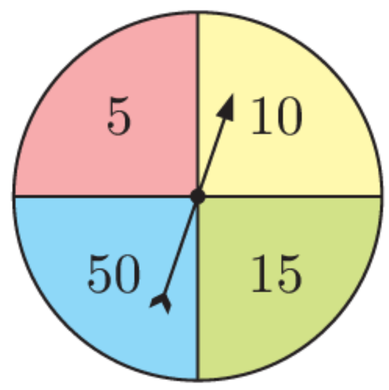
C EXPECTATION

We have already seen how probabilities can be used to predict the number of times we *expect* an event to occur when an experiment is repeated many times.

We can consider the **expected value** or **expectation** of a random variable in a similar way.

EXPECTED VALUE

When the spinner alongside is spun, players are awarded the resulting number of points. On average, how many points can we *expect* to be awarded per spin?

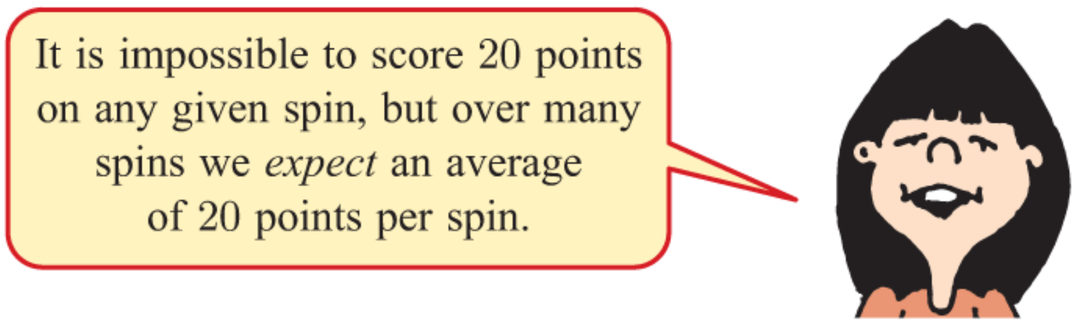


For every 4 spins, we would expect that on average, each score will be spun once. The total score in this case would be $50+15+10+5=80$, which is an average of $\frac{80}{4}=20$ points per spin.

Alternatively, we can write the average score as

$$\begin{aligned}&\tfrac{1}{4}(50+15+10+5)\\&=\tfrac{1}{4}\times 50+\tfrac{1}{4}\times 15+\tfrac{1}{4}\times 10+\tfrac{1}{4}\times 5\\&=20 \text{ points.}\end{aligned}$$

Notice that each score is multiplied by its probability of occurring.



For a random variable X with possible values $x_1, x_2, x_3, \ldots, x_n$ and associated probabilities $p_1, p_2, \ldots, p_n$, the **expected value** of X is

$$\mathbf{E}(X)=\sum_{i=1}^{n} x_i p_i = x_1p_1 + x_2p_2 + \ldots + x_np_n$$

$\mathrm{E}(X)$ is the **mean** of the probability distribution of X. It is sometimes denoted μ.

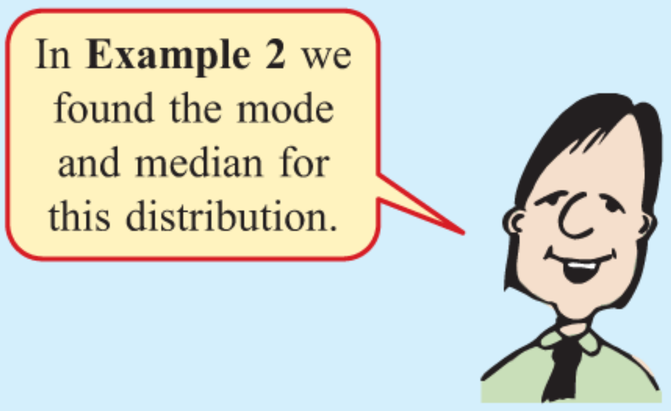
Example 4 — Self Tutor

Consider the magazine store from **Example 2**.

Find the expected number of magazines bought by each customer. Explain what this represents.

The probability table is:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05



$$\begin{aligned}\mathrm{E}(X)&=\sum_{i=1}^{n} x_i p_i\\&=1(0.23)+2(0.38)+3(0.21)+4(0.13)+5(0.05)\\&=2.39\end{aligned}$$

In the long term, the average number of magazines purchased per customer is 2.39 .

EXERCISE 27C.1

1 Find $\mathrm{E}(X)$ for the following probability distributions:

a

x_i	1	2	3
p_i	0.4	0.5	0.1

b

x_i	0	1	2	3	4
p_i	0.1	0.2	0.15	0.2	0.35

c

x_i	0	2	5	10
p_i	0.2	0.35	0.27	0.18

d

x_i	10	15	30	60
p_i	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$

2 Consider the probability distribution alongside.

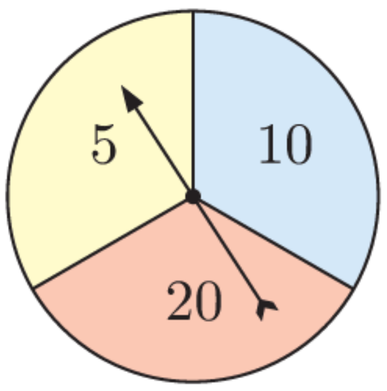
x	1	3	5
$\mathrm{P}(X = x)$	$\frac{2}{5}$	a	$\frac{1}{10}$

a Find the value of a.

b Find the mode of the distribution.

c Find the mean μ of the distribution.

3 When the spinner alongside is spun, players are awarded the resulting number of points. In the long term, how many points can we expect to be awarded per spin?



4 When Ernie goes fishing, he catches 0, 1, 2, or 3 fish, with the probabilities shown.

On average, how many fish would you expect Ernie to catch on a fishing trip?

Number of fish	0	1	2	3
Probability	0.17	0.28	0.36	0.19

5 Each time Pam visits the library, she borrows either 1, 2, 3, 4, or 5 books, with the probabilities shown.

Number of books	1	2	3	4	5
Probability	0.16	0.15	a	0.28	0.16

a Find the value of a.

b Find the mode of the distribution.

c On average, how many books does Pam borrow per visit?

6 Lachlan randomly selects a ball from a bag containing 5 red balls, 2 green balls, and 1 white ball. He is then allowed to take a particular number of lollies from a jar according to the colour of the ball.

Find the average number of lollies that Lachlan can expect to receive.

Colour	*Number of lollies*
Red	4
Green	6
White	10

7 When ten-pin bowler Jenna bowls her first bowl of a frame, she always knocks down at least 8 pins. $\frac{1}{3}$ of the time she knocks down 8 pins, and $\frac{2}{5}$ of the time she knocks down 9 pins.

a Find the probability that she knocks down all 10 pins on the first bowl.

b On average, how many pins does Jenna knock down with her first bowl?

8 When Brad's soccer team plays an offensive strategy, they win 30% of the time and lose 55% of the time. When they play a defensive strategy, they win 20% of the time and lose 30% of the time.

On the league table, teams are awarded 3 points for a win, 1 point for a draw, and no points for a loss.



a Find the probability that Brad's team will draw a match under each strategy.

b Calculate the expected number of points per game under each strategy.

c In the long run, is it better for the team to play an offensive or defensive strategy?

d Should the strategy change if teams are awarded 4 points instead of 3 points for a win?

9 Given that $\mathrm{E}(X) = 2.5$, find a and b.

x	1	2	3	4
$\mathrm{P}(X = x)$	0.3	a	b	0.2

10 Every Thursday, Zoe meets her friends in the city for dinner. There are two car parks nearby, the costs for which are shown below:

Car park A

Time	*Cost*
0 - 1 hour	\$7
1 - 2 hours	\$12
2 - 3 hours	\$15
3 - 4 hours	\$19

Car park B

Time	*Cost*
0 - 1 hour	\$6.50
1 - 2 hours	\$11
2 - 3 hours	\$16
3 - 4 hours	\$18.50

Zoe's dinner takes 1 - 2 hours 20% of the time, 2 - 3 hours 70% of the time, and 3 - 4 hours 10% of the time.

a Which car park is cheapest for Zoe if she stays:

i 1 - 2 hours **ii** 2 - 3 hours **iii** 3 - 4 hours?

b When Zoe parks her car, she does not know how long she will stay. Which car park do you recommend for her? Explain your answer.

11 An insurance policy covers a \$20 000 sapphire ring against theft and loss. If the ring is stolen then the insurance company will pay the policy owner in full. If the ring is lost then they will pay the owner \$8000. From past experience, the insurance company knows that the probability of theft is 0.0025, and the probability of loss is 0.03. How much should the company charge to cover the ring in order that their expected return is \$100?



FAIR GAMES

In gambling, the **expected gain** of the player from each game is the expected return or payout from the game, less the amount it cost them to play.

A game is said to be **fair** if the expected gain is zero.

Suppose X represents the gain of a player from each game.
The game is **fair** if $\mathrm{E}(X) = 0$.

DISCUSSION

Would you expect a gambling game to be "fair"?

Example 5 **Self Tutor**

In a game of chance, a player spins a square spinner labelled 1, 2, 3, 4. The player wins an amount of money according to the table alongside.

Number	1	2	3	4
Winnings	\$1	\$2	\$5	\$8

a Find the expected return for one spin of the spinner.

b Find the expected *gain* of the player if it costs \$5 to play each game.

c Discuss whether you would recommend playing this game.

a Let Y denote the return or payout from each spin.

Each outcome is equally likely, so the probability for each outcome is $\frac{1}{4}$

$\therefore$ the expected return $= \text{E}(Y) = \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 5 + \frac{1}{4} \times 8 = \$4$.

b Let X denote the *gain* of the player from each game.

Since it costs \$5 to play the game, the expected gain $= \text{E}(X) = \text{E}(Y) - \$5$

$= \$4 - \$5$

$= -\$1$.

c Since $\text{E}(X) \neq 0$, the game is not fair. In particular, since $\text{E}(X) = -\$1$, we expect the player to lose \$1 on average with each spin. We would not recommend that a person play the game.

EXERCISE 27C.2

1 A dice game costs \$2 to play. If an odd number is rolled, the player receives \$3. If an even number is rolled, the player receives \$1.
Determine whether the game is fair.

2 A man rolls a regular six-sided die. He wins the number of dollars shown on the uppermost face.

a Find the expected return from one roll of the die.

b Find the expected *gain* if it costs \$4 to play the game.

c Would you advise the man to play many games?

3 A roulette wheel has 18 red numbers, 18 black numbers, and 1 green number. Each number has an equal chance of occurring. I place a bet of \$2 on red. If a red is spun, I receive my \$2 back plus another \$2. Otherwise I lose my \$2.

a Calculate the expected gain from this bet.

b If the same bet is made 100 times, what is the expected result?

4 A person pays \$5 to play a game with a pair of coins. If two heads appear then \$10 is won. If a head and a tail appear then \$3 is won. If two tails appear then \$1 is won.
Let X be the gain of the person from each game. Find the expected value of X.

5 A person selects a disc from a bag containing 10 black discs, 4 blue discs, and 1 gold disc. They win \$1 for a black disc, \$5 for a blue disc, and \$20 for the gold disc. The game costs \$4 to play.

a Calculate the expected gain for this game, and hence show that the game is not fair.

b To make the game fair, the prize money for selecting the gold disc is increased. Find the new prize money for selecting the gold disc.

6 In a carnival game, a player randomly selects a ticket from a box of tickets numbered 1 to 20. If the selected number is a multiple of 3, the player wins 5 tokens. If the selected number is a multiple of 10 the player wins 10 tokens.



a Calculate the probability of a player winning:

i 5 tokens **ii** 10 tokens.

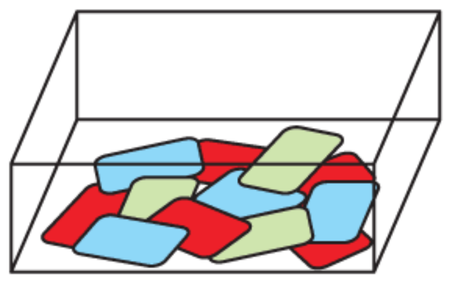
b Let X be the number of tokens won from playing this game. Find the expected value of X.

c If it costs 3 tokens to play the game, would you recommend playing the game many times? Explain your answer.

7 In a fundraising game "Lucky 11", a player selects 3 cards without replacement from a box containing 5 red, 4 blue, and 3 green cards.

The player wins \$11 if the cards drawn are all the same colour *or* are one of each colour.

If the organiser of the game wants to make an average of \$1 per game, how much should they charge to play it?



ACTIVITY GREEDY PIGS

In this Activity, we will play a variant of the dice game **Greedy Pigs**. We will use expected value to find a strategy for playing the game.

In each turn of the game, a player rolls one die a number of times, accumulating points according to the numbers rolled. After each roll, the player can either end their turn and "bank" the points accumulated so far, or continue rolling in an attempt to score more points. However, if the player rolls a 1, the player loses all of the points accumulated on that turn, and their turn is over.

What to do:

1 Play the game in pairs, so that each player has 20 turns.
Which player scored the most points in total? Discuss the strategies you used during the game. Did your strategy change during the game?

2 Expected value can be used to find a strategy that, *on average*, will maximise a player's score.

a Explain why it would not be sensible to:

i stop while you have scored less than 5 points

ii keep going if you have scored over 50 points.

b Suppose you have scored 10 points so far in your turn. Let X be the gain or loss from rolling again.

i Construct a probability distribution for X. **ii** Show that $E(X)$ is positive.

iii What does $E(X)$ tell us about whether we should roll again at 10 points?

c Find the lowest score at which, on average, it is not beneficial to continue rolling. Hence describe a strategy that will maximise your score in the long term.

d Can you think of situations where this strategy may not be the best strategy for winning the game?

3 How would the strategy for maximising your score change if your turn ended when a 6 was rolled, rather than a 1?

D VARIANCE AND STANDARD DEVIATION

In addition to the mean or expected value, we can also calculate the **variance** and **standard deviation** of a discrete random variable. These values measure the **spread** of the data values we expect to obtain when the experiment is performed many times.

Consider a discrete random variable X with possible values $x_1, x_2, ..., x_k$ and associated probabilities $p_1, p_2, ..., p_k$.

In n trials, we would expect to obtain the frequencies shown in the table.

x_i	*Frequency*
x_1	$f_1 = p_1 \times n$
x_2	$f_2 = p_2 \times n$
$\vdots$	$\vdots$
x_k	$f_k = p_k \times n$

We can calculate the variance of this data using the formula from the **Statistics Chapter** in the **Core Topics HL** book:

$$\begin{aligned}\sigma^2 &= \frac{\sum f_i(x_i - \mu)^2}{n} \\ &= \sum (x_i - \mu)^2 \left(\frac{f_i}{n}\right) \\ &= \sum (x_i - \mu)^2 p_i \\ &= \mathrm{E}[(X - \mu)^2]\end{aligned}$$

For a function $g(x)$, $\mathrm{E}[g(X)] = \sum g(x_i)p_i$.



If a discrete random variable X has k possible values $x_1, x_2, x_3, ..., x_k$ with probabilities $p_1, p_2, p_3, ..., p_k$

then:

- the **expected value** or **mean** is $\mathbf{E}(X) = \mu = \sum x_i p_i$
- the **variance** is $\mathbf{Var}(X) = \sigma^2 = \mathbf{E}[(X - \mu)^2] = \sum (x_i - \mu)^2 p_i$
- the **standard deviation** is $\sigma(X) = \sqrt{\mathbf{Var}(X)} = \sqrt{\sum (x_i - \mu)^2 p_i}$.

Example 6 — Self Tutor

In **Examples 2** and **4** we considered the probability distribution for sales in a magazine store:

x_i	1	2	3	4	5
p_i	0.23	0.38	0.21	0.13	0.05

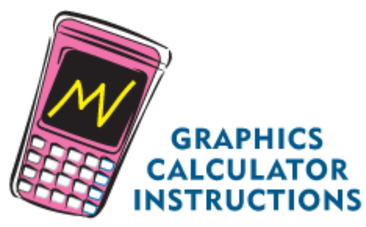
Find the standard deviation for this distribution.

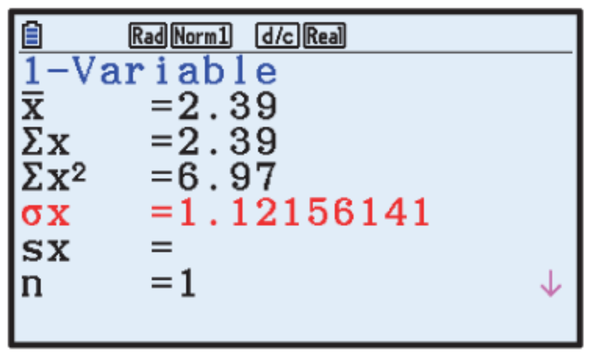
The mean $\mu = \sum x_i p_i = 2.39$ {from **Example 4**}

The standard deviation

$$\begin{aligned}\sigma &= \sqrt{\sum (x_i - \mu)^2 p_i} \\ &= \sqrt{(1 - 2.39)^2 \times 0.23 + (2 - 2.39)^2 \times 0.38 + ... + (5 - 2.39)^2 \times 0.05} \\ &\approx 1.12\end{aligned}$$

We can also use technology to find the standard deviation of a discrete random variable.





An alternative formula for the variance is

$$\sigma^2 = \mathrm{E}(X^2) - \mu^2 = \sum x_i^2 p_i - \mu^2 .$$

For example, for the roll of a die:

$$\mu = \sum x_i p_i = 1(\tfrac{1}{6}) + 2(\tfrac{1}{6}) + 3(\tfrac{1}{6}) + 4(\tfrac{1}{6}) + 5(\tfrac{1}{6}) + 6(\tfrac{1}{6}) = 3.5$$

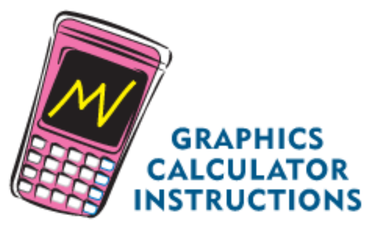
and $\sigma^2 = \sum x_i^2 p_i - \mu^2$

$$= 1^2(\tfrac{1}{6}) + 2^2(\tfrac{1}{6}) + 3^2(\tfrac{1}{6}) + 4^2(\tfrac{1}{6}) + 5^2(\tfrac{1}{6}) + 6^2(\tfrac{1}{6}) - (3.5)^2$$

$$= 2\tfrac{11}{12}$$

$$\therefore \ \sigma = \sqrt{2\tfrac{11}{12}} \approx 1.708$$

You can check these results using your calculator by generating 800 random digits from 1 to 6. The mean and standard deviation of the results should be a good approximation of the theoretical values above.



EXERCISE 27D

1 For each probability distribution, find the:

i mean μ **ii** variance σ^2 **iii** standard deviation σ.

a

x	1	2	3
$\mathrm{P}(X = x)$	0.3	0.4	0.3

b

x	0	1	2	3
$\mathrm{P}(X = x)$	0.2	0.4	0.1	0.3

Check your answers using technology.

2 For the probability distribution alongside, find the:

x_i	2	4	10	20
p_i	k	0.05	0.35	$3k$

a value of k **b** mode

c mean μ **d** standard deviation σ.

3 The probability distributions below refer to the number of aces served by Michelle and Amanda in each set of tennis they play.

Michelle:

Number of aces	0	1	2	3	4
Probability	0.1	0.15	0.45	0.25	0.05

Amanda:

Number of aces	0	1	2	3	4
Probability	0.2	0.1	0.35	0.2	0.15

a Show that each player is expected to serve an average of 2 aces per set.

b Calculate the variance and standard deviation of each probability distribution.

c Which player has the greater variation in the number of aces served?

4 A country exports crayfish to overseas markets. The buyers are prepared to pay high prices when the crayfish arrive still alive.

Let X be the number of deaths per dozen crayfish. The probability distribution for X is given by:

x	0	1	2	3	4	5	> 5
$\mathrm{P}(X = x)$	0.54	0.26	0.15	k	0.01	0.01	0.00

a Find k.

b Over a long period, what is the mean number of deaths per dozen crayfish?

c Find the standard deviation of the distribution.

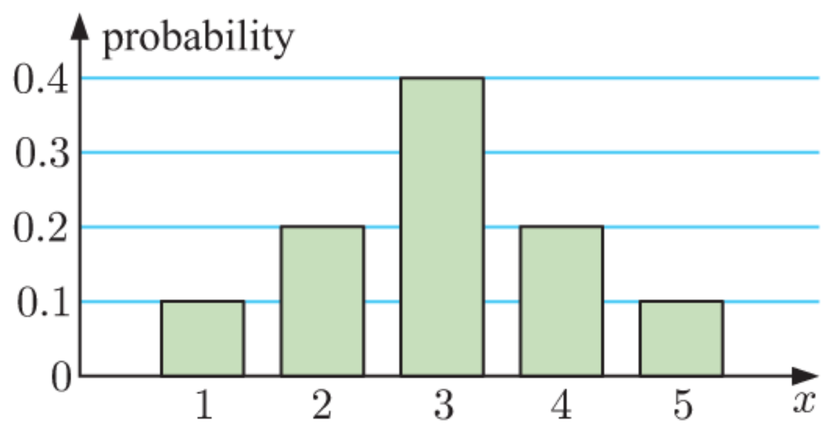
5 A die is numbered 1, 1, 2, 3, 3, 3. Let X be the result when the die is rolled once.

a Construct the probability distribution for X.

b Find the mean μ.

c Find the standard deviation of the distribution.

6 The probability distribution of a random variable X is shown in the graph.



a Copy and complete:

x_i	1	2	3	4	5
p_i					

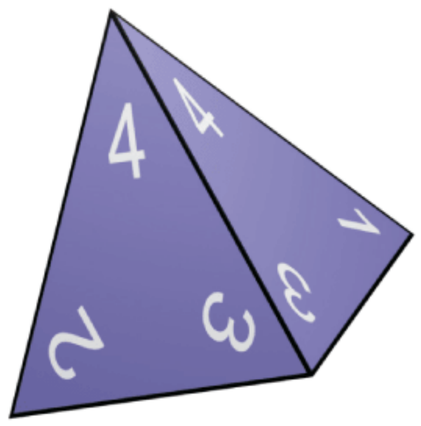
b Find the mean μ and standard deviation σ of the distribution.

7 Use $\sigma^2 = \mathrm{E}[(X - \mu)^2]$ to show that $\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$.

8 A tetrahedral die is numbered 1, 2, 3, and 4.

Let X be the number rolled when the die is rolled *once*.

Let Y be the *highest* number rolled when the die is rolled *twice*.



a Would you expect X or Y to have the greater:

i mean **ii** standard deviation?

Explain your answers.

b Construct probability distributions for X and Y.

c Calculate the mean and standard deviation of each distribution.

E PROPERTIES OF $aX + b$

If we know the expected value, variance, and standard deviation of a random variable X, we can determine these properties for a related random variable $aX + b$, where a and b are constants.

The following properties of $\mathrm{E}(X)$ are useful:

If $\mathrm{E}(X)$ is the expected value of the random variable X, then:

1 $\mathrm{E}(k) = k$ for any constant k

2 $\mathrm{E}(kX) = k\,\mathrm{E}(X)$ for any constant k

3 $\mathrm{E}[g(X) + h(X)] = \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$ for functions g and h.

Proof:

1 $\mathrm{E}(k) = \sum_{i=1}^{n} kp_i$

$= k\sum_{i=1}^{n} p_i$

$= k(1)$ {probabilities sum to 1}

$= k$

2 $\mathrm{E}(kX) = \sum_{i=1}^{n} (kx_i) \times p_i$

$= k\sum_{i=1}^{n} x_i\, p_i$

$= k\,\mathrm{E}(X)$

3 $\mathrm{E}[g(X) + h(X)]$

$= \sum_{i=1}^{n} [g(x_i) + h(x_i)]\, p_i$

$= \sum_{i=1}^{n} [g(x_i)p_i + h(x_i)p_i]$

$= \sum_{i=1}^{n} [g(x_i)p_i] + \sum_{i=1}^{n} [h(x_i)p_i]$

$= \mathrm{E}[g(X)] + \mathrm{E}[h(X)]$

The expectation of a sum is the sum of the individual expectations.



INVESTIGATION 1 PROPERTIES OF $aX + b$

The purpose of this Investigation is to discover relationships between the properties of X and those of $aX + b$.

What to do:

1 Consider the random variable X with probability distribution alongside.

x	1	2	3	4	5
$\mathrm{P}(X = x)$	0.2	0.2	0.2	0.2	0.2

a Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

b Consider the random variable $Y = 2X + 3$. The probability distribution of Y is:

y	5	7	9	11	13
$\mathrm{P}(Y = y)$	0.2	0.2	0.2	0.2	0.2

Find $\mathrm{E}(2X + 3)$ and $\mathrm{Var}(2X + 3)$.

c Repeat **b** for:

i $Y = 3X - 2$ **ii** $Y = -2X + 5$ **iii** $Y = \dfrac{X+1}{2}$ **iv** $Y = \dfrac{-X+2}{3}$

2 Make up your own probability distribution for X and repeat **1**.

3 By studying your results, predict the relationship between:

a $\mathrm{E}(X)$ and $\mathrm{E}(aX + b)$ **b** $\mathrm{Var}(X)$ and $\mathrm{Var}(aX + b)$.

4 Using the definition $\sigma(X) = \sqrt{\mathrm{Var}(X)}$ and your result for $\mathrm{Var}(aX + b)$, what can you deduce about $\sigma(aX + b)$?

From the **Investigation** you should have discovered that:

- $\mathbf{E}(aX + b) = a\,\mathbf{E}(X) + b$
- $\mathbf{Var}(aX + b) = a^2\mathbf{Var}(X)$
- $\boldsymbol{\sigma}(aX + b) = |a|\,\boldsymbol{\sigma}(X)$

Example 7 **Self Tutor**

The random variable X has mean 8.1 and standard deviation 2.37. If $Y = 4X - 7$, find the mean and standard deviation of Y.

$$\begin{aligned} \mathrm{E}(Y) &= \mathrm{E}(4X-7) \\ &= 4\mathrm{E}(X) - 7 \\ &= 4(8.1) - 7 \\ &= 25.4 \end{aligned} \qquad \begin{aligned} \sigma(Y) &= \sigma(4X-7) \\ &= |4|\,\sigma(X) \\ &= 4(2.37) \\ &= 9.48 \end{aligned}$$

For Y, the mean is 25.4 and the standard deviation is 9.48.

EXERCISE 27E

1 Use the properties of $\mathrm{E}(X)$ to prove that $\mathrm{E}(aX+b) = a\,\mathrm{E}(X) + b$.

2 The mean of X is 3. Find the mean of Y where:

a $Y = 3X + 4$ **b** $Y = -2X + 1$ **c** $Y = \dfrac{4X-2}{3}$

3 X is distributed with mean 6 and standard deviation 2. If $Y = 2X + 5$, find the mean and standard deviation of Y.

4 X is a random variable with mean 5 and standard deviation 2.
Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ for:

a $Y = 2X + 3$ **b** $Y = -5X + 3$ **c** $Y = \dfrac{X-5}{2}$

5 X has the probability distribution:

x_i	1	2	3	4
p_i	0.4	0.3	0.2	0.1

Find:

a $\mathrm{E}(X)$ **b** $\mathrm{Var}(X)$ **c** $\sigma(X)$ **d** $\mathrm{E}(X+1)$

e $\mathrm{Var}(3X+1)$ **f** $\sigma(5-X)$ **g** $\mathrm{E}\left(\dfrac{2X+5}{3}\right)$ **h** $\mathrm{Var}(20-4X)$

6 The score X obtained by rolling a biased pentagonal die has the probability distribution:

x	1	2	3	4	5
$\mathrm{P}(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	a	$\frac{1}{6}$

a Find a.

b Find the expected value, variance, and standard deviation of:

i X **ii** $X+4$ **iii** $3X-1$

7 Dominic is a door-to-door salesman. Each day he earns \$100, plus \$25 for each sale made.
From previous experience, Dominic knows that the number of sales X made per day follows the probability distribution below:

x	0	1	2	3	4	5
$\mathrm{P}(X=x)$	0.17	0.25	0.3	0.15	0.1	0.03

a Find: **i** $\mathrm{E}(X)$ **ii** $\mathrm{Var}(X)$ **iii** $\sigma(X)$

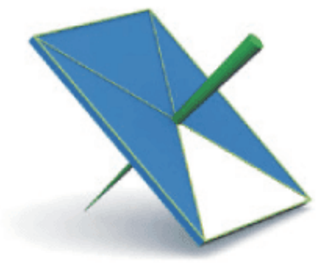
b Suppose Y is the amount of money Dominic earns, in dollars per day. Write Y in terms of X.

c Hence find: **i** $\mathrm{E}(Y)$ **ii** $\mathrm{Var}(Y)$ **iii** $\sigma(Y)$

8 Use the definition $\text{Var}(X) = \text{E}[(X-\mu)^2]$ and the result $\text{E}(aX+b) = a\,\text{E}(X)+b$ to prove that $\text{Var}(aX+b) = a^2\text{Var}(X)$.

F THE BINOMIAL DISTRIBUTION

Suppose X = the number of blues which result from spinning this spinner *once*.



The probability distribution of X is:

x	0	1
$\text{P}(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

Now suppose we spin the spinner n times and count the number of blues that result. The probability that we get a blue is the same for each spin, and each spin is independent of every other spin. This is an example of a **binomial experiment**.

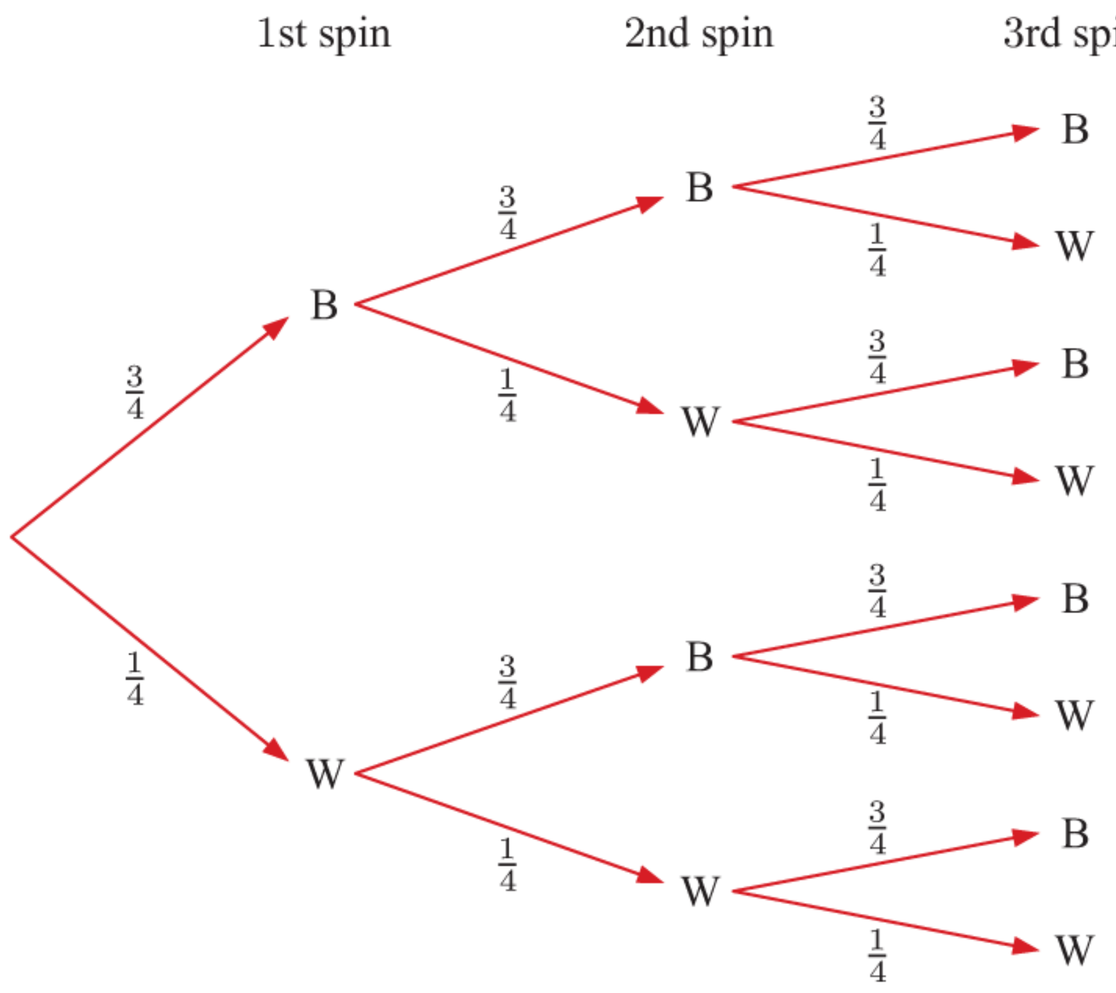
In a **binomial experiment**:

- there are a fixed number of **independent trials**
- there are only *two* possible results for each trial:
  success if some event occurs, or **failure** if the event does not occur
- the probability of success is the same for each trial.

If X is the number of successes in a binomial experiment with n trials, each with probability of success p, then X is a **binomial random variable**.

Consider the spinner above. Suppose a "success" is a blue result and let X be the number of "successes" in 3 spins of the spinner. X is a binomial random variable with $n=3$ and $p=\frac{3}{4}$, and can take the values 0, 1, 2, or 3.

To help determine the probability distribution of X, we first draw a tree diagram and find the probabilities associated with each possible outcome. We let B represent blue and W represent white.



outcome	probability
BBB	$\left(\frac{3}{4}\right)^3$
BBW	$\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)$
BWB	$\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)$
BWW	$\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2$
WBB	$\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2$
WBW	$\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$
WWB	$\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)$
WWW	$\left(\frac{1}{4}\right)^3$

The outcomes have been shaded according to the value of X.

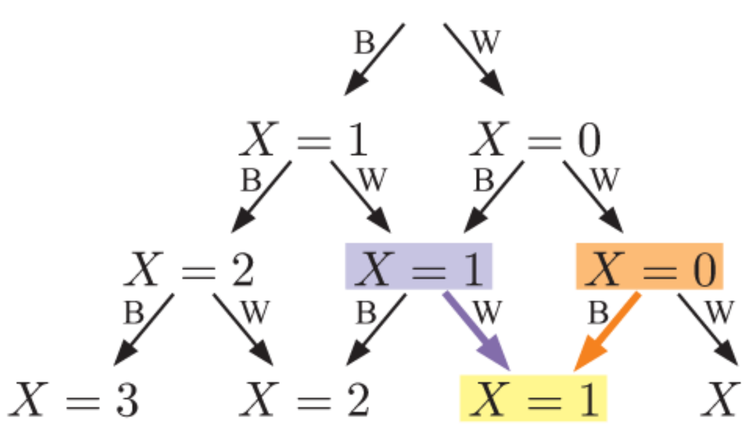
The probabilities associated with each value of X are:

$$\begin{aligned}
\mathrm{P}(X=0) &= \mathrm{P}(\mathrm{WWW}) \\
&= \left(\tfrac{1}{4}\right)^3 && \{\text{outcome shaded green}\} \\
\mathrm{P}(X=1) &= \mathrm{P}(\mathrm{BWW \text{ or } WBW \text{ or } WWB}) \\
&= \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right) \\
&= 3 \times \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)^2 && \{\text{outcomes shaded yellow}\} \\
\mathrm{P}(X=2) &= \mathrm{P}(\mathrm{BBW \text{ or } BWB \text{ or } WBB}) \\
&= \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) + \left(\tfrac{3}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right) + \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)^2 \\
&= 3 \times \left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right) && \{\text{outcomes shaded red}\} \\
\mathrm{P}(X=3) &= \mathrm{P}(\mathrm{BBB}) \\
&= \left(\tfrac{3}{4}\right)^3 && \{\text{outcome shaded blue}\}
\end{aligned}$$

Notice that for any particular value of X, each outcome with this property will have the same probability of occurring. The *order* in which blues and whites appear does not matter.

PASCAL'S TRIANGLE

We can simplify our tree diagram on the previous page to only consider the value of X after each spin:



We can use this diagram to help find the number of ways of obtaining each value of X after each spin.

Suppose that after 3 spins we have had 1 blue in total, so $X = 1$. This outcome is highlighted in yellow.

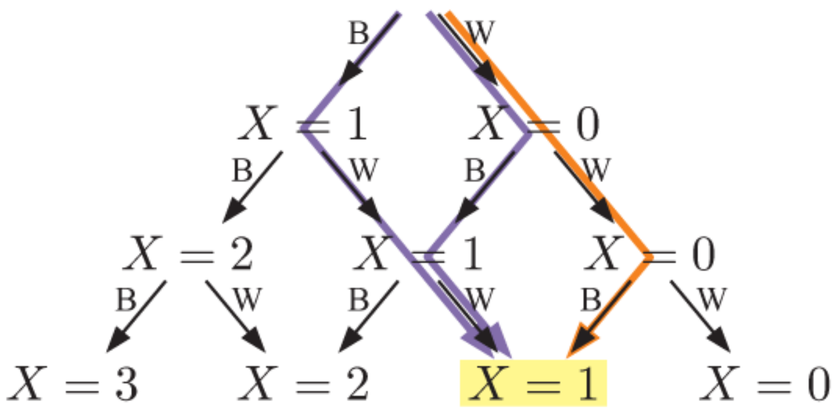
To reach this position, either:

- we had one blue after 2 spins, and we spun a white on the third spin (purple path), or
- we had no blues after 2 spins, and we spun a blue on the third spin (orange path).

$\therefore$ the total number of ways we can get 1 blue in 3 spins
$=$ number of ways of getting 1 blue in 2 spins $+$ number of ways of getting 0 blues in 2 spins
$= 2 + 1 = 3$

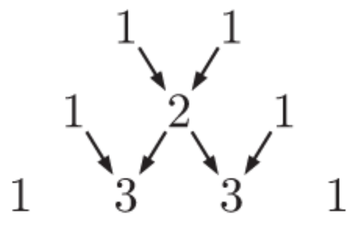
The 3 paths are highlighted alongside.





Click on the icon to observe this process for the rest of the tree diagram.

We can simplify our diagram even further by only writing the number of ways of reaching each point on the tree.



This diagram is the first three lines of a triangle of numbers called **Pascal's triangle**.

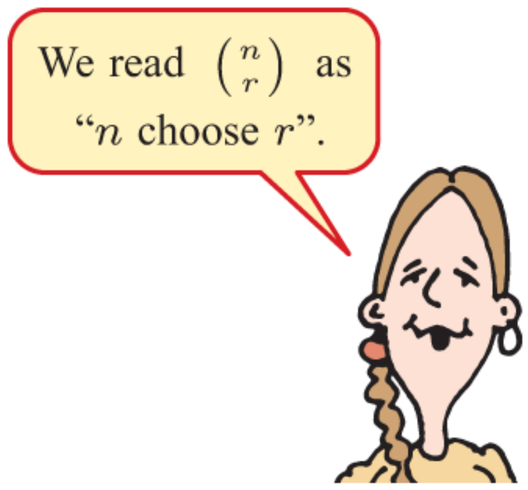
Notice that:

- the values on the end of each row are always 1
- each of the remaining values is found by adding the two values diagonally above it
- the triangle can be continued indefinitely.

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\vdots & & & & & & & & & & & & \vdots
\end{array}$$

The number of ways of getting r successes from n trials is the $(r+1)$th value in the nth row of Pascal's triangle.

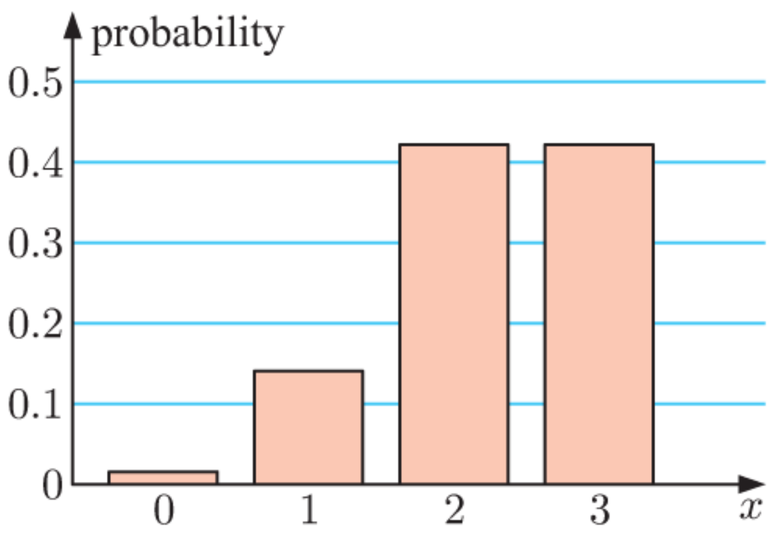
It is written as $\binom{n}{r}$ and is called the **binomial coefficient**.



Returning to the spinner with 3 blue sectors and 1 white sector, where X is the number of blues after 3 spins, we can now write the probabilities using the binomial coefficient:

$$\begin{aligned}
\text{P}(X=0) &= \left(\tfrac{1}{4}\right)^3 &&= \binom{3}{0}\left(\tfrac{3}{4}\right)^0\left(\tfrac{1}{4}\right)^3 \approx 0.0156 \\
\text{P}(X=1) &= 3\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^1 &&= \binom{3}{1}\left(\tfrac{3}{4}\right)^1\left(\tfrac{1}{4}\right)^2 \approx 0.1406 \\
\text{P}(X=2) &= 3\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^2 &&= \binom{3}{2}\left(\tfrac{3}{4}\right)^2\left(\tfrac{1}{4}\right)^1 \approx 0.4219 \\
\text{P}(X=3) &= \left(\tfrac{3}{4}\right)^3 &&= \binom{3}{3}\left(\tfrac{3}{4}\right)^3\left(\tfrac{1}{4}\right)^0 \approx 0.4219
\end{aligned}$$

So, $\text{P}(X=x) = \binom{3}{x}\left(\tfrac{3}{4}\right)^x\left(\tfrac{1}{4}\right)^{3-x}$ where $x = 0, 1, 2, 3$.

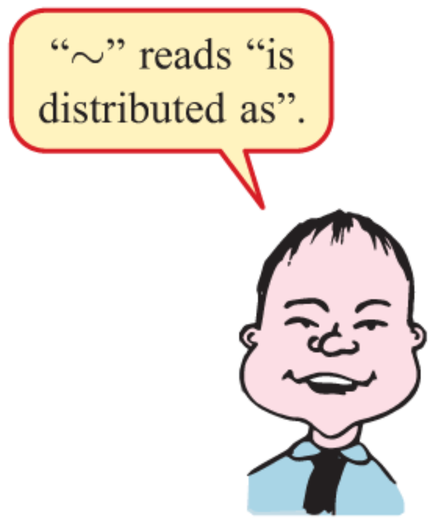


THE PROBABILITY DISTRIBUTION OF A BINOMIAL RANDOM VARIABLE

Suppose X is a binomial random variable with n independent trials and probability of success p. The probability mass function of X is:

$$P(x) = \text{P}(X=x) = \underbrace{\binom{n}{x}}_{\substack{\text{number of ways of obtaining} \\ x \text{ successes from } n \text{ trials}}} \underbrace{p^x(1-p)^{n-x}}_{\substack{\text{probability of obtaining } x \text{ successes} \\ \text{and } n-x \text{ failures in a particular order}}} \quad \text{where } x = 0, 1, 2, \ldots, n.$$

The probability distribution of X is called the **binomial distribution**, and we write $\boldsymbol{X \sim \text{B}(n, p)}$.



Example 8 **Self Tutor**

a Write down the first 5 rows of Pascal's triangle.

b An archer has a 90% chance of hitting a target with each arrow. If 5 arrows are fired, determine the chance of hitting the target:

i twice only **ii** at most 3 times.

a

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & n=1 \\ & & 1 & & 2 & & 1 & & n=2 \\ & 1 & & 3 & & 3 & & 1 & n=3 \\ 1 & & 4 & & 6 & & 4 & & 1 \quad n=4 \\ 1 & & 5 & & 10 & & 10 & & 5 \quad 1 \quad n=5 \end{array}$$

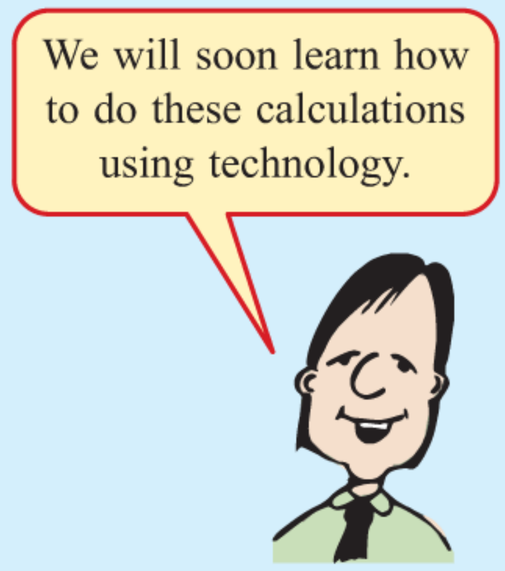
b The number of trials is $n = 5$.

The probability of success with each arrow is $p = \frac{9}{10}$.

Let X be the number of arrows that hit the target.

$\therefore \quad X \sim \text{B}\left(5, \frac{9}{10}\right)$ and the probability mass function of X is:

$$\text{P}(X = x) = \binom{5}{x}\left(\frac{9}{10}\right)^x\left(1 - \frac{9}{10}\right)^{5-x}$$
$$= \binom{5}{x}\left(\frac{9}{10}\right)^x\left(\frac{1}{10}\right)^{5-x}$$



i $\text{P}(\text{hits twice only}) = \text{P}(X = 2)$

$$= \binom{5}{2}\left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right)^{5-2}$$
$$= 10\left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right)^3$$
$$= 0.0081$$

ii $\text{P}(\text{hits at most 3 times})$

$$= \text{P}(X \leqslant 3)$$
$$= \text{P}(X = 0) + \text{P}(X = 1) + \text{P}(X = 2) + \text{P}(X = 3)$$
$$= \binom{5}{0}\left(\frac{9}{10}\right)^0\left(\frac{1}{10}\right)^{5-0} + \binom{5}{1}\left(\frac{9}{10}\right)^1\left(\frac{1}{10}\right)^{5-1} + \binom{5}{2}\left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right)^{5-2} + \binom{5}{3}\left(\frac{9}{10}\right)^3\left(\frac{1}{10}\right)^{5-3}$$
$$= \left(\frac{1}{10}\right)^5 + 5\left(\frac{9}{10}\right)\left(\frac{1}{10}\right)^4 + 10\left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right)^3 + 10\left(\frac{9}{10}\right)^3\left(\frac{1}{10}\right)^2$$
$$\approx 0.0815$$

EXERCISE 27F

1 For which of these probability experiments does the binomial distribution apply? Explain your answers.

a A coin is thrown 100 times. The variable is the number of heads.

b One hundred coins are each thrown once. The variable is the number of heads.

c A box contains 5 blue and 3 red marbles. I draw out 5 marbles one at a time, replacing the marble before the next is drawn. The variable is the number of red marbles drawn.

d A box contains 5 blue and 3 red marbles. I draw out 5 marbles without replacement. The variable is the number of red marbles drawn.

e A large bin contains ten thousand bolts, 1% of which are faulty. I draw a sample of 10 bolts from the bin. The variable is the number of faulty bolts.

2 Write down the first 6 rows of Pascal's triangle.

3 If a coin is tossed *four* times, what is the probability of getting:

a 4 heads **b** 3 heads **c** 2 heads?

4 If *five* coins are tossed simultaneously, what is the probability of getting:

a 4 heads and 1 tail in any order **b** 2 heads and 3 tails in any order

c 4 heads and then 1 tail?

5 A box of chocolates contains strawberry creams and almond centres in the ratio 2 : 1. Four chocolates are selected at random, with replacement. Find the probability of getting:

a all strawberry creams **b** two of each type

c at least 2 strawberry creams.

6 In New Zealand in 1946 there were two different coins of value one florin. These were "normal" kiwis and "flat back" kiwis, in the ratio 3 : 1. From a very large batch of 1946 florins, six were selected at random with replacement. Find the probability that:



a two were "flat backs"

b at least 3 were "flat backs"

c at most 3 were "normal" kiwis.

INVESTIGATION 2 THE GRAPH OF A BINOMIAL DISTRIBUTION

In this Investigation we will explore the graph of a binomial distribution and how its shape varies with changes to n and p.

What to do:

1 Click on the icon to access the demonstration. It shows the graph of the binomial distribution for $X \sim \text{B}(n, p)$. Set $n = 20$ and $p = 0.1$.

DEMO

a What is the mode of X?

b Describe the shape of the distribution.

2 Use the slider to change the value of p. Describe how the *shape* of the distribution changes as p changes.

3 Reset p to 0.1. Use the slider to change the value of n.
How does this affect the shape of the distribution? What happens to the shape of the binomial distribution as the number of trials n increases?

G USING TECHNOLOGY TO FIND BINOMIAL PROBABILITIES

We can quickly calculate binomial probabilities using a graphics calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

For example:

- To find the probability $P(X = k)$ that the variable takes the value k, we use the **binomial probability function**.
- To find the probability that the variable takes a *range* of values, such as $P(X \leqslant k)$ or $P(X \geqslant k)$, we use the **binomial cumulative probability function**.

Some calculator models, such as the **TI-84 Plus CE**, only allow you to calculate $P(X \leqslant k)$. To find the probability $P(X \geqslant k)$ for these models, it is often easiest to find the complement $P(X \leqslant k-1)$ and use $P(X \geqslant k) = 1 - P(X \leqslant k-1)$.

Example 9 **Self Tutor**

72% of union members are in favour of a certain change to their conditions of employment. A random sample of five members is taken. Find the probability that:

a three members are in favour of the change in conditions

b at least three members are in favour of the changed conditions.

Let X denote the number of members in the sample in favour of the change.

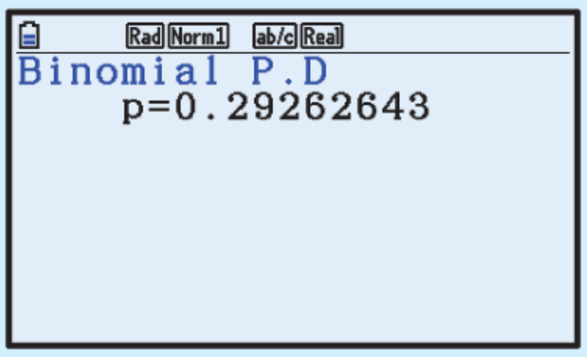
$n = 5$, so $X = 0, 1, 2, 3, 4$, or 5, and $p = 72\% = 0.72$

$\therefore \quad X \sim B(5, 0.72)$.

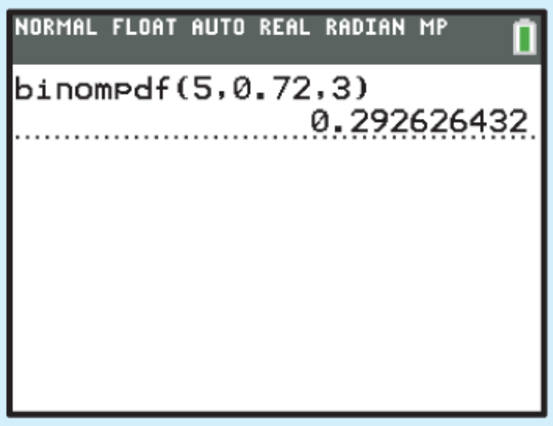
a $P(X = 3) = \binom{5}{3} (0.72)^3 (0.28)^2$

≈ 0.293

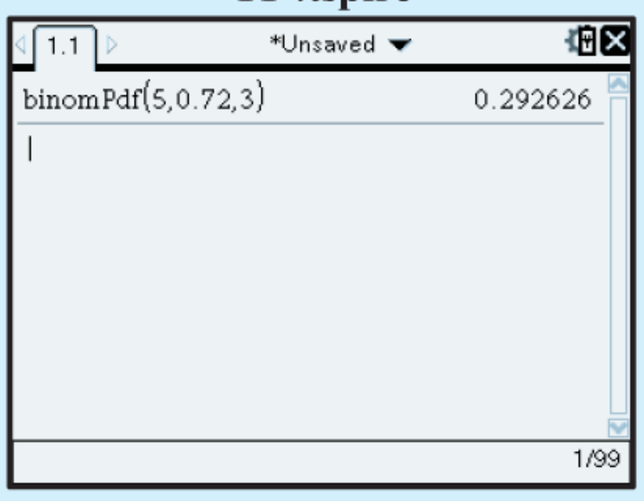
Casio fx-CG50



TI-84 Plus CE

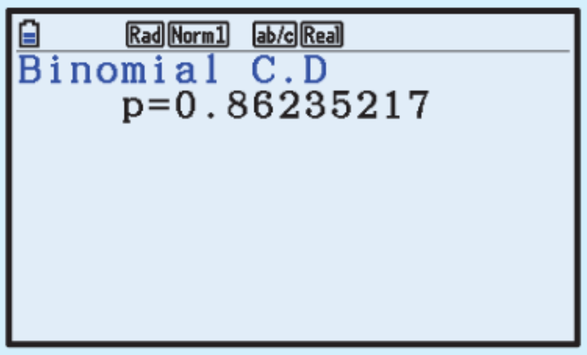


TI-*n*spire

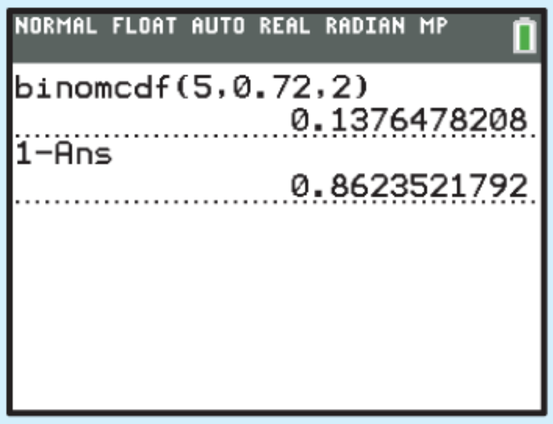


b $P(X \geqslant 3) \approx 0.862$

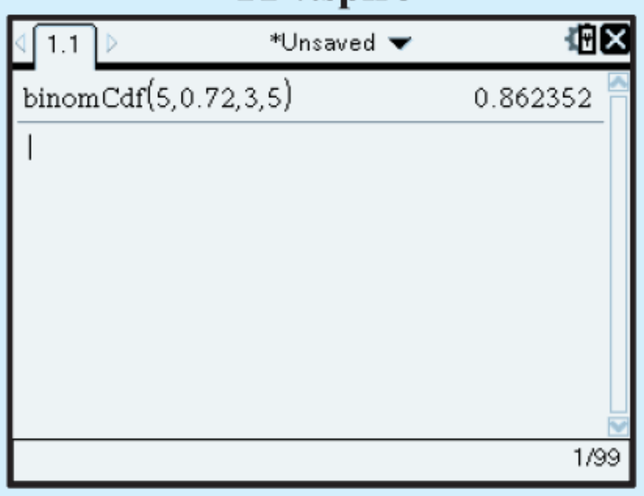
Casio fx-CG50



TI-84 Plus CE



TI-*n*spire



EXERCISE 27G

1 5% of electric light bulbs are defective at manufacture. 6 bulbs are randomly tested, with each one being replaced before the next is chosen. Determine the probability that:

a two are defective

b at least one is defective.

2 Records show that 6% of the items assembled on a production line are faulty. A random sample of 12 items is selected with replacement. Find the probability that:

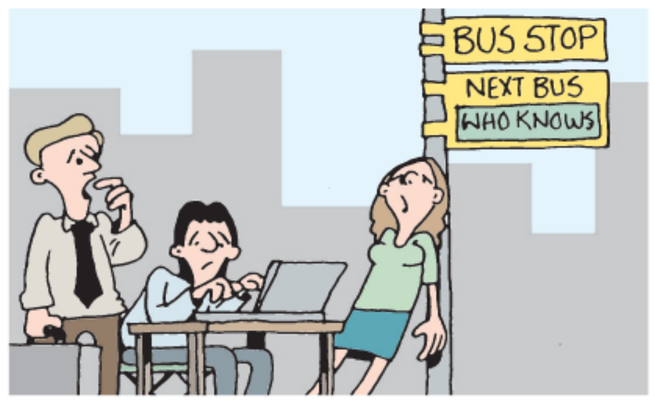
a none will be faulty **b** at most one will be faulty

c at least two will be faulty **d** less than four will be faulty.

3 The local bus service does not have a good reputation. The 8 am bus will run late on average two days out of every five. For any week of the year taken at random, find the probability of the 8 am bus being on time:

a all 7 days **b** only on Monday

c on any 6 days **d** on at least 4 days.



4 In a multiple choice test there are 10 questions. Each question has 5 choices, one of which is correct. Raj knows absolutely nothing about the subject, and guesses each answer at random. Given that the pass mark is 70%, determine the probability that he will pass.

5 An infectious flu virus is spreading through a school. The probability of a randomly selected student having the flu next week is 0.3. Mr C has a class of 25 students.

a Calculate the probability that 2 or more students from Mr C's class will have the flu next week.

b If more than 20% of the students have the flu next week, a class test will have to be cancelled. What is the probability that the test will be cancelled?

6 During a season, a basketball player has an 85% success rate in shooting from the free throw line. In one match the basketballer has 20 shots from the free throw line.
Find the probability that the basketballer is successful with:

a all 20 throws **b** at least 18 throws **c** between 14 and 17 (inclusive) throws.

7 Martina beats Jelena in 2 games out of 3 at tennis. Find the probability that Jelena wins a set of tennis 6 games to 4.

Hint: What does the score after 9 games need to be?

8 A fair coin is tossed 200 times. Find the probability of obtaining:

a between 90 and 110 (inclusive) heads **b** more than 95 but less than 105 heads.

9 **a** Find the probability of rolling double sixes with a pair of dice.

b Suppose a pair of dice is rolled 500 times. Find the probability of rolling between 10 and 20 (inclusive) double sixes.

10 Shelley must pass through 15 traffic lights on her way to work. She has probability 0.6 of being stopped at any given traffic light. If she is stopped at more than 11 traffic lights, she will be late.

a Find the probability that Shelley will be late for work on a given day.

b Find the probability that Shelley is on time for work each day of a 5 day week.

c Shelley wants to increase the probability in **b** to at least 80%. She decides to leave home a little earlier, so she must now be stopped at more than 12 traffic lights in order to be late. Has Shelley achieved her goal? Justify your answer.

11 A hot water unit relies on 20 solar components for its power, and will operate provided at least one of its 20 components is working. The probability that an individual solar component will fail in a year is 0.85, and the failure of each individual component is independent of the others.

a Find the probability that the hot water unit will fail within one year.

b Find the smallest number of solar components required to ensure that a hot water service like this one is operating at the end of one year with a probability of at least 0.98.

H THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

INVESTIGATION 3 THE MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

In this Investigation we will use a calculator to calculate the mean and standard deviation of several binomial distributions. A spreadsheet can also be used to speed up the process.



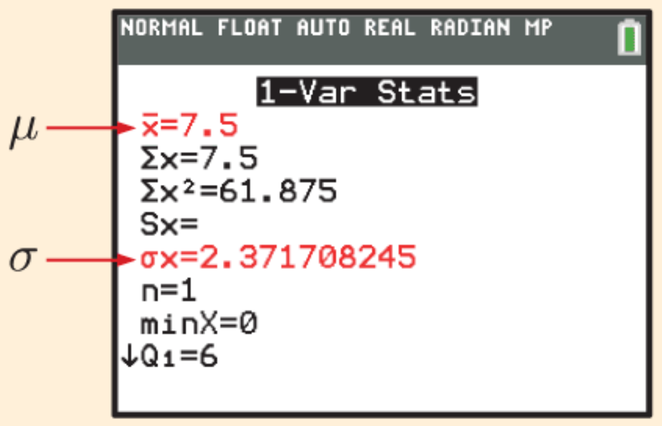
GRAPHICS CALCULATOR INSTRUCTIONS

What to do:

1 We will first calculate the mean and standard deviation for the variable $X \sim \text{B}(30, 0.25)$.

a Enter the possible values for X from $x = 0$ to $x = 30$ into **List 1**, and their corresponding binomial probabilities $\text{P}(X = x) = \binom{30}{x}(0.25)^x(0.75)^{30-x}$ into **List 2**.

b Calculate the descriptive statistics for the distribution. You should obtain the results in the screenshot.



2 Copy and complete the following table for distributions with other values of n and p.

	$p = 0.1$	$p = 0.25$	$p = 0.5$	$p = 0.7$
$n = 10$				
$n = 30$		$\mu = 7.5$ $\sigma \approx 2.3717$		
$n = 50$				

3 Compare your values with the formulae $\mu = np$ and $\sigma = \sqrt{np(1-p)}$.

From the **Investigation** you should have observed the following results:

Suppose X is a binomial random variable with parameters n and p, so $X \sim \text{B}(n, p)$.

- The **mean** of X is $\boldsymbol{\mu = np}$.
- The **variance** of X is $\boldsymbol{\sigma^2 = \text{Var}(X) = np(1-p)}$.
- The **standard deviation** of X is $\boldsymbol{\sigma = \sqrt{\text{Var}(X)} = \sqrt{np(1-p)}}$.

Example 10 Self Tutor

A fair die is rolled twelve times, and X is the number of sixes that result. Find the mean, variance, and standard deviation of X.

This is a binomial distribution with $n = 12$ and $p = \frac{1}{6}$, so $X \sim \text{B}(12, \frac{1}{6})$.

$\mu = np$ $= 12 \times \frac{1}{6}$ $= 2$

$\sigma^2 = np(1-p)$ $= 12 \times \frac{1}{6} \times \frac{5}{6}$ $= \frac{5}{3}$

and $\sigma = \sqrt{\sigma^2}$ $= \sqrt{\frac{5}{3}}$ ≈ 1.291

We expect a six to be rolled 2 times, with variance $\frac{5}{3}$ and standard deviation 1.291.

EXERCISE 27H

1 Suppose $X \sim \text{B}(6, p)$. For each of the following cases:

i Find the mean and standard deviation of X.
ii Graph the distribution using a column graph.
iii Comment on the shape of the distribution.

a $p = 0.5$ **b** $p = 0.2$ **c** $p = 0.8$

2 A coin is tossed 10 times and X is the number of heads which occur. Find the mean and variance of X.

3 Bolts produced by a machine vary in quality. The probability that a given bolt is defective is 0.04. Random samples of 30 bolts are taken from the week's production. Let X be the number of defective bolts and Y be the number of non-defective bolts in a sample.

Find the mean and standard deviation of: **a** X **b** Y.

4 A city restaurant knows that 13% of reservations are not honoured, which means the group does not arrive. Suppose the restaurant receives 30 reservations. Let the random variable X be the number of groups that do not arrive. Find the mean and standard deviation of X.

5 A new drug has a 75% probability of curing a patient within one week. Suppose 38 patients are treated using this drug. Let X be the number of patients who are cured within a week.

a Find the mean μ and standard deviation σ of X. **b** Find $\text{P}(\mu - \sigma < X < \mu + \sigma)$.

6 Let X be the number of heads which occur when a coin is tossed 100 times, and Y be the number of ones which occur when a die is rolled 300 times.

a Show that the mean of both distributions is 50.
b Calculate the standard deviation of each distribution.
c Which variable do you think is more likely to lie between 45 and 55 (inclusive)? Explain your answer.
d Find: **i** $\text{P}(45 \leqslant X \leqslant 55)$ **ii** $\text{P}(45 \leqslant Y \leqslant 55)$

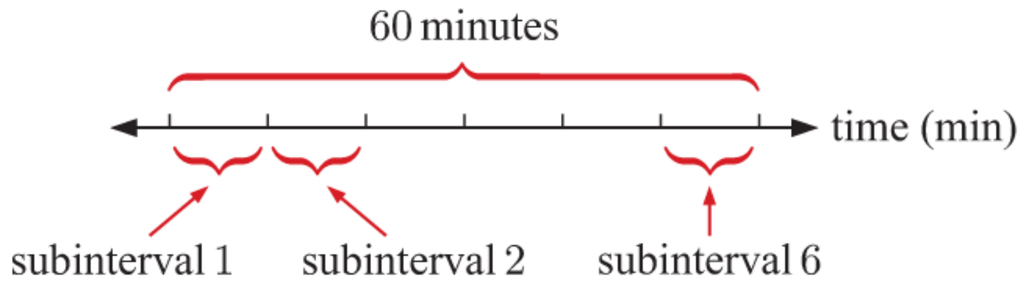
I THE POISSON DISTRIBUTION

The **Poisson distribution** generally arises when considering the number of successes within a certain *interval* (of time or space). For example, we might consider the number of:

- calls received by a call centre per hour
- fish caught in a lake per day
- misprints on a typical page of a book
- car accidents on a given road per month.

Suppose a call centre receives an average of $\lambda = 5$ calls per hour. Our aim is to find the probability of the call centre receiving X calls in a particular 60 minute interval.

Suppose we break up the 60 minute time interval into $n = 6$ equal subintervals, and consider whether or not a call happens in each subinterval.



- There are two possible outcomes for each subinterval: we either receive a call, or we do not.
- We assume that the rate of calls arriving is constant, so the probability of receiving a call in each subinterval is $\frac{\lambda}{6} = \frac{5}{6}$.
- We assume that each subinterval is independent of every other subinterval.
- By definition, we have a fixed number of trials $n = 6$, which is the number of subintervals we have.

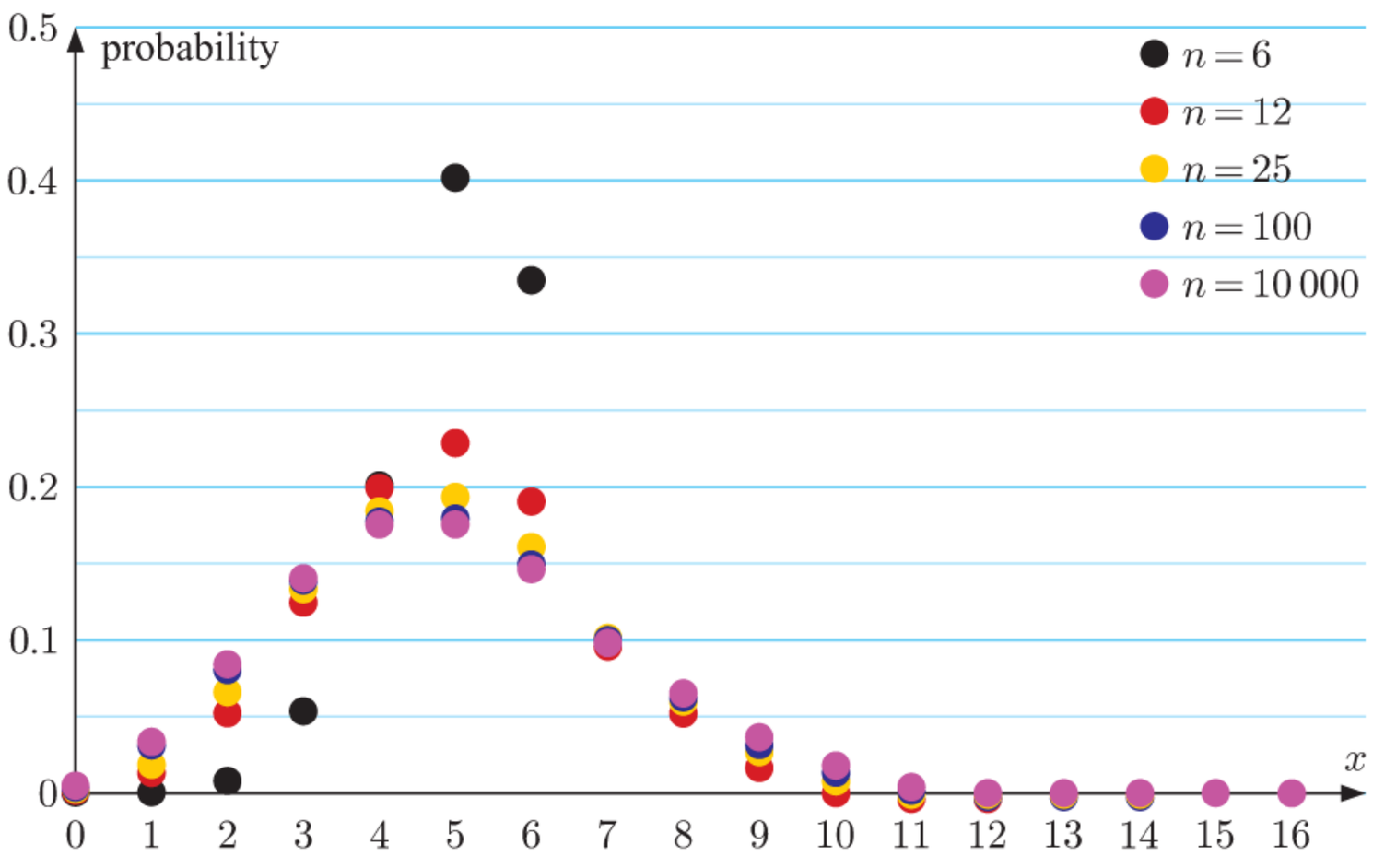
These are the conditions for a binomial distribution. So, if X_6 is the number of subintervals out of 6 in which we receive a call, then $X_6 \sim \text{B}(6, \frac{5}{6})$.

A deficiency of this model is that there may be more than 1 call in a subinterval. The binomial distribution would therefore no longer apply. Thus to improve this model, we need to increase the number of subintervals that the 60 minute interval is divided into.

Suppose we divide the 60 minute interval into n equal subintervals. We let X_n be the number of subintervals out of n in which we receive a call. Using the same reasoning as above, $X_n \sim \text{B}\left(n, \frac{\lambda}{n}\right)$.

X_n therefore has the probability mass function $\text{P}(X_n = x) = \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1 - \frac{\lambda}{n}\right)^{n-x}$.

The graph below shows the probability distribution of $X_n \sim \text{B}\left(n, \frac{5}{n}\right)$ for various values of n.



Notice how as the number of subintervals is increased, the probability distribution flattens and takes a distinct shape.

When we take the limit as $n \to \infty$, we get the **Poisson distribution**.

While the proof is beyond this course, it can be shown that $\lim_{n\to\infty} \text{P}(X_n = x) = \dfrac{\lambda^x e^{-\lambda}}{x \times (x-1) \ldots \times 3 \times 2 \times 1}$.

In factorial notation we use $x!$ to denote the sum of the first x integers $x \times (x-1) \times .... \times 3 \times 2 \times 1$. We also define $0! = 1$.

We conclude that:

If X is the **Poisson random variable** with rate $\lambda > 0$, then the **probability mass function** of X is:

$$P(x) = \mathrm{P}(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{where } x = 0, 1, 2, 3, ...$$

We write $X \sim \mathrm{Po}(\lambda)$.

THE MEAN AND VARIANCE OF A POISSON RANDOM VARIABLE

The mean and variance of a Poisson random variable are found by considering the mean and variance of $X_n \sim \mathrm{B}\left(n, \frac{\lambda}{n}\right)$ as $n \to \infty$:

$$\mathrm{E}(X_n) = n \times \frac{\lambda}{n} = \lambda \qquad \text{and} \qquad \mathrm{Var}(X_n) = n \times \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right) = \lambda\left(1 - \frac{\lambda}{n}\right) = \lambda - \frac{\lambda^2}{n}$$

$$\therefore \lim_{n \to \infty} \mathrm{Var}(X_n) = \lambda$$

If $X \sim \mathrm{Po}(\lambda)$, then $\mathbf{E}(X) = \mathbf{Var}(X) = \boldsymbol{\lambda}$.

Example 11 — Self Tutor

The number of errors that Sandra finds while proofreading follows a Poisson distribution with rate 3.21 errors per page.

a Calculate the probability that on a randomly selected page, Sandra will find:

i exactly 3 errors ii no more than 2 errors iii at least 3 errors

iv exactly 3 errors *given* that she found at least 3 errors.

b Suppose Sandra reads a 10 page exam paper.

i Write down the probability distribution for the number of errors she finds.

ii Find the probability that she will find at least 30 errors.

a Let X be the number of errors that Sandra finds on *one* randomly selected page.

$\therefore X \sim \mathrm{Po}(3.21)$

i $\mathrm{P}(X = 3) = \dfrac{(3.21)^3 e^{-3.21}}{3!} \approx 0.222$

ii $\mathrm{P}(X \leqslant 2) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1) + \mathrm{P}(X = 2) = \dfrac{(3.21)^0 e^{-3.21}}{0!} + \dfrac{(3.21)^1 e^{-3.21}}{1!} + \dfrac{(3.21)^2 e^{-3.21}}{2!} \approx 0.378$

iii $\mathrm{P}(X \geqslant 3) = \mathrm{P}(X > 2) = 1 - \mathrm{P}(X \leqslant 2) \approx 1 - 0.378$ {using ii} ≈ 0.622

iv $P(X = 3 \mid X \geqslant 3) = \dfrac{P(X = 3 \cap X \geqslant 3)}{P(X \geqslant 3)}$

$= \dfrac{P(X = 3)}{P(X \geqslant 3)}$

$\approx \dfrac{0.222}{0.622}$ {using **i** and **iii**}

≈ 0.358

b **i** The rate at which Sandra finds errors

$= 3.21$ errors per page

$= 32.1$ errors per 10 pages

Let Y be the number of errors that Sandra finds in 10 pages.

$\therefore \quad Y \sim \text{Po}(32.1)$



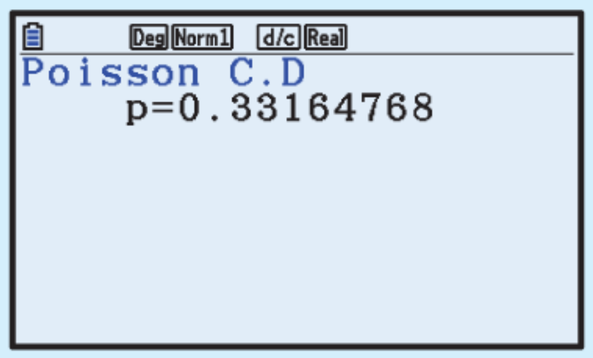
ii $P(Y \geqslant 30) = 1 - P(Y \leqslant 29)$

$\approx 1 - 0.332$

≈ 0.668



GRAPHICS CALCULATOR INSTRUCTIONS



EXERCISE 27I

1 Suppose $X \sim \text{Po}(7.12)$.

a Write down the probability mass function of X.

b Find: **i** $E(X)$ **ii** $\text{Var}(X)$ **iii** $\sigma(X)$

c Find: **i** $P(X = 2)$ **ii** $P(X \leqslant 3)$ **iii** $P(X \geqslant 5)$ **iv** $P(X \geqslant 3 \mid X \geqslant 1)$

2 Top Cars rents cars to tourists. They have four cars which are hired out on a daily basis. The number of requests each day has a Poisson distribution with rate 3. Determine the probability that on a particular day:

a none of the cars are rented **b** at least 3 of the cars are rented

c some requests will have to be refused

d all of the cars are hired out, given that at least two are.

3 A supplier of clothing materials looks for flaws before selling the materials to customers. The number of flaws follows a Poisson distribution with rate 1.7 flaws per metre.

a Find the probability of:

i exactly 3 flaws in a 1 metre length of material

ii at least one flaw in 2 metres of material.

b Find the mean, variance, and standard deviation of this Poisson distribution.

4 Experiments on a rocket aerofoil indicate a 98% chance that the aerofoil will not disintegrate on re-entry to the atmosphere. Use a Poisson distribution to find the probability that in a sample of 100 aerofoils:

a exactly one will disintegrate **b** exactly two will disintegrate

c at most two aerofoils will disintegrate.

5 Road safety figures for a large city show that any driver has a 0.002% chance of being killed each time he or she drives a car.

a Find the probability that a driver can use a car 10 times a week for a year and survive.

b For how many years can you drive in this city and have a better than 95% chance of surviving? Discuss the assumptions you make in your answer.

6 Consider a random variable $X \sim \text{Po}(\lambda)$.

a Find λ given that $\text{P}(X = 1) + \text{P}(X = 2) = \text{P}(X = 3)$.

b Hence find:

i $\text{E}(X)$ **ii** $\sigma(X)$ **iii** $\text{P}(X \geqslant 3)$ **iv** $\text{P}(X \leqslant 4 \mid X \geqslant 2)$

7 The random variable $Y \sim \text{Po}(\lambda)$ satisfies $\text{P}(Y = 3) = \text{P}(Y = 1) + 2\,\text{P}(Y = 2)$.

a Find the value of λ correct to 4 decimal places.

b Hence find: **i** $\text{P}(1 < Y < 5)$ **ii** $\text{P}(2 \leqslant Y \leqslant 6 \mid Y \geqslant 4)$

8 One gram of a radioactive substance is positioned so that each alpha-particle emission will flash on a screen. The emissions over 500 periods of 10 seconds duration are summarised in the table below.

Number per period	0	1	2	3	4	5	6	7
Frequency	91	156	132	75	33	9	3	1

a Find the mean of the data.

b Assuming the data is from a Poisson distribution, estimate the rate parameter λ.

c Find the variance of the data. Compare your answer with the variance of the Poisson distribution found in **b**.

9 The random variable U has a Poisson distribution with mean x. Let $y = \text{P}(U \leqslant 2)$.

a Write y as a function of x.

b Use technology to help sketch the graph of y for $0 \leqslant x \leqslant 3$.

c Use calculus to show that as the mean increases, $\text{P}(U \leqslant 2)$ decreases.

10 The number of eggs a butterfly lays is $Y \sim \text{Po}(20)$. Each egg has probability $p = \frac{2}{3}$ of hatching, independently of one another. Let $X \mid Y$ be the number of eggs that hatch *given* that Y eggs are laid.

a Explain why $X \mid Y \sim \text{B}(Y, \frac{2}{3})$.

b Find the probability of 10 eggs hatching given that 22 eggs are laid.

c Find $\text{E}(X \mid Y)$.

d Given the **conditional expectation** $\text{E}(X \mid Y)$, the **unconditional expectation** of X is $\text{E}(X) = \text{E}[\text{E}(X \mid Y)]$.
Find $\text{E}(X)$, the expected number of eggs laid by a butterfly which will hatch.

REVIEW SET 27A

1 Determine whether the following variables are discrete or continuous:

a the number of attempts to pass a driving test

b the length of time before a phone loses its battery charge

c the number of phone calls made before a salesperson has sold 3 products.

2 **a** State whether each of the following is a valid probability distribution:

i

x	1	2	3
$\mathrm{P}(X = x)$	0.6	0.25	0.15

ii

x	0	2	5	10
$\mathrm{P}(X = x)$	0.3	0.5	0.1	0.2

iii

x	0	1	2	3
$\mathrm{P}(X = x)$	0.4	-0.2	0.35	0.45

iv

x	2	3	4	5
$\mathrm{P}(X = x)$	0.25	0.25	0.25	0.25

v

x	2	3
$\mathrm{P}(X = x)$	0.7	0.3

vi

x	0	1
$\mathrm{P}(X = x)$	0.28	0.72

b For which of the probability distributions in **a** is X a uniform discrete random variable?

3 $\mathrm{P}(X = x) = \dfrac{a}{x^2 + 1}$, $x = 0, 1, 2, 3$ is a probability mass function.

a Find the value of a.

b Find $\mathrm{P}(X \geqslant 1)$.

4 A random variable X has the probability mass function $P(x)$ described in the table.

x	0	1	2	3	4
$P(x)$	0.10	0.30	0.45	0.10	k

a Find k.

b Find $\mathrm{P}(X \geqslant 3)$.

c Find the mode of the distribution.

d Find the expected value $\mathrm{E}(X)$ for the distribution.

5 Three green balls and two yellow balls are placed in a hat. Two balls are randomly drawn without replacement, and X is the number of green balls drawn.

a Explain why X is a discrete random variable.

b State the possible values of X.

c Construct a probability table for X.

d Find the expected number of green balls drawn.

6 The faces of a die are labelled 1, 3, 3, 4, 6, 6. Let X be the result when the die is rolled. Find the expected value of X.

7 Lakshmi rolls a regular six-sided die. She wins twice the number of dollars as the number rolled.

a How much does Lakshmi expect to win from one roll of the die?

b If it costs \$8 to play the game, would you advise Lakshmi to play many games? Explain your answer.

8 For the probability distribution alongside, find the:

x	1	2	3	4
$\mathrm{P}(X = x)$	0.15	0.1	0.35	0.4

a mean μ

b variance σ^2

c standard deviation σ.

9 Suppose X is the number of marsupials entering a park at night. It is suspected that X has a probability mass function $P(x) = a(x^2 - 8x)$ where $x = 0, 1, 2, 3, \ldots, 8$.

a Find the constant a.

b Find the expected number of marsupials entering the park on a given night.

c Find the variance of X.

10 X is a discrete random variable with mean 6 and standard deviation 2. If $Y = 4X + 3$, find the mean and standard deviation of Y.

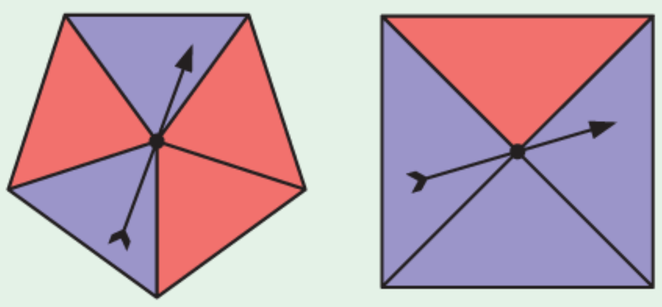
11 With every attempt, Jack has an 80% chance of kicking a goal. In one quarter of a match he has 5 kicks for goal. Use Pascal's triangle to help determine the probability that he scores:

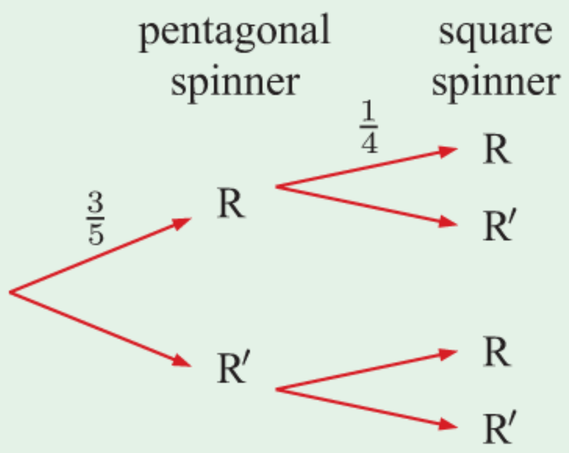
a 3 goals then misses twice **b** 3 goals and misses twice.

12 It is observed that 3% of all batteries produced by a company are defective.
For a random sample of 20 batteries, calculate the probability that:

a none are defective **b** at least one is defective.

13





a Copy and complete the tree diagram which shows all possible results when these two spinners are spun together.

b Calculate the probability that exactly one red will occur.

c The spinners are spun 10 times. Let X be the number of times that exactly one red occurs.

i State the distribution of X.

ii Write down expressions for $P(X = 1)$ and $P(X = 9)$. Hence determine which of these outcomes is more likely.

iii Find the mean and standard deviation of X.

14 Patients arrive at random to a hospital Emergency room at a rate of 14 per hour.
Find the probability that:

a more than 10 patients arrive between 8:00 am and 9:00 am

b exactly five patients arrive between 9:00 am and 9:45 am

c fewer than seven patients arrive between 10:00 am and 10:30 am.

REVIEW SET 27B

1 Sally's number of hits in each softball match has the probability distribution shown.

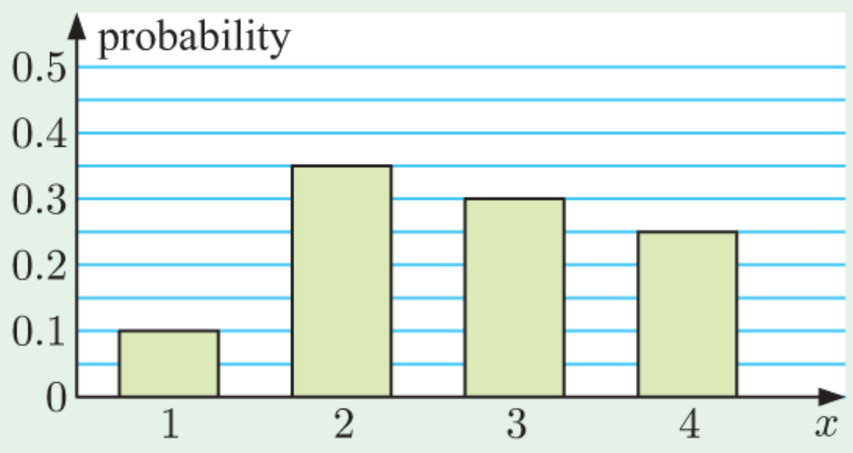
x	0	1	2	3	4	5
$P(X = x)$	0.07	0.14	k	0.46	0.08	0.02

a State clearly what the random variable represents.

b Find: **i** k **ii** $P(X \geqslant 2)$ **iii** $P(1 \leqslant X \leqslant 3)$

c Find the mode and median number of hits.

2 The probability distribution of a random variable X is graphed alongside. Find:



a the mode of X

b the median of X

c the expected value of X.

3 Show that the following are valid probability mass functions:

a $P(x) = \dfrac{e^x}{1+e}$, $x = 0, 1$ **b** $P(x) = \dfrac{x^2 + x}{40}$, $x = 1, 2, 3, 4$

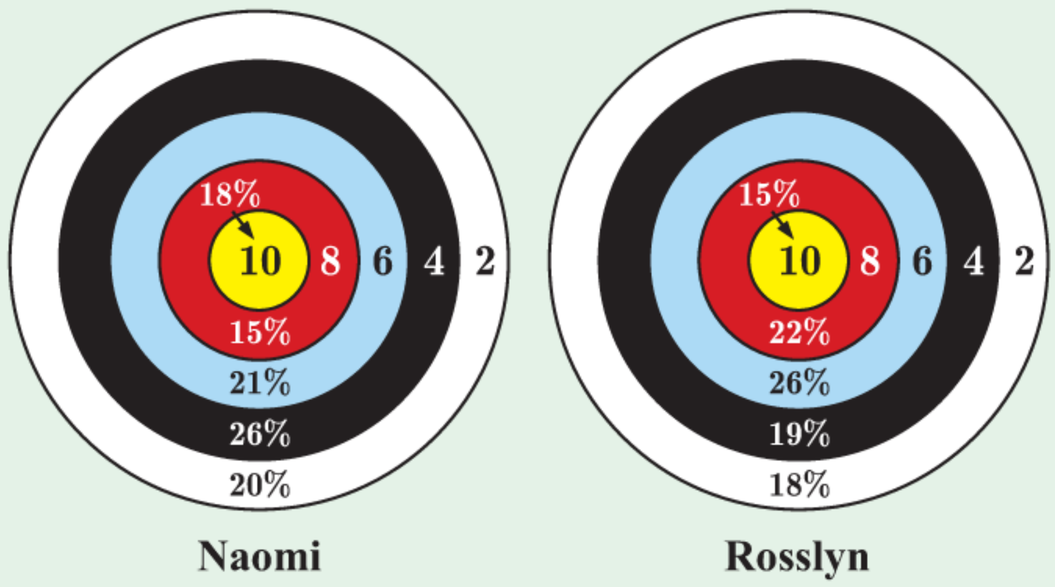
4 The probabilities of Naomi and Rosslyn hitting each section of an archery target is shown alongside.

a On a single shot, who is more likely to score:

i 10 points

ii at least 6 points?

b In the long run, who would you expect to score more points per shot?



Naomi **Rosslyn**

5 The numbers from 1 to 20 are written on tickets and placed in a bag. A person draws out a number at random. The person wins \$3 if the number is even, \$6 if the number is a square number, and \$9 if the number is both even and square.

a Calculate the probability that the player wins:

i \$3 **ii** \$6 **iii** \$9

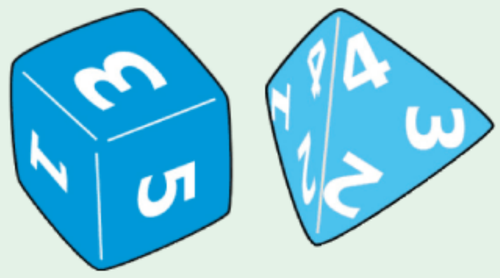
b How much should be charged to play the game so that it is fair?

6 A 6-sided and 4-sided die are rolled simultaneously. Let X be the number of twos rolled.

a Explain why X is not a binomial random variable.

b Find the probability distribution of X.

c Find the mean of X.



7 Suppose X has the probability distribution alongside.

x_i	1	2	3	4
p_i	0.2	a	0.3	b

a Given that $\mathrm{E}(X) = 2.8$, find a and b.

b Find $\mathrm{Var}(X)$.

8 Caleb is thrown a baseball 4 times. Let X be the number of times Caleb catches the ball. The probability distribution of X is shown alongside.

x	0	1	2	3	4
$\mathrm{P}(X = x)$	0.1	0.2	0.3	0.3	0.1

a Find: **i** $\mathrm{E}(X)$ **ii** $\mathrm{Var}(X)$

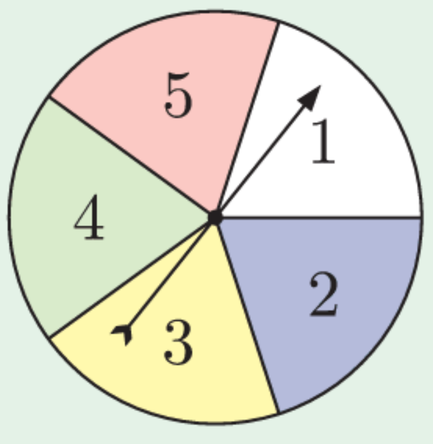
b Let Y be the number of times Caleb *drops* the ball.

i Write Y in terms of X. **ii** Hence find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$.

9 The spinner alongside is spun 20 times. Let X be the number of threes spun.

a Explain why X is a binomial random variable.

b Find the mean and standard deviation of X.



10 24% of visitors to a museum make voluntary donations. On a certain day the museum has 175 visitors.

a Find the expected number of donations.

b Find the probability that:

i less than 40 visitors make a donation

ii between 50 and 60 (inclusive) visitors make a donation.

11 A school volleyball team has 9 players, each of whom has a 75% chance of coming to any given game. The team needs at least 6 players to avoid forfeiting the game.

a Find the probability that for a randomly chosen game, the team will:

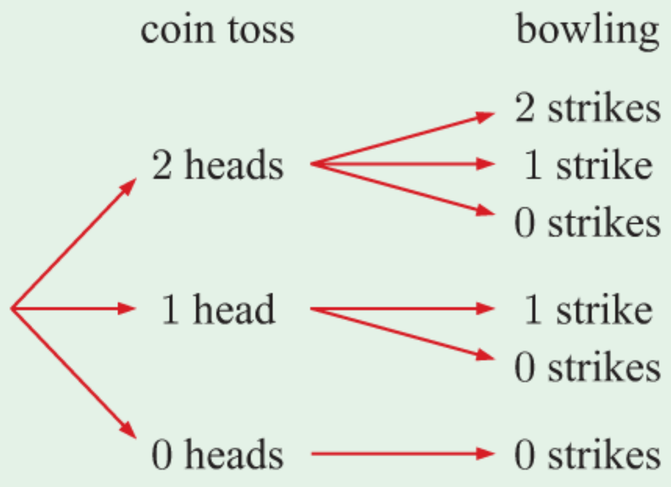
i have all of its players **ii** have to forfeit the game.

b The team plays 30 games for the season. How many games would you expect the team to forfeit?

12 Suvi plays a game involving 2 coins and a set of bowling pins. The coins are flipped and the number of heads that result is the number of attempts she gets to knock down the pins. If she knocks all of the pins down on a given attempt, it is called a "strike".

Suvi wins a prize worth \$10 multiplied by the number of strikes she gets. On each attempt, the probability that Suvi gets a strike is $\frac{1}{3}$.

a Copy and complete this tree diagram of possible outcomes:



b Let X be the number of strikes that Suvi gets. Find the probability distribution of X.

c Calculate Suvi's expected return per game.

d Find Suvi's expected gain if the game costs \$5 to play. Would you advise her to play the game many times?

13 During peak period, customers arrive at random at a fish and chip shop at the rate of 20 customers every 15 minutes.

a Find the probability that during peak period, exactly 15 customers will arrive in the next 15 minutes.

b If the probability that more than 10 customers will arrive during 10 minutes of peak period is greater than 80%, the manager will employ an extra shop assistant. Will the manager hire an extra shop assistant?

14 A Poisson random variable X is such that $\mathrm{P}(X = 1) = \mathrm{P}(2 \leqslant X \leqslant 4)$.

a Find the mean and standard deviation of X.

b Find $\mathrm{P}(X \geqslant 2)$.

Chapter 28

The normal distribution

Contents:

A Introduction to the normal distribution
B Calculating probabilities
C The standard normal distribution
D Quantiles



OPENING PROBLEM

A salmon breeder is interested in the distribution of the *weight of female adult salmon*, w.

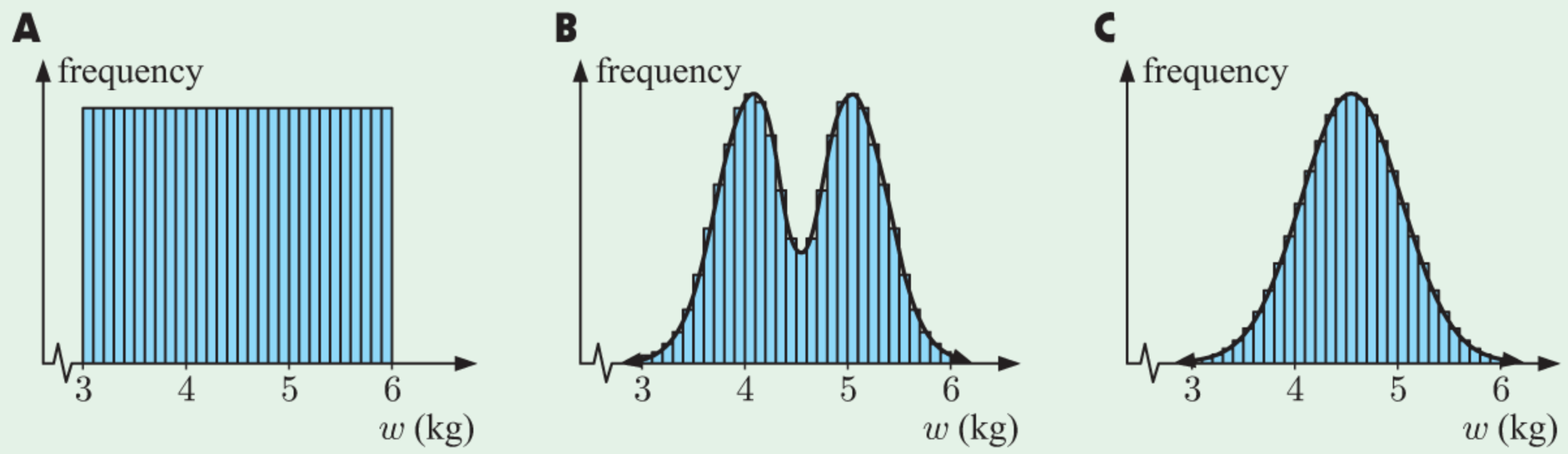


He catches hundreds of female adult fish and records their weights in a frequency table with class intervals $3 \leqslant w < 3.1$ kg, $3.1 \leqslant w < 3.2$ kg, $3.2 \leqslant w < 3.3$ kg, and so on.

The mean weight is 4.73 kg, and the standard deviation is 0.53 kg.

Things to think about:

a Which of these do you think is the most likely distribution for the weights of the female adult salmon?



b How can we use the mean and standard deviation to estimate the proportion of salmon that weigh:

i more than 6 kg **ii** between 4 kg and 6 kg?

c How can we find the weight which:

i 90% of salmon weigh less than **ii** 25% of salmon weigh more than?

In the previous Chapter we looked at discrete random variables and examined binomial probability distributions where the random variable X could take the non-negative integer values $x = 0, 1, 2, 3, 4, ..., n$.

For a **continuous random variable** X, x can take any real value within some reasonable domain. There are infinitely many values X can take, and even if a measuring device enabled us to measure X exactly, the measurements of X from any two members of the population would never be *identical*. This means that the probability that X is exactly equal to any particular value is zero.

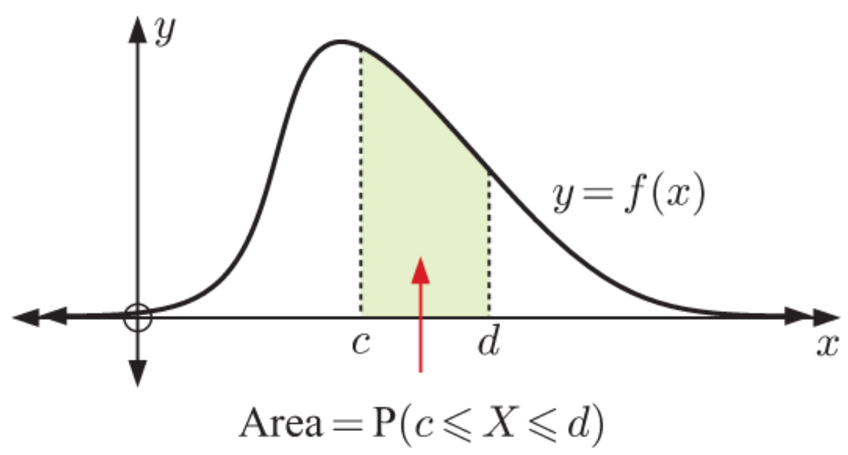
For a continuous variable X, $\quad P(X = x) = 0 \quad$ for all x.

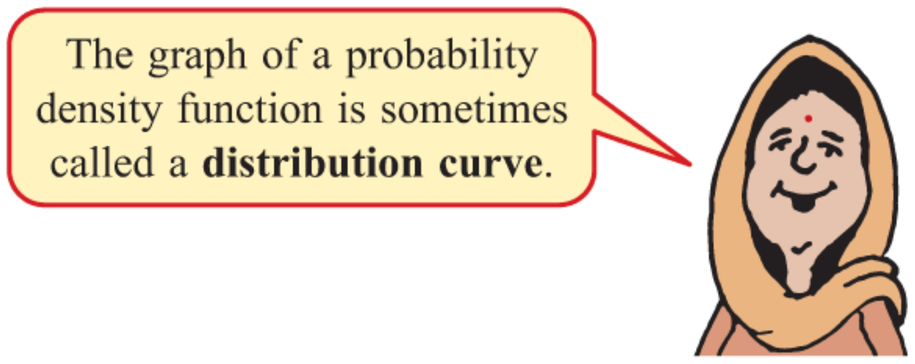
For example, the probability that an egg will weigh *exactly* 72.9 g is zero. If you were to weigh an egg on scales that measure to the nearest 0.1 g, a reading of 72.9 g means the weight lies somewhere between 72.85 g and 72.95 g. No matter how accurate your scales are, you can only ever know the weight of an egg within a range.

So, for a continuous variable X, we can only talk about the probability that a measured value lies in an **interval**.

Remembering that $P(X = x) = 0$ for all x,
$P(c \leqslant X \leqslant d) = P(c < X \leqslant d) = P(c \leqslant X < d) = P(c < X < d)$.

Since $P(X = x) = 0$ for all x, we cannot use a probability mass function to describe the distribution. Instead we use a function called a **probability density function**. The value of the function is not a probability. Rather, probabilities are found by calculating areas under the probability density function curve for a particular interval.





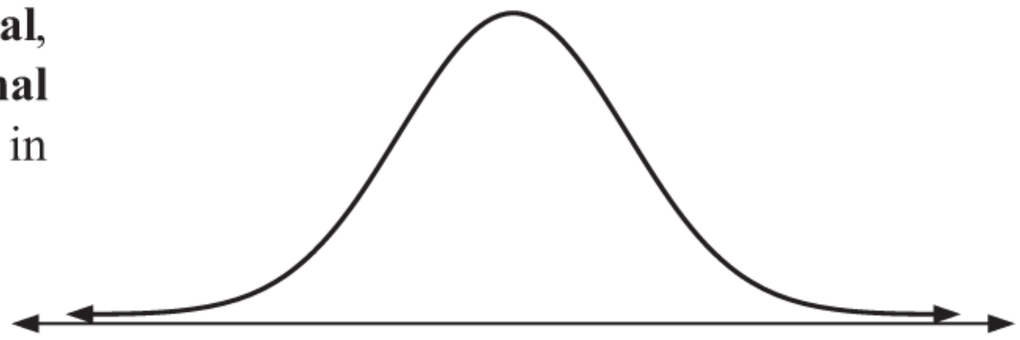
DISCUSSION

Can you explain why:

- the probability density function is always positive
- the *total* area under the curve is 1?

A INTRODUCTION TO THE NORMAL DISTRIBUTION

In this Chapter, we consider variables with **symmetrical**, **bell-shaped** distribution curves. We call this a **normal distribution**. It is the most important distribution in statistics.



The normal distribution arises in nature when many different factors affect the value of the variable.

For example, consider the apples harvested from an apple orchard. They do not all have the same weight. This variation may be due to genetic factors, the soil, the amount of sunlight reaching the leaves and fruit, weather conditions, and so on.

The result is that most of the fruit will have weights centred about the mean weight, and there will be fewer apples that are much heavier or much lighter than this mean.



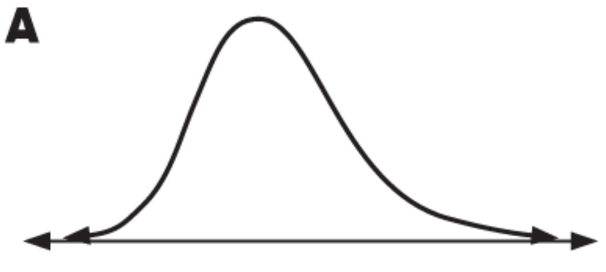
Some examples of quantities that may be normally distributed or approximately normally distributed are:

- the heights of 16 year old boys
- the lengths of adult sharks
- the yields of corn or wheat
- the volumes of liquid in soft drink cans
- the weights of peaches in a harvest
- the life times of batteries

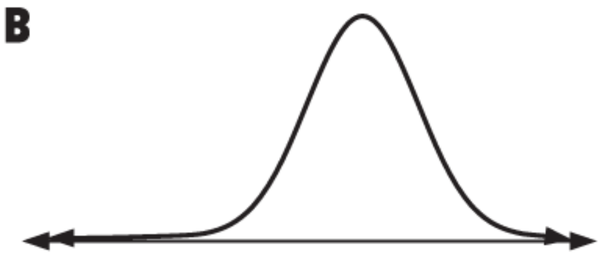
EXERCISE 28A.1

1 Which of the following appear to be normal distribution curves?

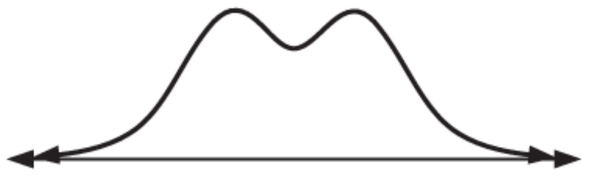
A



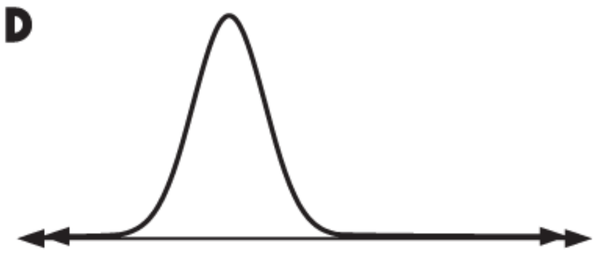
B



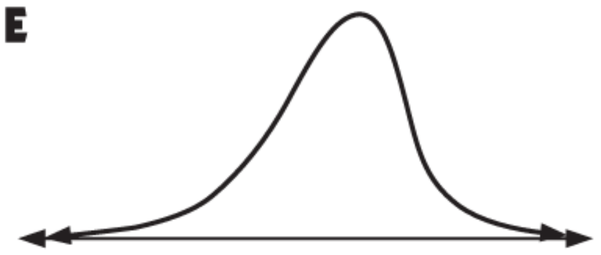
C



D



E



F



2 Explain why it is likely that the following variables will be normally distributed:

a the diameter of wooden rods cut using a lathe

b scores for tests taken by a large population

c the amount of time a student takes to walk to school each day.

3 Discuss whether the following variables are likely to be normally distributed. Sketch a graph to illustrate the possible distribution of each variable.

a the ages of people at a football match

b the distances recorded by a long jumper

c the numbers drawn in a lottery

d the lengths of carrots in a supermarket

e the amounts of time passengers spend waiting in a queue at an airport

f the numbers of brown eggs in a sample of cartons which each contain a dozen eggs

g the numbers of children in families living in Cardiff, Wales

h the heights of buildings in a city.



THE NORMAL DISTRIBUTION CURVE

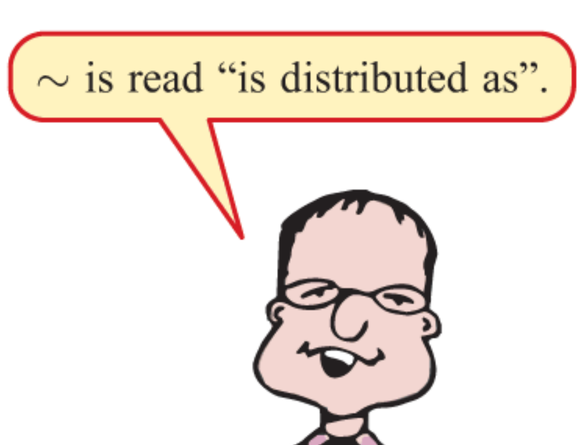
Although all normal distributions have the same general bell-shaped curve, the exact location and shape of the curve is determined by:

- the **mean** μ which measures the **centre** of the distribution
- the **standard deviation** σ which measures the **spread** of the distribution.

If X is a normally distributed random variable with mean μ and standard deviation σ, we write $\boldsymbol{X \sim N(\mu, \sigma^2)}$.

We say that μ and σ are the **parameters** of the distribution.

The probability density function of X is called the **normal distribution curve** or **normal curve**.



INVESTIGATION 1 PROPERTIES OF THE NORMAL CURVE

In this Investigation, we will look at some interesting properties of the normal distribution curve with the help of **graphing software**.

Click on the icon to explore the normal distribution curve and how it changes when μ and σ are altered.

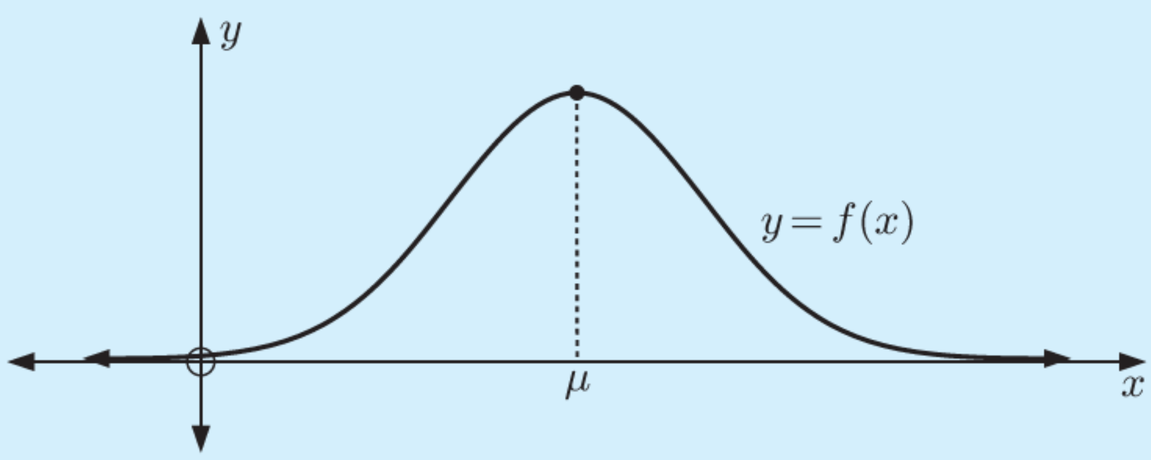


What to do:

1. What effects do variations in μ and σ have on the curve? How do these relate to what μ and σ represent?
2. Does the curve have a line of symmetry? If so, what is it?
3. Is the function ever negative? Why is this important?
4. Discuss the behaviour of the normal curve as $x \to \pm\infty$.
5. What do you think happens to the *area* under the curve as you change μ and σ?

From the **Investigation**, you should have found that:

- The normal curve is symmetrical about the vertical line $x = \mu$.
- $f(x) > 0$ for all x.
- The x-axis is a horizontal asymptote.



EXERCISE 28A.2

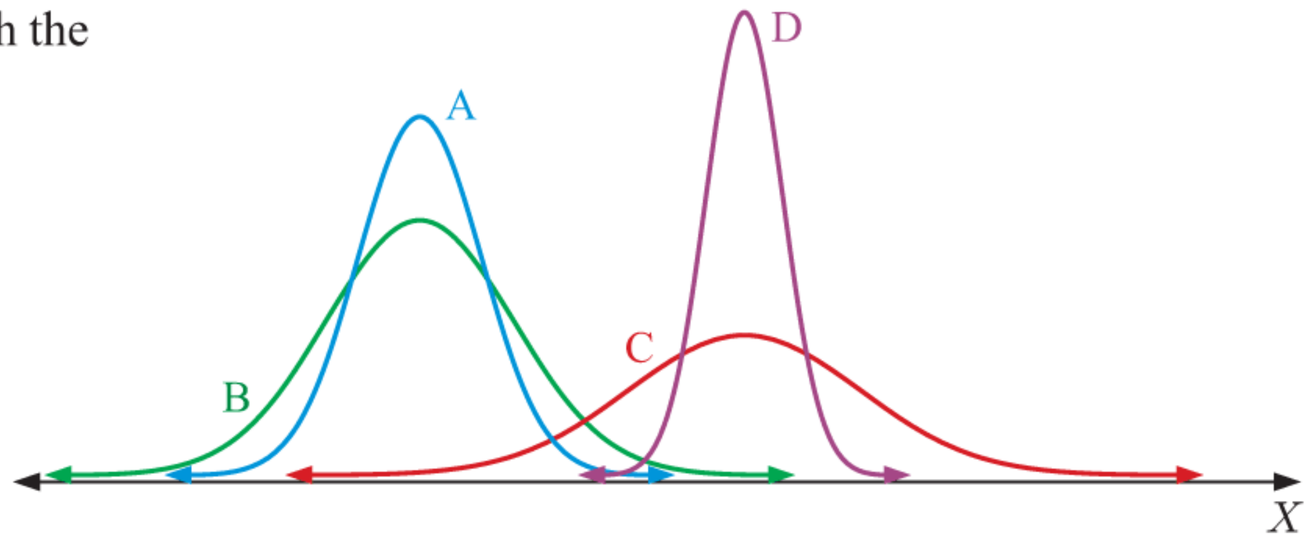
1 Match each pair of parameters with the correct normal distribution curve:

a $\mu = 5,\ \sigma = 2$

b $\mu = 15,\ \sigma = 0.5$

c $\mu = 5,\ \sigma = 1$

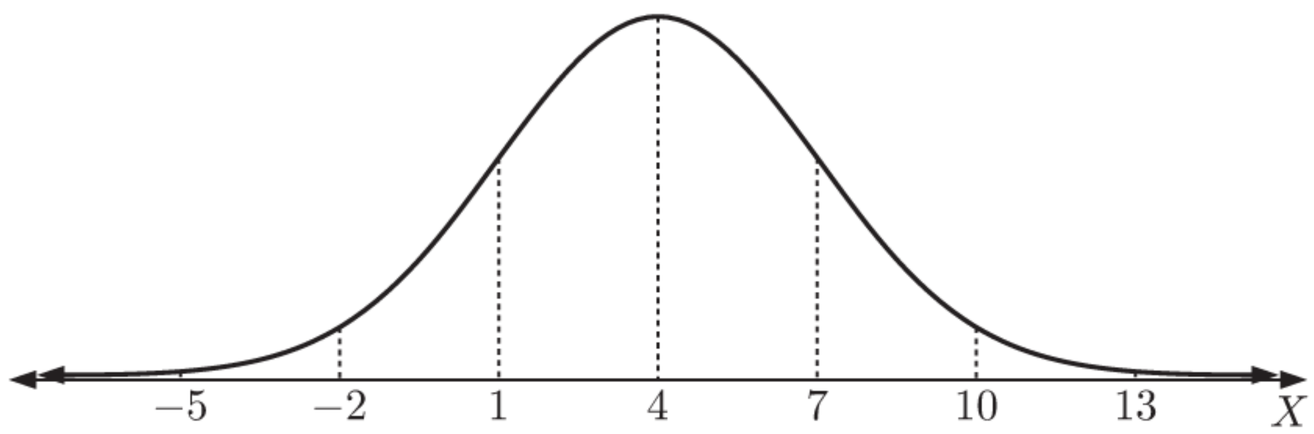
d $\mu = 15,\ \sigma = 3$



2 Sketch the following normal distributions on the same set of axes.

Distribution	*Mean* (mL)	*Standard deviation* (mL)
A	25	5
B	30	2
C	21	10

3 Consider the distribution curve of $X \sim \text{N}(\mu, \sigma^2)$ where $\mu = 4$ and $\sigma = 3$ shown:



Copy the above graph, and on the same set of axes sketch the distribution curve for:

PRINTABLE GRAPH

a $\text{N}(\mu + 2, \sigma^2)$ **b** $\text{N}(\mu, (2\sigma)^2)$ **c** $\text{N}(\mu + 2, (2\sigma)^2)$

d $\text{N}\left(\mu - 1, \left(\frac{\sigma}{3}\right)^2\right)$ **e** $\text{N}\left(3 + \mu, \frac{\sigma^2}{4}\right)$

HISTORICAL NOTE

The normal distribution was first characterised by **Carl Friedrich Gauss** in 1809 as a way to rationalise his **method of least squares** for linear regression. In fact, the normal distribution curve is a special case of the **Gaussian function**.

Since the normal distribution has such strong ties to Gauss, it is sometimes called the **Gaussian distribution**.



Carl Friedrich Gauss

B CALCULATING PROBABILITIES

INVESTIGATION 2 PROPORTIONS FROM A NORMAL DISTRIBUTION

In this Investigation we find the proportions of normally distributed data which lie within σ, 2σ, and 3σ of the mean.

What to do:

1 Click on the icon to run a demonstration which randomly generates 1000 data values from a normal distribution with mean μ and standard deviation σ. Set $\mu = 0$ and $\sigma = 1$.

DEMO

2 Find the endpoints of the interval:

a $\mu - \sigma$ to $\mu + \sigma$ **b** $\mu - 2\sigma$ to $\mu + 2\sigma$ **c** $\mu - 3\sigma$ to $\mu + 3\sigma$

3 Use the frequency table provided to find the proportion of data values which lie between:

a $\mu - \sigma$ and $\mu + \sigma$ **b** $\mu - 2\sigma$ and $\mu + 2\sigma$ **c** $\mu - 3\sigma$ and $\mu + 3\sigma$

4 Repeat **2** and **3** for values of μ and σ of your choosing. Summarise your answers in a table like the one below.

μ	σ	$\mu-\sigma$ to $\mu+\sigma$		$\mu-2\sigma$ to $\mu+2\sigma$		$\mu-3\sigma$ to $\mu+3\sigma$	
		Interval	*Proportion*	*Interval*	*Proportion*	*Interval*	*Proportion*
0	1	$-1 \leqslant X \leqslant 1$					

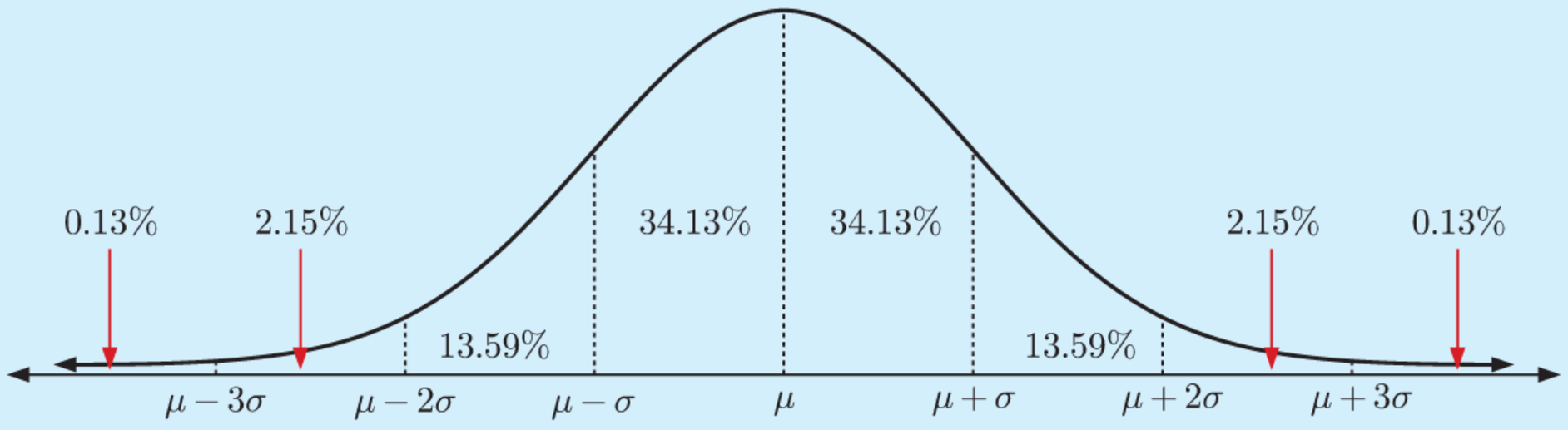
5 What do you notice about the proportion of data values in each interval?

From the **Investigation**, you should have found that:

For any population that is normally distributed with mean μ and standard deviation σ:

- approximately **0.68** or **68%** of the population will lie between $\mu-\sigma$ and $\mu+\sigma$
- approximately **0.95** or **95%** of the population will lie between $\mu-2\sigma$ and $\mu+2\sigma$
- approximately **0.997** or **99.7%** of the population will lie between $\mu-3\sigma$ and $\mu+3\sigma$.

The proportion of data values that lie within different ranges relative to the mean are:



For any variable that is normally distributed, we can use the mean and standard deviation to estimate the proportion of data that will lie in a given interval. This proportion tells us the probability that a randomly selected member of the population will be in that interval.

Example 1 **Self Tutor**

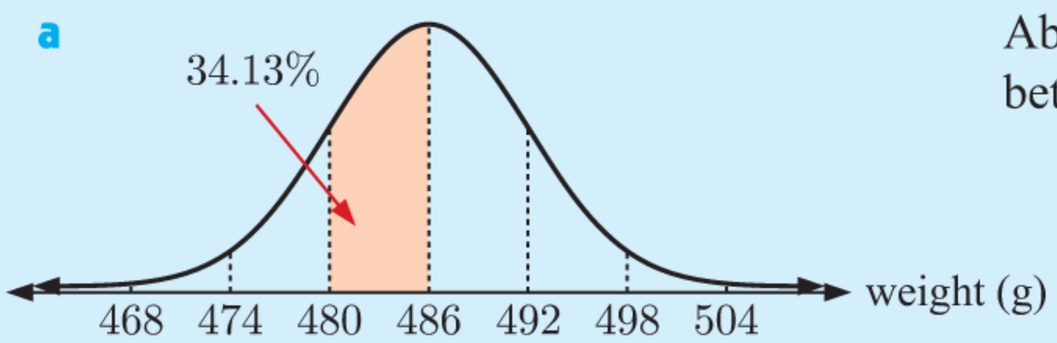
A sample of cans of peaches was taken from a warehouse, and the contents of each can was weighed. The sample mean was 486 g with standard deviation 6 g.

State the proportion of cans that weigh:

a between 480 g and 486 g **b** more than 492 g.

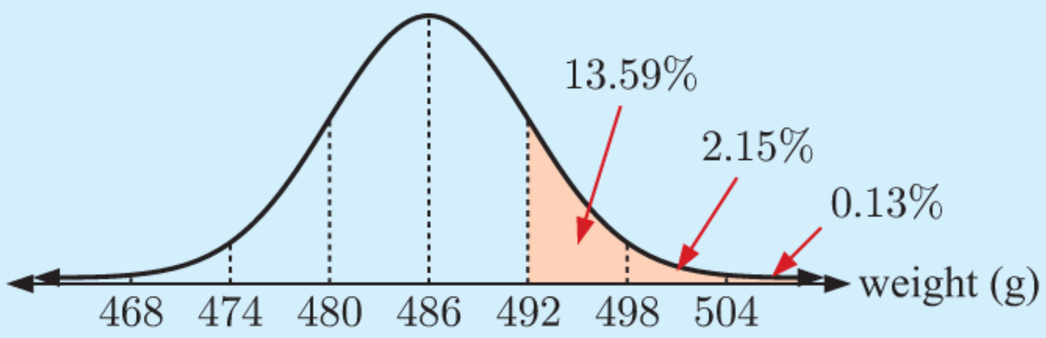
For a manufacturing process such as this, the distribution of weights is approximately normal.

a

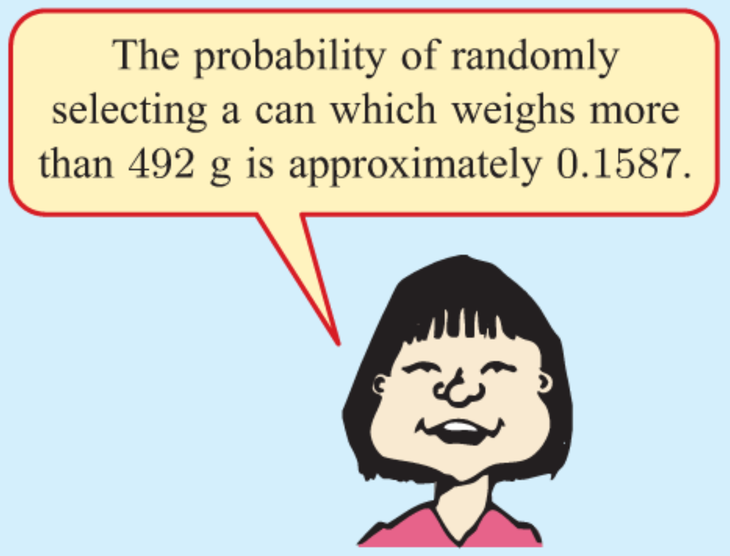


About 34.13% of the cans are expected to weigh between 480 g and 486 g.

b



About $13.59\% + 2.15\% + 0.13\% = 15.87\%$ of the cans are expected to weigh more than 492 g.



EXERCISE 28B.1

1 Suppose X is normally distributed with mean 30 and standard deviation 5.

a State the value which is:

i 2 standard deviations above the mean **ii** 1 standard deviation below the mean.

b Describe the following values in terms of the number of standard deviations above or below the mean:

i 35 **ii** 20 **iii** 45

c Draw a curve to illustrate the distribution of X.

d What proportion of values of X are between 25 and 30?

e Find the probability that a randomly selected member of the population will measure between 35 and 40.

2 Suppose the variable X is normally distributed according to the curve shown.

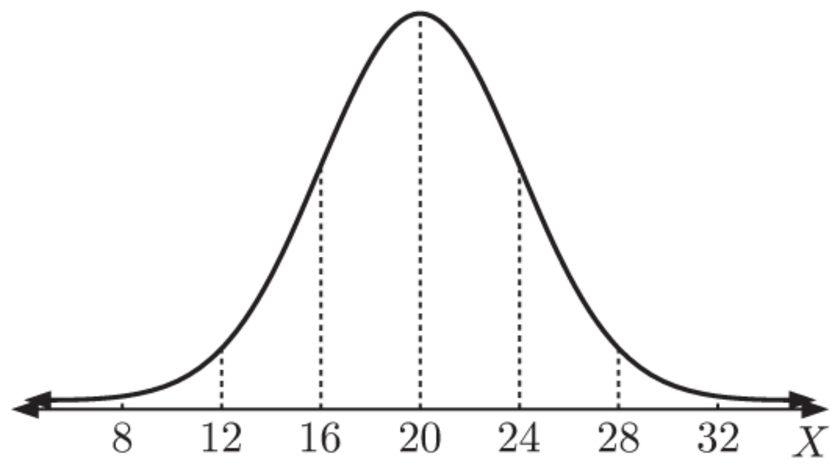
a State the mean and standard deviation of X.

b Find the proportion of values of X which are:

i between 20 and 24

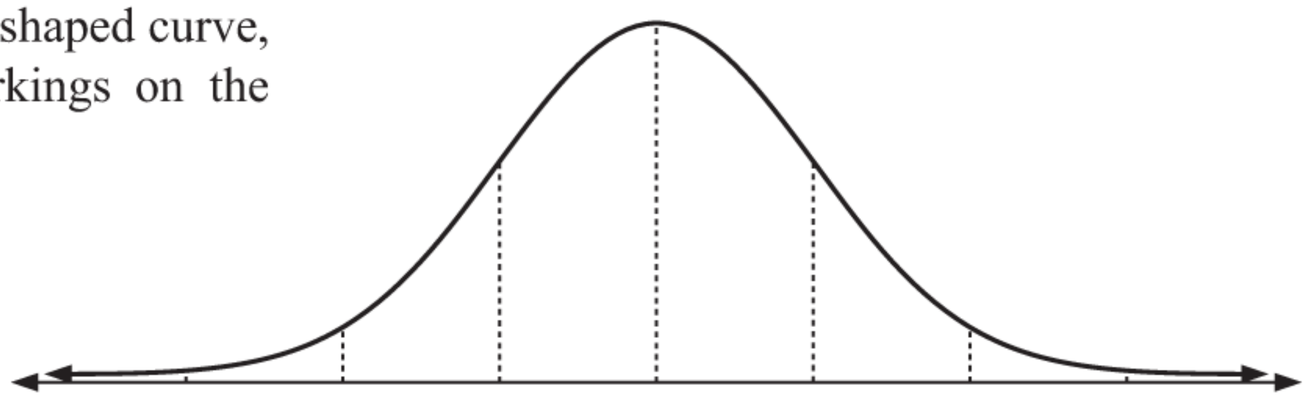
ii between 12 and 16

iii greater than 28.



3 A school's Grade 12 students sat for a Mathematics examination. Their marks were approximately normally distributed with mean 75 and standard deviation 8.

a Copy and complete this bell-shaped curve, assigning scores to the markings on the horizontal axis.



b What proportion of students would you expect to have scored:

i more than 83 **ii** less than 59 **iii** between 67 and 91?

4 State the probability that a randomly selected, normally distributed value:

a lies within one standard deviation either side of the mean

b is more than two standard deviations above the mean.

Example 2 **Self Tutor**

The chest measurements of 18 year old male rugby players are normally distributed with mean 95 cm and standard deviation 8 cm.

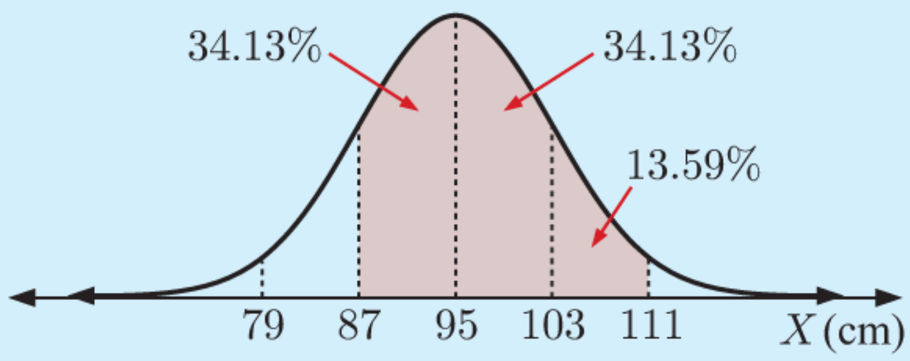
a From a group of 200 18 year old male rugby players, how many would you expect to have a chest measurement between 87 cm and 111 cm?

b Find the value of k such that approximately 16% of chest measurements are below k cm.



a About $34.13\% + 34.13\% + 13.59\% = 81.85\%$ of the rugby players have a chest measurement between 87 cm and 111 cm.

So, we would expect 81.85% of $200 \approx 164$ of the rugby players to have a chest measurement between 87 cm and 111 cm.

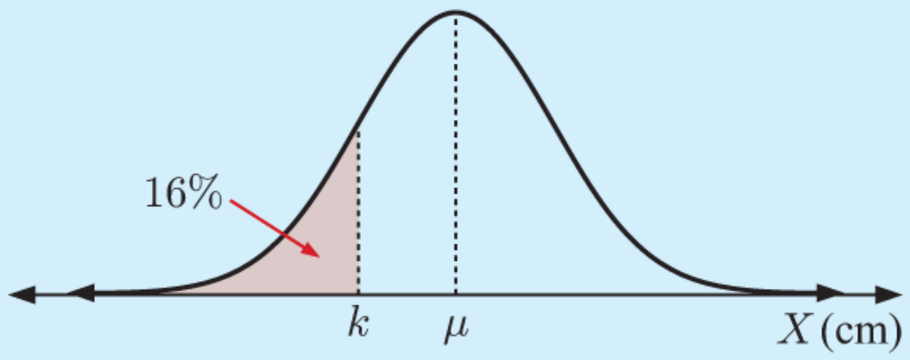


b Approximately 16% of data lies more than one standard deviation below the mean.

$\therefore$ k is σ below the mean μ

$\therefore$ $k = 95 - 8$

$= 87$



5 The height of female students at a university is normally distributed with mean 170 cm and standard deviation 8 cm.

a Find the percentage of female students whose height is:

i between 162 cm and 170 cm **ii** between 170 cm and 186 cm.

b Find the probability that a randomly chosen female student has a height:

i less than 154 cm **ii** greater than 162 cm.

c From a group of 500 female university students, how many would you expect to be between 178 cm and 186 cm tall?

d Estimate the value of k such that 16% of the female students are taller than k cm.

6 The lengths of adult female frilled sharks are normally distributed with mean 1.4 m and standard deviation 15 cm. Find the proportion of adult female frilled sharks that measure:

a more than 1.25 m **b** between 1.1 m and 1.55 m.

7 The weights of the 545 babies born at a maternity hospital last year were normally distributed with mean 3.0 kg and standard deviation 200 grams. Estimate the number that weighed:

a less than 3.2 kg **b** between 2.8 kg and 3.4 kg.

8 An industrial machine fills an average of 20 000 bottles each day with standard deviation 2000 bottles. Assuming that production is normally distributed and the year comprises 260 working days, estimate the number of working days on which:

a under 18 000 bottles are filled **b** over 16 000 bottles are filled

c between 18 000 and 24 000 bottles are filled.

9 Two hundred lifesavers competed in a swimming race. Their times were normally distributed with mean 10 minutes 30 seconds and standard deviation 15 seconds. Estimate the number of competitors who completed the race in a time:

a longer than 11 minutes **b** less than 10 minutes 15 seconds

c between 10 minutes 15 seconds and 10 minutes 45 seconds.

10 The weights of Jason's oranges are normally distributed. 84% of the crop weighs more than 152 grams and 16% weighs more than 200 grams.

a Find μ and σ for the crop.

b What percentage of the oranges weigh between 152 grams and 224 grams?

11 When a particular variety of radish is grown without fertiliser, the weights of the radishes produced are normally distributed with mean 40 g and standard deviation 10 g.
When these radishes are grown in the same conditions but with fertiliser added, their weights are also normally distributed, but with mean 140 g and standard deviation 40 g.

a Determine the proportion of radishes grown:

i without fertiliser which weigh less than 50 grams

ii with fertiliser which weigh less than 60 grams.

b Find the probability that a randomly selected radish weighs between 20 g and 60 g, if it is grown:

i with fertiliser **ii** without fertiliser.

c One radish grown with fertiliser and one radish grown without fertiliser are selected at random. Find the probability that *both* radishes weigh more than 60 g.

USING TECHNOLOGY

When calculating normal distribution probabilities, we have so far only considered numbers that are a whole number of standard deviations from the mean.

To calculate other probabilities, we could use definite integrals of the normal probability density function. However, this function does not have an indefinite integral, so we need to use a numerical approximation. Your **graphics calculator** has built-in functions to calculate these integrals.


GRAPHICS CALCULATOR INSTRUCTIONS

Example 3 **Self Tutor**

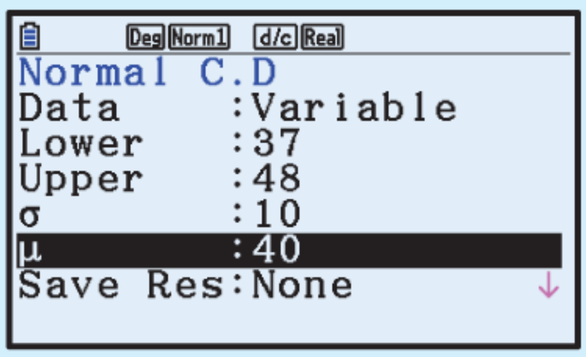
The variable X is normally distributed with mean 40 and standard deviation 10. Find:

a $P(37 < X < 48)$ **b** $P(X > 45)$ **c** $P(X < 26)$

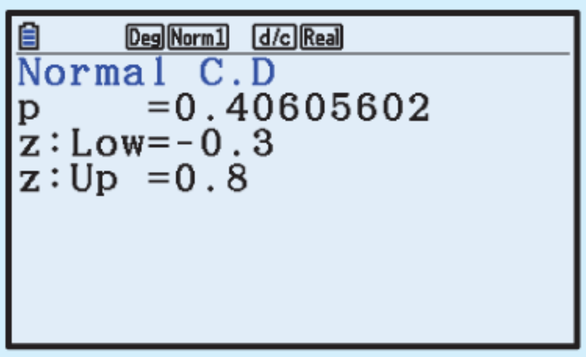
Illustrate your answers.

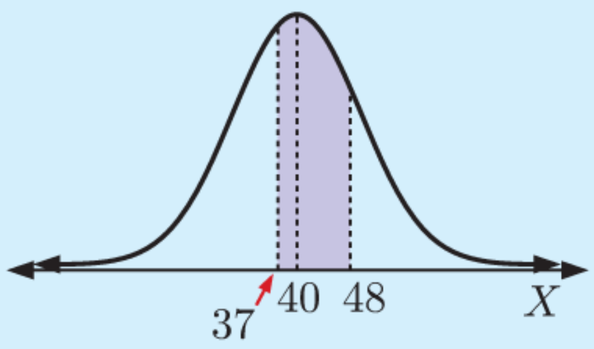
a To find $P(37 < X < 48)$, we set the lower bound to 37 and the upper bound to 48.

Casio fx-CG50



Casio fx-CG50

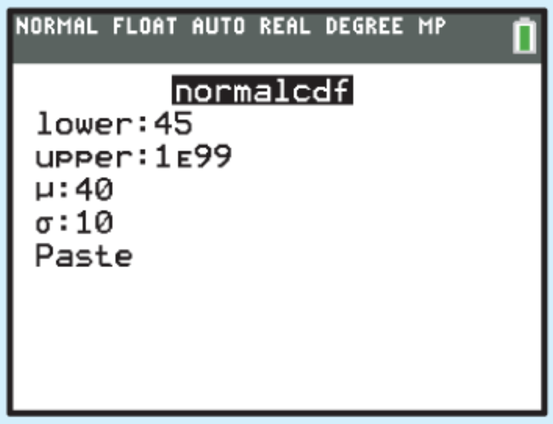




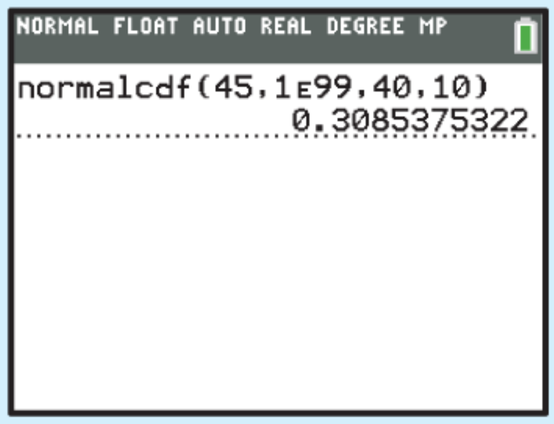
$P(37 < X < 48) \approx 0.406$

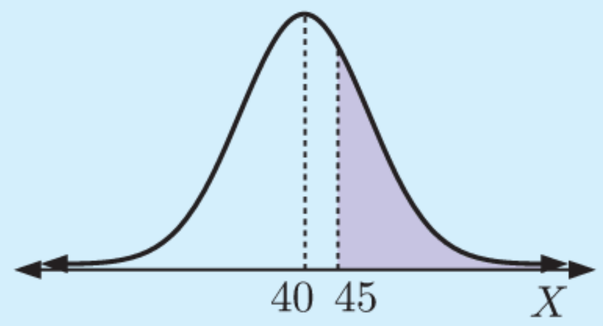
b To find $P(X > 45)$, we use a very high value such as 10^{99} to represent the upper bound.

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TI-84 Plus CE

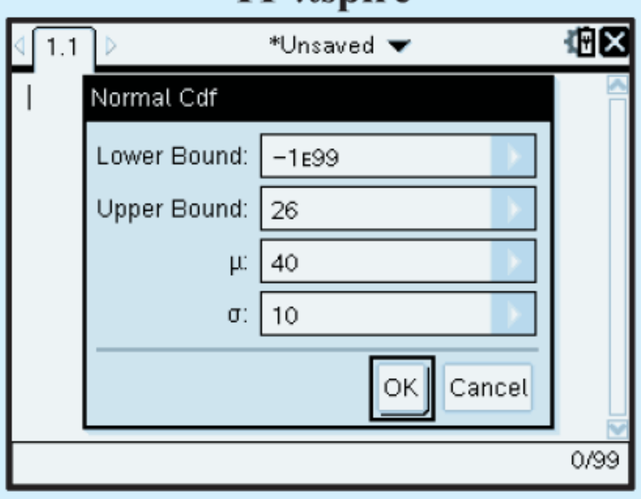




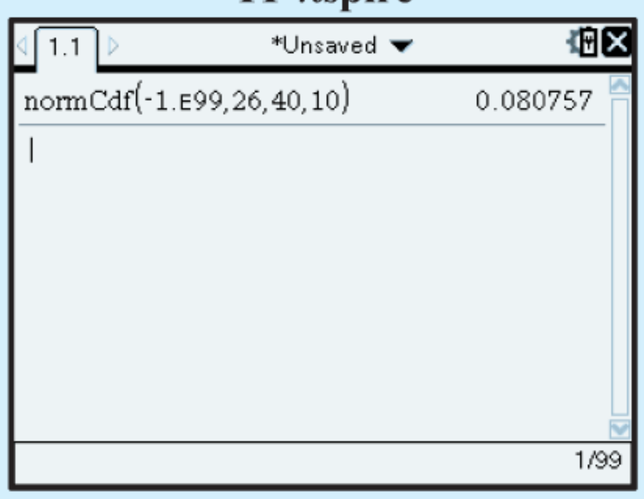
$P(X > 45) \approx 0.309$

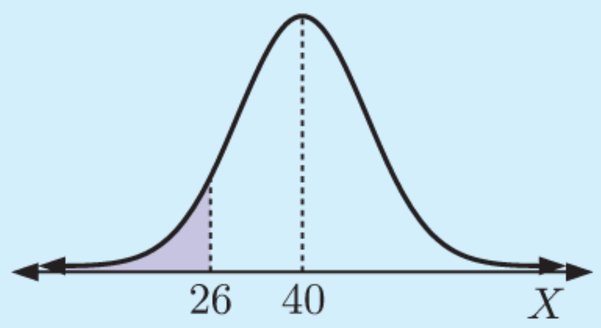
c To find $P(X < 26)$, we use a very low value such as -10^{99} to represent the lower bound.

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$P(X < 26) \approx 0.081$

EXERCISE 28B.2

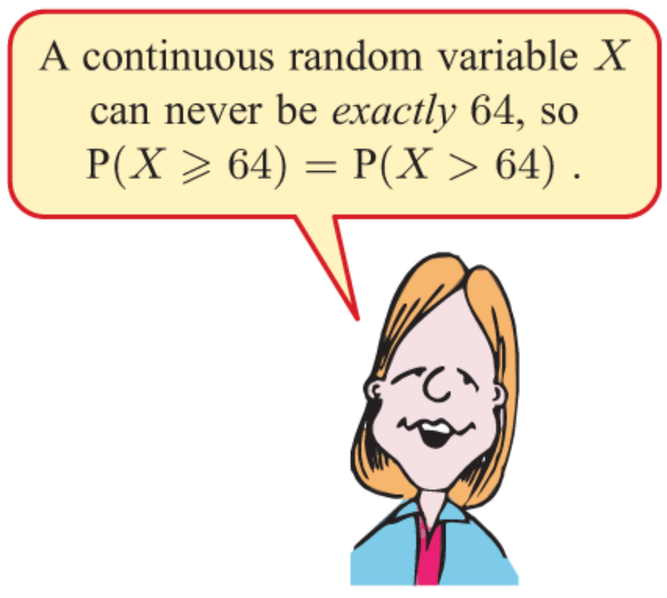
1 Suppose X is normally distributed with mean 60 and standard deviation 5. Find:

a $P(60 \leqslant X \leqslant 65)$ **b** $P(62 \leqslant X \leqslant 67)$

c $P(X \geqslant 64)$ **d** $P(X \leqslant 68)$

e $P(X \leqslant 61)$ **f** $P(57.5 \leqslant X \leqslant 62.5)$

Illustrate your answers.



2 Suppose X is normally distributed with mean 37 and standard deviation 7.

a Use technology to find $P(X > 40)$.

b Hence find $P(37 \leqslant X \leqslant 40)$ without technology.

3 A machine produces metal bolts. The lengths of these bolts have a normal distribution with mean 19.8 cm and standard deviation 0.3 cm. If a bolt is selected at random from the machine, find the probability that it will have a length between 19.7 cm and 20 cm.

4 The speed of cars passing a supermarket is normally distributed with mean 46.3 km h^{-1} and standard deviation 7.4 km h^{-1}. Find the probability that a randomly selected car is travelling:

a between 50 and 65 km h^{-1} **b** slower than 60 km h^{-1} **c** faster than 50 km h^{-1}.

5 Eels are washed onto a beach after a storm. Their lengths have a normal distribution with mean 41 cm and standard deviation 5.5 cm.

a If an eel is randomly selected, find the probability that it is at least 50 cm long.

b Find the percentage of eels measuring between 40 cm and 50 cm long.

c How many eels from a sample of 200 would you expect to measure at least 45 cm in length?

6 Max's customers put money for charity into a collection box on the front counter of his shop. The weekly collection is approximately normally distributed with mean \$40 and standard deviation \$6.

a On what percentage of weeks would Max expect to collect:

i between \$30 and \$50 **ii** at least \$50?

b How much money would you expect Max to collect in two years?

7 The amount of petrol bought by customers at a petrol station is normally distributed with mean 36 L and standard deviation 7 L.

a What percentage of customers buy:

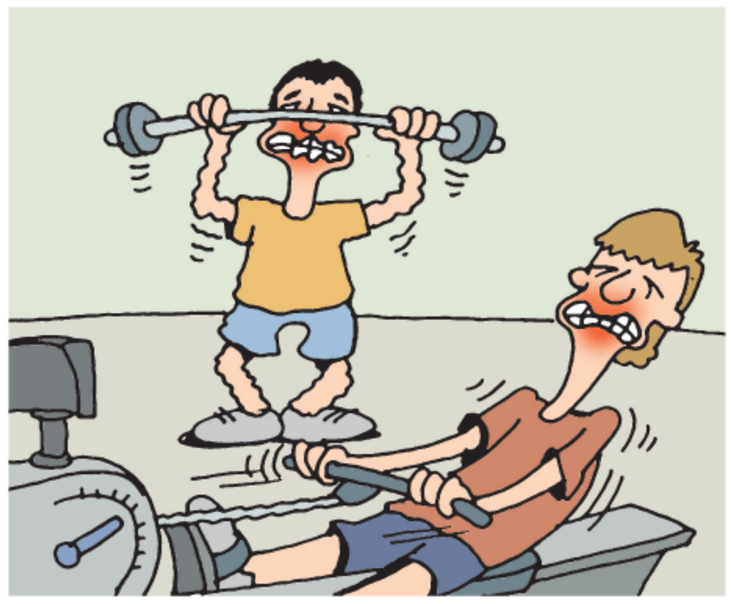
i less than 28 L of petrol **ii** between 30 L and 40 L of petrol?

b On a particular day, the petrol station has 600 customers.

i How much petrol would you expect the petrol station to sell on this day?

ii How many customers would you expect to buy at least 44 L of petrol?

8 The times Enrique and Damien spend working out at the gym each day are both normally distributed with mean 45 minutes. The standard deviation of Enrique's times is 9 minutes, and the standard deviation of Damien's times is 6 minutes.



a On what percentage of days does:

i Enrique spend between 32 and 40 minutes at the gym

ii Damien spend less than 55 minutes at the gym?

b Tomorrow, who do you think is more likely to spend:

i at least 1 hour at the gym **ii** between 40 minutes and 50 minutes at the gym?

Explain your answers.

c Perform calculations to check your answers to **b**.

Example 4 **Self Tutor**

The times taken by students to complete a puzzle are normally distributed with mean 28.3 minutes and standard deviation 3.6 minutes. Calculate the probability that:

a a randomly selected student took at least 30 minutes to complete the puzzle

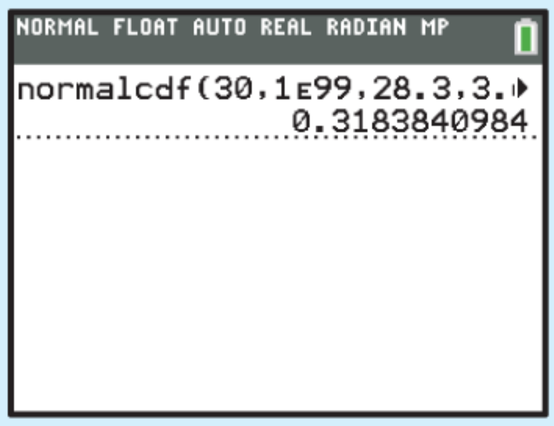
b out of 10 randomly selected students, 5 or fewer of them took at least 30 minutes to complete the puzzle.

a Let X denote the time for a student to complete the puzzle.

$X \sim \text{N}(28.3,\ 3.6^2)$

$\therefore\ \text{P}(X \geqslant 30) \approx 0.318\,38$

≈ 0.318



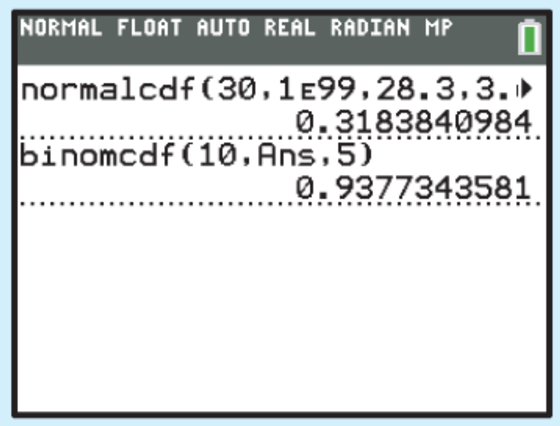
b Let Y denote the number of students who took at least 30 minutes to complete the puzzle.

$Y \sim \text{B}(10,\ 0.318\,38)$

$\therefore \quad \text{P}(Y \leqslant 5) \approx 0.938$



$\text{B}(n,\ p)$ is the binomial distribution with n independent trials, each with probability of success p.



9 Apples from a grower's crop were normally distributed with mean 173 grams and standard deviation 34 grams. Apples weighing less than 130 grams were too small to sell.

a Find the percentage of apples from this crop which were too small to sell.

b Find the probability that in a picker's basket of 100 apples, more than 10 apples were too small to sell.

10 People found to have high blood pressure are prescribed a course of tablets. They have their blood pressure checked at the end of 4 weeks. The drop in blood pressure over the period is normally distributed with mean 5.9 units and standard deviation 1.9 units.

a Find the proportion of people who show a drop of more than 4 units.

b Eight people taking the course of tablets are selected at random. Find the probability that at least six of them will show a drop in blood pressure of more than 4 units.

11 The lengths of red snapper are normally distributed with mean 58 cm and standard deviation 18 cm. Fish measuring less than 38 cm long must be released back into the water.

a Find the probability that a single red snapper caught must be released back into the water.

b Rodney went fishing and caught 14 red snapper. Find the probability that Rodney can keep at least 10 fish from his catch.

C THE STANDARD NORMAL DISTRIBUTION

Suppose a random variable X is normally distributed with mean μ and standard deviation σ.

For each value of x we can calculate a **z-score** using the algebraic transformation $z = \dfrac{x - \mu}{\sigma}$.

This algebraic transformation is known as the **Z-transformation**.

INVESTIGATION 3 z-SCORES

In this Investigation we consider how the z-scores for a distribution are themselves distributed.

What to do:

1 Consider the x-values: 1, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 5, 6, 6, 7.

a Draw a histogram of the x-values to check that the distribution is approximately normal.

b Find the mean μ and standard deviation σ of the x-values.

c Calculate the z-score for each x-value.

d Find the mean and standard deviation of the z-scores.

2 Click on the icon to access a demo which randomly generates data values from a normal distribution with given mean and standard deviation. The z-score of each data value is calculated, and histograms of the original data and the z-scores are also shown.

DEMO

a Generate samples using various values of μ and σ of your choosing.

b Record the mean and standard deviation of the z-scores in a table like the one below.

x-values		*z-scores*	
Mean	*Standard deviation*	*Mean*	*Standard deviation*

c How does the histogram of the z-scores generally compare with the histogram of the x-values?

d What conclusions can you make about the *distribution* of the z-scores?

From the **Investigation**, you should have found that the z-scores were normally distributed with mean 0 and standard deviation 1.

If $X \sim \mathrm{N}(\mu, \sigma^2)$ and $\boldsymbol{Z = \dfrac{X - \mu}{\sigma}}$ then $\boldsymbol{Z \sim \mathrm{N}(0, 1^2)}$.

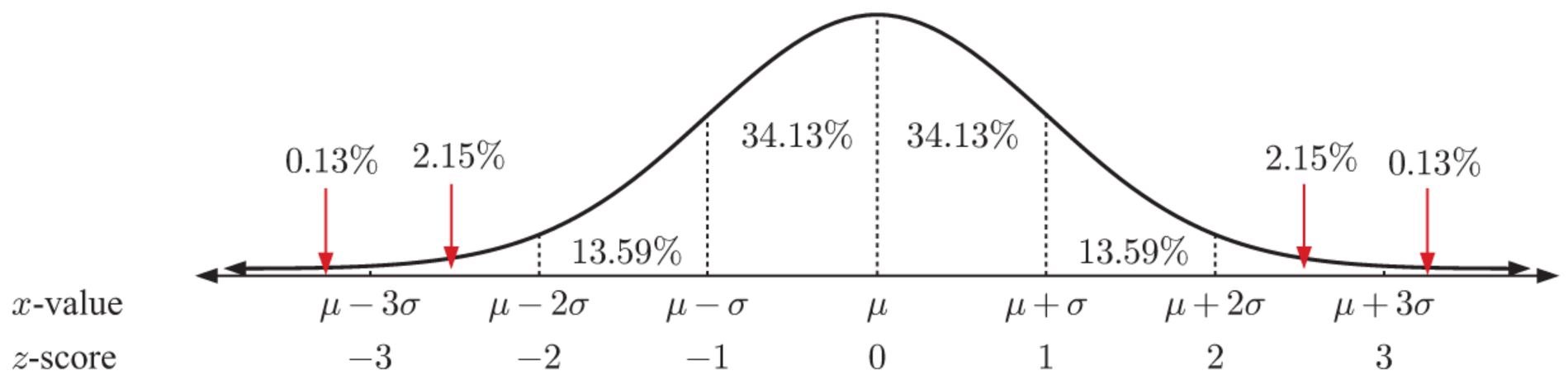
No matter what the parameters μ and σ of the original X-distribution are, we always end up with the same Z-distribution, $Z \sim \mathrm{N}(0, 1^2)$.

THE Z-TRANSFORMATION

Click on the icon to see why the Z-transformation works.

The distribution $\boldsymbol{Z \sim \mathrm{N}(0, 1^2)}$ is called the **standard normal distribution** or **Z-distribution**.

The diagram below shows how z-scores are related to a general normal curve:



Notice that the value of the z-score corresponds to the coefficient of σ in the x-value.

The **z-score** of x is the number of standard deviations x is from the mean.

For example:

- if $z = 1.84$, then x is 1.84 standard deviations to the *right* of the mean
- if $z = -0.273$, then x is 0.273 standard deviations to the *left* of the mean.

z-scores are particularly useful when comparing two populations with different μ and σ. However, these comparisons will only be reasonable if both distributions are approximately normal.

Example 5 **Self Tutor**

Kelly scored 73% in History, where the class mean was 68% and the standard deviation was 10.2%. In Mathematics she scored 66%, where the class mean was 62% and the standard deviation was 6.8%.

In which subject did Kelly perform better compared with the rest of her class?

Assume the scores for both subjects were normally distributed.

Kelly's z-score for History $= \dfrac{73-68}{10.2} \approx 0.490$

Kelly's z-score for Mathematics $= \dfrac{66-62}{6.8} \approx 0.588$

Kelly's result in Mathematics was 0.588 standard deviations above the mean, whereas her result in History was 0.490 standard deviations above the mean.

$\therefore$ Kelly's result in Mathematics was better compared to her class, even though her percentage was lower.

EXERCISE 28C.1

1 In Emma's classes, the exam results for each subject are normally distributed with the mean μ and standard deviation σ shown in the table.

Subject	*Emma's score*	μ	σ
English	48	40	4.4
Mandarin	81	60	9
Geography	84	55	18
Biology	68	50	20
Mathematics	84	50	15

a Find the z-score for each of Emma's scores.

b Arrange Emma's subjects from best to worst in terms of the z-scores.

c Explain why the z-scores are a reasonable way to compare Emma's performances with the rest of her class.

2

Subject	*Sergio's score*	μ	σ
Physics	73%	78%	10.8%
Chemistry	77%	72%	11.6%
Mathematics	76%	74%	10.1%
German	91%	86%	9.6%
Biology	58%	62%	5.2%

The table alongside shows Sergio's results in his final examinations, along with the class means and standard deviations.

a Find Sergio's z-score for each subject.

b Hence arrange Sergio's performances in each subject from best to worst.

3 At a swimming competition, Frederick competed in the 50 m freestyle, 100 m backstroke, 200 m breaststroke, and 100 m butterfly events. His times are summarised in the table, along with the event means and standard deviations.

Event	*Time* (seconds)	μ (seconds)	σ (seconds)
50 m freestyle	32.1	27.8	2.2
100 m backstroke	53.5	58.1	4.3
200 m breaststroke	140.0	143.7	6.4
100 m butterfly	59.6	57.7	5.5

a Calculate the z-scores for each of Frederick's times.

b Explain why in this case a lower z-score indicates a better performance.

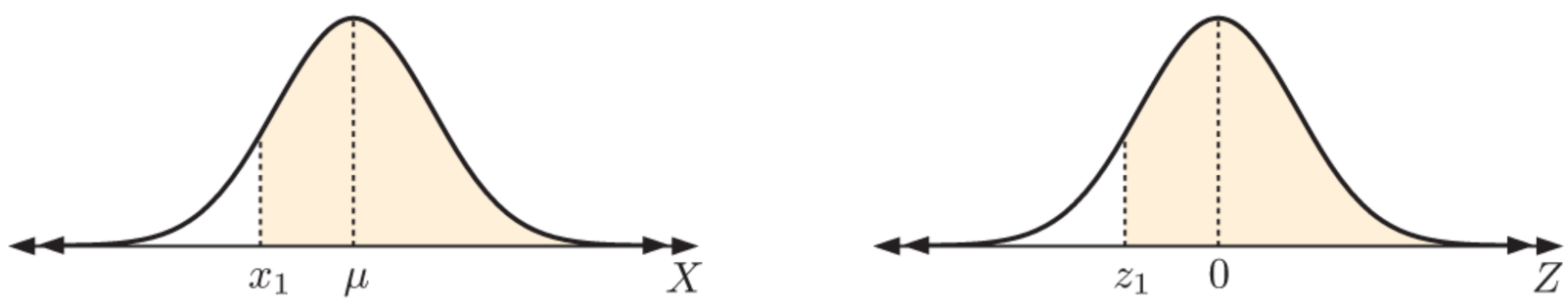
c Hence arrange Frederick's performances in each event from best to worst.

CALCULATING PROBABILITIES USING THE Z-DISTRIBUTION

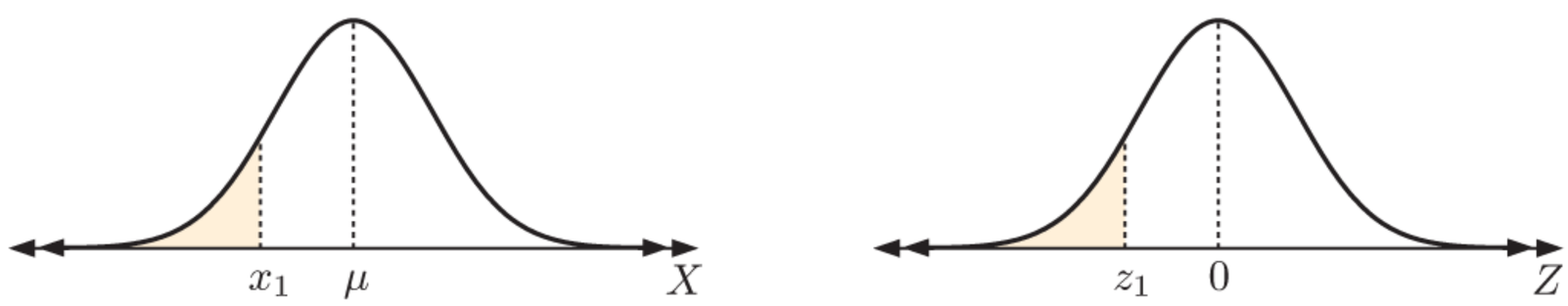
Consider the variables $X \sim \mathrm{N}(\mu, \sigma^2)$ and $Z \sim \mathrm{N}(0, 1^2)$.

For any $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$, with corresponding z-scores $z_1 = \dfrac{x_1 - \mu}{\sigma}$ and $z_2 = \dfrac{x_2 - \mu}{\sigma}$:

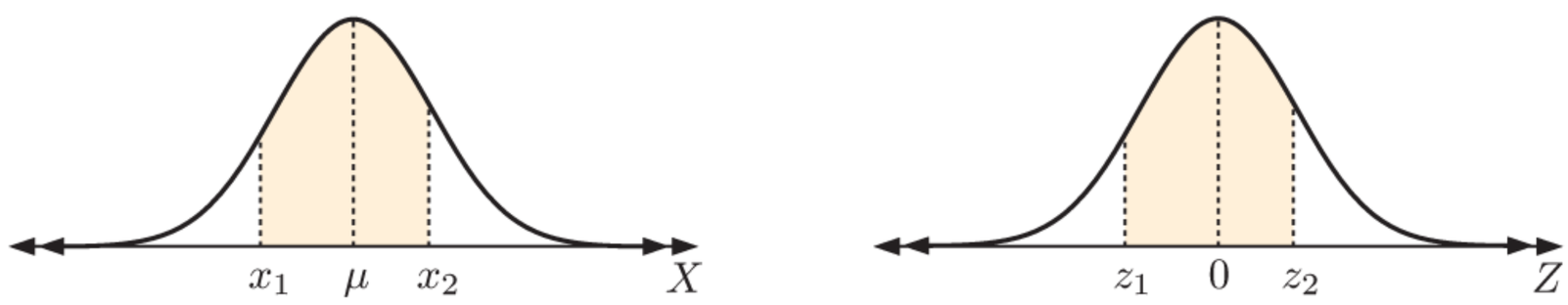
- $\mathrm{P}(X \geqslant x_1) = \mathrm{P}(Z \geqslant z_1)$



- $\mathrm{P}(X \leqslant x_1) = \mathrm{P}(Z \leqslant z_1)$



- $\mathrm{P}(x_1 \leqslant X \leqslant x_2) = \mathrm{P}(z_1 \leqslant Z \leqslant z_2)$



Example 6

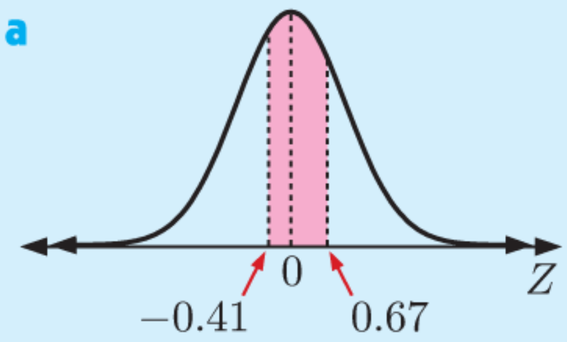
Self Tutor

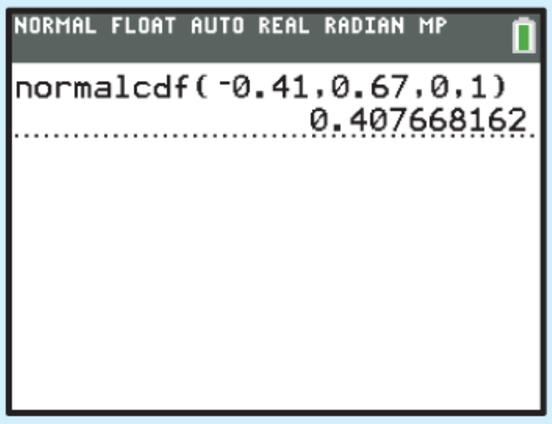
Use technology to illustrate and calculate:

a $\mathrm{P}(-0.41 \leqslant Z \leqslant 0.67)$ **b** $\mathrm{P}(Z \leqslant 1.5)$ **c** $\mathrm{P}(Z > 0.84)$

$Z \sim \mathrm{N}(0, 1^2)$

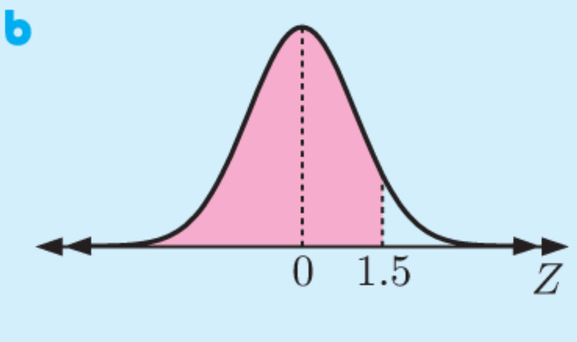
a

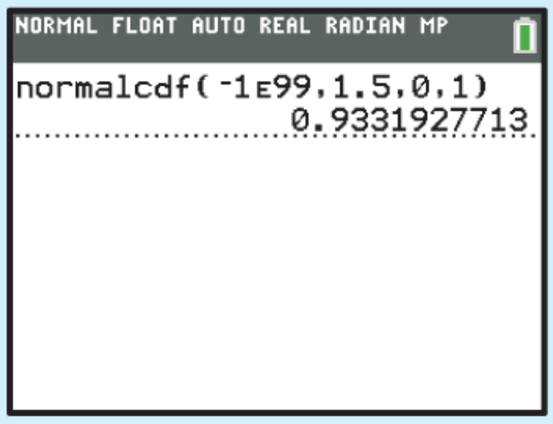




$\mathrm{P}(-0.41 \leqslant Z \leqslant 0.67) \approx 0.408$

b

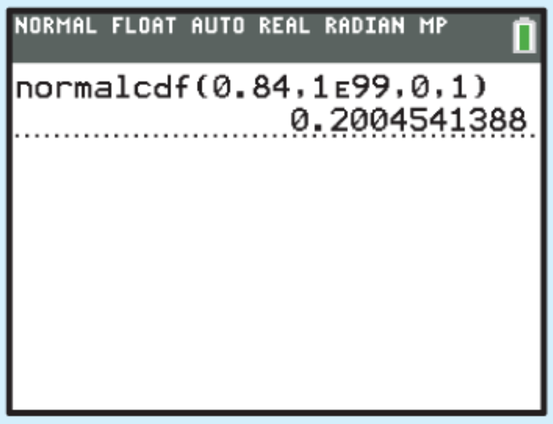




$\mathrm{P}(Z \leqslant 1.5) \approx 0.933$

c

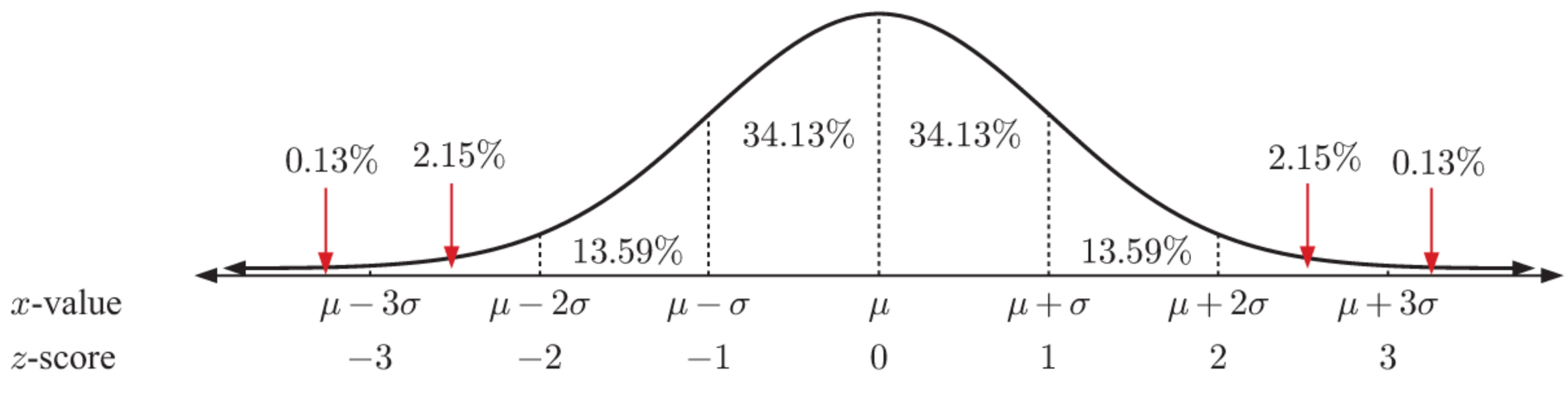




$\mathrm{P}(Z > 0.84) \approx 0.200$

EXERCISE 28C.2

1 Consider the normal distribution curve below.



Use the diagram to calculate the following probabilities.

a $P(-1 < Z < 1)$ **b** $P(-1 \leqslant Z \leqslant 3)$ **c** $P(-1 < Z < 0)$

d $P(Z < 2)$ **e** $P(-1 < Z)$ **f** $P(Z \geqslant 1)$

2 Given $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0, 1^2)$, determine the values of a and b such that:

a $P(\mu - \sigma < X < \mu + 2\sigma) = P(a < Z < b)$ **b** $P(\mu - 0.5\sigma < X < \mu) = P(a < Z < b)$

c $P(0 \leqslant Z \leqslant 3) = P(\mu - a\sigma \leqslant X \leqslant \mu + b\sigma)$

3 If $Z \sim N(0, 1^2)$, find the following probabilities using technology.

a $P(0.5 \leqslant Z \leqslant 1)$ **b** $P(-0.86 \leqslant Z \leqslant 0.32)$ **c** $P(-2.3 \leqslant Z \leqslant 1.5)$

d $P(Z \leqslant 1.2)$ **e** $P(Z \leqslant -0.53)$ **f** $P(Z \geqslant 1.3)$

g $P(Z \geqslant -1.4)$ **h** $P(Z > 4)$ **i** $P(-0.5 < Z < 0.5)$

j $P(-1.960 \leqslant Z \leqslant 1.960)$ **k** $P(-1.645 \leqslant Z \leqslant 1.645)$ **l** $P(|Z| > 1.645)$

4 **a** Suppose X is normally distributed with mean μ and standard deviation σ.

i Explain why $P(\mu - 3\sigma < X < \mu + 2\sigma) = P(-3 < Z < 2)$.

ii Hence find $P(\mu - 3\sigma < X < \mu + 2\sigma)$.

b For a random variable $X \sim N(\mu, \sigma^2)$, find:

i $P(\mu - 2\sigma < X < \mu + 1.5\sigma)$ **ii** $P(\mu - 2.5\sigma < X < \mu - 0.5\sigma)$

5 Suppose X is normally distributed with mean $\mu = 58.3$ and standard deviation $\sigma = 8.96$.

a Let the z-score of $x_1 = 50.6$ be z_1 and the z-score of $x_2 = 68.9$ be z_2.

i Calculate z_1 and z_2. **ii** Find $P(z_1 \leqslant Z \leqslant z_2)$.

b Check your answer by calculating $P(50.6 \leqslant X \leqslant 68.9)$ directly using technology.

HISTORICAL NOTE

The normal distribution has two parameters μ and σ, whereas the standard normal distribution has no parameters. This means that a unique table of probabilities can be constructed for the standard normal distribution.

Before graphics calculators and computer packages, it was impossible to calculate probabilities for a general normal distribution $N(\mu, \sigma^2)$ directly.

Instead, all data was transformed using the Z-transformation, and the standard normal distribution table was consulted for the required probabilities.

D QUANTILES

Consider a population of crabs where the length of a shell, X mm, is normally distributed with mean 70 mm and standard deviation 10 mm.



A biologist wants to protect the population by allowing only the largest 5% of crabs to be harvested. He therefore wants to know what length corresponds to the 95th percentile of crabs.

To answer this question we need to find k such that $\mathrm{P}(X \leqslant k) = 0.95$.

The number k is known as a **quantile**. In this case it is the 95% quantile.

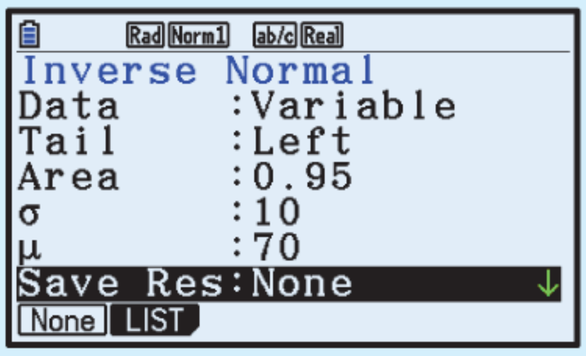
When finding quantiles, we are given a probability and are asked to calculate the corresponding measurement. This is the *inverse* of finding probabilities, so we use the **inverse normal function** on our calculator.

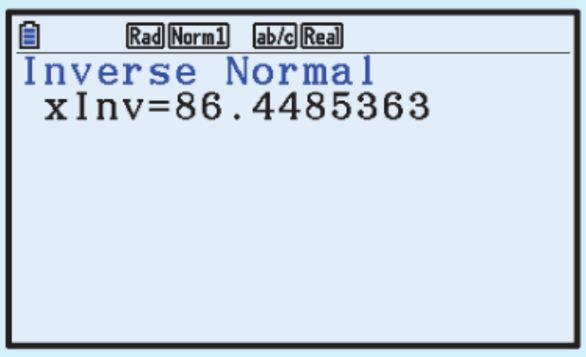


Example 7 — Self Tutor

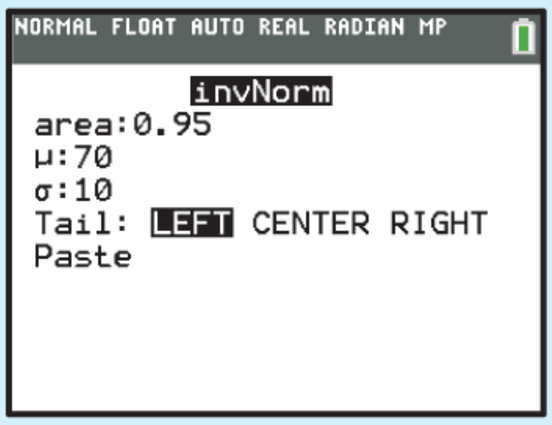
A population of crabs has shell length X mm. X is normally distributed with mean 70 and standard deviation 10. Find k for which $\mathrm{P}(X < k) = 0.95$.

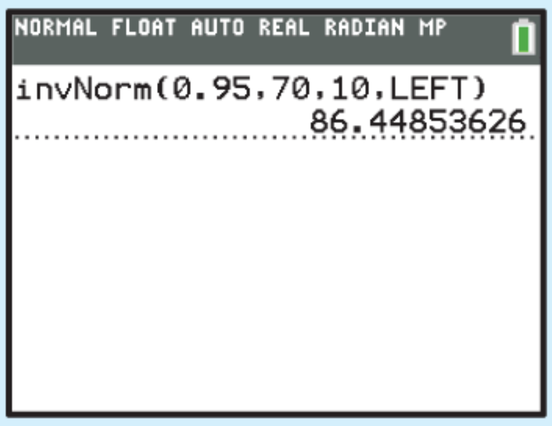
Casio fx-CG50



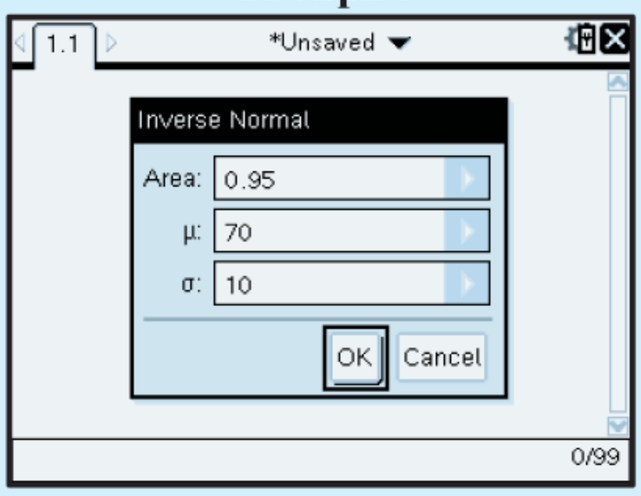


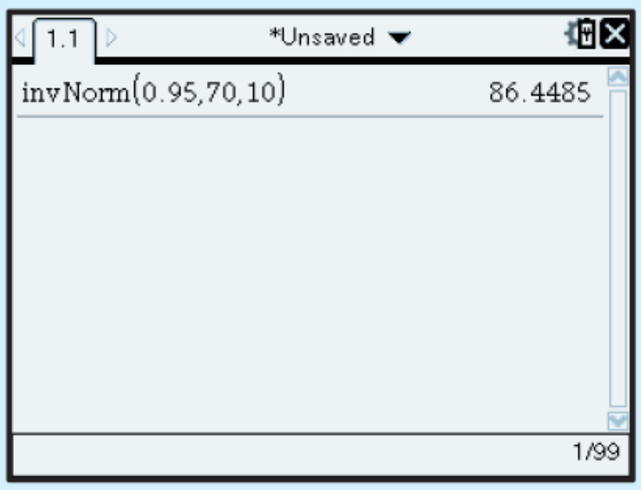
TI-84 Plus CE





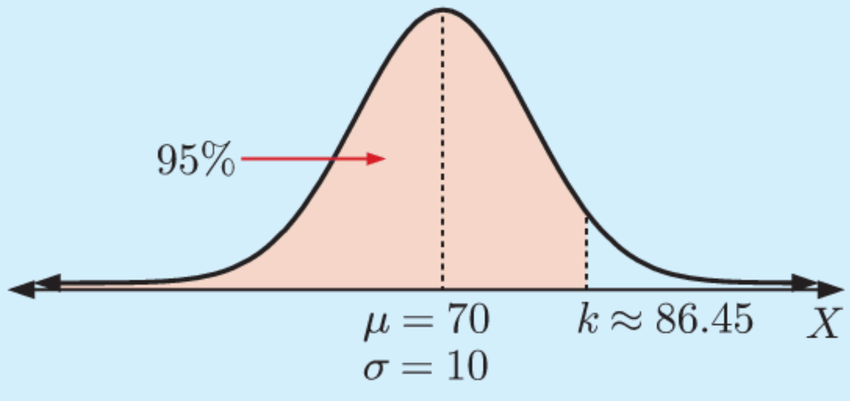
TI-*n*spire



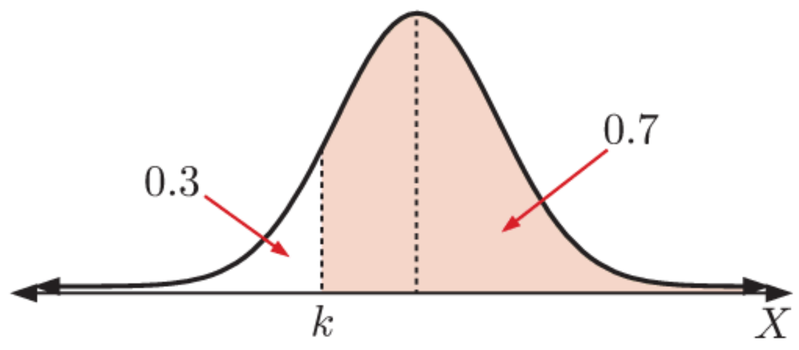


If $\mathrm{P}(X < k) = 0.95$ then
$k \approx 86.45$

The 95% quantile corresponds to a shell width of 86.45 mm.



When using the **HP Prime**, **TI-84 Plus CE**, or **TI-*n*spire** calculators, we must always use the area to the *left* of k. Therefore, to find k such that $P(X > k) = 0.7$, we instead find k such that $P(X < k) = 1 - 0.7 = 0.3$.



EXERCISE 28D

1 Suppose X is normally distributed with mean 20 and standard deviation 3. Illustrate with a sketch and find k such that:

a $P(X \leqslant k) = 0.3$ **b** $P(X \leqslant k) = 0.9$ **c** $P(X \leqslant k) = 0.5$

d $P(X > k) = 0.2$ **e** $P(X < k) = 0.62$ **f** $P(X \geqslant k) = 0.13$

2 Given that $Z \sim N(0, 1^2)$, find k such that:

a $P(Z \leqslant k) = 0.81$ **b** $P(Z \leqslant k) = 0.58$ **c** $P(Z \leqslant k) = 0.17$

d $P(Z \geqslant k) = 0.95$ **e** $P(Z \geqslant k) = 0.9$ **f** $P(Z \geqslant k) = 0.41$

3 Suppose X is normally distributed with mean 30 and standard deviation 5, and $P(X \leqslant a) = 0.57$.

a Using a diagram, determine whether a is greater or less than 30.

b Use technology to find a.

c Without using technology, find:

i $P(X \geqslant a)$ **ii** $P(30 \leqslant X \leqslant a)$

4 Given that X is normally distributed with mean 15 and standard deviation 3, find k such that:

a $P(X < k) = 0.2$ **b** $P(X > k) = 0.1$

c $P(15 - k < X < 15 + k) = 0.9$

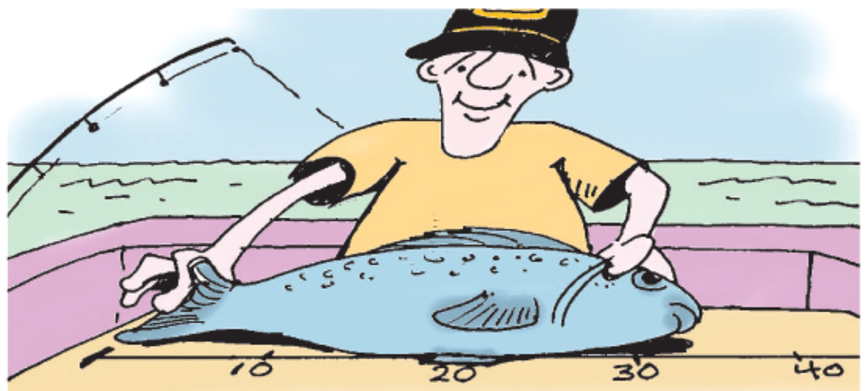
5 Suppose X is normally distributed with mean 80 and standard deviation 10.

a Find $P(X \leqslant 72)$. **b** Hence find k such that $P(72 \leqslant X \leqslant k) = 0.1$.

6 Given that $X \sim N(45, 8^2)$, find a such that:

a $P(35 \leqslant X \leqslant a) = 0.25$ **b** $P(a \leqslant X \leqslant 50) = 0.15$ **c** $P(a \leqslant X \leqslant 54) = 0.6$

7 The lengths of a fish species are normally distributed with mean 35 cm and standard deviation 8 cm. The fisheries department has decided that the smallest 10% of the fish are not to be harvested. What is the size of the smallest fish that can be harvested?



8 The lengths of screws produced by a machine are normally distributed with mean 75 mm and standard deviation 0.1 mm. 1% of the screws are rejected because they are too long. What is the length of the smallest screw to be rejected?

9 The marks X for a Mathematics examination are normally distributed with mean 57 and standard deviation 10. 60% of students passed the exam and 10% received an A grade.

a Find:

i the passing mark k **ii** the mark l required for an A.

b Check that $P(k \leqslant X \leqslant l) = 0.5$.

10 The volumes of cool drink in bottles filled by a machine are normally distributed with mean 503 mL and standard deviation 0.5 mL. 1% of the bottles are rejected because they are underfilled, and 2% are rejected because they are overfilled. They are otherwise kept for sale. Find, correct to 1 decimal place, the range of volumes in the bottles that are kept.



11 Abbey goes for a morning walk as long as the temperature is not too cold and not too hot. The morning temperatures are normally distributed with mean 20°C and standard deviation 5°C. Given that the lower limit of Abbey's walking temperatures is 11°C, and that she goes for a walk 95% of the time, find the upper limit of Abbey's walking temperatures.

GAME

Click on the icon to play a card game for the normal distribution.



INVESTIGATION 4 THE NORMAL APPROXIMATION TO THE BINOMIAL DISTRIBUTION

In the previous Chapter, we saw how the **binomial distribution** arises from considering the number of "successes" in a fixed number of independent trials of an experiment.

Suppose $X \sim \text{B}(n, p)$ is the number of successes in n independent trials, each with probability of success p. The probability of getting x successes is:

$$\mathbf{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{where } x = 0, 1, 2, ..., n$$

You will have used this formula to calculate probabilities for binomial random variables in cases where the number of trials n is relatively small. As n increases, the probability becomes more difficult to calculate. This is because the binomial coefficient $\binom{n}{x}$ becomes very large.

What to do:

1 Click on the icon to access a demonstration which draws the probability distribution of $X \sim \text{B}(n, p)$.



a Set $p = 0.5$ and use the sliders to change the value of n. Describe what happens to the distribution of X as n increases.

b Repeat **a** for p equal to:

i 0.25 **ii** 0.1 **iii** 0.75 **iv** 0.9

Comment on your observations.

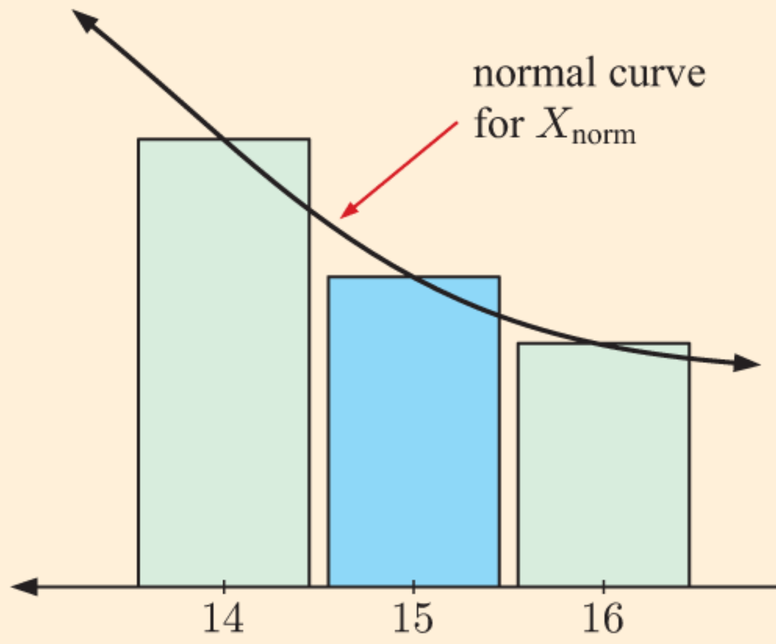
c Do you think that it would be *reasonable* to approximate the binomial distribution with a normal distribution? Explain your answer.

d What should be the mean and standard deviation of this normal distribution?

2 Consider $X \sim \text{B}(50,\ 0.2)$ and its normal approximation $X_{\text{norm}} \sim \text{N}(\mu,\ \sigma^2)$.

a Write down expressions for μ and σ.

b Suppose we want to calculate the probability of 15 successes. The diagram alongside shows part of the probability distribution of X with the normal distribution curve of X_{norm} drawn over the top. Explain why $\text{P}(X = 15) \approx \text{P}(14.5 \leqslant X_{\text{norm}} \leqslant 15.5)$.



c Describe how you would estimate the following using the normal approximation:

i $\text{P}(X \leqslant 10)$ **ii** $\text{P}(X < 25)$

iii $\text{P}(10 \leqslant X < 25)$

3 It is known that 2% of tyres manufactured by a company are unfit for sale. A quality inspector randomly sampled 500 tyres. Use a normal approximation to estimate the probability that at least 10 tyres in the sample will be unfit for sale.

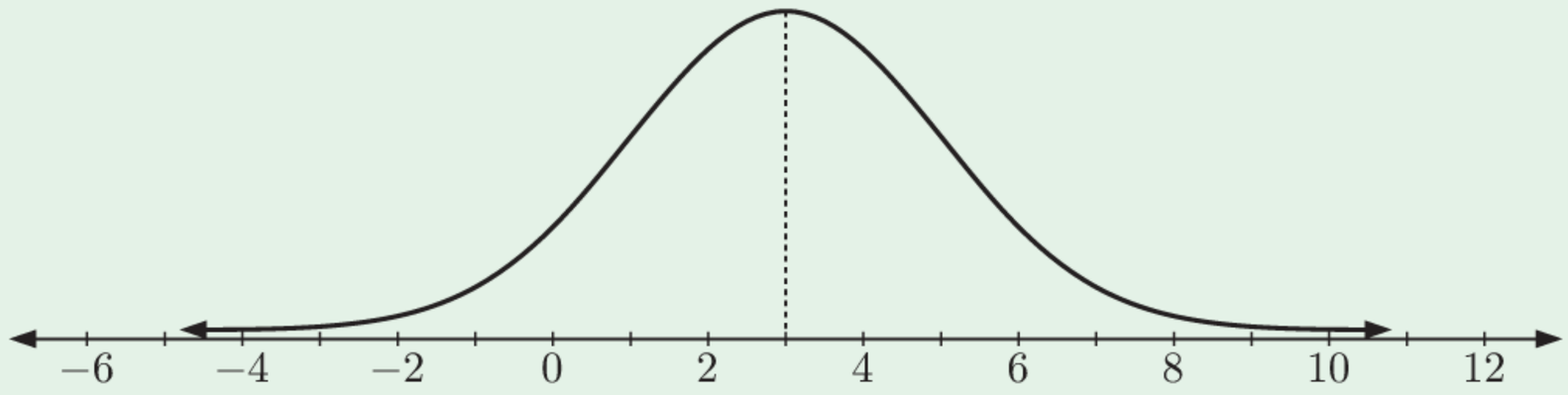
REVIEW SET 28A

1 Discuss whether the following variables will be normally distributed:

a the time students take to read a novel

b the amount spent on groceries at a supermarket.

2 The normal curve of the $\text{N}(3,\ 2^2)$ distribution is shown below.



PRINTABLE GRAPH

On the same set of axes, sketch the normal curves for:

a $\text{N}(5,\ 2^2)$ **b** $\text{N}(1,\ 4^2)$ **c** $\text{N}(0,\ 1^2)$

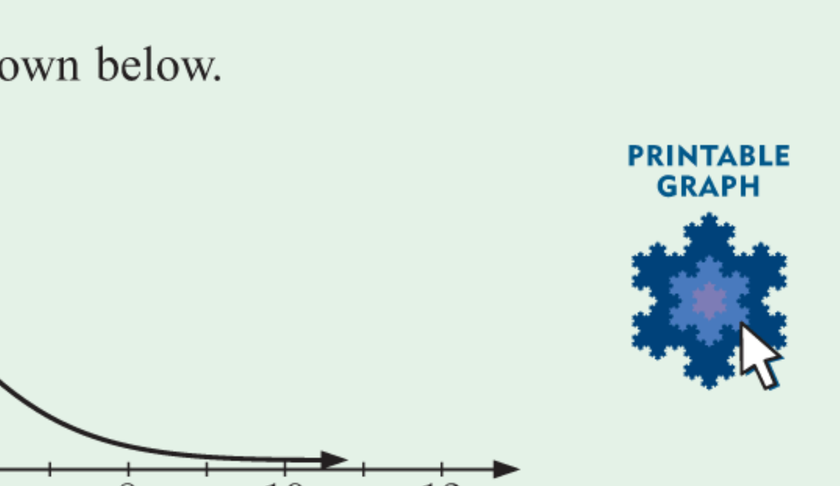
3 The amount of juice Simon can squeeze from his lemons is normally distributed with mean 35 mL and standard deviation 5 mL.

a Copy and complete this curve.

b What percentage of the lemons will produce:

i between 25 mL and 35 mL of juice

ii at least 45 mL of juice?



4 The average height of 17 year old boys is normally distributed with mean 179 cm and standard deviation 8 cm. Calculate the percentage of 17 year old boys whose heights are:

a more than 195 cm **b** between 171 cm and 187 cm

c between 163 cm and 195 cm.

5 The weight of the edible part of a batch of Coffin Bay oysters is normally distributed with mean 38.6 grams and standard deviation 6.3 grams.

a Find the percentage of oysters that weigh between 30 g and 40 g.

b From a sample of 200 oysters, how many would you expect to weigh more than 50 g?



6 The results of a test are normally distributed. The z-score of Harri's test score is -2.

a Interpret this z-score with regard to the mean and standard deviation of the test scores.

b What percentage of students obtained a better score than Harri?

c The mean test score was 61 and Harri's actual score was 47. Find the standard deviation of the test scores.

7 A random variable X is normally distributed with mean 20.5 and standard deviation 4.3. Find:

a $P(X \geqslant 22)$ **b** $P(18 \leqslant X \leqslant 22)$ **c** k such that $P(X \leqslant k) = 0.3$.

8 Let X be the weight in grams of bags of sugar filled by a machine. Bags less than 500 grams are considered underweight.

Suppose $X \sim N(503, 2^2)$.

a What percentage of bags are underweight?

b If a quality inspector randomly selects 20 bags, what is the probability that 2 or fewer bags are underweight?

9 The daily energy intake of Canadian adults is normally distributed with mean 8700 kJ and standard deviation 1000 kJ.

Find the proportion of Canadian adults whose daily energy intake is:

a greater than 8000 kJ **b** less than 7500 kJ

c between 9000 and 10 000 kJ.

10 Suppose $X \sim N(30, 8^2)$. Find k such that:

a $P(X \leqslant k) = 0.1$ **b** $P(X \geqslant k) = 0.6$

11 The life of a Xenon-brand battery is normally distributed with mean 33.2 weeks and standard deviation 2.8 weeks.

a Find the probability that a randomly selected battery will last at least 35 weeks.

b For how many weeks can the manufacturer expect the batteries to last before 8% of them fail?

12 Machines A and B both produce nails whose lengths are normally distributed. The lengths of nails from machine A have mean 50.2 mm and standard deviation 1.1 mm. The lengths of nails from machine B have mean 50.6 mm and standard deviation 0.8 mm. Nails which are longer than 52 mm or shorter than 48 mm are rejected.

a Find the probability of randomly selecting a nail that has to be rejected from:

i machine A **ii** machine B.

b A quality inspector randomly selects a nail from a randomly chosen machine. Find the probability that the nail was made by machine A *given* that it should be rejected.

13 Suppose X is normally distributed with mean 25 and standard deviation 6. Find k such that:

a $P(X \leqslant k) = 0.7$ **b** $P(X \geqslant k) = 0.4$ **c** $P(20 \leqslant X \leqslant k) = 0.3$

REVIEW SET 28B

1 Sketch the following normal distributions on the same set of axes.

Distribution	*Mean* (cm)	*Standard deviation* (cm)
A	22	2
B	20	4
C	25	7

2 The variable X is normally distributed with graph shown.

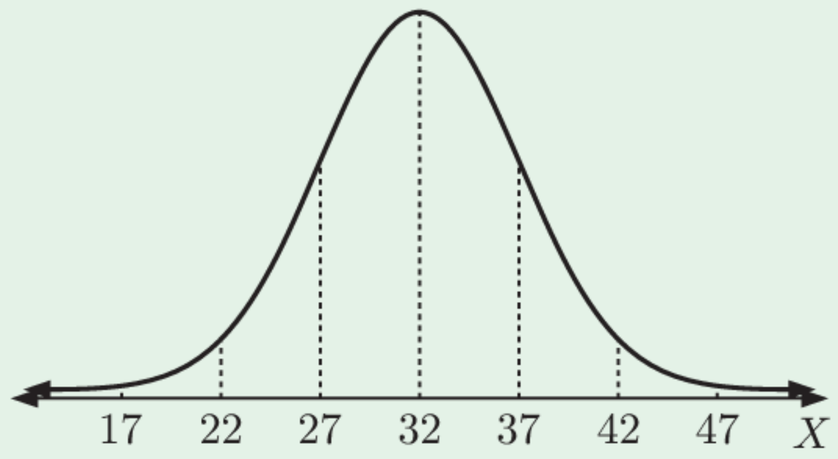
a State the mean and standard deviation of X.

b What percentage of values of X are:

i between 27 and 32

ii less than 37

iii greater than 42?



3 The random variable Z has the standard normal distribution $N(0, 1^2)$. Find the value of k for which $P(-k \leqslant Z \leqslant k) = 0.95$.

4 The contents of soft drink cans are normally distributed with mean 327 mL and standard deviation 4.2 mL.

a Find the percentage of cans with contents:

i less than 318.6 mL **ii** between 322.8 mL and 339.6 mL.

b Find the probability that a randomly selected can contains between 327 mL and 331.2 mL.

5 The arm lengths of 18 year old females are normally distributed with mean 64 cm and standard deviation 4 cm.

a Find the percentage of 18 year old females whose arm lengths are:

i between 61 cm and 73 cm **ii** greater than 57 cm.

b Find the probability that a randomly chosen 18 year old female has an arm length in the range 50 cm to 65 cm.

c The arm lengths of 70% of the 18 year old females are more than x cm. Find the value of x.

6 Suppose X is normally distributed with mean 50 and standard deviation 7. Find:

a $P(46 \leqslant X \leqslant 55)$ **b** $P(X \geqslant 60)$ **c** k such that $P(X > k) = 0.23$.

7 In a competition to see who could hold their breath underwater the longest, the times in the preliminary round were normally distributed with mean 150 seconds and standard deviation 12 seconds. If the top 15% of contestants went through to the finals, what time was required to advance?

8 For $X \sim N(12, 2^2)$, find a such that:

a $P(X < a) = 0.07$ **b** $P(X > a) = 0.2$ **c** $P(a \leqslant X \leqslant 11) = 0.1$

9 The weights of suitcases at an airport are normally distributed with mean 17 kg and standard deviation 3.4 kg.

a Find the probability that a randomly selected suitcase weighs between 10 kg and 15 kg.

b 300 suitcases are presented for check-in over a one hour period. How many of these suitcases would you expect to be lighter than 20 kg?

c 3.9% of the suitcases are rejected because they exceed the maximum weight limit. Find the maximum weight limit.

10 Kerry's marks for an English essay and a Chemistry test were 26 out of 40, and 82% respectively.

a Explain briefly why the information given is not sufficient to determine whether Kerry's results are better in English than in Chemistry.

b Suppose that the marks of all students in the English essay and the Chemistry test were normally distributed as $N(22, 4^2)$ and $N(75, 7^2)$ respectively. Use this information to determine which of Kerry's two marks is better.

11 On an ostrich farm the weights of the birds are found to be normally distributed. The weights of the females have mean 78.6 kg and standard deviation 5.03 kg. The weights of the males have mean 91.3 kg and standard deviation 6.29 kg.



a Find the probability that a randomly selected:

i male will weigh less than 80 kg

ii female will weigh less than 80 kg

iii female will weigh between 70 and 80 kg.

b 20% of females weigh less than k kg. Find k.

c The middle 90% of the males weigh between a kg and b kg. Find the values of a and b.

d In one field there are 82% females and 18% males. One of these ostriches is selected at random.
Calculate the probability that the ostrich weighs less than 80 kg.

12 The weight of an apple in an apple harvest is normally distributed with mean 300 grams and standard deviation 50 grams. Only apples with weights between 250 grams and 350 grams are considered fit for sale.

a Find the percentage of apples fit for sale.

b In a sample of 100 apples, what is the probability that at least 75 are fit for sale?

13 Giovanni and Beppe are both carrot farmers. The lengths of Giovanni's carrots are normally distributed with mean 22 cm and standard deviation 3.4 cm. The lengths of Beppe's carrots are also normally distributed, with mean 23.5 cm and standard deviation 4.2 cm.

a Find the probability that a carrot is longer than 20 cm, given it comes from:

i Giovanni's farm **ii** Beppe's farm.

b A buyer randomly selects a carrot from each farmer's crop. Calculate the probability that *neither* carrot is longer than 20 cm.

Chapter 29

Estimation and confidence intervals

Contents:

A Linear combinations of random variables
B The sum of two independent Poisson random variables
C Linear combinations of normal random variables
D The Central Limit Theorem
E Confidence intervals for a population mean with known variance
F Confidence intervals for a population mean with unknown variance



OPENING PROBLEM

Reg is a pumpkin grower. Historically, the pumpkins Reg has produced have had an average weight of 5 kg, with standard deviation 0.5 kg.

Recently, Reg has trialled a new farming technique which he claims increases the weight of his pumpkins. To support his claim, he takes a sample of the new pumpkins, and he finds their average weight is 5.2 kg.

Things to think about:

a Does Reg's sample prove that the new mean weight of his pumpkins is *exactly* 5.2 kg?

b In what *range* of values is it *likely* that the new mean weight of pumpkins lies?

c Does it matter whether Reg chose to test a small sample or a large sample of pumpkins?

d How can we establish the likelihood of whether Reg's claim is correct?

In the last two Chapters we have studied discrete and continuous random variables. We have considered properties of these random variables including:

- probability distributions
- expectation (or mean)
- variance and standard deviation.

So far, we have only considered one random variable at a time. However, in the real world it is much more realistic to consider multiple random variables simultaneously.

For example, when we take a sample of measurements from a population, we need to treat each measurement as a random variable.

In statistics, we formalise this idea by defining a **random sample**:

Consider a population with distribution X.

A **random sample** of size n $\{X_1, X_2, ..., X_n\}$ is a set of independent observations from the **same** population.

- Each X_i is a random variable that has the same distribution as the population.
- $X_1, X_2, ..., X_n$ are **independent random variables**.

For example, suppose the population distribution $X \sim \text{N}(4, 5.1^2)$.

If $\{X_1, X_2, X_3\}$ is a random sample of size 3 drawn from this population, then X_1, X_2, and X_3 are all normally distributed with mean 4 and standard deviation 5.1 .

Random variables are **independent** if the outcome of one does *not* affect the other.



In this Chapter, we will use the properties of random samples to explore the statistical theory behind **estimation**.

A LINEAR COMBINATIONS OF RANDOM VARIABLES

A **linear combination** of the random variables $X_1, X_2, ..., X_n$ has the form

$$a_1X_1 + a_2X_2 + ... + a_nX_n$$

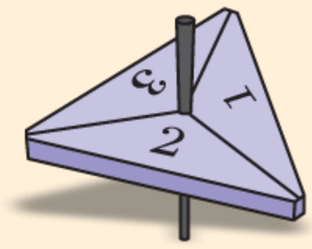
where $a_1, a_2, ..., a_n$ are constants.

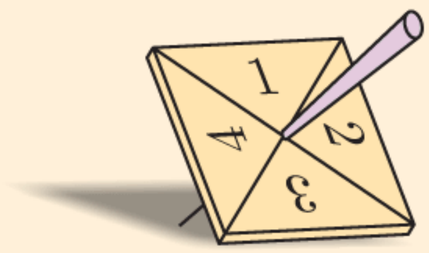
For example $X_1 + 2X_2 - X_3$ is a linear combination of the random variables X_1, X_2, and X_3 where $a_1 = 1$, $a_2 = 2$, and $a_3 = -1$.

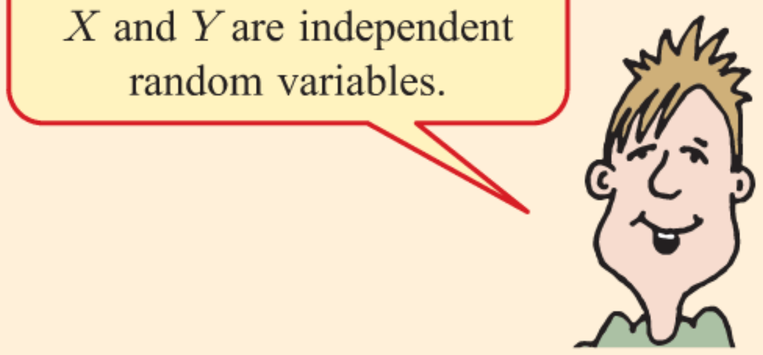
A linear combination of random variables is itself a random variable for which we can find a probability distribution, and calculate an expectation and variance.

INVESTIGATION 1 THE EXPECTATION AND VARIANCE OF $X + Y$

Suppose X is the number spun with the triangular spinner and Y is the number spun with the square spinner.







In this Investigation we will consider the distribution of the random variable $W = X + Y$.

What to do:

1 **a** Write down the probability distributions of X and Y.

b Copy and complete:

Variable	*Expectation*	*Variance*
X		
Y		

2 Use a table to find the possible values of $W = X + Y$.

3 Calculate the probability of each possible value of W.

4 Hence find: **a** $\mathrm{E}(W)$ **b** $\mathrm{Var}(W)$.

5 Calculate: **a** $\mathrm{E}(X) + \mathrm{E}(Y)$ **b** $\mathrm{Var}(X) + \mathrm{Var}(Y)$.
What do you notice?

6 Repeat **1** to **5** for discrete random variables X and Y of your choosing. Comment on your results.

From the **Investigation**, you should have seen that:

For two random variables X and Y:

1 $\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$

2 If X and Y are independent, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

The results above can be extended to any linear combination of random variables.

For n random variables $X_1, X_2, ..., X_n$, and constants $a_1, a_2, ..., a_n$:

1 $\mathrm{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i \mathrm{E}(X_i)$

2 If $X_1, X_2, ..., X_n$ are all independent of one another, then $\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathrm{Var}(X_i)$.

Proof:

1 $$\mathrm{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} \mathrm{E}(a_i X_i) \qquad \{\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)\}$$
$$= \sum_{i=1}^{n} a_i \,\mathrm{E}(X_i) \qquad \{\mathrm{E}(kX) = k\,\mathrm{E}(X)\}$$

2 $$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(a_i X_i) \qquad \{\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)\}$$
$$= \sum_{i=1}^{n} {a_i}^2 \,\mathrm{Var}(X_i) \qquad \{\mathrm{Var}(kX) = k^2\,\mathrm{Var}(X)\}$$

EXERCISE 29A

1 Suppose X and Y are two independent discrete random variables with $\mathrm{E}(X) = 3$, $\mathrm{E}(Y) = 2$, $\mathrm{Var}(X) = \frac{3}{2}$, and $\mathrm{Var}(Y) = \frac{5}{4}$. Calculate:

a $\mathrm{E}(X+Y)$ **b** $\mathrm{E}(2X-Y)$ **c** $\mathrm{Var}(X+Y)$ **d** $\mathrm{Var}(2X-3Y)$

2 Consider two independent random variables X and Y with means and standard deviations shown.

	μ	σ
X	3.8	0.323
Y	5.7	1.02

Find the mean and standard deviation of:

a $X+2Y$ **b** $Y-X$ **c** $3X-2Y$.

3 Let X be the number of heads obtained when a coin is tossed twice.
Let Y be the result when an ordinary 6-sided die is rolled once.

a Write down the probability distribution of:

i X **ii** Y.

b Find the mean and variance of:

i X **ii** Y **iii** $X+Y$ **iv** $4X-2Y$.

4 In a gambling game, the player bets on the outcomes of two spinners. These outcomes are X and Y with the following probability distributions:

x	0	1	3	6
$\mathrm{P}(X=x)$	0.25	0.25	0.25	0.25

y	0	2	8
$\mathrm{P}(Y=y)$	0.5	0.3	0.2

The game costs \$5 to play. The player wins the number of dollars equal to the sum of the numbers spun.

a Explain why X and Y have well defined probability distributions.

b Find the mean and standard deviation of: **i** X **ii** Y.

c Calculate the expected gain from playing this game.

d Would you advise people to play this game many times? Explain your answer.

5 The **Bernoulli distribution** is a special case of the binomial distribution where there is only a single trial of the experiment. So, if X is a **Bernoulli random variable** then $X \sim \mathrm{B}(1,\, p)$.

a Write $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ in terms of p.

b The binomial random variable $Y \sim \mathrm{B}(n,\, p)$ is the number of successes that occur in n independent trials of the experiment.

i Explain why Y can be considered as the *sum* of n independent Bernoulli random variables, each with probability of success p.

ii Use the statistical properties of linear combinations to show that $\mathrm{E}(Y) = np$ and $\mathrm{Var}(Y) = np(1-p)$.

B THE SUM OF TWO INDEPENDENT POISSON RANDOM VARIABLES

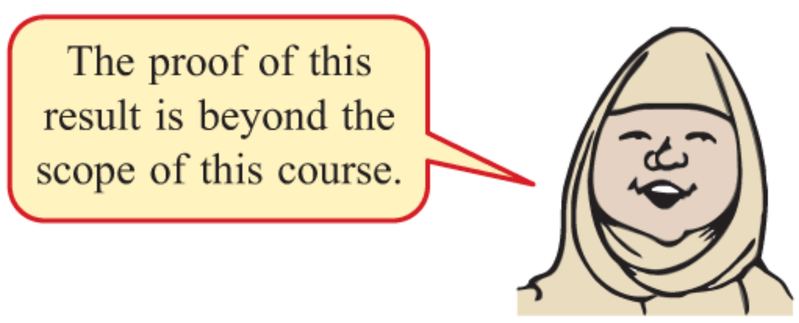
Suppose we have two *independent* Poisson random variables $X \sim \text{Po}(\lambda_1)$ and $Y \sim \text{Po}(\lambda_2)$.

Using the results from the previous Section:

- $\text{E}(X+Y) = \text{E}(X) + \text{E}(Y)$
  $= \lambda_1 + \lambda_2$
- $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$ $\{X \text{ and } Y \text{ are independent}\}$
  $= \lambda_1 + \lambda_2$

We therefore see that $\text{E}(X+Y) = \text{Var}(X+Y)$, which is one of the properties of a Poisson random variable.

If $X \sim \text{Po}(\lambda_1)$ and $Y \sim \text{Po}(\lambda_2)$ are independent Poisson random variables, then $\boldsymbol{(X+Y) \sim \text{Po}(\lambda_1 + \lambda_2)}$.

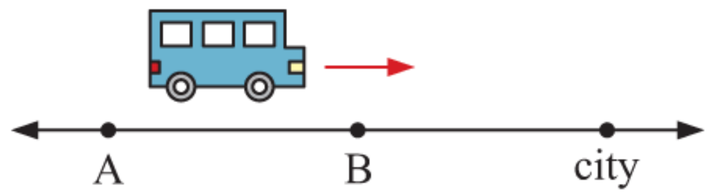


EXERCISE 29B

1 A bus stops at 2 major interchanges, A and B, before arriving in the city centre.

Let $X \sim \text{Po}(10)$ be the number of people who get on the bus at interchange A.

Let $Y \sim \text{Po}(17)$ be the number of people who get on the bus at interchange B.



a Write down the distribution of T, the total number of people who get on the bus at an interchange.

b Find the probability that at least 30 people get on the bus at an interchange.

2 Suppose $X \sim \text{Po}(3)$ and $(X+Y) \sim \text{Po}(7)$. Given that X and Y are independent:

a State the distribution of Y.

b Find $\text{P}(Y \geqslant 2)$.

3 Suppose $X \sim \text{Po}(2)$ and $Y \sim \text{Po}(3)$ are independent Poisson random variables.

a Find $\text{P}(X+Y=4)$.

b Write an *expression* for $\text{P}(Y \leqslant 1 \cap (X+Y)=4)$ in terms of probabilities of X and Y only.

c Hence calculate $\text{P}(Y \leqslant 1 \cap (X+Y)=4)$.

d Find $\text{P}(Y \leqslant 1 \mid (X+Y)=4)$.

4 Josie and Callie are collecting signatures for a petition.

The number of signatures that Josie collects follows a Poisson distribution with rate 10 signatures per hour.

The number of signatures that Callie collects follows a Poisson distribution with rate 8 signatures per 40 minutes.

If they begin collecting signatures at 8 am, find the probability that they will collect at least 250 signatures by 6 pm.

5 X and Y are independent Poisson random variables, each with rate λ.
Given that $\mathrm{P}(X = 1) = 2\,\mathrm{P}(X + Y = 1)$, find the value of λ.

6 **a** Suppose $X_1 \sim \mathrm{Po}(\lambda_1)$, $X_2 \sim \mathrm{Po}(\lambda_2)$, and $X_3 \sim \mathrm{Po}(\lambda_3)$ are independent Poisson random variables. State the distribution of:

i $X_1 + X_2$ **ii** $X_1 + X_2 + X_3$.

b Suppose $\{X_k \sim \mathrm{Po}(\lambda_k)\}$, $k = 1, 2, 3, ..., n$ is a set of n independent Poisson random variables.

Predict the distribution of $Y = \sum_{k=1}^{n} X_k$. Explain your answer.

C LINEAR COMBINATIONS OF NORMAL RANDOM VARIABLES

We have already seen that the sum of two independent Poisson random variables is itself a Poisson variable. The same is true of independent normal random variables. In fact, it can be shown that:

> Any linear combination of independent normally distributed random variables is itself a normally distributed random variable.

For example, suppose $X_1 \sim \mathrm{N}(1, 5^2)$, $X_2 \sim \mathrm{N}(3, 1^2)$, and $X_3 \sim \mathrm{N}(2, 4^2)$ are independent normal random variables. The linear combination $Y = 2X_1 + 3X_2 - 4X_3$ is also a normal random variable, with:

$$\begin{aligned}\mathrm{E}(Y) &= \mathrm{E}(2X_1 + 3X_2 - 4X_3) \\ &= 2\,\mathrm{E}(X_1) + 3\,\mathrm{E}(X_2) - 4\,\mathrm{E}(X_3) \\ &= 2(1) + 3(3) - 4(2) \\ &= 3\end{aligned}$$

and

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(2X_1 + 3X_2 - 4X_3) \\ &= 4\,\mathrm{Var}(X_1) + 9\,\mathrm{Var}(X_2) + 16\,\mathrm{Var}(X_3) \\ &= 4(5^2) + 9(1^2) + 16(4^2) \\ &= 365\end{aligned}$$

So, $Y \sim \mathrm{N}(3, 365)$.

Example 1 **Self Tutor**

The weights of male employees in a bank are normally distributed with mean $\mu = 81.5$ kg and standard deviation $\sigma = 7.3$ kg. The bank has an elevator which will carry a maximum load of 494 kg.

a Six male employees enter the elevator. Calculate the probability that their combined weight exceeds the maximum load.

b If there is to be at most a 0.1% chance of the total weight exceeding 494 kg, recommend the maximum number of males who should use the lift together.

a Let the weights of the employees be the independent random variables $X_1, X_2, ..., X_6$.

Let the sum of their weights be $Y_6 = \sum_{k=1}^{6} X_k$,

where $X_k \sim \mathrm{N}(81.5, (7.3)^2)$, $k = 1, 2, ..., 6$.

Now $\mathrm{E}(Y_6) = \sum_{k=1}^{6} \mathrm{E}(X_k)$ and $\mathrm{Var}(Y_6) = \sum_{k=1}^{6} \mathrm{Var}(X_k)$

$= 6 \times 81.5$ $= 6 \times (7.3)^2$

$= 489$ kg $= 319.74$ kg^2

$\therefore \quad Y_6 \sim \mathrm{N}(489,\ 319.74)$

Now $\mathrm{P}(Y_6 > 494) \approx 0.390$

So, there is a 39.0% chance that their combined weight will exceed 494 kg.

b Six men is too many, as there is a 39.0% chance of overload.

Instead we try 5 male employees with total weight $Y_5 = \sum_{k=1}^{5} X_5$.

Now $\mathrm{E}(Y_5) = 5 \times 81.5$ and $\mathrm{Var}(Y_5) = 5 \times (7.3)^2$

$= 407.5$ kg ≈ 266.45 kg^2

$\therefore \quad Y_5 \sim \mathrm{N}(407.5,\ 266.45)$

$\therefore \quad \mathrm{P}(Y_5 > 494) \approx 5.82 \times 10^{-8}$

For $n = 5$, there is much less than a 0.1% chance of the total weight exceeding 494 kg.

We recommend that at most 5 men use the elevator together.

EXERCISE 29C

1 The weights of pears are normally distributed with mean $\mu = 118$ g and standard deviation $\sigma = 4.6$ g.

5 pears are chosen at random. Find the probability that their combined weight is:

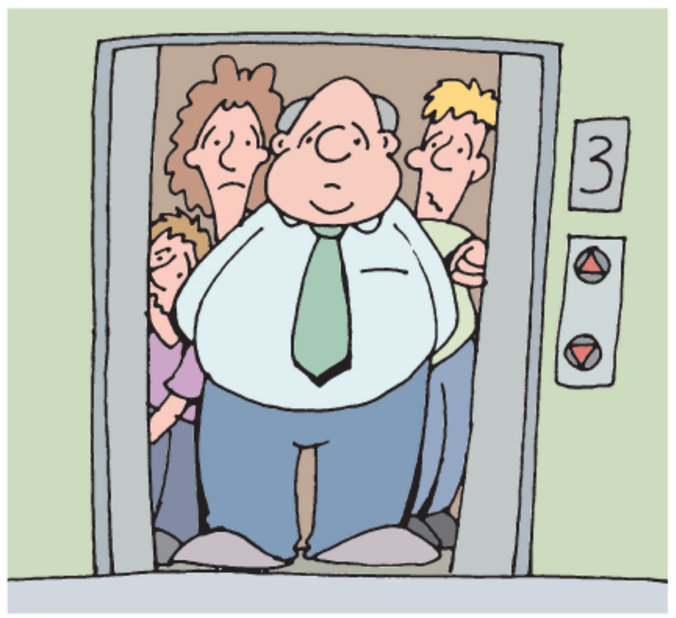
a less than 570 g **b** more than 605 g **c** between 580 g and 600 g.

2 The maximum load of an elevator is 440 kg. The weights of adults are normally distributed with mean 81 kg and standard deviation 11 kg. The weights of children are normally distributed with mean 48 kg and standard deviation 4 kg.

Suppose the elevator contains 4 adults and 3 children.

a Find the probability that the maximum load will be exceeded.

b What assumption have you made in your calculation?



3 A coffee machine dispenses a cappuccino containing black coffee distributed normally with mean 120 mL and standard deviation 7 mL, and froth distributed normally with mean 28 mL and standard deviation 4.5 mL.

The café manager has decided that if the total volume is less than 135.5 mL, the customer will receive their cappuccino free of charge. However, the manager does not wish to give away more than 1% of cappuccinos made.

Determine whether or not the manager needs to adjust the settings on her machine.

4 X and Y are independent normal random variables with $X \sim \mathrm{N}(-10,\ 1)$ and $Y \sim \mathrm{N}(25,\ 25)$.

a Find the mean and standard deviation of the random variable $U = 3X + 2Y$.

b Find $\mathrm{P}(U < 0)$.

5 A drinks manufacturer produces bottles of drink in two sizes, small (S) and large (L). The distributions for the contents are independent, and normally distributed with $S \sim \text{N}(280 \text{ mL}, 4 \text{ mL}^2)$ and $L \sim \text{N}(575 \text{ mL}, 16 \text{ mL}^2)$.

a A bottle of each size is selected at random. Find the probability that the large bottle contains less than two times the amount in the small bottle.

b One large and two small bottles are selected at random. Find the probability that the large bottle contains less than the total amount in the two small bottles.

6 Chocolate bars are produced independently in two sizes, small and large. The weights of the bars are distributed normally and independently as $S \sim \text{N}(32, 5)$ and $L \sim \text{N}(92, 15)$ respectively.

a One of each size bar is selected at random. Find the probability that the large bar weighs more than three times the small bar.

b One large and three small bars are selected at random. Find the probability that the large bar weighs more than the total of the three small bars.

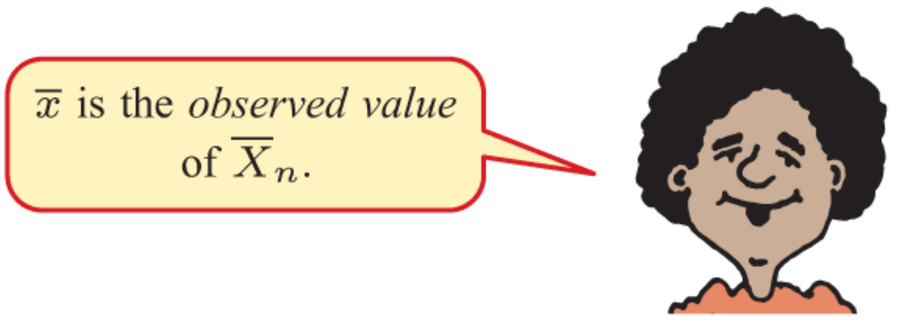
D THE CENTRAL LIMIT THEOREM

If we want to make inferences about a population mean μ, the most obvious thing to do is to use the **sample mean** $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ to estimate μ.

However, it is unlikely that two samples from the same population will be exactly the same, so the value of $\overline{x}$ will *vary* depending on the sample chosen. The sample mean is therefore a **random variable**.

For the random sample $\{X_1, ..., X_n\}$, the **sample mean** is

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$



INVESTIGATION 2 THE DISTRIBUTION OF SAMPLE MEANS

In this Investigation we will explore the distribution of $\overline{X}_n$ for different types of populations. To help us we will use a computer simulation which generates the distribution of sample means of size n for a given population. The population distribution may be "normal" (bell-shaped) or be far from normal.



What to do:

1 Consider a population with mean μ and standard deviation σ.

a Use the properties of linear combinations of random variables to show that:

i $\text{E}(\overline{X}_n) = \mu$ **ii** $\text{Var}(\overline{X}_n) = \frac{\sigma^2}{n}$ **iii** $\sigma(\overline{X}_n) = \frac{\sigma}{\sqrt{n}}$

b Suppose the population is normally distributed. Explain why $\overline{X}_n$ is itself normally distributed and hence state its distribution.

2 In the simulation, set the population distribution to *normal* with $\mu = 0$ and $\sigma = 1$.

a Use the slider to change the sample size n. Comment on how the *shape* of the distribution of $\overline{X}_n$ changes as n increases.

b Use the information underneath the histogram in the software to copy and complete this table:

Normal distribution, $\mu = 0$, $\sigma = 1$

n	*Mean of* $\overline{X}_n$	*Standard deviation of* $\overline{X}_n$	$\frac{\sigma}{\sqrt{n}}$
4			
9			
16			
25			
100			

c Explain how your observations relate to your answers in **1**.

3 Change the population distribution to *negatively skewed*. Keep $\mu = 0$ and $\sigma = 1$.

a Copy and complete this table:

Negatively skewed, $\mu = 0$, $\sigma = 1$

n	*Mean of* $\overline{X}_n$	*Standard deviation of* $\overline{X}_n$	$\frac{\sigma}{\sqrt{n}}$
4			
9			
16			
25			
100			

b For large values of n, does the distribution of $\overline{X}_n$ resemble that of the population?

c Comment on the similarities and differences from your observations of the normal distribution in **2**.

PRINTABLE TABLES



4 Repeat **3** for the other non-normal distributions. Try experimenting with different values of μ and σ, and comment on your findings.

From the **Investigation**, you should have found that for a population with mean μ and standard deviation σ:

- The distribution of the sample means has:
  - $\mathrm{E}(\overline{X}_n) = \mu$
  - $\mathrm{Var}(\overline{X}_n) = \dfrac{\sigma^2}{n}$
  - $\sigma(\overline{X}_n) = \dfrac{\sigma}{\sqrt{n}}$

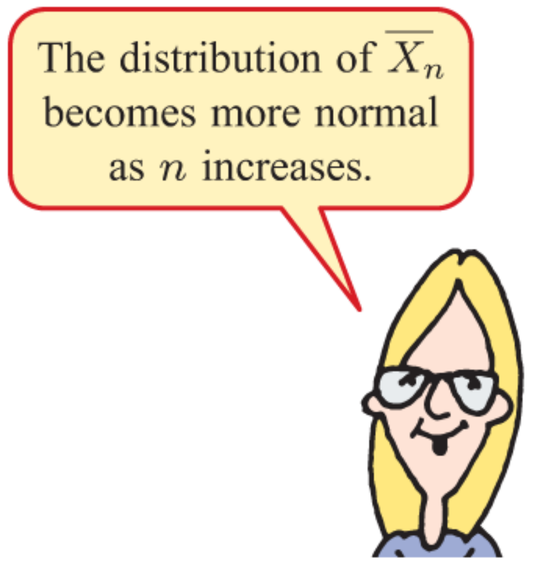
  Since $\mathrm{E}(\overline{X}_n) = \mu$, we say $\overline{X}_n$ is an **unbiased estimator** for μ.
- If the population is normally distributed, then $\overline{X}_n \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ for all n.

You should also have observed the remarkable result known as the **Central Limit Theorem**:

For any population, even one not normally distributed, $\overline{X}_n$ is approximately normally distributed for sufficiently large n, with $\overline{X}_n \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$.

We note that:

- It is sensible to ask what is meant by "sufficiently large n". Many texts suggest a "rule of thumb" of $n \geqslant 30$ to indicate n is large enough. However, this value depends on the "normality" of the population, which means how close it is to being bell-shaped.
- If the population is finite and the n observations are chosen without replacement, then n must also be small compared to the size of the population. This is to ensure that the observations are approximately independent.



The Central Limit Theorem allows us to estimate probabilities associated with $\overline{X}_n$ using what we learned in **Chapter 28**.

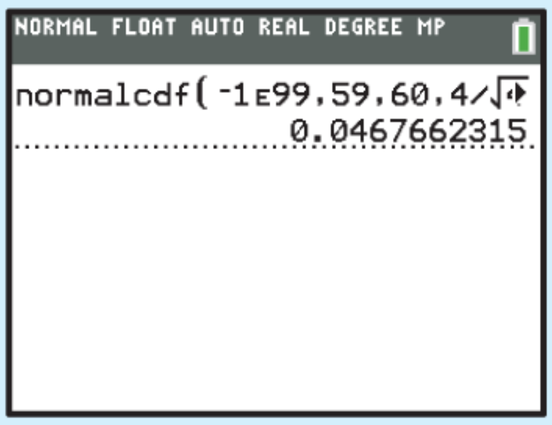
Example 2 **Self Tutor**

A random variable X has mean 60 and standard deviation 4. A sample of size 45 is taken from X. Find the probability that the mean of the sample is less than 59.

Since the sample has size $n = 45$, we apply the Central Limit Theorem.

$\therefore$ $\overline{X}_{45}$ is approximately normally distributed with mean 60 and standard deviation $\dfrac{4}{\sqrt{45}}$.

$\therefore$ $P(\overline{X}_{45} < 59) \approx 0.0468$



EXERCISE 29D

1 X is a random variable with mean 70 and standard deviation 12. Let $\overline{X}_{64}$ be the sample mean of 64 observations of X.

a Find the mean and standard deviation of $\overline{X}_{64}$.

b Is $\overline{X}_{64}$ approximately normally distributed? Explain your answer.

2 The random variable X is normally distributed with mean 50 and standard deviation 9.

a For the sample means of size 5, $\overline{X}_5$, find the:

i mean **ii** standard deviation.

b Can we say that $\overline{X}_5$ is normally distributed? Explain your answer.

3 Suppose X has mean 80 and standard deviation 9.

a Find the mean and standard deviation of $\overline{X}_{36}$.

b Is $\overline{X}_{36}$ approximately normally distributed? Explain your answer.

c Find:

i $P(75 \leqslant \overline{X}_{36} \leqslant 81)$ **ii** $P(\overline{X}_{36} > 82)$ **iii** $P(\overline{X}_{36} < 79)$

d Can we find $P(70 < X < 85)$? Explain your answer.

4 Suppose Y has mean 3.6 and standard deviation 0.7.

a Find the mean and standard deviation of $\overline{Y}_{32}$.

b Is $\overline{Y}_{32}$ approximately normally distributed? Explain your answer.

c Find:

i $P(3.5 < \overline{Y}_{32} < 3.7)$ **ii** $P(\overline{Y}_{32} > 3.7)$ **iii** $P(\overline{Y}_{32} < 3.3)$

5 The random variable X is normally distributed with mean 90 and standard deviation 20.

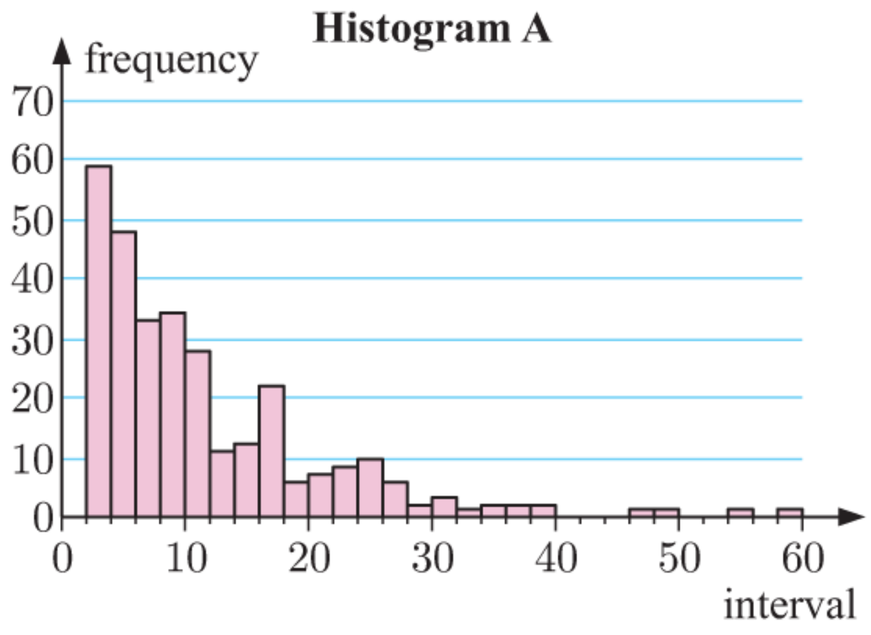
a Is $\overline{X}_5$ normally distributed? Explain your answer.

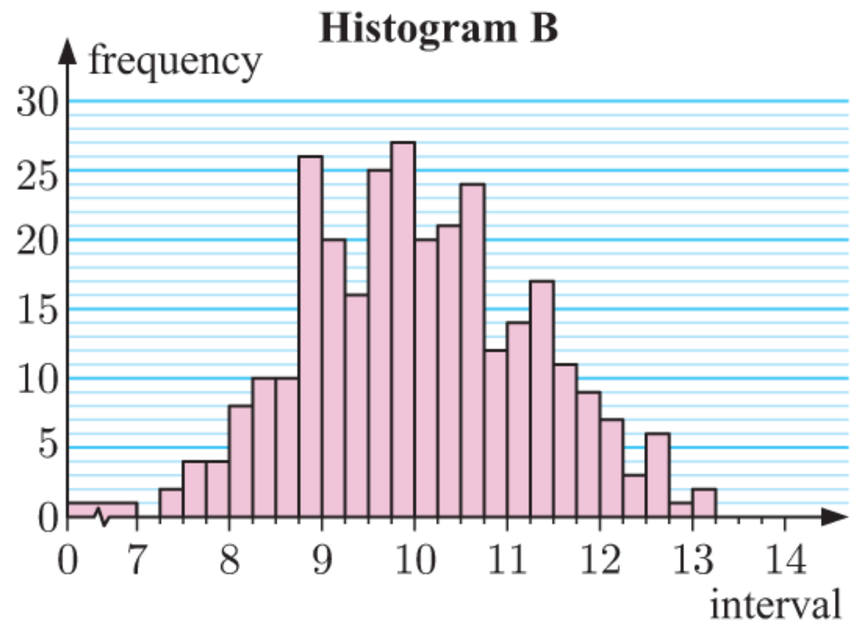
b Would you expect X or $\overline{X}_5$ to have a greater proportion of scores over 100? Explain your answer.

c Find:

i $P(X > 100)$ **ii** $P(\overline{X}_5 > 100)$

6 Two histograms are shown below. One is from a distribution X with mean 10 and standard deviation 10. The other is from the distribution of the sample means $\overline{X}_{64}$ selected from the distribution X. Note that the scales are not the same in the two histograms.





a Which of the two histograms is from $\overline{X}_{64}$? Give reasons for your answer.

b Use the appropriate histogram to estimate $P(\overline{X}_{64} < 9)$.

c Use the appropriate histogram to estimate the probability that $\overline{X}_{64}$ is within one standard deviation from the mean. How does this answer compare with using the normal approximation?

Example 3 **Self Tutor**

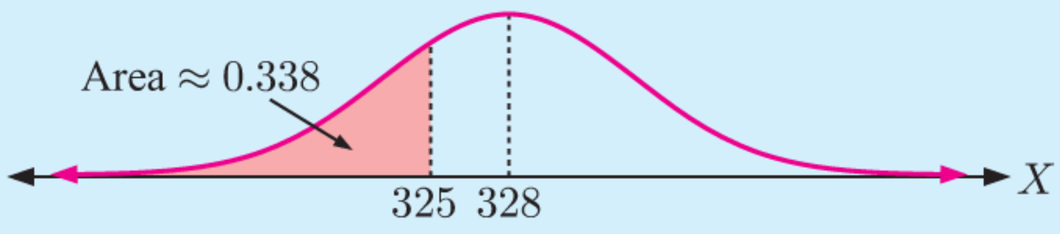
The contents of soft drink cans is normally distributed with mean 328 mL and standard deviation 7.2 mL. Find the probability that:

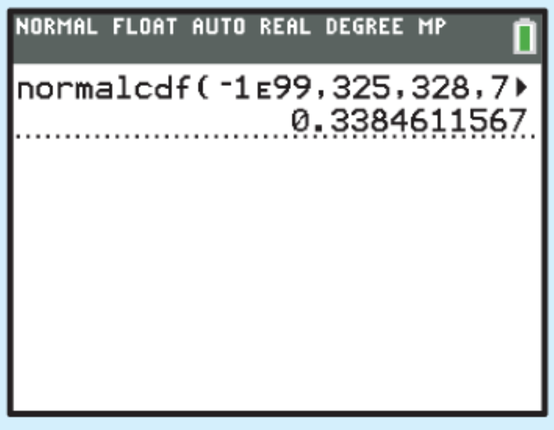
a an individual can contains less than 325 mL

b a box of 36 cans has average contents less than 325 mL per can.

a The contents X of an individual can is normally distributed with mean 328 mL and standard deviation 7.2 mL.

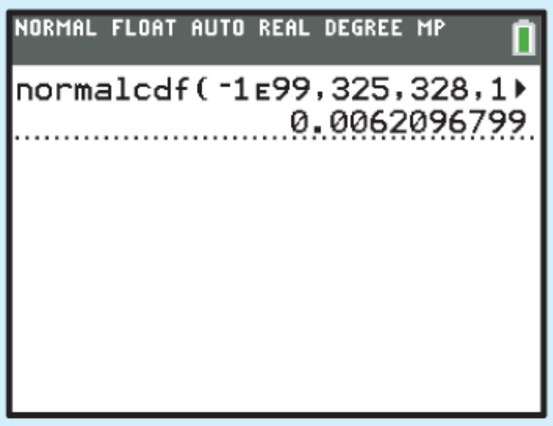
$\therefore \quad P(X < 325) \approx 0.338$

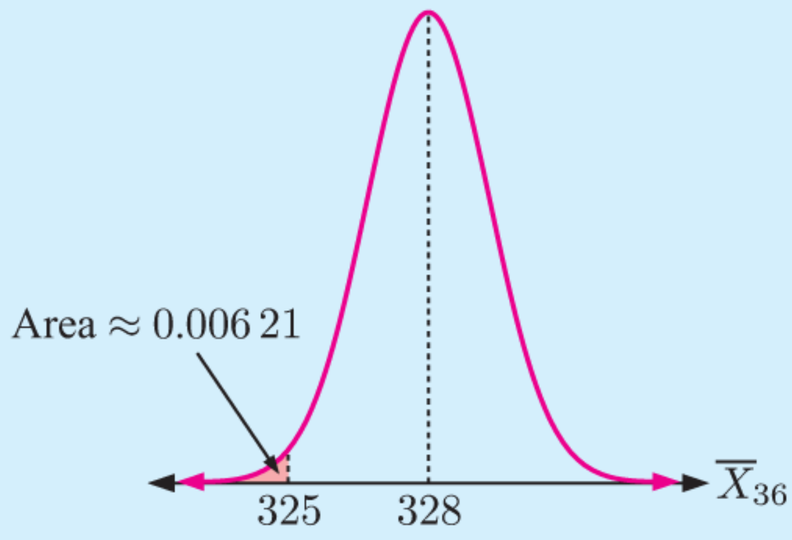




b The average contents $\overline{X}_{36}$ of a box of 36 cans is normally distributed with mean 328 mL per can and standard deviation $\dfrac{7.2}{\sqrt{36}} = 1.2$ mL.

$\therefore \quad P(\overline{X}_{36} < 325) \approx 0.006\,21$





7 Let the random variable X be the IQ of 17 year old girls. Suppose X is normally distributed with mean 105 and standard deviation 15.

a Find the probability that an individual 17 year old girl has an IQ of more than 110.

b Find the probability that the mean IQ of a class of twenty 17 year old girls is greater than 110.

8 The energy content of a fruit bar is normally distributed with mean 1067 kJ and standard deviation 61.7 kJ. Find the probability that:

a an individual bar has more than 1050 kJ of energy

b the mean energy content of a sample of 30 fruit bars is more than 1050 kJ.

9 Customers at a clothing store are in the shop for a mean time of 18 minutes with standard deviation 5.3 minutes. What is the probability that, in a sample of 37 customers, the mean stay in the shop is between 17 and 20 minutes?

10 The mean sodium content of a box of cheese rings is 1183 mg with standard deviation 88.6 mg. Find the probability that the mean sodium content per box for a sample of 50 boxes lies between 1150 mg and 1200 mg.

11 The weight of oranges is normally distributed with mean 130 g and standard deviation 15 g. The oranges are sold in bags of 9.

a Find the probability that an individual orange weighs more than 140 g.

b Find the probability that the average weight of oranges in a bag is more than 140 g.

12 Suppose the duration of human pregnancies can be modelled by a normal distribution with mean 267 days and standard deviation 15 days. If a pregnancy lasts longer than 267 days it is said to be "overdue". If a pregnancy lasts less than 267 days it is said to be "premature".

a What percentage of pregnancies will be overdue by between 1 and 2 weeks?

b A certain obstetrician is providing prenatal care for 64 pregnant women.

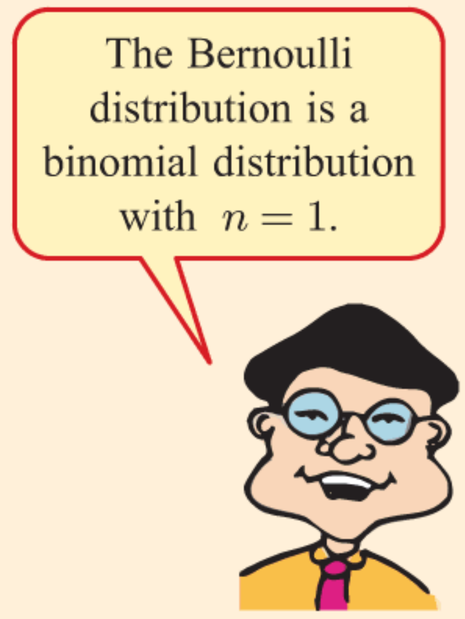
i Describe the distribution of $\overline{X}_{64}$.

ii Find the probability that the mean duration of these 64 patients' pregnancies will be premature by at least one week.

c Suppose the duration of human pregnancies is not actually normally distributed, but is positively skewed. Does this impact the answers to parts **a** and **b** above?

INVESTIGATION 3 APPLYING THE CENTRAL LIMIT THEOREM TO THE BINOMIAL DISTRIBUTION

In **Exercise 29A** question **5** we saw that a binomial random variable $X \sim \text{B}(n,\, p)$ can be thought of as the sum of n independent and identically distributed **Bernoulli random variables** with probability of success p.



So, $X = \sum_{i=1}^{n} X_i$ where $X_i \sim \text{B}(1,\, p)$ independently.

Let $\widehat{p} = \dfrac{X}{n}$.

What to do:

1 Explain why $\widehat{p}$ is a sample mean.

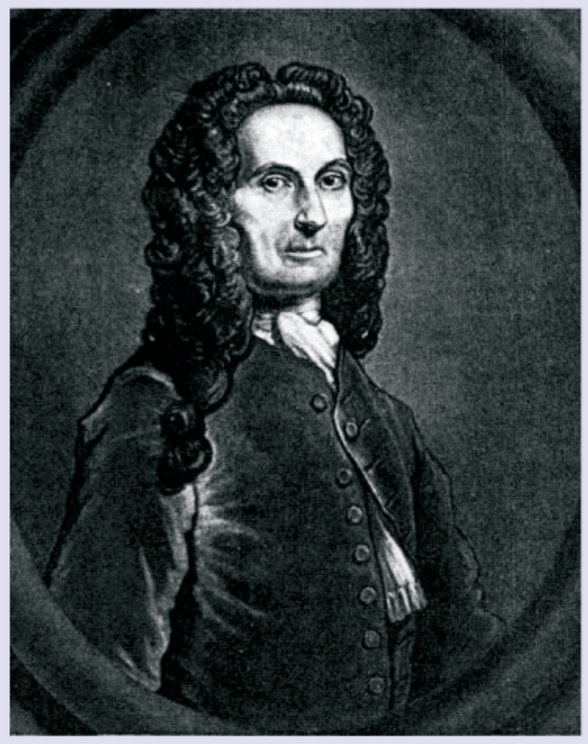
2 Find $\text{E}(\widehat{p})$ and $\text{Var}(\widehat{p})$.

3 Assuming n is large, use the Central Limit Theorem to find an appropriate distribution for:

a $\widehat{p}$ **b** X.

THEORY OF KNOWLEDGE

The French mathematician **Abraham de Moivre** is most famous for his formula linking complex numbers and trigonometry. However, he was also responsible for the early development of the Central Limit Theorem. In an article written in 1733, he used the normal distribution to approximate the distribution of the number of heads resulting from many tosses of a fair coin.



Abraham de Moivre

The work of de Moivre was extended in 1812 by country-man **Pierre-Simon Laplace**, but the theorem was not formalised and rigorously proven until the early 20th century work of the Russian mathematicians **Pafnuty Chebyshev**, **Andrey Markov**, and **Aleksandr Lyapunov**.

In 1889, **Sir Francis Galton** wrote about the normal distribution, with particular relevance to the Central Limit Theorem:

> *"I know of scarcely anything so apt to impress the imagination as the wonderful form of cosmic order expressed by the law of frequency of error. The law would have been personified by the Greeks if they had known of it. It reigns with serenity and complete self-effacement amidst the wildest confusion. The larger the mob, the greater the apparent anarchy, the more perfect is its sway. It is the supreme law of unreason."*

1 How is mathematical order linked to our sense of beauty?

The name "Central Limit Theorem" was first used in 1920 by the Hungarian mathematician **George Pólya** (1887 - 1985). He used the word "central" because of the importance of the theorem in probability theory.

2 What makes a theorem "fundamental"? Could the Central Limit Theorem reasonably be referred to as the Fundamental Theorem of Statistics?

3 Discuss the statement: "Without the Central Limit Theorem, there could be no statistics of any value within the human sciences."

The philosophies of **rationalism** and **empiricism** concern the extent to which we depend on sensory experience in order to gain knowledge.

Rationalists claim that knowledge can be gained through logic independent of sensory experience. They argue about the limitations of what sensory experience can provide, and how reason in other forms contributes additional information about the world.

By contrast, empiricists claim that sensory experience is most important for knowledge, arguing that we cannot regard something as knowledge unless observation and experience can provide it.

4 Is it more important to rationalise the Central Limit Theorem by mathematical proof, or confirm its truth by empirical application?

E CONFIDENCE INTERVALS FOR A POPULATION MEAN WITH KNOWN VARIANCE

We often hear claims about a population mean.

For example, it is claimed that the mean protein content of a 1 litre carton of milk is 39 grams. The truth of this claim could only be known by measuring the protein content of *every* 1 litre carton of milk. This is clearly an impossible task. However, it is possible to draw reasonable conclusions from measuring the protein content of a random selection of cartons.



Suppose we randomly select a sample of milk cartons, and the mean protein content for the sample is $\overline{x} = 38$ g. This could be seen as evidence to dispute the claim that $\mu = 39$ g. However, it is possible that the average protein content is indeed 39 g, and that the sampled cartons had lower than average protein.

Our sample cannot tell us with certainty the exact population mean μ. However, we can use it to predict an **interval** in which we are **confident** the population mean will lie. This interval is called a **confidence interval**.

A **confidence interval** for a population mean μ, is an interval of values between two limits, together with a percentage indicating our confidence that μ lies in that interval.

Consider the statement "We are 95% confident that the mean protein content of milk cartons is between 37 g and 40 g".

The statement indicates that the probability that the interval $[37,\ 40]$ grams contains the population mean μ is 0.95.

THE 95% CONFIDENCE INTERVAL

The Central Limit Theorem is used as a basis for finding confidence intervals.

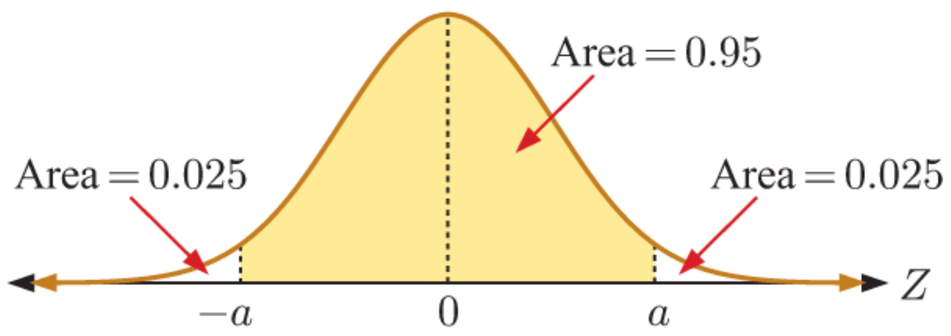
Consider the distribution $\overline{X}_n$ of all sample means $\overline{x}$ from sufficiently large samples of size n taken from population X with population mean μ and known population standard deviation σ. $\overline{X}_n$ is normally distributed with mean μ and standard deviation $\dfrac{\sigma}{\sqrt{n}}$.

The corresponding standard normal random variable is $Z = \dfrac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$ and $Z \sim \mathrm{N}(0, 1^2)$.



For a **95% confidence level** we need to find a for which $\mathrm{P}(-a \leqslant Z \leqslant a) = 0.95$.

Using the symmetry of the normal distribution, the statement reduces to
$\mathrm{P}(Z \leqslant -a) = 0.025$ or $\mathrm{P}(Z \leqslant a) = 0.975$.



Using technology, we find that $a \approx 1.96$.

$$\therefore \quad \mathrm{P}(-1.96 \leqslant Z \leqslant 1.96) = 0.95$$

$$\therefore \quad \mathrm{P}\left(-1.96 \leqslant \frac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \leqslant 1.96\right) = 0.95$$

$$\therefore \quad \mathrm{P}\left(-1.96\frac{\sigma}{\sqrt{n}} \leqslant \overline{X}_n - \mu \leqslant 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

$$\therefore \quad \mathrm{P}\left(-\overline{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \leqslant -\mu \leqslant -\overline{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

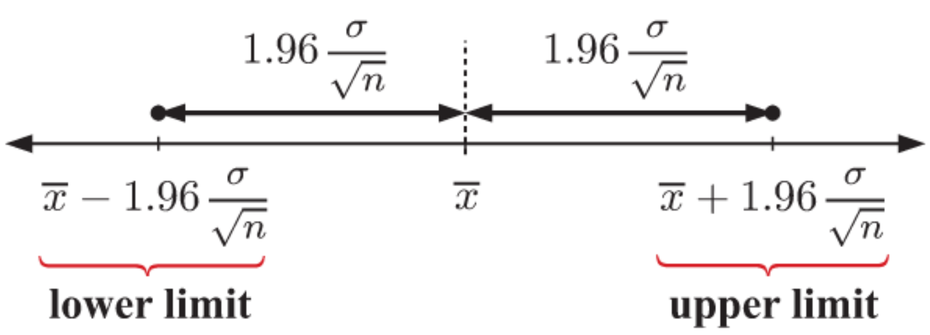
$$\therefore \quad \mathrm{P}\left(\overline{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$

To estimate this interval, we replace $\overline{X}_n$ with its observed value $\overline{x}$ to obtain the **95% confidence interval** for μ.

The **95% confidence interval for μ** given a known population variance σ^2 and a data set with sample mean $\overline{x}$ is

$$\overline{x} - 1.96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\frac{\sigma}{\sqrt{n}},$$

which can also be written as $\overline{x} \pm 1.96\dfrac{\sigma}{\sqrt{n}}$ or $\left[\overline{x} - 1.96\dfrac{\sigma}{\sqrt{n}},\ \overline{x} + 1.96\dfrac{\sigma}{\sqrt{n}}\right]$.



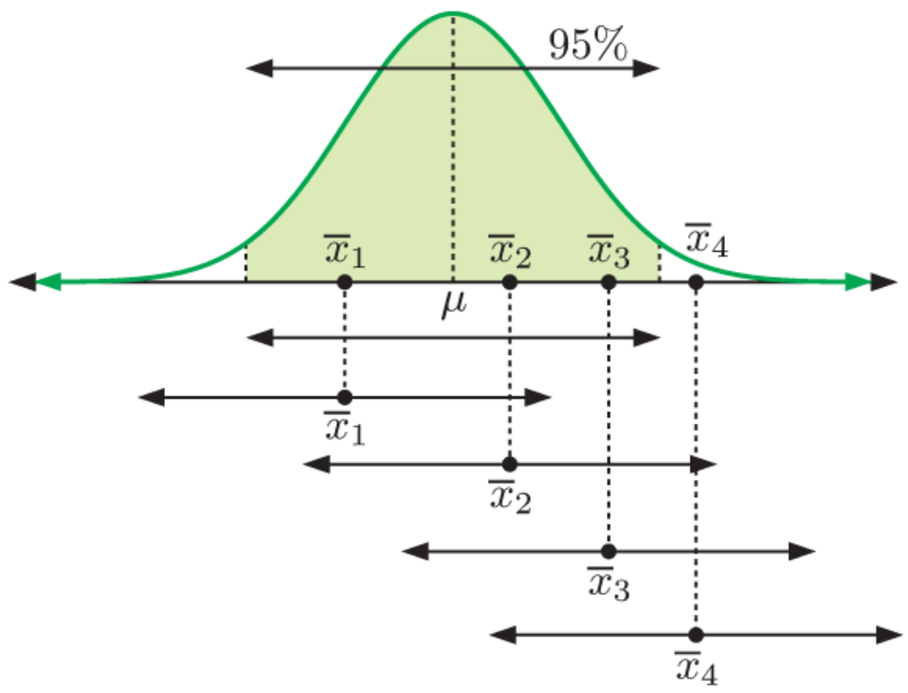
The value $1.96\dfrac{\sigma}{\sqrt{n}}$ is called the **margin of error**.



We notice that:

- The exact **centre** or midpoint of the confidence interval is the value of $\overline{x}$ for the sample taken.
- The **width** of the 95% confidence interval for μ is $2 \times 1.96 \dfrac{\sigma}{\sqrt{n}}$.
- We cannot be certain that the confidence interval contains μ, but there is probability 0.95 that it will.

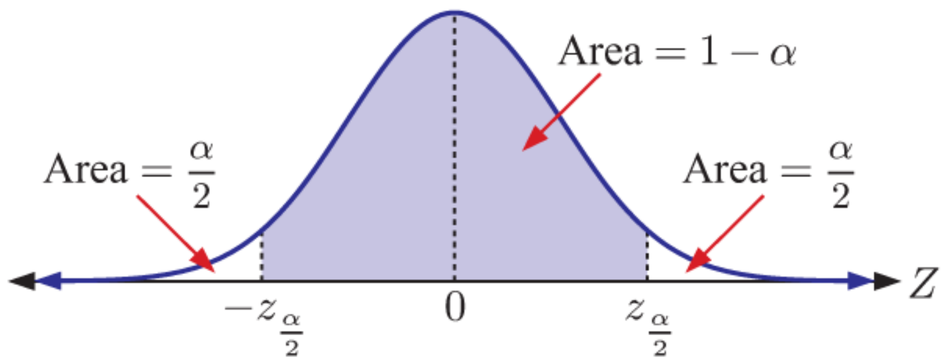
  For example, the diagram alongside shows several values for $\overline{x}$, and the corresponding confidence intervals. The confidence intervals for $\overline{x}_1$, $\overline{x}_2$, and $\overline{x}_3$ all contain μ, whereas the confidence interval for $\overline{x}_4$ does not.



OTHER CONFIDENCE INTERVALS FOR μ

If we want a confidence interval that contains μ with probability $1 - \alpha$, we require a such that $\mathrm{P}(-a \leqslant Z \leqslant a) = 1 - \alpha$.

Since the normal distribution is symmetric, this means $\mathrm{P}(Z \leqslant -a) = \dfrac{\alpha}{2}$ and $\mathrm{P}(Z \geqslant a) = \dfrac{\alpha}{2}$.



The particular value of a we seek is therefore labelled $z_{\frac{\alpha}{2}}$.

By following the same steps which we used for the 95% confidence interval, we find that

$$\mathrm{P}\left(\overline{X}_n - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{X}_n + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

By replacing $\overline{X}_n$ in the interval with its observed value $\overline{x}$, we obtain the **general confidence interval** for μ.

The **general confidence interval for μ** given a known population variance σ^2 and a data set with sample mean $\overline{x}$ is

$$\overline{\boldsymbol{x}} - \boldsymbol{z}_{\frac{\alpha}{2}} \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{n}}} \leqslant \boldsymbol{\mu} \leqslant \overline{\boldsymbol{x}} + \boldsymbol{z}_{\frac{\alpha}{2}} \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{n}}}$$

The interval contains μ with probability $1 - \alpha$, so the **confidence level** of the interval is $(1 - \alpha) \times 100\%$.

Notice that:

- We can determine the value of $z_{\frac{\alpha}{2}}$ for any α using technology.
- The observed sample mean $\overline{x}$ is the **centre** or midpoint of the confidence interval.
- The **margin of error** is $z_{\frac{\alpha}{2}} \dfrac{\sigma}{\sqrt{n}}$.
- The **width** of the confidence interval is $2 \times (\text{margin of error}) = 2 \times z_{\frac{\alpha}{2}} \dfrac{\sigma}{\sqrt{n}}$.

ACTIVITY PROPERTIES OF CONFIDENCE INTERVALS

1 The table alongside shows the values of α and $z_{\frac{\alpha}{2}}$ for several common choices of confidence levels.

By performing your own calculations, check that the values of $z_{\frac{\alpha}{2}}$ are correct for each confidence level.

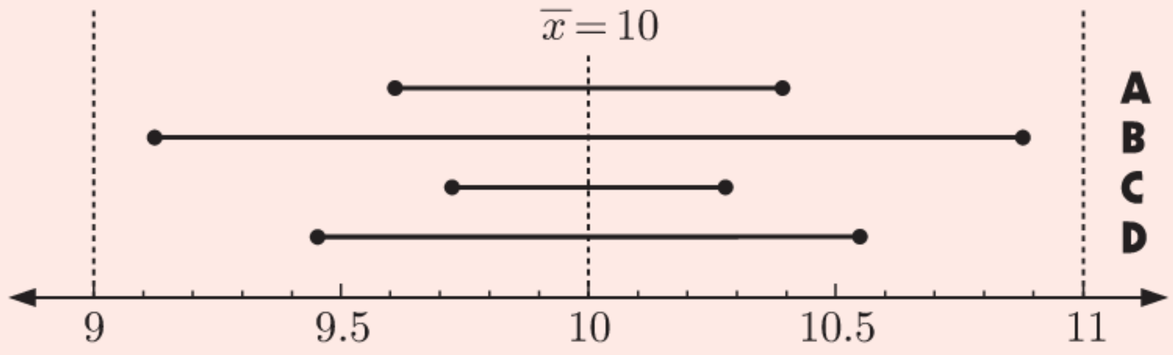
Confidence level	α	$z_{\frac{\alpha}{2}}$
90%	0.1	1.645
95%	0.05	1.960
98%	0.02	2.326
99%	0.01	2.576

2 Consider samples of different size but all with sample mean 10 and population standard deviation 2. The 95% confidence interval is $10 - 1.960 \times \frac{2}{\sqrt{n}} \leqslant \mu \leqslant 10 + 1.960 \times \frac{2}{\sqrt{n}}$.

a Copy and complete the table alongside:

n	95% *confidence interval*
20	$9.123 \leqslant \mu \leqslant 10.877$
50	
100	
200	

b Hence identify which illustrated interval corresponds to each value of n in the diagram below:



c Copy and complete:

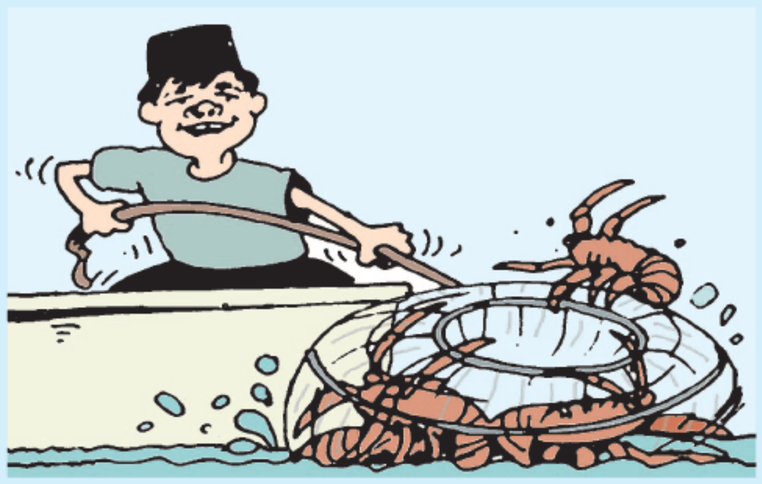
Increasing the sample size n produces confidence intervals of ..... width.

Example 4 — Self Tutor

A sample of 60 yabbies was taken from a dam. The sample mean weight of the yabbies was 84.6 grams.

Given the population standard deviation is 16.8 grams, construct a 95% confidence interval for the population mean.

Check your answer using technology.



We are given that $\overline{x} = 84.6$ and $\sigma = 16.8$.

The 95% confidence interval is $\overline{x} - 1.96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + 1.96\frac{\sigma}{\sqrt{n}}$

$$\therefore \quad 84.6 - 1.96 \times \frac{16.8}{\sqrt{60}} \leqslant \mu \leqslant 84.6 + 1.96 \times \frac{16.8}{\sqrt{60}}$$

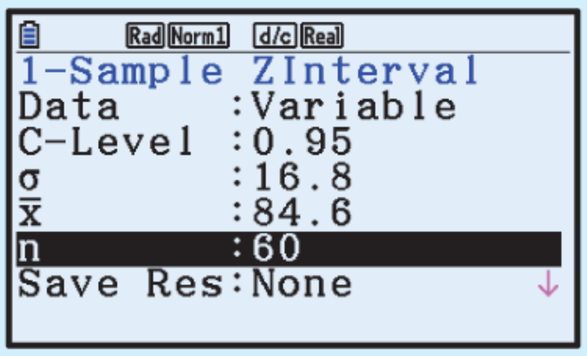
$$\therefore \quad 80.3 \leqslant \mu \leqslant 88.9$$

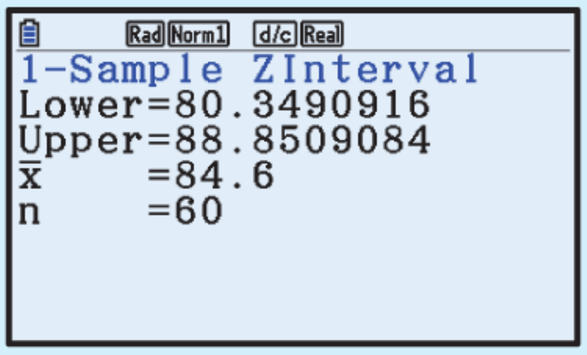
So, we are 95% confident that the population mean weight of yabbies lies between 80.3 grams and 88.9 grams.



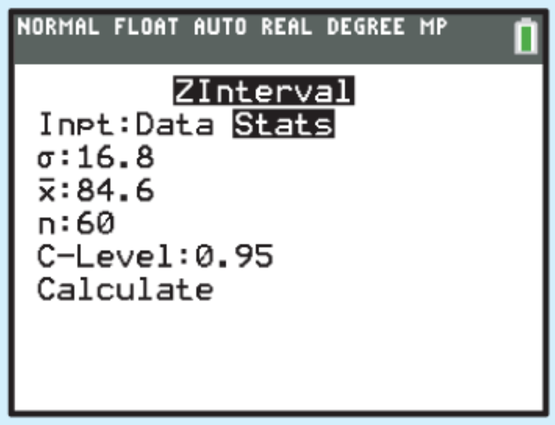
GRAPHICS CALCULATOR INSTRUCTIONS

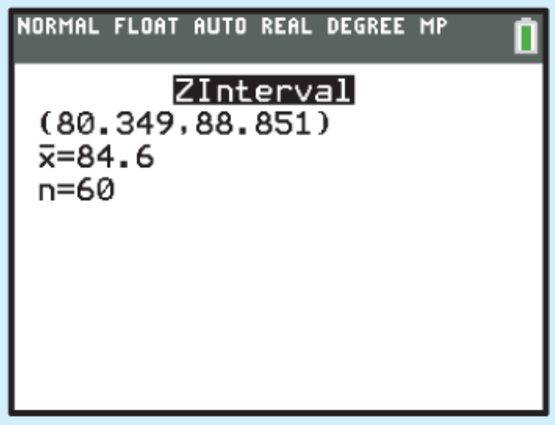
Casio fx-CG50





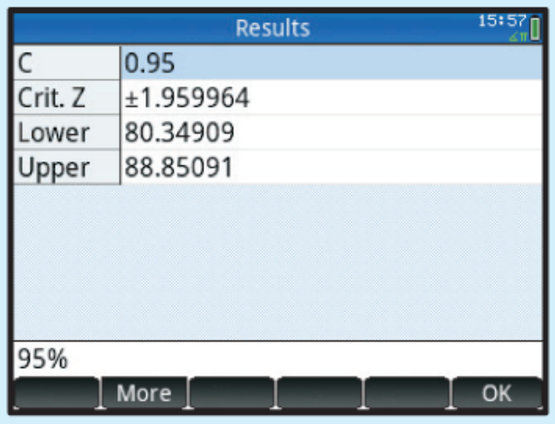
TI-84 Plus CE





HP Prime





EXERCISE 29E.1

1 A sample of 50 is selected from a population with known standard deviation 8. The sample mean is 73.

Find a 95% confidence interval for the population mean. Use technology to check your answer.

2 A random sample of 106 individuals is selected from a population with known standard deviation 11. The sample mean is 81.6.

a Determine the value of $z_{\frac{\alpha}{2}}$ required for a confidence level of 92%.

b Hence construct a 92% confidence interval for μ.

3 A sample of 80 is selected from a population with known standard deviation 10. The sample mean is 42.5.

a Find a:

i 90% confidence interval for μ **ii** 95% confidence interval for μ

iii 99% confidence interval for μ.

b What effect does increasing the confidence level have on the margin of error?

4 Sophie took a sample of measurements of a random variable X. Her 95% confidence interval for the population mean was $27.6 \leqslant \mu \leqslant 31.8$.

a Find the sample mean $\overline{x}$.

b Find the confidence interval Sophie would have calculated if she had used a 99% confidence level.

5 While peaches are being canned, 250 mg of preservative should be added to each can by a dispensing device. It is known that the standard deviation of preservative added is 7.3 mg.

To check the machine, the quality controller obtains 60 random samples of dispensed preservative, and finds that the mean preservative added is 242.6 mg.

a Construct a 95% confidence interval for the population mean amount of preservative.

b Find the width of the confidence interval.

c Explain how the quality controller could obtain a 95% confidence interval with a smaller width.

DISCUSSION

When performing a statistical analysis, we can choose the confidence level for a confidence interval. Why might a statistician choose a confidence level less than 99%?

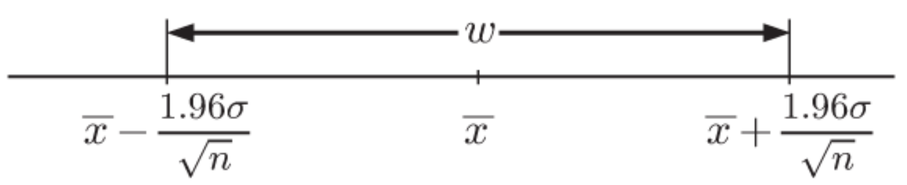
DETERMINING HOW LARGE A SAMPLE SHOULD BE

When using confidence intervals to estimate population means, we generally want a narrow confidence interval.

For example, when estimating the population mean weight μ g of apples, a 95% confidence interval $80 \leqslant \mu \leqslant 85$ is more useful than a 95% confidence interval $70 \leqslant \mu \leqslant 95$.

For a given confidence level, we can decrease the width of the confidence interval by increasing the sample size.

Consider the 95% confidence interval shown in the diagram. Its width $w = 2 \times \dfrac{1.96\sigma}{\sqrt{n}}$.



By taking a sufficiently large sample size n, we can make w as small as we like.

Rearranging $w = 2 \times \dfrac{1.96\sigma}{\sqrt{n}}$, we find $n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2$.

When we wish to estimate the population mean to lie with 95% confidence within an interval of width w, the sample size required is given by $\boldsymbol{n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2}$ where σ is the population standard deviation.

Example 5 **Self Tutor**

In **Example 4** we considered a population of yabbies whose weights had population standard deviation 16.8 grams.

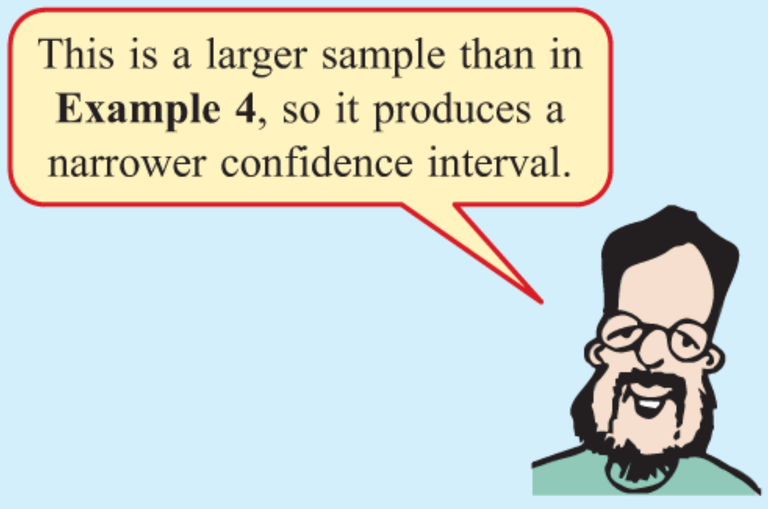
What sample size is required to find a 95% confidence interval for the mean weight μ with width 5 g?

The required sample size $n = \left(\dfrac{2 \times 1.96\sigma}{w}\right)^2$

$= \left(\dfrac{2 \times 1.96 \times 16.8}{5}\right)^2$

≈ 173.5

Rounding up, a sample size of 174 yabbies is required.



As we increase the sample size, we obtain a narrower and hence more accurate confidence interval. However, it may be expensive or time consuming to obtain a large sample. We must therefore strike a balance between an accurate confidence interval and a practical sample size.

EXERCISE 29E.2

1 A population is known to have standard deviation $\sigma = 34$. Find the sample size n that should be taken to find a 95% confidence interval for the population mean μ, of width:

a $w = 5$ **b** $w = 1$ **c** $w = 0.1$.

2 A researcher wishes to estimate the mean weight of adult crayfish in Australian waters. She knows that the population standard deviation σ is 250.5 grams. How large must the sample be to obtain a 95% confidence interval of width 140 g?

3 A porridge manufacturer samples 80 packets of porridge. The population standard deviation σ of the contents' weight is 17.8 grams. How many packets must be sampled to obtain a 95% confidence interval for the population mean weight μ, of width:

a 6 g **b** 4 g?

4 **a** If the size n of a sample is doubled, by how much will the width of a 95% confidence interval decrease?

b How much larger do you have to make a sample size to halve the width of a 95% confidence interval?

5 A manufacturer of bottled water knows that the machine dispenses water into 1 litre bottles with a standard deviation of 2.3 mL. The machine needs to be checked regularly to ensure it is still delivering the correct volume.

A sample of 30 bottles produces the sample mean $\overline{x} = 1001$ mL.

a Construct a 95% confidence interval for the population mean contents.

b Find the width of the confidence interval in **a**.

c It is decided that the accuracy of the testing procedure must be improved so that the 95% confidence interval has width $\leqslant 0.5$ mL.

i What sample size is required?

ii What problems may result from obtaining a narrower confidence interval?

USING A CONFIDENCE INTERVAL TO TEST A CLAIM ABOUT μ

Confidence intervals can be used to test claims about a population mean.

To test the claim that a population mean takes a particular value μ_0, we construct a confidence interval for the true population mean μ.

- If μ_0 does *not* lie within the confidence interval, we reject the claim that $\mu = \mu_0$.
- If μ_0 lies within the confidence interval, we do not have enough evidence to reject the claim that $\mu = \mu_0$.

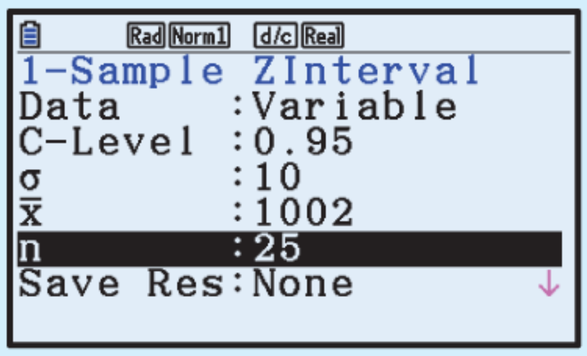
Example 6 **Self Tutor**

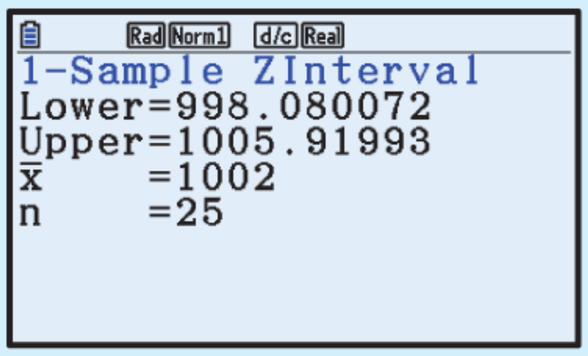
The volume of milk dispensed into cartons by a machine is normally distributed with mean μ which can be adjusted, and standard deviation 10 mL which is fixed. The value of μ is supposed to be 1005 mL. A quality controller tests 25 cartons, and finds that their mean volume is 1002 mL.

a Construct a 95% confidence interval for the mean volume μ dispensed by the machine.

b Use the 95% confidence interval to assess the claim that the volume dispensed by the machine is 1005 mL.

a The 95% confidence interval is $998.1 \leqslant \mu \leqslant 1005.9$.





b 1005 mL lies within the 95% confidence interval. Despite the fact that the sample mean is less than 1005 mL, we do not have enough evidence to reject the claim that $\mu = 1005$ mL.

Remember that not all confidence intervals will contain the population mean. If the claimed population mean does not lie in the confidence interval, it does not mean for certain that the claim is false. Likewise, if the claimed population mean *does* lie in the confidence interval, it does not mean for certain that the claim is true.

Confidence intervals cannot be used to make definitive statements about whether a claim is true or false. Their purpose is to give us a guide to whether a claim is *reasonable*.

EXERCISE 29E.3

1 A random variable X has population standard deviation $\sigma = 7$. To test the claim that the population mean $\mu = 50$, a random sample of 60 is taken, and the sample mean is 49.1.

a Construct a 95% confidence interval for μ.

b Use the 95% confidence interval to assess the claim that $\mu = 50$.

2 Bags of salted cashew nuts are advertised to have a net weight of 100 g. The manufacturer knows that the standard deviation of the population is 1.6 g.
A customer claims that the bags have been lighter in recent purchases, so the factory quality control manager decides to investigate. He samples 40 bags and finds that their mean net weight is 99.4 g.

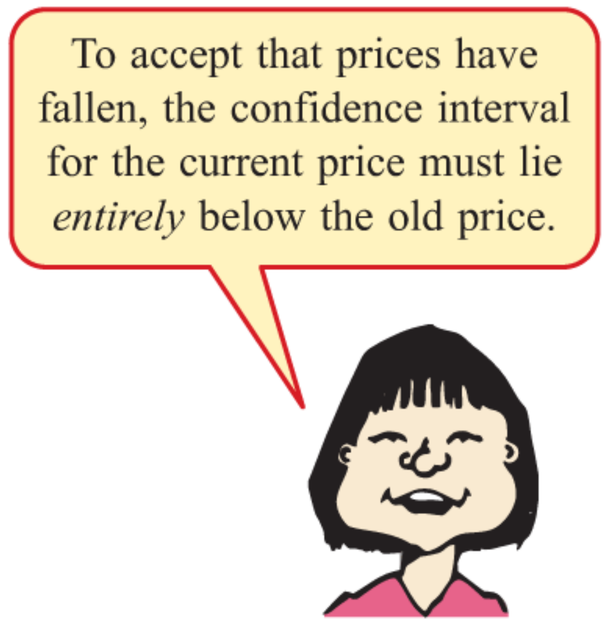
a Construct a 95% confidence interval for the population mean net weight μ.

b Use the confidence interval to test the claim that the mean net weight of the bags is 100 g.

3 A liquor store claimed that the mean price of a bottle of wine had fallen from what it was 12 months previously. Records show that 12 months ago the mean price was \$13.45 with standard deviation \$0.25. A random sample of prices of 389 different bottles of wine is now taken from the store. The mean price for the sample is \$13.30.

a Construct a 95% confidence interval for the mean price, assuming the standard deviation has not changed.

b Is there sufficient evidence to accept that the mean price has fallen?



4 The time it takes Joan to run 100 metres is normally distributed with mean $\mu = 12.46$ seconds and standard deviation 0.3 seconds. To improve her time, Joan goes on a training program. After 6 weeks, Joan finds that her mean time from 12 trial runs is now 12.18 seconds.

a Assuming the same standard deviation, construct a 95% confidence interval for Joan's mean.

b Hence assess the claim that:

i Joan's time to run 100 metres has improved

ii Joan is now faster than Betty, whose time for the 100 metres is 12.26 seconds.

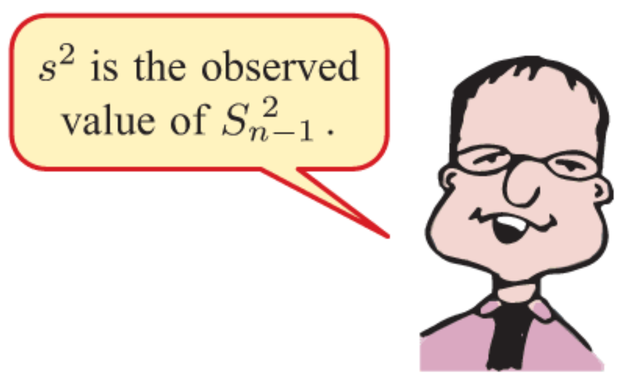
F CONFIDENCE INTERVALS FOR A POPULATION MEAN WITH UNKNOWN VARIANCE

In most cases we will not know the population variance σ^2, so we cannot substitute it into $Z = \dfrac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$.

Instead, we *estimate* σ^2 with the **sample variance** $s^2 = \dfrac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$.

We have seen that when we consider a random sample $\{X_1, X_2, ..., X_n\}$, the sample mean $\overline{X}_n$ is itself a random variable. The same is true for the sample variance.

For a random sample $\{X_1, X_2, ..., X_n\}$, the **sample variance** is $\boldsymbol{S^2_{n-1} = \dfrac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2}$.



We can use s^2 to estimate σ^2 because $\mathrm{E}(S^2_{n-1}) = \sigma^2$. Click on the icon to obtain the proof. We say that S^2_{n-1} is an **unbiased estimator** for σ^2.

PROOF

If we substitute S^2_{n-1} for σ^2 into $Z = \dfrac{\overline{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}}$, we obtain the statistic $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}}$ with observed value $t = \dfrac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}}$.

INVESTIGATION 4 THE DISTRIBUTION OF T

In this Investigation, we will explore the distribution of T for samples from a normal distribution.

What to do:

1 Click on this icon to access a simulation that generates random samples of size n from a normal distribution. The t-value is calculated for each sample.

SIMULATION

a Set $n = 10$ and use the sliders to change the mean μ and standard deviation σ of the normal distribution. Describe the distribution of the t-values.

b Use the slider to increase the value of n. What do you notice?

2 Describe how likely it is to obtain a t-value in the interval:

a $-1 \leqslant t \leqslant 1$ **b** $-2 \leqslant t \leqslant 2$ **c** $-3 \leqslant t \leqslant 3$

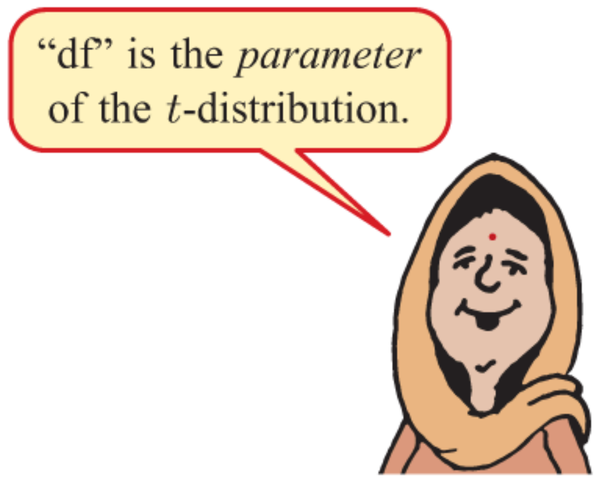
The characteristics you observed in the **Investigation** might suggest that the distribution of T is a normal distribution with mean 0. However, the distribution of T actually has a different curve.

THE t-DISTRIBUTION

Suppose a population is normally distributed with mean μ. Let $\overline{X}_n$ be the sample mean and S_{n-1}^2 be the sample variance of a sample of size n from this population.

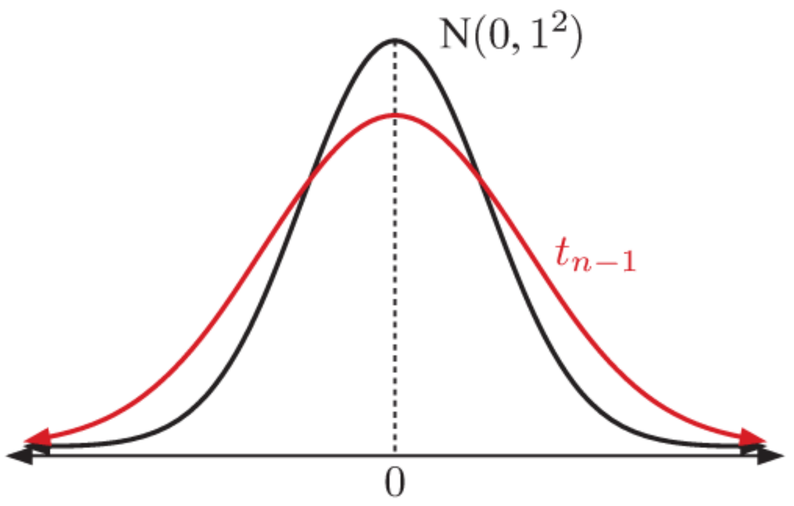
The distribution of the statistic $T = \dfrac{\overline{X}_n - \mu}{\dfrac{S_{n-1}}{\sqrt{n}}}$ is a **t-distribution** with $n-1$ **degrees of freedom** (df).

We write $\boldsymbol{T \sim t_{n-1}}$.



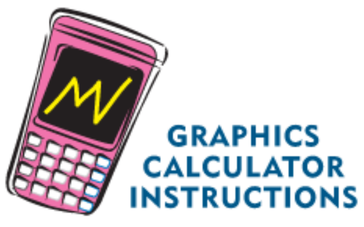
The t-distribution is slightly "flatter" than the standard normal distribution $\mathrm{N}(0, 1^2)$.

As the sample size and therefore the degrees of freedom increases, the shape of the t-distribution approaches that of $\mathrm{N}(0, 1^2)$.





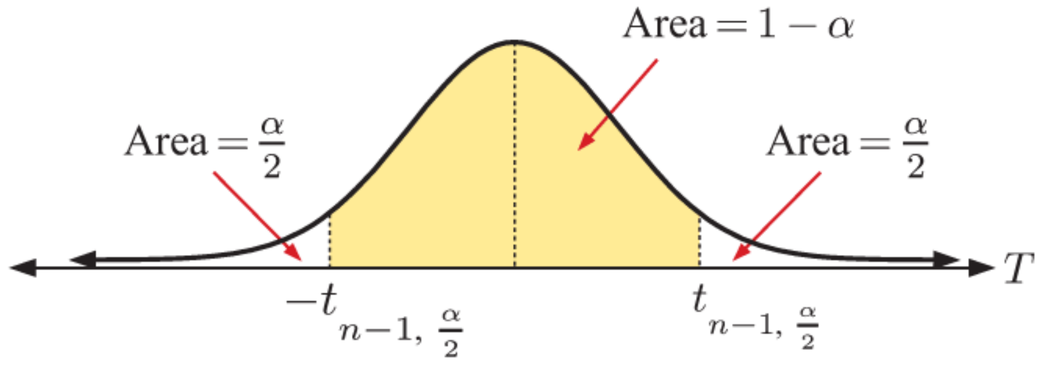
Like with the normal distribution, you can calculate probabilities and quantiles for the t-distribution using your **graphics calculator**.



CONFIDENCE INTERVALS FOR A POPULATION MEAN WHERE THE VARIANCE IS UNKNOWN

Now that we know the distribution for T, we can construct a general confidence interval using T in a similar way as we did using Z.

For a confidence interval with probability $1-\alpha$, we require a such that $\mathrm{P}(-a \leqslant T \leqslant a) = 1 - \alpha$.



Using the symmetry of the t-distribution, this is equivalent to $\mathrm{P}(T \leqslant -a) = \dfrac{\alpha}{2}$ and $\mathrm{P}(T \geqslant a) = \dfrac{\alpha}{2}$.

We denote the particular value of a we seek with the notation $t_{n-1,\,\frac{\alpha}{2}}$.

df probability

Using the same procedure as in **Section E**, but with $t_{n-1,\,\frac{\alpha}{2}}$ instead of $z_{\frac{\alpha}{2}}$ and S_{n-1} instead of σ, we find that:

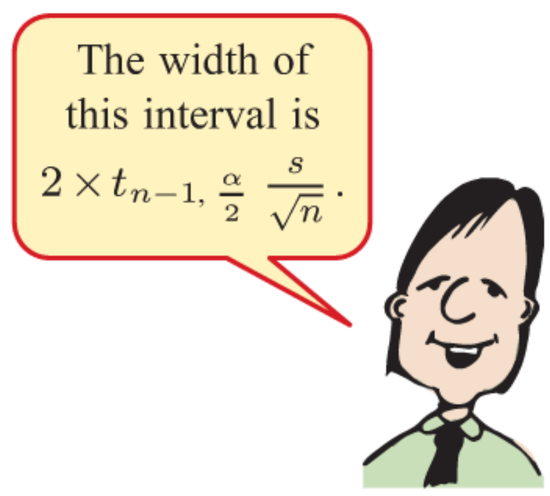
$$\mathrm{P}\left(\overline{X}_n - t_{n-1,\,\frac{\alpha}{2}} \frac{S_{n-1}}{\sqrt{n}} \leqslant \mu \leqslant \overline{X}_n + t_{n-1,\,\frac{\alpha}{2}} \frac{S_{n-1}}{\sqrt{n}}\right) = 1 - \alpha$$

Replacing $\overline{X}_n$ and S_{n-1} with their observed values $\overline{x}$ and s respectively, we define:

The **general confidence interval for μ** given a data set with sample mean $\overline{x}$ and sample standard deviation s is

$$\overline{x} - t_{n-1,\,\frac{\alpha}{2}} \frac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{n-1,\,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

The interval contains μ with probability $1-\alpha$, so the confidence level of the interval is $(1-\alpha) \times 100\%$.



Example 7 **Self Tutor**

The sugar content, in grams, of 30 randomly selected loaves of bread at the local bakery was measured as follows:

15.1	14.8	13.7	15.6	15.1	16.1	16.6	17.4	16.1	13.9	17.5	15.7	16.2	16.6	15.1
12.9	17.4	16.5	13.2	14.0	17.2	17.3	16.1	16.5	16.7	16.8	17.2	17.6	17.3	14.7

Determine a 98% confidence interval for the average sugar content of all loaves of bread baked.

Using technology, $\overline{x} \approx 15.897$ and $s \approx 1.365$.

Since $n = 30$ is sufficiently large, $\overline{X}_n$ is approximately normally distributed.

{Central Limit Theorem}

σ is unknown and $T = \dfrac{\overline{X}_n - \mu}{\frac{S_{n-1}}{\sqrt{n}}} \sim t_{29}$.

For a 98% confidence interval, $\alpha = 0.02$. $\therefore \quad \dfrac{\alpha}{2} = 0.01$.

Using technology, $t_{29,\,0.01} \approx 2.462$

The 98% confidence interval for μ is

$$\overline{x} - t_{29,\,0.01} \frac{s}{\sqrt{n}} \leqslant \mu \leqslant \overline{x} + t_{29,\,0.01} \frac{s}{\sqrt{n}}$$

$$\therefore \quad 15.897 - 2.462 \times \frac{1.365}{\sqrt{30}} \leqslant \mu \leqslant 15.897 + 2.462 \times \frac{1.365}{\sqrt{30}}$$

$$\therefore \quad 15.283 \leqslant \mu \leqslant 16.511$$

Alternatively, using technology, a 98% confidence interval for μ is $15.28 \leqslant \mu \leqslant 16.51$.



EXERCISE 29F

1 Neville works for a software company. He keeps records of the times customers wait to receive telephone support for their software. During a six month period he logs 167 calls. The mean waiting time is 8.7 minutes, and s is 2.08 minutes. Construct a 95% confidence interval for estimating the mean waiting time for all customer calls for software support.

2 A breakfast cereal manufacturer uses a machine to deliver the cereal into plastic packets which then go into cardboard boxes. The quality controller randomly samples 75 packets and obtains a sample mean of 513.8 grams with standard deviation 14.9 grams. Construct a 90% confidence interval in which the true population mean should lie.

3 A sample of 42 patients from a drug rehabilitation program showed a mean length of stay on the program of 38.2 days with standard deviation 4.7 days. Estimate, with a 90% confidence interval, the average length of stay for all patients on the program.

4 A sample of 30 torch globes is randomly selected, and their lifetime in hours is measured:

82.5	84.7	77.6	80.5	84.4	87.0	79.8	85.6	81.4	75.6
83.0	89.6	81.5	74.2	79.8	81.5	82.7	77.7	86.2	82.3
84.1	82.5	87.7	78.6	83.2	83.5	85.9	85.7	76.2	79.8

a Find the sample mean $\overline{x}$ and standard deviation s.

b Construct a 95% confidence interval for the mean lifetime of the globes.

5 A complaint was made that a call centre took a mean time of 12 minutes before a caller was put through to an operator. After changes were made, the call centre claimed that the service had improved. To check this claim, a consumer group made 40 calls to the centre. They found the mean waiting time $\overline{x} = 8$ minutes and sample standard deviation $s = 3$ minutes.

a Assuming that 40 is large enough for the Central Limit Theorem to apply, construct a 95% confidence interval for the mean waiting time μ.

b Does the confidence interval support the call centre's claim?

6 Fabtread manufactures motorcycle tyres. Using their regular tyre tread, the stopping time for motorcycles travelling at 60 km h^{-1} is 3.45 seconds. Their production team has just designed and manufactured a new tyre tread, which they claim reduces average stopping times to 3.1 seconds. 36 stopping time measurements are taken with the new tyre tread. They are listed below in seconds:

3.26	3.14	3.07	3.15	3.09	3.38	3.41	3.32	3.48	3.27	3.02	3.05
3.15	3.22	3.01	2.96	3.14	3.27	3.02	3.17	3.22	3.04	2.96	3.18
3.21	3.06	3.34	3.27	3.10	3.02	3.16	2.94	3.15	3.29	3.21	3.13

a Construct a 95% confidence interval for the mean stopping time μ.

b On the basis of your confidence interval:

i Does it appear that the new tyre tread improves stopping times?

ii Does the production team's claim appear to be justified?

REVIEW SET 29A

1 X_1, X_2, and X_3 are independent random variables, each with mean μ and standard deviation σ. Find the mean and standard deviation of:

a $X_1 + 2X_2 + 3X_3$ **b** $2X_1 - 3X_2 + X_3$.

2 A soft drink manufacturer produces small and large bottles of drink. The volumes of both sizes of drink are independent and normally distributed with means and standard deviations given in the table.

	μ (mL)	σ (mL)
small bottles	338	3
large bottles	1010	12

a Find the probability that one large bottle selected at random will contain more than the combined contents of three smaller bottles selected at random.

b Find the probability that one large bottle selected at random will contain three times more than one smaller bottle selected at random.

3 Suppose $X \sim \text{Po}(a)$ and $Y \sim \text{Po}(b)$ are independent random variables.
Given that $\text{E}(X+Y) = a^2 + b^2$ and $\text{Var}(X+Y) = a^2 - b^2 + 2$, find the values of a and b.

4 A random variable X has mean $\mu = 35$ and standard deviation $\sigma = 8$. Find the mean and standard deviation of:

a $\overline{X}_4$ **b** $\overline{X}_{30}$.

5 The mature heights of a particular species of plant have mean 21 cm and standard deviation $\sqrt{90}$ cm. A random sample of 40 mature plants is taken and the mean height is calculated. Find the probability that this sample mean lies between 19.5 cm and 24 cm.

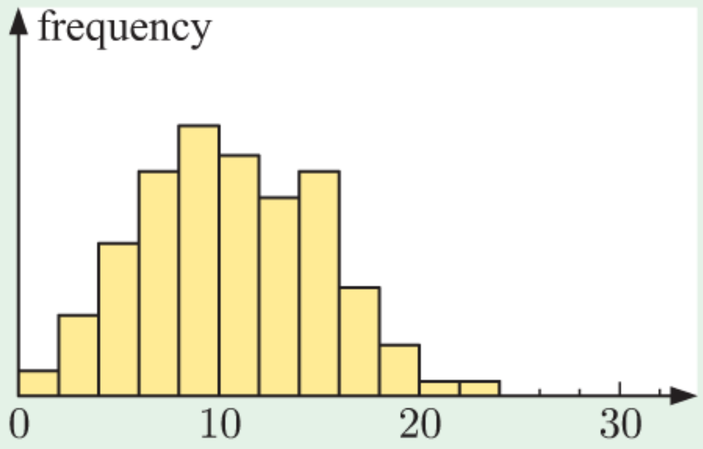
6 In order to estimate the copper content of a potential mine, drill core samples are used. The drill core is crushed and mixed before samples are removed.

Suppose X is the copper content in grams per kilogram of core, and that X has mean $\mu = 11.4$ and standard deviation $\sigma = 9.21$.

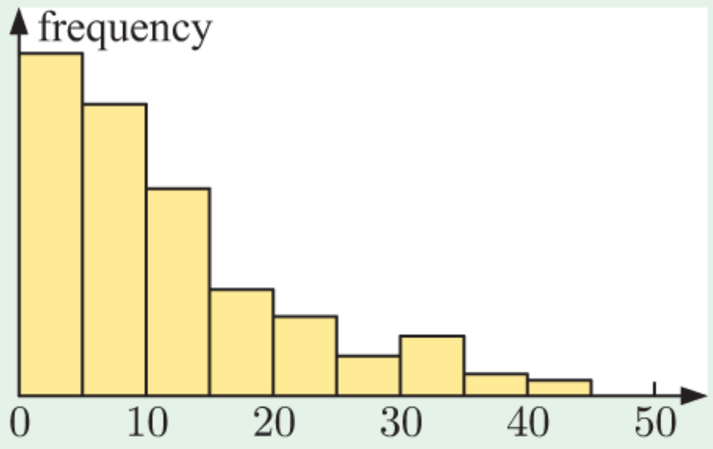
Samples of 9 drill cores are randomly chosen.

Histograms **A** and **B** show the distributions of X and $\overline{X}_9$, not necessarily in that order.

Histogram A



Histogram B



a Which of the histograms is the histogram for X? Explain your answer.

b Find the mean and standard deviation of $\overline{X}_9$.

7 The manufacturer of the breakfast cereal Maxiweet knows that the net contents of each packet has standard deviation 12.3 g. A sample of 120 packets is chosen at random, and their mean weight is found to be 596.7 grams per packet.

Construct a 95% confidence interval for the true mean weight of the population. Interpret your answer.

8 A factory which cans apricots uses a machine to deliver the fruit and syrup into cans. The quality controller randomly samples 65 cans and finds that the mean mass of contents is 828.2 grams.

a Given that the population standard deviation is 16.3 grams, construct a 95% confidence interval in which the true population mean should lie.

b What should the sample size be to construct a 95% confidence interval with half the width of that in **a**?

9 The random variable X has a normal distribution with mean μ. A randomly selected sample of size 15 is taken, and it is found that $\sum_{i=1}^{15} (x_i - \overline{x})^2 = 230$.

a Find the sample variance for this sample.

b A $k\%$ confidence interval for μ taken from this sample is $[124.94, 129.05]$.

i Find the 95% confidence interval for μ taken from this sample.

ii Find the value of k.

REVIEW SET 29B

1 X_1, X_2, and X_3 are independent random variables where $X_1 \sim \text{N}(2, \frac{1}{8})$, $X_2 \sim \text{N}(3, \frac{1}{16})$, and $X_3 \sim \text{N}(a, b)$.

a If $Y = 2X_3 - 2X_2 - X_1$, find: **i** $\text{E}(Y)$ **ii** $\text{Var}(Y)$

b Given that $\text{E}(Y) = 0$ and $\text{Var}(Y) = 1$, find the values of a and b.

c State the nature of the distribution of Y.

d Find $\text{P}(Y \geqslant 8b)$.

2 Rowan is a musician. His music is hosted on two separate music streaming services, A and S. The number of times Rowan's music is listened to per day on service A follows a Poisson distribution with mean 105.

On service S, it follows a Poisson distribution with mean 250, independently of service A.

Let T be the total number of times Rowan's music is listened to per day across both streaming services.

a State the distribution of T.

b Find the probability that Rowan's music will be listened to more than 400 times in a day.

3 Suppose X has mean 50 and standard deviation 6.

a Find the mean and standard deviation of $\overline{X}_{60}$.

b Is $\overline{X}_{60}$ approximately normally distributed? Explain your answer.

c Find $\text{P}(\overline{X}_{60} > 49)$.

4 The weight X of a packet of rice is normally distributed with mean 500 g and standard deviation 3.5 g. Let $\overline{X}_{20}$ be the average weight of a sample of 20 packets.

a Find the mean and standard deviation of $\overline{X}_{20}$.

b Is $\overline{X}_{20}$ normally distributed? Explain your answer.

5 The weight W of gourmet sausages produced by Hans is normally distributed with mean 63.4 g and standard deviation 6.33 g.

Suppose $\overline{W}_{10}$ is the mean weight of 10 randomly selected sausages.

a Find the probability that a randomly chosen sausage will weigh 60 grams or less.

b Find the mean and standard deviation of $\overline{W}_{10}$.

c Hence calculate the probability that $\overline{W}_{10} \leqslant 60$ g. Interpret your result.



6 Damien recorded the scores from a sample of 45 games at a ten-pin bowling centre:

185	126	115	191	86	103	122	107	94	135	108	113	190	152	76
87	119	125	142	106	94	131	127	103	149	110	81	72	126	145
122	141	137	95	68	101	147	150	121	100	84	72	113	171	83

a Find the sample mean $\overline{x}$ and the sample standard deviation s.

b Construct a 95% confidence interval for the mean score at the bowling centre.

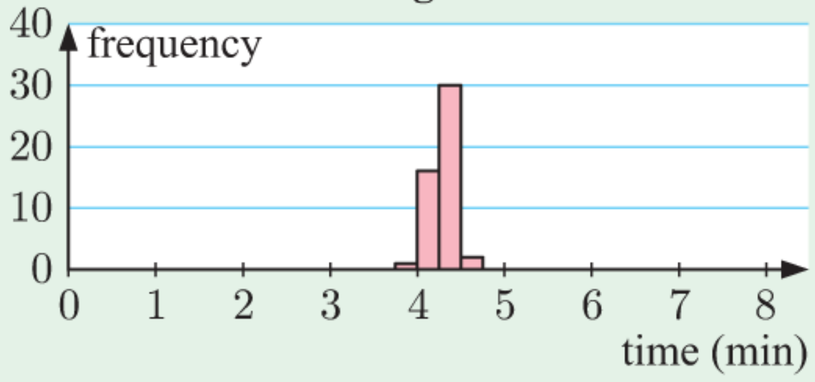
c Find the width of this confidence interval.

7 A telephone call centre handles many calls each day. Let T be the time in minutes taken to answer a call. It has been found that T has mean $\mu = 4.3$ minutes and standard deviation $\sigma = 1.2$ minutes.

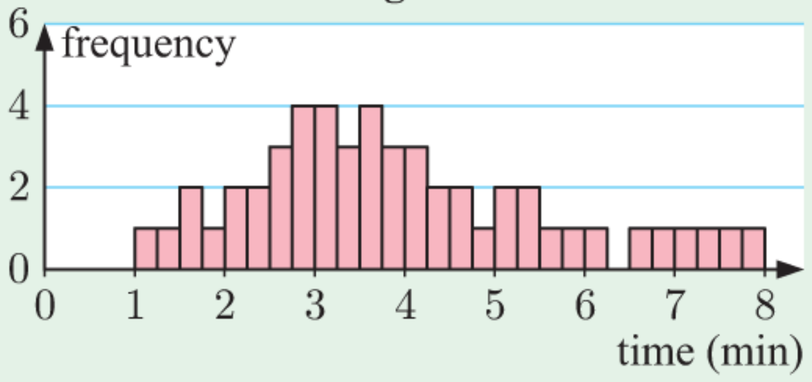
Let $\overline{T}_{100}$ be the mean time taken to answer a random sample of 100 calls.

a The histograms below show data from T and $\overline{T}_{100}$. Note that the horizontal scale and the interval widths are the same in both histograms, but the vertical scales are different.

Histogram A



Histogram B



Identify the histogram that represents a sample from $\overline{T}_{100}$. Explain your answer.

b **i** Assuming that $n = 100$ is sufficiently large, explain why the distribution of $\overline{T}_{100}$ is approximately normal with mean 4.3 minutes and standard deviation 0.12 minutes.

ii Calculate $\text{P}(\overline{T}_{100} \leqslant 4.35)$.

iii Hence calculate the probability that an operator in the call centre can answer 100 calls in less than seven and a quarter hours.

8 Free range chickens have mean meat protein content 24.9 units per kg. 50 chickens are randomly chosen from a battery cage. These chickens are fed special meals which are supposed to increase their protein content. Following the feeding program, the sample had mean meat protein content 26.1 units per kg with standard deviation 6.38 units per kg.

a Construct a 95% confidence interval for the mean meat protein content of battery cage chickens on the feeding program.

b Based on the confidence interval, is there sufficient evidence to claim that battery cage chickens on the feeding program have higher meat protein content than free range chickens?

9 A drink manufacturer produces iced tea in bottles advertised as containing 600 mL. The machines producing these drinks are set so that the average volume dispensed into each bottle is 601 mL with standard deviation 1.84 mL. The volumes dispensed into each bottle are distributed normally.

a Find the probability that an individual randomly selected bottle contains less than 598 mL.

b Find the probability that a randomly selected pack of 12 bottles has average volume less than 600 mL.

c To ensure companies meet their advertising claims, the government specifies that a randomly selected pack of 12 bottles must have average contents no less than the advertised amount.

The company wants less than 1% chance of failing to meet the requirement.

Find, to the nearest 0.1 mL, what the average volume dispensed by the machine needs to be, assuming the standard deviation is unchanged.

Chapter 30

Hypothesis testing



Contents:

A Statistical hypotheses
B The Z-test
C Critical values and critical regions
D Student's t-test
E Paired t-tests
F The two-sample t-test for comparing population means
G Hypothesis tests for the mean of a Poisson population
H Hypothesis tests for a population proportion
I Hypothesis tests for a population correlation coefficient
J Error probabilities and statistical power

OPENING PROBLEM

Frank is a market gardener who grows tomatoes.

Last year, Frank weighed a sample of 50 tomatoes and found that the mean weight was 105.3 g with standard deviation 12.41 g.

This year, Frank used a new fertiliser to try to increase the weight of his crop. Frank collected a random sample of 65 tomatoes and found the mean weight of the sample to be 110.1 g with standard deviation 13.1 g.



Using this year's data, a 95% confidence interval for the mean weight μ of Frank's crop is $106.9 \leqslant \mu \leqslant 113.4$. The previous mean of 105.3 g is entirely below this interval, suggesting the new fertiliser was effective.

Things to think about:

a Assuming the change in fertiliser had *no* effect, how can we use last year's sample to find the probability that this year's sample would have mean $\mu \geqslant 110.1$ g?

b How can this probability be used to decide whether the fertiliser was effective?

c The confidence interval for this year's sample only uses data from *one* sample. Why would it be useful to use the data from *both* samples in our analysis?

We often hear claims about **population parameters** such as the **population mean** μ or a **population proportion** p.

For example, a manufacturer of insect repellent might claim that on average, their new product is effective for longer than 6 hours. In statistics, we call this a **statistical hypothesis**.

In the previous Chapter, we tested claims about a population mean using confidence intervals. We now look at a more formal procedure to determine whether a statistical hypothesis is reasonable. We call it a **hypothesis test**.

The hypothesis tests we will study include:

- the ***Z*-test** used to test hypotheses about the mean of a normally distributed population with *known* variance
- various ***t*-tests** used instead of the Z-test when the population variance is *unknown*
- hypothesis tests that consider other parameters such as a population **proportion** or the population **correlation coefficient**.

All of these tests follow the same general procedure:

Step 1: Formulate **statistical hypotheses**.

Step 2: Choose a **significance level** for the test. This is a threshold for making a decision, like the confidence levels we saw previously.

Step 3: Use data from a sample to calculate a **test statistic**.

Step 4: Calculate a ***p*-value** for the test statistic. This is the probability of that test statistic occurring under the assumptions of one of the hypotheses.

Step 5: **Make decisions** about the hypotheses.

Step 6: **Interpret** the decision in the context of the problem.

A STATISTICAL HYPOTHESES

Suppose a claim is made that a population mean μ has the value μ_0. We call this the **null hypothesis** H_0, and we write

$$H_0: \ \mu = \mu_0.$$

This statement is assumed to be true unless we have enough evidence to reject it.

If H_0 is not rejected, we accept that the population mean is μ_0. So, the null hypothesis is a statement that there is *no difference* between μ and μ_0.

If H_0 is rejected, we accept that there *is a difference* between μ and μ_0. This statement is called the **alternative hypothesis** H_1.

ONE-TAILED AND TWO-TAILED ALTERNATIVE HYPOTHESES

Given the null hypothesis $H_0: \ \mu = \mu_0$, the alternative hypothesis could be:

- $H_1: \ \mu > \mu_0$ **(one-tailed hypothesis)**
- $H_1: \ \mu < \mu_0$ **(one-tailed hypothesis)**
- $H_1: \ \mu \neq \mu_0$ **(two-tailed hypothesis**, as $\mu \neq \mu_0$ could mean $\mu > \mu_0$ or $\mu < \mu_0$).

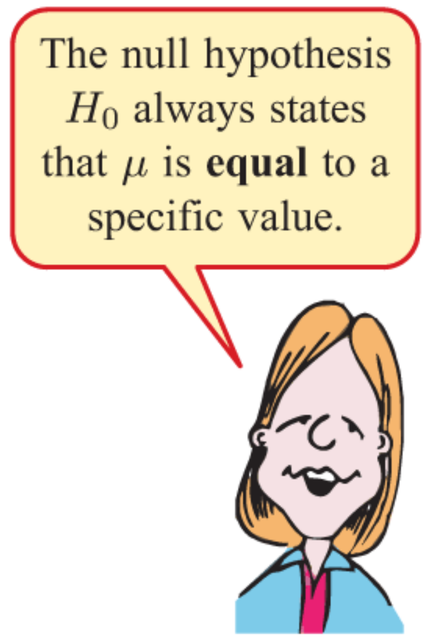
Consider the insect repellent example on the previous page:

- If the manufacturer of the new brand wants evidence that the new product is *superior* in protection time, the hypotheses would be:

  $H_0: \ \mu = 6$ {the new product gives the same protection as the old ones}
  $H_1: \ \mu > 6$ {the new product protects for longer than the old ones}.

- If a competitor wants evidence that the new product has an *inferior* protection time, the hypotheses would be:

  $H_0: \ \mu = 6$ {the new product gives the same protection as the old ones}
  $H_1: \ \mu < 6$ {the new product protects for less time than the old ones}.



- If a market researcher studying all products on the market wants to show that the new product *differs* from the old ones, but is not concerned whether the protection time is more or less, the hypotheses would be:

  $H_0: \ \mu = 6$ {the new product gives the same protection as the old ones}
  $H_1: \ \mu \neq 6$ {the new product gives a different protection time compared with the old ones}.

ERROR TYPES

In the decision to accept or reject the null hypothesis H_0, there are two possible errors:

- A **Type I error** is when we make the mistake of rejecting H_0 when H_0 is in fact true.
- A **Type II error** is when we make the mistake of accepting H_0 when H_0 is in fact false.

For example, if a coin is fair then the population proportion of *heads* it produces is $p = 0.5$.

- A Type I error would be deciding a fair coin is biased because we observe 10 heads in 10 tosses. Although improbable, it is still possible to obtain this result with a fair coin.

- A Type II error would be accepting that a biased coin is fair because we observe 6 heads in 10 tosses, and we know this can occur with a fair coin with reasonable probability.

EXERCISE 30A

1 Explain what is meant by:

a a Type I error **b** a Type II error

c the null hypothesis **d** the alternative hypothesis.

2 A statistician wishes to test H_0: $\mu = 20$ against H_1: $\mu > 20$. State the type of error made if:

a the mean is actually 20 but the statistician concludes that the mean exceeds 20

b the population mean is actually 21.8 but the statistician concludes that the mean is 20.

3 A researcher wishes to test H_0: $\mu = 40$ against H_1: $\mu \neq 40$. State the type of error made if she concludes that:

a the mean is 40 when it is in fact 38.1 **b** the mean is not 40 when it actually is 40.

4 In many countries where juries are used in trials, a person is *presumed innocent until proven guilty.*
In this court system, the null hypothesis is
H_0: the person on trial is innocent.



a State the alternative hypothesis H_1.

b If an innocent person is judged guilty, what type of error has been made?

c If a guilty person is judged as innocent, what type of error has been made?

5 A researcher conducts experiments to determine the effectiveness of two anti-dandruff shampoos X and Y. He tests the hypotheses:

H_0: X and Y have the same effectiveness H_1: X is more effective than Y.

What decision would cause:

a a Type I error **b** a Type II error?

6 Current torch globes have a mean life of 80 hours. Globe Industries are considering mass production of a new globe they believe will last longer.

a If Globe Industries wants to demonstrate that their new globe lasts longer, what set of hypotheses should they consider?

b The new globe costs less to produce, so Globe Industries will adopt it unless it has an inferior lifespan to the old type. What set of hypotheses should they now consider?

7 The top speed of submarines currently produced by a manufacturer is 26.3 knots. When their engineers modify the design to reduce drag, they believe that the maximum speed will be increased. What set of hypotheses should they consider to test whether or not the new design is faster?

8 Whitex produce copy paper, and the weight of their copy paper is given as 80 g per m^2. The company wants to determine whether this information is correct. What set of hypotheses should be considered?

9 The average peak-hour travel time along a particular stretch of road is currently 27 minutes. To help reduce travel times, electronic signs displaying real-time information are erected. If the travel times improve, the signs will be more widely implemented. What set of hypotheses should be considered?

10 Brand A's muesli bars have 3 g of fat. Brand B claims that their muesli bars have 10% less fat than Brand A's muesli bars. Brand A wants evidence that Brand B's muesli bars have more fat than is claimed. What set of hypotheses should they consider?

11 In 2010, 21% of the British population smoked. A researcher wants to know if there has been any significant decrease in the proportion of smokers in the population since 2010. What set of hypotheses should the researcher consider?

12 A pharmaceutical company conducts a clinical trial to test the effectiveness of a new cancer treatment. Only 30% of patients using the best currently available treatment survive more than 5 years after diagnosis. What set of hypotheses should the company consider?

13 In the last election, 42% of voters voted for Party A. An independent polling agency wants to gauge voting trends for the next election. What set of hypotheses should the polling agency consider if they are interested:

a only in an increase in the proportion of Party A supporters

b in *any* change in the proportion of Party A supporters?

DISCUSSION

- If H_0 is false, does this mean H_1 must be true?
- Can H_0 *ever* be true for a continuous random variable?
- Why do we talk about *accepting* or *rejecting* H_0, rather than deciding whether it is true or false?

B THE Z-TEST

The **Z-test** is used to test hypotheses about a **population mean μ** when:

- the population is **normally distributed**
- the **population variance σ^2 is known**.

We have just seen how to formulate hypotheses about population means.

We now move on to the remaining steps of the test procedure, remembering that the concepts are applicable to *all* of the hypothesis tests we will study.

THE TEST STATISTIC

A **test statistic** is a *random variable* that summarises the information in a sample.

The distribution of the test statistic under the assumptions of H_0 is called the **null distribution**.

In the context of the Z-test, we have seen that for a population which is normally distributed with mean μ and standard deviation σ, the sample mean $\overline{X}_n \sim \text{N}\left(\mu, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$.

If the population is *not* normally distributed but the sample size is sufficiently large, we can apply the Central Limit Theorem to assume $\overline{X}_n$ is normal.

Under the assumptions of H_0, $\mu = \mu_0$, so $\overline{X}_n \sim \mathrm{N}\left(\mu_0, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$.

$\therefore$ the Z-transformation of $\overline{X}_n$, $Z = \dfrac{\overline{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}}$, has the standard normal distribution $\mathrm{N}(0, 1^2)$.

Consider a statistical hypothesis test of H_0: $\mu = \mu_0$ for a normally distributed population with *known* standard deviation σ. Given a sample of size n with observed sample mean $\overline{x}$:

- the **test statistic** is $\boldsymbol{Z = \dfrac{\overline{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}}}$ which has **observed value** $\boldsymbol{z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}}$
- the **null distribution** is $\boldsymbol{Z \sim \mathrm{N}(0, 1^2)}$.

For example, consider the insect repellent example from the previous Section.

Suppose the population standard deviation $\sigma = 15$ minutes $= 0.25$ hours is known.

When a researcher takes a random sample of 50 bottles of the new product, they find that the mean protection time $\overline{x} = 6.12$ hours. The observed value of the test statistic is $z = \dfrac{6.12 - 6}{\frac{0.25}{\sqrt{50}}} \approx 3.39$.

DISCUSSION

Why do we use $Z = \dfrac{\overline{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}}$ as the test statistic instead of $\overline{X}_n$ directly?

CALCULATING THE p-VALUE

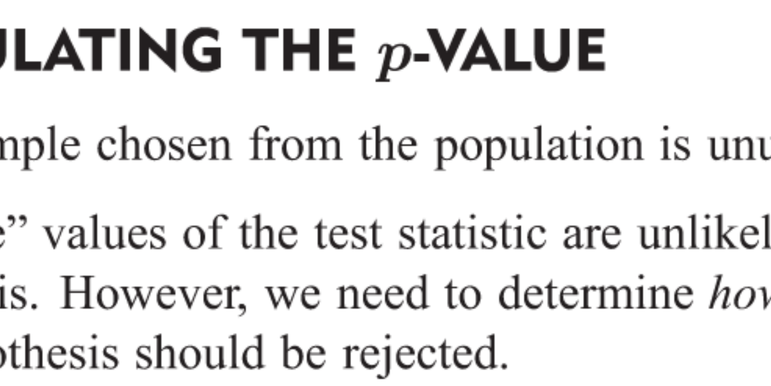
If the sample chosen from the population is unusual, the observed sample mean may be "extreme".

"Extreme" values of the test statistic are unlikely, so observing such a value is evidence against the null hypothesis. However, we need to determine *how* extreme a value should be before we conclude that the null hypothesis should be rejected.

The ***p*-value** of a test statistic is the probability of that result being observed if H_0 is true.

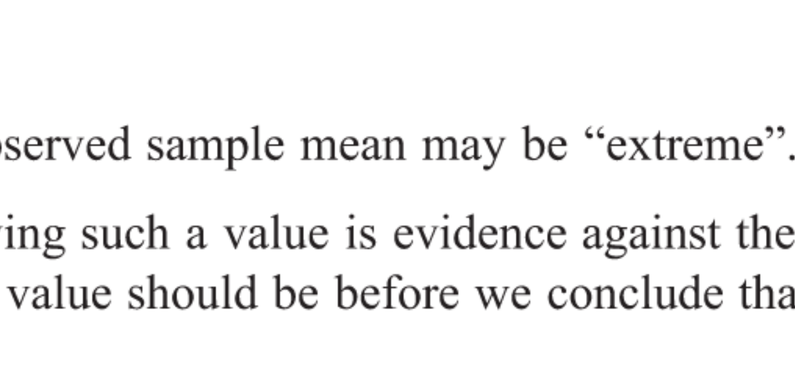
The meaning of "extreme" depends on whether the alternative hypothesis is one-tailed or two-tailed:

- If H_1 is a one-tailed alternative hypothesis, we use *one* tail of the null distribution to calculate the p-value.
  - If H_1: $\mu < \mu_0$, we use the lower tail.



***p*-value** $= \mathbf{P}(Z \leqslant z)$

  - If H_1: $\mu > \mu_0$, we use the upper tail.

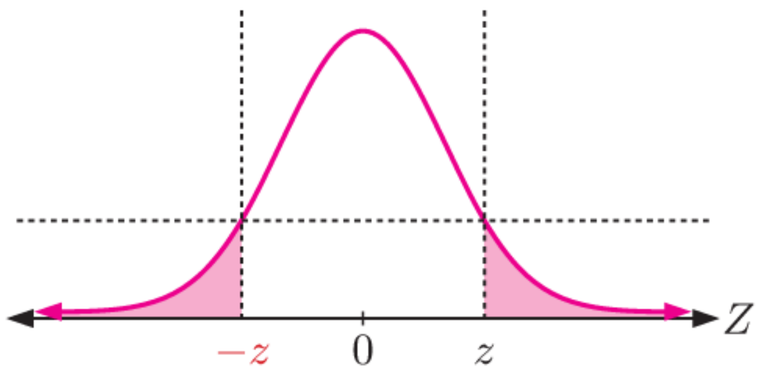


***p*-value** $= \mathbf{P}(Z \geqslant z)$

- If H_1 is the two-tailed alternative hypothesis $H_1: \mu \neq \mu_0$, then we must consider *both* tails of the null distribution. We define "extreme" values as those which have probability less than or equal to that of the test statistic.

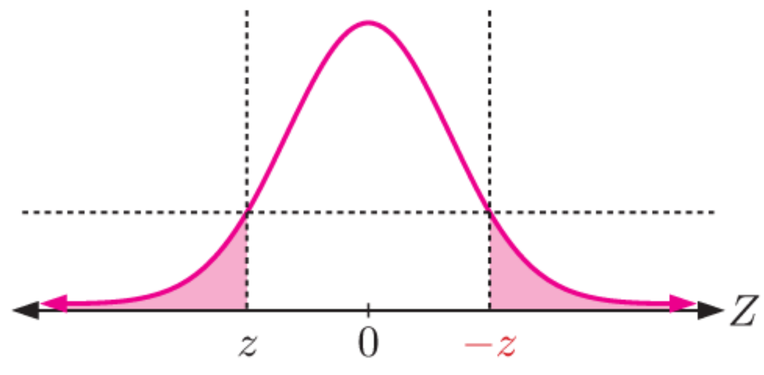
If $z \geqslant 0$,

$$p\text{-value} = \mathrm{P}(Z \geqslant z \text{ or } Z \leqslant -z)$$
$$= 2 \times \mathrm{P}(Z \geqslant z) \quad \{\text{symmetry}\}$$



If $z < 0$,

$$p\text{-value} = \mathrm{P}(Z \geqslant -z \text{ or } Z \leqslant z)$$
$$= 2 \times \mathrm{P}(Z \geqslant -z) \quad \{\text{symmetry}\}$$



So, for a two-tailed alternative hypothesis, $\boldsymbol{p\text{-value} = 2 \times \mathrm{P}(Z \geqslant |z|)}$.

MAKING DECISIONS

Although "extreme" values are unlikely to occur, it is still *possible* to observe such values in a sample. We therefore need a rule which defines how much evidence is required to reject the null hypothesis.

The **significance level** α of a statistical hypothesis test is the largest p-value that would result in rejecting H_0. Any p-value less than or equal to α results in H_0 being rejected.

If a statistical hypothesis test has significance level α, the probability of a Type I error is α.

The significance level may be given as a decimal or a percentage.

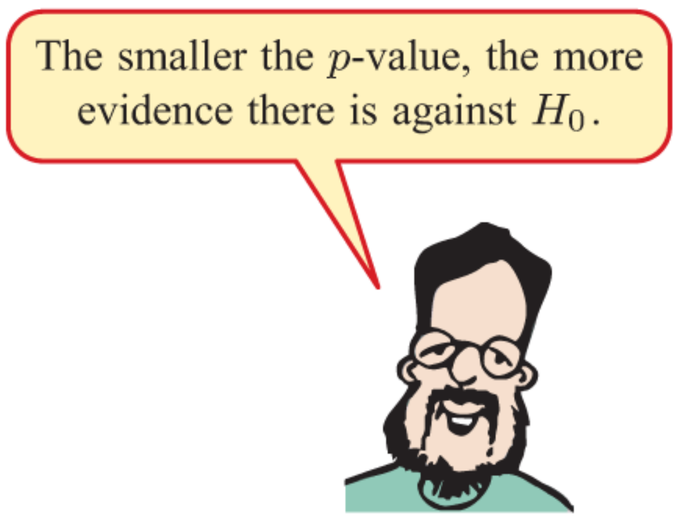
In our insect repellent example, suppose the researcher was concerned about any change in the average protection time. The researcher decides to test the hypothesis $H_0: \mu = 6$ against $H_1: \mu \neq 6$ at a significance level $\alpha = 0.01$.

Using the test statistic $z \approx 3.39$ and $Z \sim \mathrm{N}(0, 1^2)$,

the p-value $\approx 2 \times \mathrm{P}(Z \geqslant |3.39|)$

$$\approx 6.89 \times 10^{-4}$$

Since the p-value is less than the significance level α, the researcher has enough evidence to reject H_0.



THE Z-TEST FOR THE MEAN OF A POPULATION WITH KNOWN VARIANCE

Step 1: State the **null hypothesis** $H_0: \mu = \mu_0$ and **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the observed value of the **test statistic**: $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$.

Step 4: Calculate the ***p*-value** using $Z \sim \mathrm{N}(0, 1^2)$ as follows:

- If $H_1: \mu > \mu_0$, p-value $= \mathrm{P}(Z \geqslant z)$.
- If $H_1: \mu < \mu_0$, p-value $= \mathrm{P}(Z \leqslant z)$.
- If $H_1: \mu \neq \mu_0$, p-value $= 2 \times \mathrm{P}(Z \geqslant |z|)$.

Step 5: Reject H_0 if p-value $< \alpha$, otherwise accept H_0.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Important:

We **must** choose a significance level **before** we test the hypothesis. Otherwise, we can be tempted to choose a significance level to give the test outcome that we desire. For example, it is *not appropriate* to calculate a p-value and then select a value of α so that H_0 will be rejected.

Example 1 **Self Tutor**

The manager of a restaurant chain goes to a seafood wholesaler and inspects a large catch of over 50 000 prawns. It is known that the population is normally distributed with standard deviation 4.2 grams. He will buy the catch if the mean weight exceeds 55 grams per prawn.

A random sample of 60 prawns is taken, and the mean weight is 56.2 grams. At a 5% significance level, should the manager purchase the catch?

Step 1: Let μ be the population mean weight per prawn.
The hypotheses that should be considered are:
H_0: $\mu = 55$ {the mean weight is 55 grams per prawn}
H_1: $\mu > 55$ {the mean weight exceeds 55 grams per prawn}

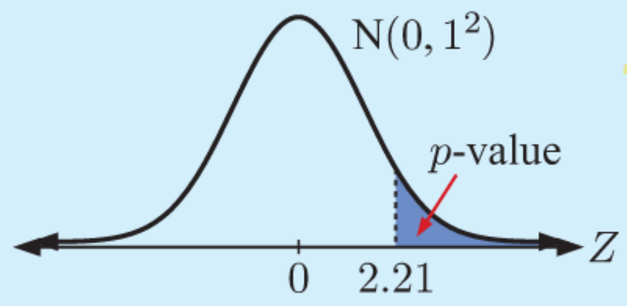
Step 2: The significance level is $\alpha = 0.05$.

Step 3: $\overline{x} = 56.2$ grams, $\sigma = 4.2$ grams, $n = 60$
The observed value of the test statistic is $z = \dfrac{56.2 - 55}{\frac{4.2}{\sqrt{60}}} \approx 2.21$

A sketch of the null distribution curve is useful to remind us what the p-value represents.



Step 4: Since H_1: $\mu > 55$,
p-value $= \text{P}(Z \geqslant z)$ where $Z \sim \text{N}(0, 1^2)$
$\approx \text{P}(Z \geqslant 2.21)$
≈ 0.0134



Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% significance level.

Step 6: Since we have accepted H_1, we conclude that the mean weight exceeds 55 grams per prawn. The manager should purchase the catch.

EXERCISE 30B

1 A population has known standard deviation $\sigma = 3.97$. A sample of size 36 is taken and the sample mean $\overline{x} = 23.75$. We are required to test the hypothesis H_0: $\mu = 25$ against H_1: $\mu < 25$.

a Find: **i** the test statistic **ii** the p-value.

b What decision should be made at a 5% level?

2 A statistician believes that a population which has a standard deviation of 12.9, has a mean μ that is greater than 80. To test this he takes a random sample of 200 measurements, and finds the sample mean is 83.1. He then performs a hypothesis test with significance level $\alpha = 0.01$.

a Write down the null and alternative hypotheses.

b Find the observed value of the test statistic and state the null distribution.

c Calculate the p-value.

d Make a decision to accept or reject H_0.

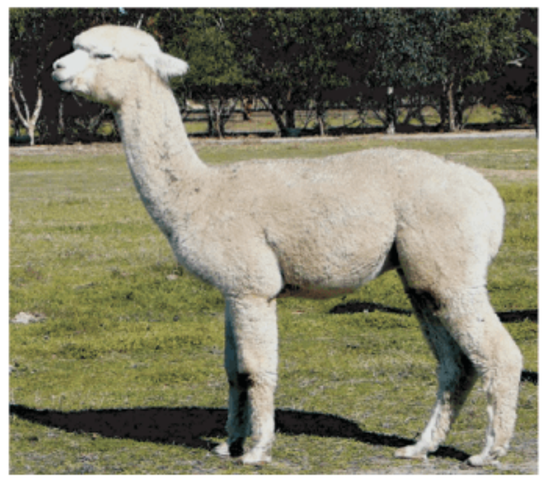
e State the conclusion for the test.

3 Bags of salted cashew nuts state that their net contents is 100 g. The manufacturer knows that the standard deviation of the population is 1.6 g. A customer claims that the bags have been lighter in recent purchases, so the factory quality control manager decides to investigate. He samples 40 bags and finds that their mean weight is 99.4 g.

Perform a hypothesis test at the 5% level of significance to determine whether the customer's claim is valid.

4 An alpaca breeder wants to produce fleece which is extremely fine. In 2013, his herd had mean fineness 20.3 microns with standard deviation 2.89 microns. The standard deviation remains relatively constant over time. In 2017, a sample of 80 alpacas from the herd was randomly selected, and the mean fineness was 19.2 microns.

Perform a two-tailed Z-test at the 5% level of significance to determine whether the herd fineness has changed.



INVESTIGATION 1 MULTIPLE TESTING AND STATISTICAL FALLACY

In many applications of hypothesis testing, it is common to conduct multiple identical or very similar hypothesis tests simultaneously. For example, in genetics an experiment called a **DNA microarray** is used to measure expression levels of thousands of genes, each with their own set of hypotheses to test.

In this Investigation, we will explore the effects of conducting multiple hypothesis tests on the interpretation of individual outcomes.

What to do:

1 A normally distributed population has mean μ and standard deviation $\sigma = 5$.
Consider the following hypotheses for this population: H_0: $\mu = 2$
H_1: $\mu \neq 2$

Click on the icon to run a computer simulation which generates samples of size 10 from the $N(2, 5^2)$ distribution. The above hypotheses are tested for each sample at a significance level of α, and a p-value is calculated.



a Write down the formula used to calculate the test statistic given a sample mean $\overline{x}$ and population standard deviation σ.

b Set $\alpha = 0.05$. Copy and complete the following table by generating m samples and counting the number of times H_0 is rejected.

m	*Number of times H_0 is rejected*	*Proportion of samples where H_0 was rejected*
20		
50		
100		
400		
1000		
10 000		

c Repeat **b** for:

i $\alpha = 0.1$ **ii** $\alpha = 0.025$ **iii** $\alpha = 0.01$.

Comment on your results.

2 For the simulation in **1**, explain why:

- **a** H_0 is true in every sample that the simulation generates
- **b** P(incorrectly reject H_0) $= \alpha$ for each sample
- **c** the *expected number* of samples where H_0 is incorrectly rejected is $m\alpha$.

3 Sabeen is a psychologist. She is writing a journal article about the effect of diet cola on a person's ability to concentrate. To account for possible confounding factors, Sabeen divides her data into 10 different age groups for each gender. For each age group and gender, she conducts a hypothesis test and obtains a p-value. Her results are:

Age group	10 - 14	15 - 19	20 - 24	25 - 29	30 - 34
Male	0.296	0.143	0.305	0.378	0.169
Female	0.814	0.022	0.125	0.301	0.432

Age group	35 - 39	40 - 44	45 - 49	50 - 54	55+
Male	0.699	0.221	0.078	0.790	0.423
Female	0.987	0.643	0.448	0.672	0.789

- **a** Which age group and gender do you think Sabeen is most likely to report on in her journal article? Explain your answer.
- **b** Sabeen's colleague Mysha repeats Sabeen's experiment. She samples people exclusively from the group you identified in **a**. Do you think Mysha is likely to replicate Sabeen's results for this group? Explain your answer.

THEORY OF KNOWLEDGE

Sir Ronald Aylmer Fisher was an English statistician and biologist. He was known for his work in both agriculture and statistics, combining the disciplines with his work in classical statistics and significance testing.

In 1952 Fisher published a book titled *Statistical Methods for Research Workers* which is best known for the following statement about p-values:

> "*The value for which* $p = 0.05$, *or* 1 *in* 20 [....] *it is convenient to take this point as a limit in judging whether a deviation is to be considered significant or not.*"

Today, a significance level of $\alpha = 0.05$ is still widely quoted in scientific journals when testing for significance.

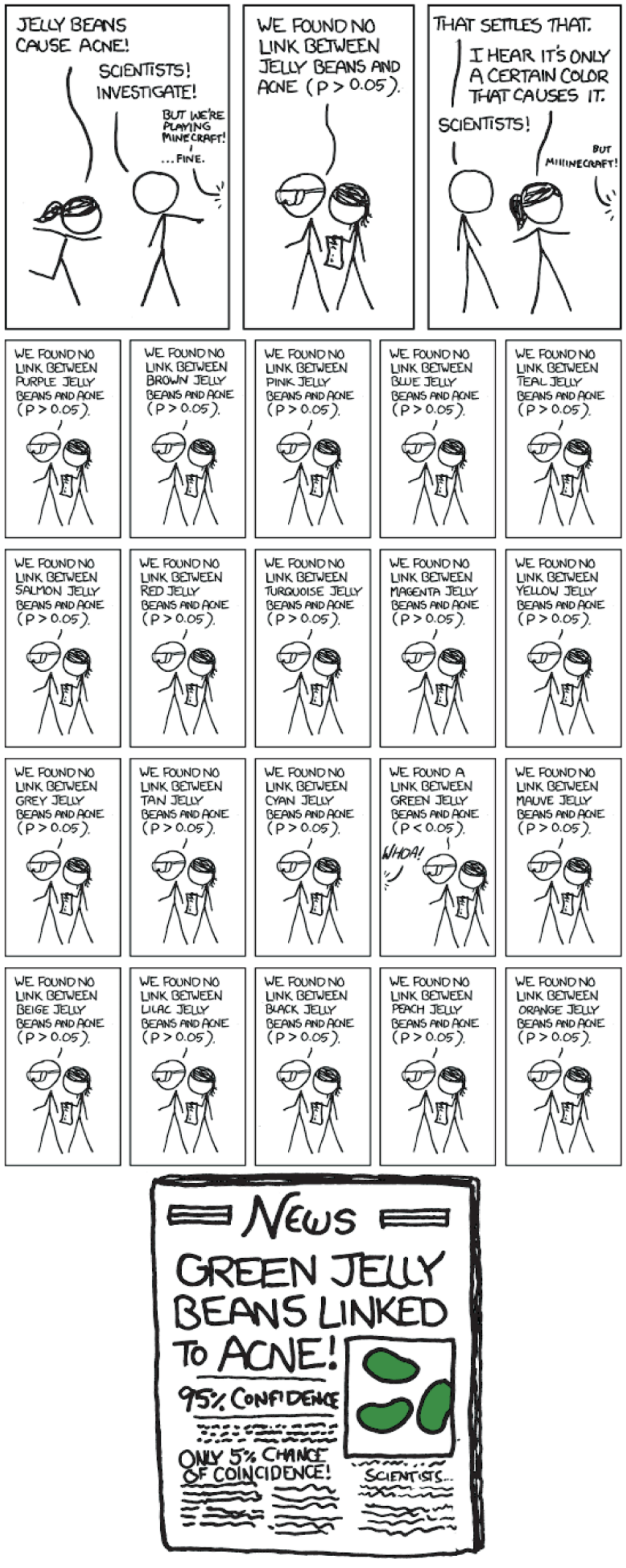


Sir Ronald Fisher

1 If someone tells you that they are 95% confident in something, do you normally stop to consider the 5% chance that the person is wrong? Do you think you *should*?

2 Suppose a researcher is writing a report for a medical journal.

- **a** Is it realistic to expect the researcher to be 100% confident in their findings?
- **b** What is *your* expectation of how confident a researcher should be, in order that they publish their results?
- **c** Do you think it is important that the researcher tells you how confident they are?


JELLY BEANS CAUSE ACNE!
SCIENTISTS! INVESTIGATE!
BUT WE'RE PLAYING MINECRAFT!
... FINE.
WE FOUND NO LINK BETWEEN JELLY BEANS AND ACNE (P > 0.05).
THAT SETTLES THAT.
I HEAR IT'S ONLY A CERTAIN COLOR THAT CAUSES IT.
SCIENTISTS!
BUT MIIIINECRAFT!
WE FOUND NO LINK BETWEEN PURPLE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BROWN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN PINK JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BLUE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TEAL JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN SALMON JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN RED JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TURQUOISE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN MAGENTA JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN YELLOW JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN GREY JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN CYAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND A LINK BETWEEN GREEN JELLY BEANS AND ACNE (P < 0.05).
WHOA!
WE FOUND NO LINK BETWEEN MAUVE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BEIGE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN LILAC JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BLACK JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN PEACH JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN ORANGE JELLY BEANS AND ACNE (P > 0.05).
News
GREEN JELLY BEANS LINKED TO ACNE!
95% CONFIDENCE
ONLY 5% CHANCE OF COINCIDENCE!
SCIENTISTS...

C CRITICAL VALUES AND CRITICAL REGIONS

In a statistical hypothesis test:

- The **critical region** $\mathcal{C}$ is the set of all values of the test statistic which result in H_0 being rejected.
- The **acceptance region** $\mathcal{A}$ is the set of all values of the test statistic which result in H_0 being accepted.
- We can make a decision about H_0 using the test statistic directly by comparing it to a **critical value** c which is the value in the critical region which has the largest p-value associated with it.

In a Z-test with significance level α, the **critical value** is the value of the test statistic for which the p-value $= \alpha$.

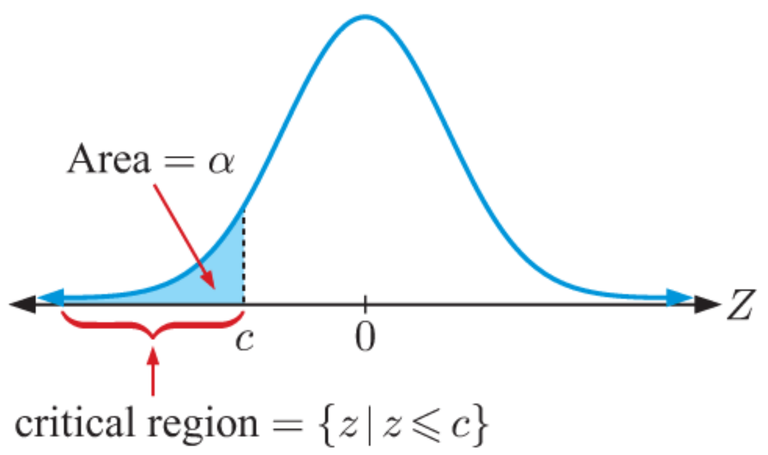
ONE-TAILED ALTERNATIVE HYPOTHESES

For H_1: $\mu < \mu_0$, the critical value c satisfies:

$$\mathrm{P}(Z \leqslant c) = \alpha$$

$$\therefore \quad c = -z_\alpha$$

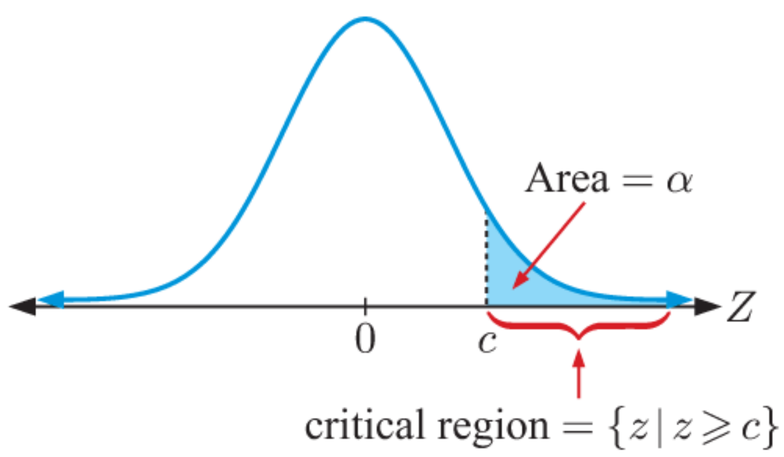
Any value in the shaded area has a p-value less than or equal to that of c, so the shaded area corresponds to the critical region.



Similarly, for H_1: $\mu > \mu_0$, the critical value c satisfies:

$$\mathrm{P}(Z \geqslant c) = \alpha$$

$$\therefore \quad c = z_\alpha$$



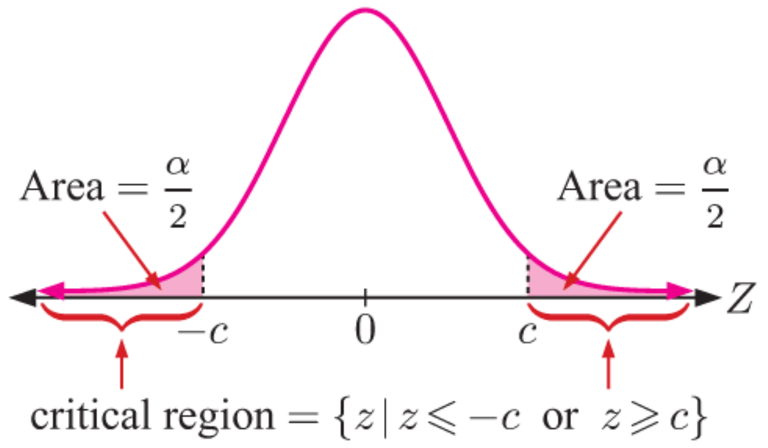
TWO-TAILED ALTERNATIVE HYPOTHESES

If H_1: $\mu \neq \mu_0$, then the critical value c satisfies:

$$2 \times \mathrm{P}(Z \geqslant |c|) = \alpha$$

$$\therefore \quad \mathrm{P}(Z \geqslant |c|) = \frac{\alpha}{2}$$

$$\therefore \quad c = \pm z_{\frac{\alpha}{2}}$$



There are *two* critical values for a two-tailed alternative. However, because of symmetry, we only need to perform one calculation.

Example 2 **Self Tutor**

Suppose μ is the mean time taken by students to travel to school in minutes. It is known that the population is normally distributed with standard deviation $\sigma = 2.3$ minutes. Miro wants to test the following hypotheses at the 5% level:

H_0: $\mu = 5$ {the mean travel time is 5 minutes}
H_1: $\mu > 5$ {the mean travel time is *greater* than 5 minutes}

a For Miro's Z-test, find:

i the critical value c **ii** the critical region $\mathcal{C}$ **iii** the acceptance region $\mathcal{A}$.

b Miro randomly selects 34 students and finds that the mean travel time for his sample is $\overline{x} = 6.4$ minutes.

i Calculate the observed value of the test statistic.

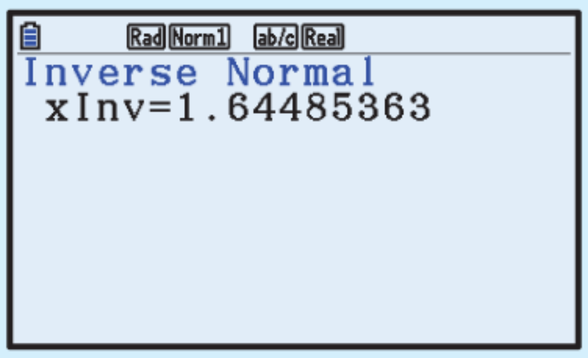
ii Is this sufficient evidence to reject H_0?

a **i** We require c such that $P(Z \geqslant c) = 0.05$

$\therefore \quad c = z_{0.05} \approx 1.64$

ii $\mathcal{C} = \{z \mid z \geqslant c\} \approx \{z \mid z \geqslant 1.64\}$

iii $\mathcal{A} = \{z \mid z < c\} \approx \{z \mid z < 1.64\}$

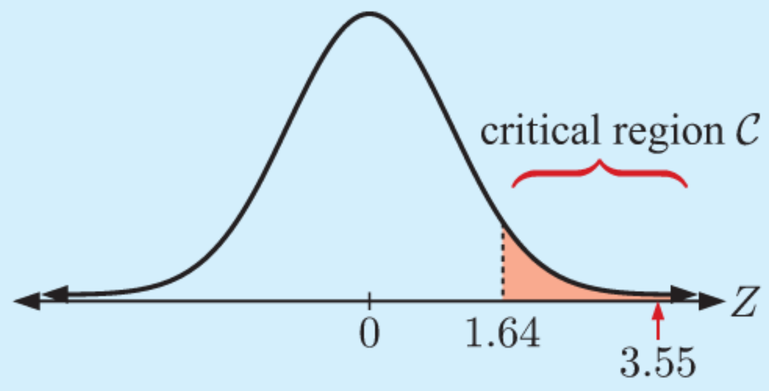


b **i** $\overline{x} = 6.4$ minutes, $\sigma = 2.3$ minutes, $n = 34$

The observed value of the test statistic is

$$z = \frac{6.4 - 5}{\frac{2.3}{\sqrt{34}}} \approx 3.55$$

ii $3.55 \in \mathcal{C}$, so there is sufficient evidence to reject H_0.



EXERCISE 30C

1 **a** Explain why the following sets of hypotheses, each tested at the same significance level α, have the same critical value, critical region, and acceptance region.

H_0: $\mu = 0.1$
H_1: $\mu < 0.1$

and

H_0: $\mu = 120$
H_1: $\mu < 120$

b Could we make the same conclusion in **a** if the alternative hypotheses were both two-tailed instead? Explain your answer.

2 The tread depth of car tyres is known to be normally distributed with standard deviation $\sigma = 0.5$ mm.

An average tread depth of at least 1.6 mm is required by law. A police officer wants to determine if the mean tyre tread depth of a taxi company's fleet satisfies the legal requirement.

a State the hypotheses that should be tested.

b For this test, and a 2.5% significance level, find:

i the critical value **ii** the critical region **iii** the acceptance region.

c The police officer randomly selects 10 taxis and measures the tread depth of each taxi's four tyres. The sample mean is found to be $\overline{x} = 1.51$ mm. Is this sufficient evidence to claim that the mean tread depth of the fleet does not meet the legal requirement?

3 The daily energy intake of students at a school is normally distributed with standard deviation $\sigma = 300$ kJ. Last year the mean energy intake of the students was 9210 kJ. To test whether the mean has changed, 50 students were selected, and their mean energy intake was $\overline{x} = 9100$ kJ.

a For this test, and a 2% significance level, find:

i the critical region **ii** the acceptance region.

b Is there sufficient evidence to claim that the mean energy intake has changed?

Example 3 **Self Tutor**

A population is normally distributed with mean μ and standard deviation $\sigma = 7$. To test the hypothesis H_0: $\mu = 40$ against H_1: $\mu \neq 40$, a random sample of size 60 was taken and found to have mean $\overline{x}$.

For what values of $\overline{x}$ will the null hypothesis be rejected at the 5% level?

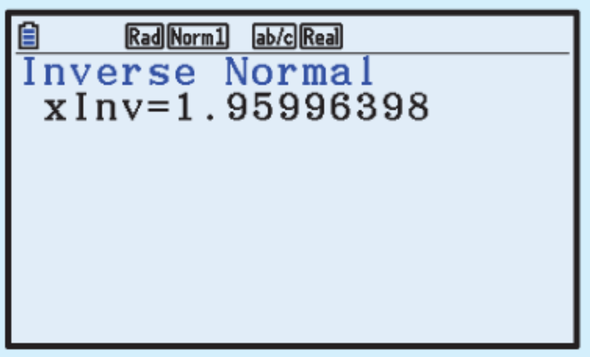
The observed test statistic is $z = \dfrac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \dfrac{\overline{x} - 40}{\frac{7}{\sqrt{60}}}$.

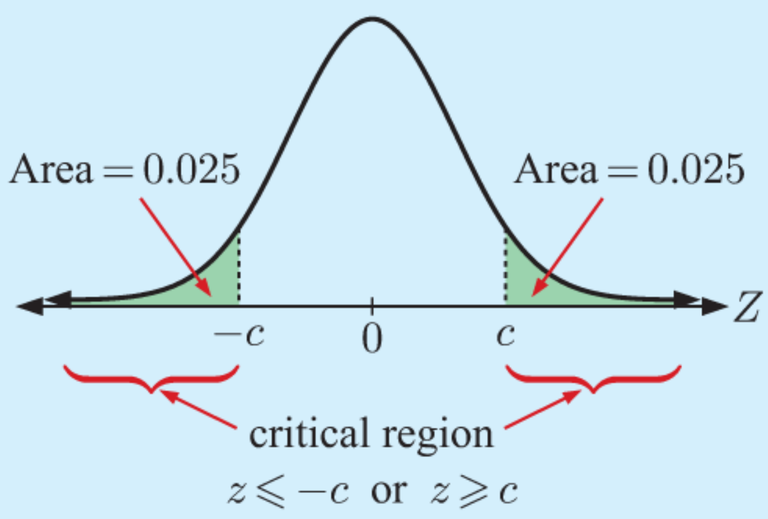
The critical value c satisfies $\mathrm{P}(Z \leqslant -c \text{ or } Z \geqslant c) = 0.05$

$\therefore \mathrm{P}(Z \geqslant c) = 0.025$

$\therefore c = z_{0.025}$

≈ 1.96





H_0 will be rejected if: $z \leqslant -1.96$ or $z \geqslant 1.96$

$\therefore \dfrac{\overline{x} - 40}{\frac{7}{\sqrt{60}}} \leqslant -1.96$ or $\dfrac{\overline{x} - 40}{\frac{7}{\sqrt{60}}} \geqslant 1.96$

$\therefore \overline{x} \leqslant 40 - 1.96 \times \frac{7}{\sqrt{60}}$ or $\overline{x} \geqslant 40 + 1.96 \times \frac{7}{\sqrt{60}}$

$\therefore \overline{x} \leqslant 38.2$ or $\overline{x} \geqslant 41.8$

So, the null hypothesis will be rejected if $\overline{x} \leqslant 38.2$ or $\overline{x} \geqslant 41.8$.

4 A normally distributed population has mean μ and standard deviation $\sigma = 4$. To test the hypothesis H_0: $\mu = -23$ against H_1: $\mu \neq -23$, a random sample of size 100 was taken and found to have mean $\overline{x}$.

For what values of $\overline{x}$ will the null hypothesis be rejected at the 5% level?

5 The volume of orange juice dispensed by a machine is normally distributed with standard deviation 3 mL. A quality controller is required to adjust the machine if the mean volume dispensed is not 504 mL. In a routine test, the quality controller finds the mean volume $\overline{x}$ of 20 randomly selected bottles.

For what values of $\overline{x}$ should the quality controller *not* adjust the machine? Use a 2% level of significance.

DISCUSSION

Consider how we tested claims about a population mean μ using **confidence intervals** in the previous Chapter.

1. How is this approach related to the Z-test procedure using a critical region?
2. Is a $(1-\alpha)\times 100\%$ confidence interval for μ equivalent to the acceptance region for a Z-test of H_0: $\mu=\mu_0$ versus H_1: $\mu\neq\mu_0$ with significance level α?

D STUDENT'S t-TEST

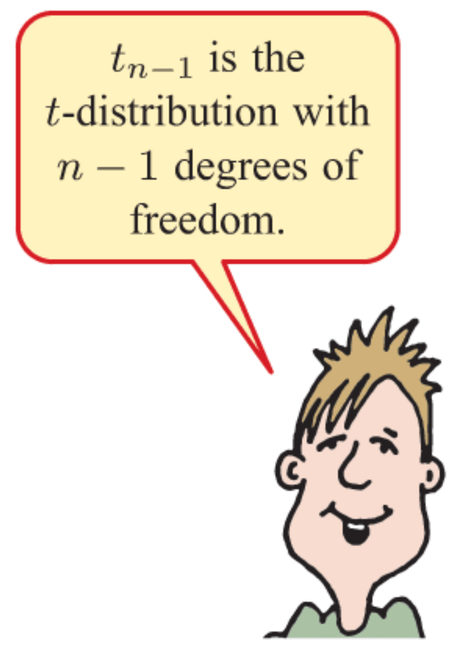
When the population variance σ^2 is unknown, we must use the sample variance S_{n-1}^2 to estimate it. We have already seen that when we substitute S_{n-1} for σ in the test statistic of the Z-test, we obtain the statistic $T=\dfrac{\overline{X}_n-\mu_0}{\frac{S_{n-1}}{\sqrt{n}}}$ which has a ***t*-distribution**.

Consider a statistical hypothesis test of H_0: $\mu=\mu_0$ for a normally distributed population with an **unknown** standard deviation. Given a sample of size n, with observed sample mean $\overline{x}$ and sample standard deviation s:

- the **test statistic** is $T=\dfrac{\overline{X}_n-\mu_0}{\frac{S_{n-1}}{\sqrt{n}}}$ which has

  observed value $t=\dfrac{\overline{x}-\mu_0}{\frac{s}{\sqrt{n}}}$

- the **null distribution** is $T\sim t_{n-1}$.



HISTORICAL NOTE

William Sealy Gosset (1876 - 1937) studied chemistry and mathematics at New College, Oxford University. In 1899 he moved to Dublin, Ireland, to work for the brewery of Arthur Guinness & Son. His desire was to improve the production process by selecting the best yielding varieties of barley. Through his study and work with **Karl Pearson** from University College, London, he devised a **test statistic**. His work was published in 1908 in the journal *Biometrika*, but using the name **Student** because of concern by Guinness that other brewers may use his work to their advantage.

Gosset's test statistic was revised by **Sir Ronald Aylmer Fisher** (1890 - 1962) who recognised the importance of Gosset's work. Fisher called the new statistic t, completing the name **Student's *t*-test**.

The t-distribution is sometimes **Student's *t*-distribution** due to its association with Student's t-test.



William Sealy Gosset

STUDENT'S t-TEST FOR THE MEAN OF A POPULATION WITH UNKNOWN VARIANCE

Step 1: State the **null hypothesis** H_0: $\mu = \mu_0$ and **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the observed value of the **test statistic** $t = \dfrac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n}}}$.

Step 4: Calculate the ***p*-value** using $T \sim t_{n-1}$ as follows:

- If H_1: $\mu > \mu_0$, p-value $= \mathrm{P}(T \geqslant t)$.
- If H_1: $\mu < \mu_0$, p-value $= \mathrm{P}(T \leqslant t)$.
- If H_1: $\mu \neq \mu_0$, p-value $= 2 \times \mathrm{P}(T \geqslant |t|)$.

Alternatively, find the critical region $\mathcal{C}$ using α and the null distribution.

Step 5: Reject H_0 if p-value $< \alpha$ or if $t \in \mathcal{C}$, otherwise accept H_0.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Example 4 **Self Tutor**

The fat content (in grams) of 30 randomly selected pasties at the local bakery was recorded:

15.1	14.8	13.7	15.6	15.1	16.1	16.6	17.4	16.1	13.9
17.5	15.7	16.2	16.6	15.1	12.9	17.4	16.5	13.2	14.0
17.2	17.3	16.1	16.5	16.7	16.8	17.2	17.6	17.3	14.8

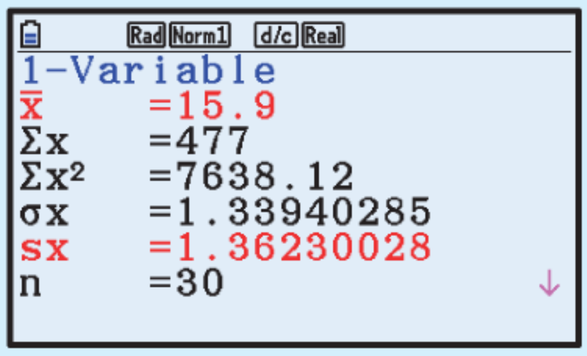
For a mean fat content of pasties made at this bakery μ, conduct a two-tailed t-test of H_0: $\mu = 16$ grams on a 10% level of significance.

Step 1: H_0: $\mu = 16$ {the mean fat content is 16 grams}
H_1: $\mu \neq 16$ {the mean fat content is *not* 16 grams}

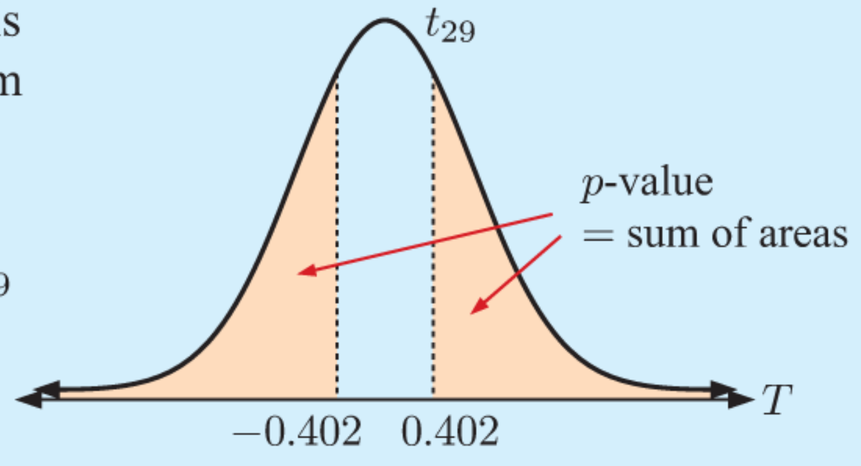
Step 2: The significance level is $\alpha = 0.1$.

Step 3: The sample size $n = 30$.
Using technology, $\overline{x} = 15.9$ and $s \approx 1.36$.
The observed value of the test statistic is

$$t \approx \frac{15.9 - 16}{\frac{1.36}{\sqrt{30}}} \approx -0.402$$



Step 4: Assuming that a sample size $n = 30$ is large enough for the Central Limit Theorem to apply, we can assume normality.
Since H_1: $\mu \neq 16$,
p-value $= 2 \times \mathrm{P}(T \geqslant |t|)$ where $T \sim t_{29}$
$\approx 2 \times \mathrm{P}(T \geqslant |-0.402|)$
≈ 0.691



Step 5: Since p-value $> 0.1 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 10% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude that the mean fat content is appreciably different from 16 grams.

Your **graphics calculator** can perform the calculations in a t-test automatically. However, the t-test functionality on most calculator models will only give you the p-value and observed t, but not the critical value(s) of the test.



Example 5 **Self Tutor**

In 2017, the average house price in a suburb was \$235 000. In 2019, a random sample of 200 houses in the suburb was taken. The sample mean was $\overline{x} = \$215\,000$, and the sample standard deviation was $s = \$30\,000$.

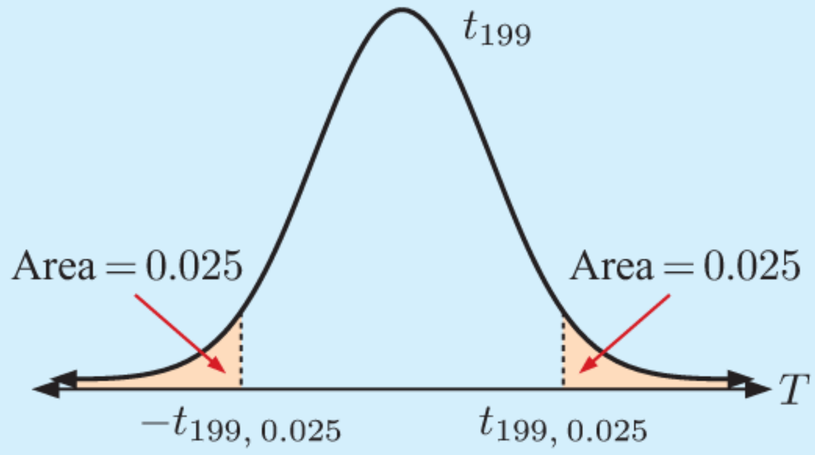
Use a critical value to determine whether there is evidence at the 5% level, that the average house price has changed.

Step 1: Let μ be the population mean of house prices in this suburb.
The hypotheses that should be considered are:
H_0: $\mu = 235\,000$ {the mean house price has stayed the same}
H_1: $\mu \neq 235\,000$ {the mean house price has changed}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: The sample size is $n = 200$.
The observed value of the test statistic is $t = \dfrac{215\,000 - 235\,000}{\frac{30\,000}{\sqrt{200}}} \approx -9.428$

Step 4: The null distribution is t_{199}.
The critical region $\mathcal{C}$ is
$t \leqslant -t_{199,\,0.025}$ or $t \geqslant t_{199,\,0.025}$
Using technology $t_{199,\,0.025} \approx 1.972$.
$\therefore \quad t \leqslant -1.972$ or $t \geqslant 1.972$



Step 5: Since $t \approx -9.428 \in \mathcal{C}$, we reject H_0 on a 5% level of significance.

Step 6: There is sufficient evidence at the 5% level of significance to suggest that $\mu \neq 235\,000$. We conclude that the average house price in 2019 was different from the average house price in 2017.

EXERCISE 30D

1 A population has unknown variance σ^2. A sample of size 24 has sample mean $\overline{x} = 17.14$ and sample standard deviation $s = 4.365$. We are required to test the hypothesis H_0: $\mu = 18.5$ against H_1: $\mu \neq 18.5$.

a Find:

i the test statistic **ii** the null distribution **iii** the p-value.

b What decision should be made at a 5% level using:

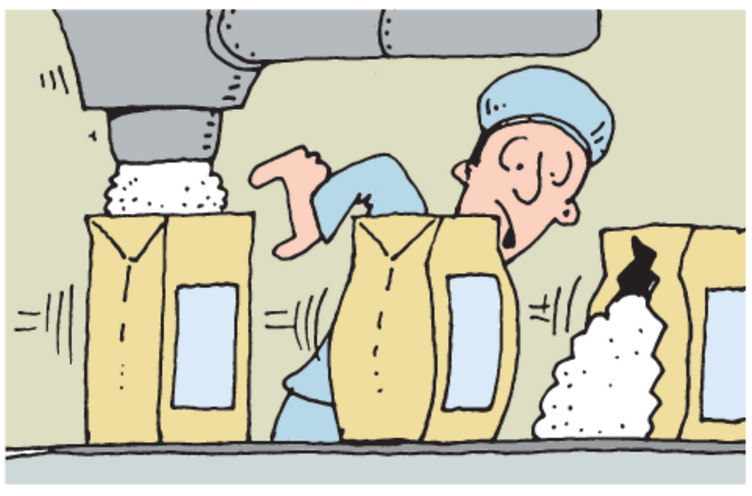
i the p-value **ii** the critical region?

2 A store claims that the mean price of a bottle of wine has fallen over the last 12 months. Records show that 12 months ago the mean price was \$13.45 for a 750 mL bottle. A random sample of prices of 389 different bottles of wine is now taken from the store. The sample mean is \$13.30 with sample standard deviation \$0.25. Use a p-value to test the store's claim at a 2% level of significance.

3 A machine is used to fill bottles with 500 mL of vinegar. A sample of ten random measurements of the volumes put in different bottles had mean 499 mL and standard deviation 1.2 mL. Assuming that the volumes of vinegar are normally distributed, test at the 1% level whether the machine is underfilling the bottles.

4 The length of screws produced by a machine is normally distributed. The machine is supposed to produce screws with mean length $\mu = 2.00$ cm. A quality controller selects a random sample of 15 screws. She finds that the mean length of the 15 screws is $\overline{x} = 2.04$ cm with sample standard deviation $s = 0.09$ cm. Does this justify the need to adjust the machine on a 2% level of significance?

5



A machine packs sugar into 1 kg bags. It is known that the masses of the bags of sugar are normally distributed. A random sample of eight filled bags was taken and the masses of the bags measured to the nearest gram. Their masses in grams were:

1001, 998, 999, 1002, 1001, 1003, 1002, 1002.

Perform a test at the 1% level, to determine whether the machine under-fills the bags.

6 A market gardener claims that the carrots in his field have a mean weight of more than 50 grams. A prospective buyer will purchase the crop if the market gardener's claim is true. To test this she pulls 20 carrots at random, and finds that their individual weights in grams are:

57.6	34.7	53.9	52.5	61.8	51.5	61.3	49.2	56.8	55.9
57.9	58.8	44.3	58.3	49.3	56.0	59.5	47.0	58.0	47.2

a Explain why it is reasonable to assume that the carrots' weights are normally distributed.

b Determine whether the buyer will purchase the crop using a 5% level of significance.

7 The management of a golf club claims that the mean income of its members is more than €95 000, and its members can therefore afford to pay increased fees. To show that this claim is invalid, the members seek the help of a statistician. From a random sample of 113 club members, the statistician finds an average income of €96 318 with standard deviation €14 268.

a Test at a 0.02 significance level whether the management's claim should be rejected.

b If the conclusion made by the statistician is incorrect, what type of error is made?

E PAIRED t-TESTS

We are often interested in comparing sets of results for the same group of individuals.

For example, we might consider:

- race times for a class of students at the start and finish of the athletics season
- test results for students in a class at the start and end of a semester.

In this situation we choose a random sample of individuals from the population, then use the data for this group to provide two sets of results for comparison. The data can therefore be **matched in pairs** for each individual in the group.

Notice that the two sets of results are not independent, since they come from the same sample group. However, the scores in each set of results must be independent for the analysis to be meaningful.

Suppose we have 2 sets of results X and Y with population means μ_1 and μ_2 respectively.

The null hypothesis might state that the means are equal:

$$H_0: \ \mu_1 = \mu_2$$

or equivalently, that the *difference* between the means is zero:

$$H_0: \ \mu_1 - \mu_2 = 0.$$

A randomly selected sample of 10 individuals is chosen, and the results from these individuals are used to form samples for X and Y:

x_i	5	6	0	5	7	2	5	8	8	4
y_i	1	4	0	8	4	1	4	5	5	2

For each individual in the sample, we can calculate the difference d between its x-value and its y-value:

x_i	5	6	0	5	7	2	5	8	8	4
y_i	1	4	0	8	4	1	4	5	5	2
$d_i = x_i - y_i$	4	2	0	-3	3	1	1	3	3	2

These values d_i are observed values of the random variable $D = X - Y$.

Notice that
$$\begin{aligned}\overline{D} &= \frac{1}{n}\sum_{i=1}^{n}(X_i - Y_i)\\ &= \frac{1}{n}\sum_{i=1}^{n}X_i - \frac{1}{n}\sum_{i=1}^{n}Y_i\\ &= \overline{X} - \overline{Y}\end{aligned}$$

so
$$\begin{aligned}\mathrm{E}(\overline{D}) &= \mathrm{E}(\overline{X} - \overline{Y})\\ &= \mathrm{E}(\overline{X}) - \mathrm{E}(\overline{Y})\\ &= \mu_1 - \mu_2\end{aligned}$$

So, the population mean difference $\mu_D = \mu_1 - \mu_2$ is the population mean of the differences d_i.

We can therefore use a t-test to test $H_0: \ \mu_D = 0,$ using the differences d_i to calculate the test statistic and p-value.

For the example above, the test statistic is $t = \dfrac{\overline{d} - 0}{\frac{s_D}{\sqrt{n}}} = \dfrac{1.6 - 0}{\frac{2.01}{\sqrt{10}}} \approx 2.52\,.$

We would then calculate the p-value using the t-distribution with $10 - 1 = 9$ degrees of freedom.

Alternatively, we could find the critical region $\mathcal{C}$ using α and the null distribution.

EXERCISE 30E

1 The performance of 20 elite sprinters was monitored over a twelve month period. Below is the best time in seconds for each athlete in 100 m trials at the start and end of the year.

Athlete	A	B	C	D	E	F	G	H	I	J
Start	10.3	10.5	10.6	10.4	10.8	11.1	9.9	10.6	10.6	10.8
End	10.2	10.3	10.8	10.1	10.8	9.7	9.9	10.6	10.4	10.6

Athlete	K	L	M	N	O	P	Q	R	S	T
Start	11.2	11.4	10.9	10.7	10.7	10.9	11.0	10.3	10.5	10.6
End	10.8	11.2	11.0	10.5	10.7	11.0	11.1	10.5	10.3	10.2

a Calculate the difference $d_i = \text{end time} - \text{start time}$ for each athlete.

b State the hypotheses that should be considered for the population mean difference μ_D if we are interested in whether:

i the athletes have improved

ii there is any change in the athletes' performance.

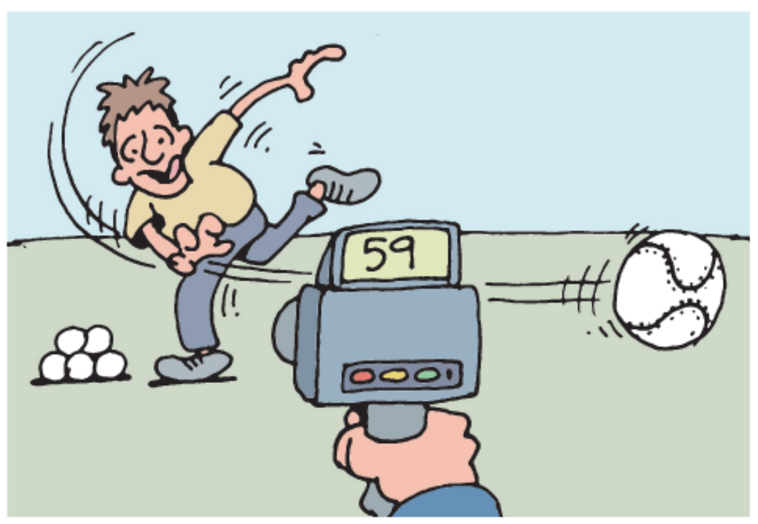
c Conduct a t-test at a 5% level of significance to test whether the athletes have improved.

2 A mathematics tutor claims to significantly increase students' test results with a week of tutoring. To test this claim, 12 students were tested prior to receiving tutoring, and their results recorded. However, the students were not given the answers or their results. After a week of tutoring, the students repeated the test to see whether they had improved. The results were:

Before tutoring	15	17	25	11	28	20	23	34	27	14	26	26
After tutoring	20	16	25	18	28	19	26	37	31	13	27	20

Conduct a paired t-test at a 5% level of significance to test the tutor's claim.

3 A group of 12 year old children were asked to throw a baseball as fast as they could. A radar was used to measure the speed of each throw. One year later, the same group was asked to repeat the experiment. The results are shown below, with the speeds given in km h^{-1}.



Child	A	B	C	D	E	F	G	H	I	J	K
Age 12	76	81	59	67	90	74	78	71	69	72	82
Age 13	79	82	66	72	93	76	77	82	75	77	86

a Calculate the difference in speed d_i for each child's throws.

b A sports commission report suggests that an average throwing speed difference of $5\ \text{km h}^{-1}$ is expected between these ages.

Conduct a hypothesis test at a 5% level of significance to determine whether the report's claim is valid.

Hint: The null value for μ_D does not have to be zero.

F THE TWO-SAMPLE t-TEST FOR COMPARING POPULATION MEANS

We have just seen how two sets of results from a single sample can be compared. We now consider how population means are compared if they come from *different samples*.

For example, in the **Opening Problem** on page **784**, Frank has a sample of results from each year, but he cannot *pair* the data because it comes from two completely different samples of tomatoes.

Frank could perhaps test the effectiveness of the fertiliser by letting the mean of last year's sample be μ_0 in a t-test of this year's sample. However, there are two problems with this approach:

- In general, the sample mean will not be exactly equal to the population mean.
- The variation of last year's sample is ignored.

If we are given the data for **two** samples, we use the **two-sample *t*-test** instead.

Suppose we are comparing two populations with means μ_1 and μ_2.

Our null hypothesis states that the means are equal:

$$H_0:\ \mu_1 = \mu_2$$

or equivalently, the *difference* between the means is zero:

$$H_0:\ \mu_1 - \mu_2 = 0$$

The formula for the test statistic t and the distribution used to calculate the p-values are beyond the scope of this course, but we can use technology to calculate t and the p-value.



The two-sample t-test follows the same general procedure as the single sample version on page **798**.

Example 6 **Self Tutor**

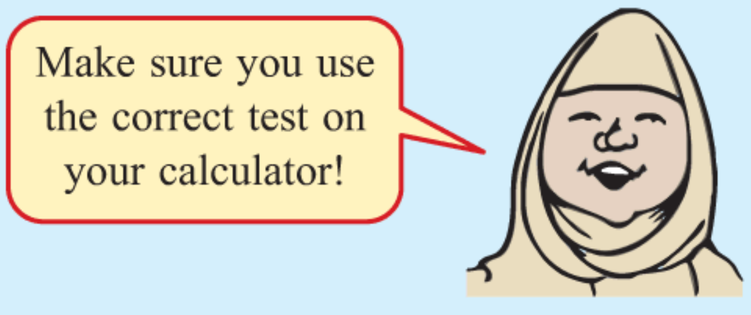
A pharmaceutical company has developed a new medication to help lower cholesterol levels.

To test the effectiveness of the new medication, volunteers were separated into two groups. 15 patients were given the old medication and 17 patients were given the new medication. The *reduction* in cholesterol levels after 4 weeks was measured for both samples. The results are shown below:

	Sample mean	*Sample standard deviation*
Old medication	0.351	0.058
New medication	0.497	0.077

Conduct a two-sample t-test to determine whether the new medication is *better* than the old medication using a 5% level of significance.

Step 1: Let μ_1 be the population mean cholesterol level reduction of the old medication group, and μ_2 be the population mean cholesterol level reduction of the new medication group.



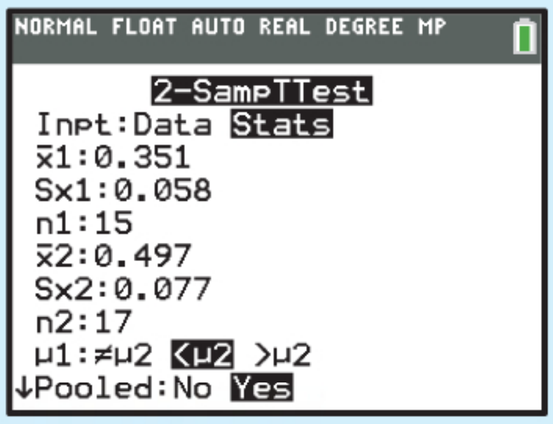
The hypotheses that should be considered are:

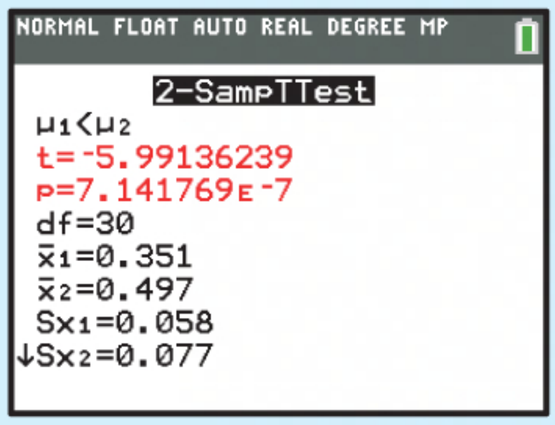
H_0: $\mu_1 = \mu_2$ {new medication is no better than the old medication}

H_1: $\mu_1 < \mu_2$ {new medication is better than the old medication}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: Using technology, the value of the test statistic is $t \approx -5.99$.





Step 4: From the screenshots above, the p-value $\approx 7.14 \times 10^{-7}$.

Step 5: Since the p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Step 6: Since we have accepted H_1, we conclude that the new medication is better than the old medication.

POOLED VERSUS UNPOOLED TESTS

You should have noticed that your calculator will give you the option of performing a "pooled" or "unpooled" test.

In a pooled test, it is assumed that the variances of the two populations are the same. The samples are therefore combined or *pooled* to estimate this common variance when calculating t and the p-value.

In an unpooled test, the variances are assumed to be different. We therefore use a different formula to calculate t and the p-value.

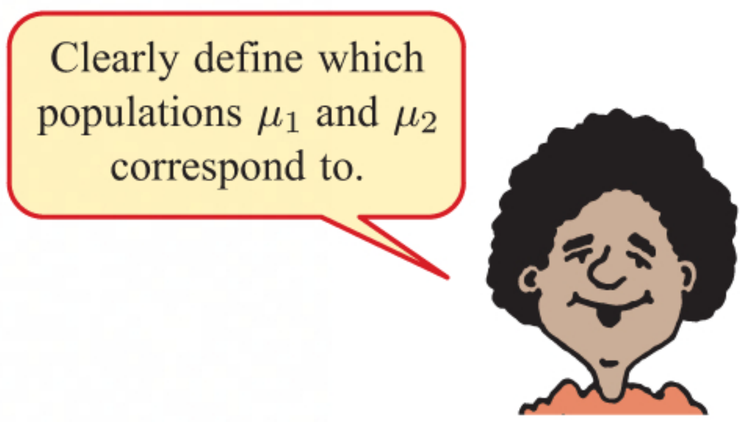
In this course you are expected to assume **equal variances** and hence use the **pooled two-sample t-test**.

EXERCISE 30F

1 Consider Frank's tomatoes in the **Opening Problem** on page **784**.

a Write down the hypotheses that Frank should consider.

b Conduct a two-sample t-test to determine whether the fertiliser was effective on a 1% level of significance.



2 A researcher claims that on average, high school students sleep less than middle school students. The researcher recorded the sleeping times of 49 middle school students and 55 high school students.

For the middle school students, the mean daily time asleep was 8.0 hours with standard deviation 0.2 hours.

For the high school students, the mean daily time asleep was 7.9 hours with standard deviation 0.5 hours.

Conduct a two-sample t-test to test the researcher's claim on the 5% level.



3 Two groups of seedlings were grown with different brands of compost: A and B. After a period of time, the height (in cm) of each seedling was measured.

Brand A:	12.1	14.6	10.1	8.7	13.2	15.1	16.5	14.6
Brand B:	12.3	15.2	9.9	9.5	13.4	14.9	17.0	14.8

The manufacturer of *Brand B* guarantees it will improve the growth of seedlings more than *Brand A* compost. Conduct a two-sample t-test at a 5% level, to determine whether the claim is valid.

4 The following data show the times (in seconds) that Jesiah and Billy recorded for the 100 m sprint over the last fortnight:

Jesiah:	12.3	13.0	11.3	11.5	14.3	12.0	14.5	13.7	12.9	11.9	12.1	12.6	11.3	12.7
Billy:	10.4	12.0	11.7	13.2	11.2	11.1	11.1	12.0	11.4	12.5	12.1	12.0		

a Conduct a two-sample t-test to determine whether there is a significant difference between the runners' times on the 5% level.

b Which runner do you think is faster? Explain your answer.

G HYPOTHESIS TESTS FOR THE MEAN OF A POISSON POPULATION

So far we have studied hypothesis tests for the population mean of a normally distributed population, or where we can use the Central Limit Theorem to assume normality when calculating probabilities or quantiles associated with the test statistic.

We now consider a hypothesis test for the population mean of a population with a **Poisson distribution**.

POISSON RANDOM SAMPLES

Suppose the population has a Poisson distribution with an unknown mean λ. We want to test the null hypothesis H_0: $\lambda = \lambda_0$.

Consider a random sample of n independent and identically distributed observations from this population, $\{X_1, ..., X_n\}$.

We have already seen that an estimator for the population mean is the sample mean $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

We will therefore consider $n\overline{X}_n = \sum_{i=1}^{n} X_i$ when we choose our test statistic.

THE TEST STATISTIC

In **Exercise 29B** you should have shown that:

> The sum of n independent Poisson random variables is itself a Poisson random variable.

For our Poisson random sample above, the statistic $T = n\overline{X}_n = \sum_{i=1}^{n} X_i$ is a Poisson random variable with mean $\mathrm{E}(T) = \mathrm{E}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{E}(X_i) = \sum_{i=1}^{n} \lambda = n\lambda$.

Under the assumptions of H_0, $\lambda = \lambda_0$, so $T \sim \mathrm{Po}(n\lambda_0)$.

Consider a statistical hypothesis test of H_0: $\lambda = \lambda_0$ for a Poisson distributed population. Given a sample of size n with observed sample mean $\overline{x}$:

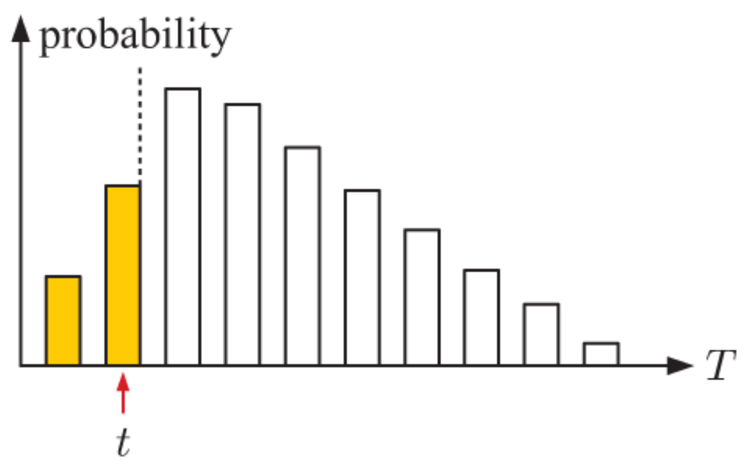
- the **test statistic** is $T = n\overline{X}_n$ which has observed value $t = n\overline{x}$
- the **null distribution** is $T \sim \text{Po}(n\lambda_0)$.

THE p-VALUE

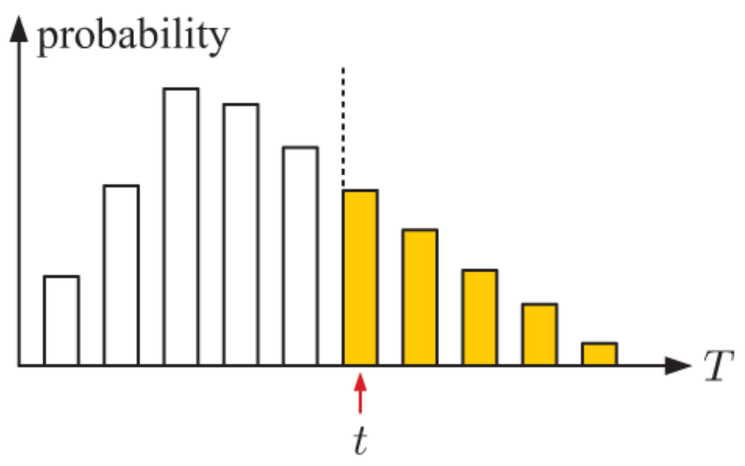
The p-value is calculated using the **null distribution** $\text{Po}(n\lambda_0)$.

In this course we only consider Poisson tests with a one-tailed alternative hypothesis H_1. In this case the calculation of the p-value is analogous to what we saw with the normal distribution and t-distribution:

- If H_1: $\lambda < \lambda_0$, we use the **lower tail**.
- If H_1: $\lambda > \lambda_0$, we use the **upper tail**.



p-value $= \mathbf{P}(T \leqslant t)$



p-value $= \mathbf{P}(T \geqslant t)$

Important:

Because the Poisson distribution is *discrete*, we must be careful with the inequality sign.

Since the p-value is defined as the probability of observing something *as or more extreme* than the test statistic, we must *include* the observed test statistic in the probability calculation.

THE HYPOTHESIS TEST FOR THE POPULATION MEAN FOR A POISSON DISTRIBUTION

Step 1: State the **null hypothesis** H_0: $\lambda = \lambda_0$ and **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the observed value of the **test statistic** $t = \sum_{i=1}^{n} x_i = n\overline{x}$.

Step 4: Calculate the ***p*-value** using $T \sim \text{Po}(n\lambda_0)$ as follows:

- If H_1: $\lambda > \lambda_0$, p-value $= \mathrm{P}(T \geqslant t)$.
- If H_1: $\lambda < \lambda_0$, p-value $= \mathrm{P}(T \leqslant t)$.

Step 5: Reject H_0 if p-value $< \alpha$, otherwise accept H_0.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Example 7 **Self Tutor**

A sample of 5 observations is drawn from a Poisson distribution with unknown mean λ.

The sum of the data values is $\sum_{i=1}^{5} x_i = 6$.

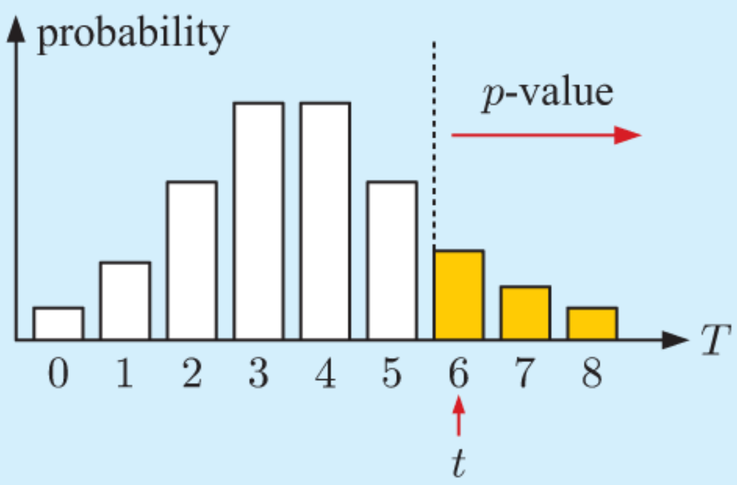
Is there enough evidence on a 10% level of significance to conclude that $\lambda > 0.8$?

Step 1: The hypotheses to be considered are:
H_0: $\lambda = 0.8$
H_1: $\lambda > 0.8$

Step 2: The significance level is $\alpha = 0.10$.

Step 3: The test statistic is $t = 6$.

Step 4: $n \times \lambda_0 = 5 \times 0.8 = 4$
So the null distribution is $T \sim \text{Po}(4)$
$\therefore$ p-value $= \text{P}(T \geqslant t)$
$= \text{P}(T \geqslant 6)$
≈ 0.215



Step 5: Since p-value $> 0.10 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 10% level of significance. We therefore accept H_0.

Step 6: Since we have accepted H_0, we cannot conclude that $\lambda > 0.8$.

EXERCISE 30G

1 In a hypothesis test for the mean of a Poisson distribution, explain why the observed test statistic must be a non-negative integer.

2 To find the age of archaeological specimens, scientists measure the emission of radioactive particles. The older the specimen is, the *fewer* particles it will emit.

The number of particles emitted per minute follows a Poisson distribution with an unknown rate λ.

A scientist wishes to determine whether a particular specimen is older than 5000 years old. If the specimen is 5000 years old, then $\lambda = 2$.

a Write down the hypotheses the scientist should consider.

b The scientist finds that in $n = 5$ minutes, 9 particles were emitted. Test the hypotheses on a 5% level of significance.



3 A statistician wishes to test the following hypotheses for a Poisson population with unknown rate λ on an $\alpha = 0.05$ level of significance:

H_0: $\lambda = 5.1$
H_1: $\lambda > 5.1$

The *mean* of a sample of size 16 drawn from this population is 5.375.

a Calculate the observed value of the test statistic for the statistician's hypothesis test.

b Calculate the p-value for the test and hence determine whether or not H_0 should be rejected.

4 A local council is concerned about the build up of traffic at a particular intersection. The council proposes that if, in peak time, the number of cars passing through the intersection per minute exceeds 30 cars, then the intersection should be upgraded to help improve traffic flow. The number of cars passing through the intersection per minute follows a Poisson distribution with unknown rate λ.

a Write down the hypotheses the council should consider.

b A traffic surveyor found that the total number of cars passing through the intersection over a 30 minute peak time interval is 1100. Test the hypotheses with a 10% level of significance, to determine whether the intersection should be upgraded.

c What assumption did you make when you conducted your hypothesis test? Describe how it might affect your conclusion.

5 The data below shows the number of siblings of each student in Emily's class:

0	1	0	2	1	3	3	0	1	3	2	2	1	1	1
0	1	0	2	2	1	2	2	1	3	2	1	0	2	2

Emily wants to determine whether the mean number of siblings per student at her school is greater than 1.5.

a Write down the hypotheses that Emily should consider.

b Conduct a one-tailed hypothesis test using a Poisson distribution with a 5% level of significance.

c Conduct a one-sample t-test with a 5% level of significance.

d Compare the results of the tests. Discuss which test you think is more appropriate for Emily.

Example 8 — Self Tutor

The hypothesis test in **Example 7** had hypotheses H_0: $\lambda = 0.8$ versus H_1: $\lambda > 0.8$ and significance level $\alpha = 0.05$.

a Find:

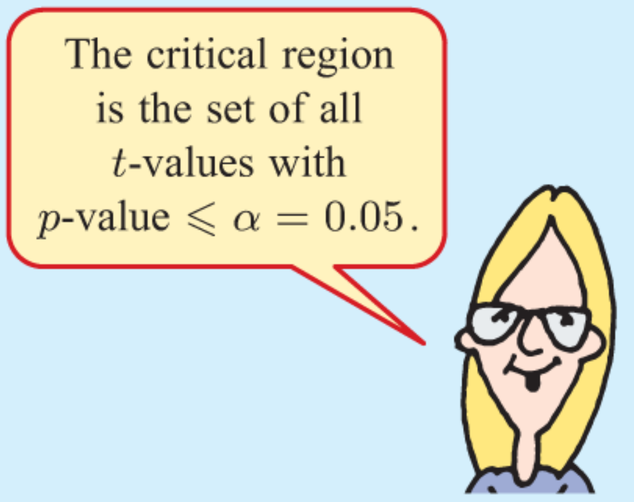
i the critical region $\mathcal{C}$ **ii** the acceptance region $\mathcal{A}$ **iii** the critical value c.

b A sample of 5 observations is drawn, and the sum of the data values is $\sum_{i=1}^{5} x_i = 6$.

Use the critical region to determine whether there is sufficient evidence to reject H_0.

a The null distribution is $T \sim \text{Po}(4)$.

t	p-value $= \text{P}(T \geqslant t)$	
0	1	p-value $> \alpha$
1	≈ 0.982	
2	≈ 0.908	
3	≈ 0.762	
4	≈ 0.567	
5	≈ 0.371	
6	≈ 0.215	
7	≈ 0.111	
8	≈ 0.0511	
9	≈ 0.0214	p-value $\leqslant \alpha$
$\vdots$	$\vdots$	



i $\mathcal{C} = \{t \mid t \geqslant 9\}$

ii $\mathcal{A} = \{t \mid 0 \leqslant t \leqslant 8\}$
$= \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

iii The value in $\mathcal{C}$ with the largest p-value is 9.
$\therefore$ the critical value $c = 9$.

b The test statistic $t = \sum_{i=1}^{5} x_i = 6$.

$6 \notin \mathcal{C}$, so we do not have sufficient evidence to reject H_0.

6 For each hypothesis test, find:

i the critical region $\mathcal{C}$ **ii** the acceptance region $\mathcal{A}$ **iii** the critical value c.

a H_0: $\lambda = 2$ against H_1: $\lambda < 2$ using a sample of 3 individuals and the significance level $\alpha = 0.05$.

b H_0: $\lambda = 1.5$ against H_1: $\lambda > 1.5$ using a sample of 5 individuals and the significance level $\alpha = 0.01$.

c H_0: $\lambda = 0.95$ against H_1: $\lambda < 0.95$ using a sample of 10 individuals and the significance level $\alpha = 0.1$.

7 A call centre receives an average of λ calls per hour. One morning it received 20 calls between 9 am and 12 pm.

Test the hypotheses H_0: $\lambda = 6$ against H_1: $\lambda > 6$ with an $\alpha = 0.05$ significance level. Use a critical region to make your decision.

8 A hypothesis test for the mean of a Poisson population has null hypothesis H_0: $\lambda = 4$ and alternative hypothesis H_1: $\lambda < 4$.

When the sample size $n = 2$, the critical region is $\mathcal{C} = \{0, 1, 2, 3\}$.

Explain why the significance level of the test is $0.0424 \leqslant \alpha < 0.0996$.

H HYPOTHESIS TESTS FOR A POPULATION PROPORTION

Suppose we wish to determine the proportion p of a population with a particular property. We therefore wish to consider the null hypothesis

$$H_0: \ p = p_0.$$

THE TEST STATISTIC AND THE NULL DISTRIBUTION

Consider a random sample of size n in which each member has probability p of having the property of interest.

If X is the number of members in the sample with the property of interest, then X follows the **binomial distribution** $\mathrm{B}(n, p)$.

Under the null hypothesis, $p = p_0$, so $X \sim \mathrm{B}(n, p_0)$.

Consider a statistical hypothesis test of H_0: $p = p_0$. Given a sample of size n:

- the **test statistic** is X, the number of members in the sample with the property of interest
- the **null distribution** is $X \sim \mathrm{B}(n, p_0)$.

THE p-VALUE

Like the Poisson distribution, the binomial distribution is discrete. We calculate the p-value using the same principles.

THE HYPOTHESIS TEST FOR A POPULATION PROPORTION

Step 1: State the **null hypothesis** H_0: $p = p_0$ and **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Identify the observed value of the **test statistic** x as the number of "successes" in the sample.

Step 4: Calculate the ***p*-value** using $X \sim \text{B}(n, p_0)$ as follows:

- If H_1: $p > p_0$, p-value $= \text{P}(X \geqslant x)$.
- If H_1: $p < p_0$, p-value $= \text{P}(X \leqslant x)$.

Step 5: Reject H_0 if p-value $< \alpha$, otherwise accept H_0.

Step 6: Interpret your decision in context, and write your conclusion as a sentence.

Example 9 **Self Tutor**

Theresa flipped a coin 20 times and got 15 heads. Theresa suspects that the coin is biased towards heads.

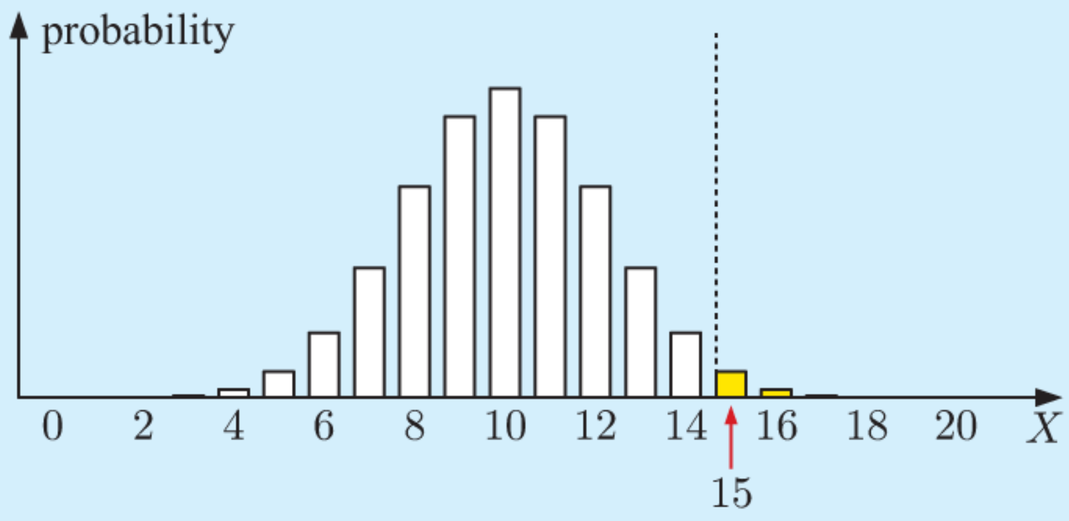
Conduct a one-tailed test with significance level $\alpha = 0.05$ to determine whether Theresa's claim is justified.

Step 1: Let p be the probability of obtaining a head when the coin is tossed.
The hypotheses that should be considered are:
H_0: $p = 0.5$ {the coin is fair}
H_1: $p > 0.5$ {the coin is biased towards heads}

Step 2: The significance level is $\alpha = 0.05$.

Step 3: The observed value of the test statistic is $x = 15$.

Step 4: The null distribution is $X \sim \text{B}(20, 0.5)$. The alternative hypothesis is H_1: $p > 0.5$, so we use the upper tail of the null distribution.



p-value $= \text{P}(X \geqslant 15)$
≈ 0.0207 {using technology}

Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on the $\alpha = 0.05$ significance level.

Step 6: Since we have accepted H_1, we conclude that the coin is indeed biased towards heads. Theresa's claim is justified.

EXERCISE 30H

1 A magician claims that he can roll a six with a fair die on average nine times out of ten. To test the magician's claim, he is invited to roll the die six times. His claim will be accepted if he rolls at least four sixes.

a State the hypotheses to be tested. **b** Identify the significance level of this test.

2 A student suspects that her six-sided die is biased towards the number four. She decides to conduct a statistical hypothesis test with a 5% level of significance. She rolls the die 30 times and 8 fours appear.

a State the null and alternative hypotheses to be tested.

b State the significance level α of the test.

c Define the test statistic and write down the null distribution.

d Calculate the p-value.

e Does the student have sufficient evidence to conclude that her die is biased towards the number four?

3 To test whether a coin is biased towards tails, it was tossed 80 times. 37 heads appeared.

a State the null and alternative hypotheses to be tested.

b Define the test statistic and write down the null distribution.

c Calculate the p-value.

d Is there enough evidence to conclude that the coin is biased with a 5% level of significance?

4 Sixty migraine sufferers were asked to change from their old medication to a new one. Thirty eight said the medication improved their condition. The remaining participants said it made their condition worse. Is there enough evidence on a 5% level to say that the new medication is better than the old?

Hint: If the medication made no difference, the probability of a person saying they improved is the same as them saying they became worse.

5 A lotto company produces instant scratch lottery tickets. They claim that 5% of its tickets win a prize. A consumer group suspects that the actual percentage of winning tickets is less than 5%. In a randomly selected sample of 200 tickets, 8 were found to be winning tickets. Does this support the consumer group's suspicions on a 2% significance level?

6 In 2008, 20.6% of adults said they expected to be better off financially in 5 years' time. To see if this proportion had improved in 2018, a pollster surveyed a random selection of 300 adults. 66 said they expected to be better off in 5 years' time. Does this support the hypothesis that people's feelings about their future financial situation have improved on a 1% significance level?

Example 10 **Self Tutor**

Let p be the probability of rolling a one with a four-sided die.

Consider the hypotheses H_0: $p = 0.25$ and H_1: $p > 0.25$.

The hypotheses will be tested using a sample of 6 rolls of the die, with significance level $\alpha = 0.05$.

a Find:

i the critical region $\mathcal{C}$ **ii** the acceptance region $\mathcal{A}$ **iii** the critical value c.

b The die was rolled 6 times and 3 ones appeared. Is this sufficient evidence to reject H_0?

a The test statistic is the number of ones that appear in 6 rolls of the die, X.
$\therefore$ the null distribution is $X \sim \text{B}(6, 0.25)$

x	p-value $= \text{P}(X \geqslant x)$	
0	1	p-value $> \alpha$
1	≈ 0.822	
2	≈ 0.466	
3	≈ 0.169	
4	≈ 0.0376	p-value $\leqslant \alpha$
5	$\approx 0.004\,64$	
6	$\approx 2.44 \times 10^{-4}$	

i $\mathcal{C} = \{4, 5, 6\}$

ii $\mathcal{A} = \{0, 1, 2, 3\}$

iii The value in $\mathcal{C}$ with the largest p-value is 4.
$\therefore$ the critical value $c = 4$

b $3 \notin \mathcal{C}$, so we do not have sufficient evidence to reject H_0.

7 For each hypothesis test, find:

i the critical region $\mathcal{C}$ **ii** the acceptance region $\mathcal{A}$ **iii** the critical value c.

a H_0: $p = 0.6$ against H_1: $p < 0.6$ using a sample of 5 individuals and a significance level of $\alpha = 0.05$.

b H_0: $p = 0.35$ against H_1: $p > 0.35$ using a sample of 8 individuals and a significance level of $\alpha = 0.01$.

c H_0: $p = 0.74$ against H_1: $p < 0.74$ using a sample of 10 individuals and a significance level of $\alpha = 0.1$.

8 Judeau wishes to test if a coin is biased. Let p be the probability of a head.
Judeau's hypotheses are: H_0: $p = 0.5$
H_1: $p > 0.5$

To test these hypotheses on the 5% significance level, Judeau counts the number of heads in 4 tosses of the coin.

a For Judeau's test, find: **i** the critical region **ii** the acceptance region.

b Is it *possible* to reject H_0 in this situation?

c What should Judeau do if he wants to test his hypotheses more accurately?

ACTIVITY USING A Z-TEST TO TEST HYPOTHESES ABOUT A POPULATION PROPORTION

In previous Chapters, we saw that for large values of n:

- the distribution of $X \sim \text{B}(n, p)$ can be approximated with $\text{N}(np, np(1-p))$
- the distribution of $\widehat{p} = \dfrac{X}{n}$ can be approximated with $\text{N}\left(p, \dfrac{p(1-p)}{n}\right)$.

In this Activity, we will see how these *approximations* can be used to test hypotheses about a population proportion p:

$$H_0: \ p = p_0$$
$$H_1: \ p \neq p_0$$

Assume that n is large enough for the Central Limit Theorem to apply.

What to do:

1 Under the null hypothesis H_0: $p = p_0$, state the approximate distribution of:

a X **b** $\widehat{p}$

2 Suppose we test the hypotheses using a Z-test with $\widehat{p}$ as $\overline{X}_n$, the sample mean.

a For this Z-test, state the value of:

i μ_0 **ii** σ

b Show that the test statistic for this Z-test is: $Z = \dfrac{\widehat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$

3 Redo questions **4** to **6** in **Exercise 30H** using a Z-test. How do your p-values from a Z-test compare to the p-values calculated with the binomial distribution?

INVESTIGATION 2 RANDOMISED TESTS

Since binomial random variables are discrete, even though we may state a significance level of α, the actual significance level for the hypothesis tests we have seen may not be *exactly* α.

In this Investigation we consider **randomised tests** used to achieve a significance level of exactly α.



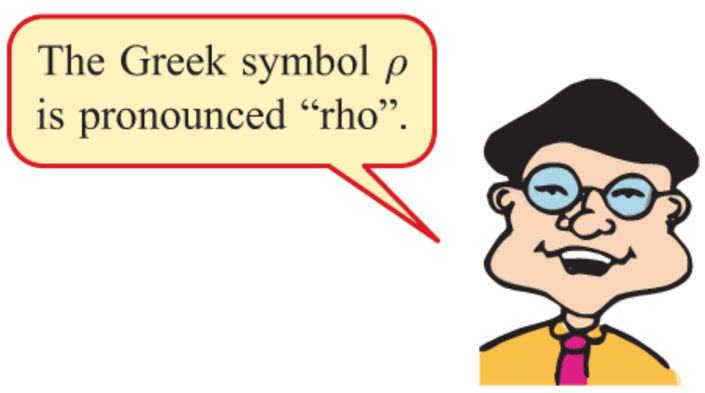
I HYPOTHESIS TESTS FOR A POPULATION CORRELATION COEFFICIENT

In **Chapter 7** we saw how the relationship or **correlation** between two numerical variables can be described using a number called **Pearson's product moment correlation coefficient** r.

Important properties of Pearson's product-moment correlation coefficient are summarised on pages **141** and **142**.

In **Chapter 7** you used technology to calculate r for a *sample* of bivariate data. In this Section, we will refer to r as the **sample product moment correlation coefficient**.

When we talk about the correlation between two variables in a *population*, we use the **population product moment correlation coefficient** ρ.



HYPOTHESES FOR ρ

If no relationship or association exists between two variables, then there is **no correlation** between them and $\rho = 0$.

In a hypothesis test, this is equivalent to stating a null hypothesis H_0: $\rho = 0$. The alternative hypothesis would then depend on the *direction* of the correlation that we would like to detect.

If we are interested in:

- **positive** correlation, we use H_1: $\rho > 0$
- **negative** correlation, we use H_1: $\rho < 0$
- **any** correlation, whether positive or negative, we use H_1: $\rho \neq 0$.

THE BIVARIATE NORMAL DISTRIBUTION

If we are given a set of data points (x, y), we can think of the x and y-values as two random variables X and Y respectively. Since we are considering these random variables simultaneously, we can talk about the **joint distribution** of X and Y, which is the probability distribution of the vector $\begin{pmatrix} X \\ Y \end{pmatrix}$.

> If $X \sim \mathrm{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$, then the joint distribution of X and Y is called a **bivariate normal distribution**.

We observe bivariate normal distributions when we consider the relationship between physical characteristics such as height, weight, and lung capacity.

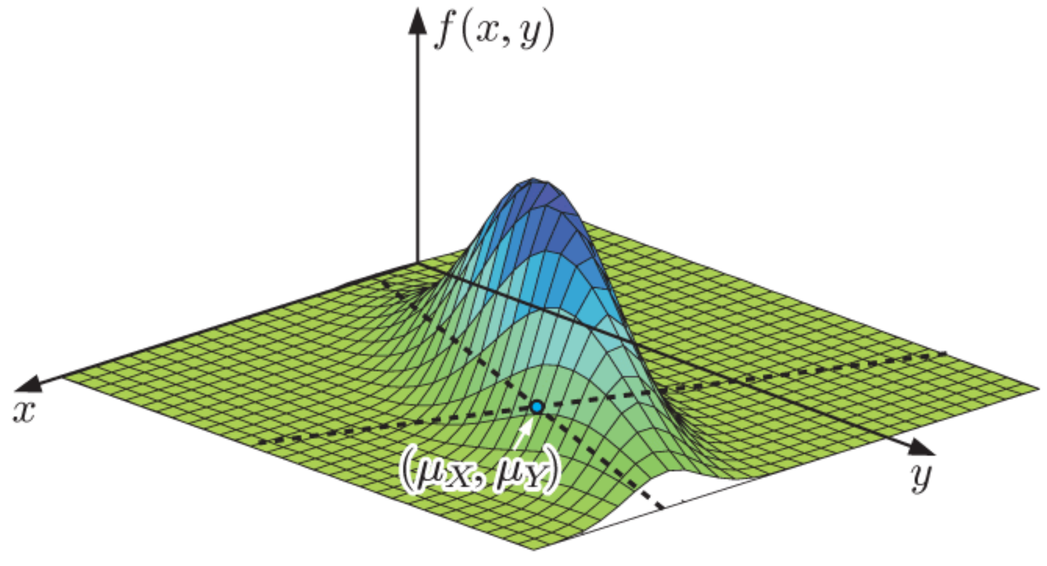
For example, a cat's weight (X) and the length of its whiskers (Y) are both normally distributed, so their joint distribution is a bivariate normal distribution.

We can visualise the bivariate normal distribution as a "hill" whose:

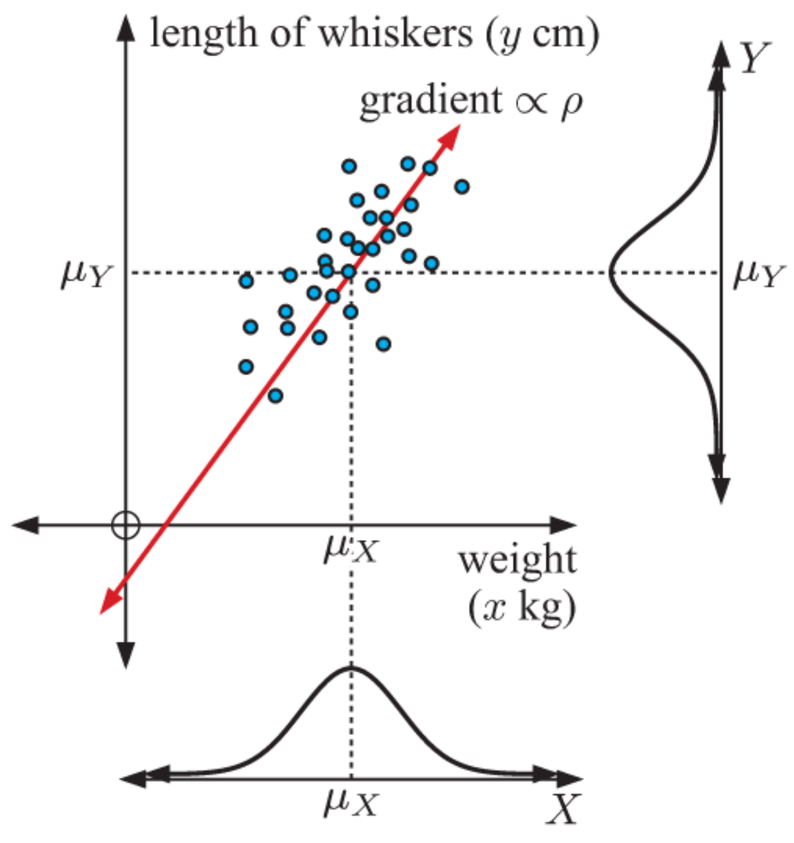
- **peak** is at (μ_X, μ_Y)
- **scale** is controlled by σ_X in the X direction and σ_Y in the Y direction
- **shape** is controlled by the correlation ρ between X and Y.



Probability distribution



Scatter diagram

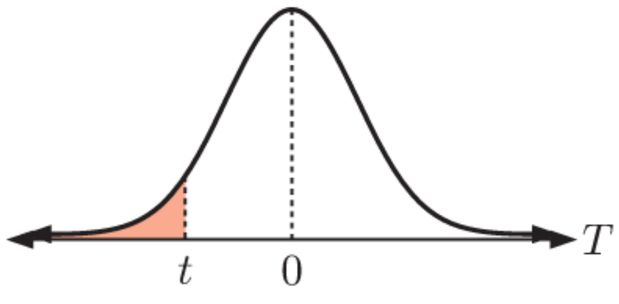


THE TEST STATISTIC AND p-VALUE

The formula for the test statistic for the hypothesis test of H_0: $\rho = 0$ against H_1 given a bivariate normal distribution is beyond the scope of this course.

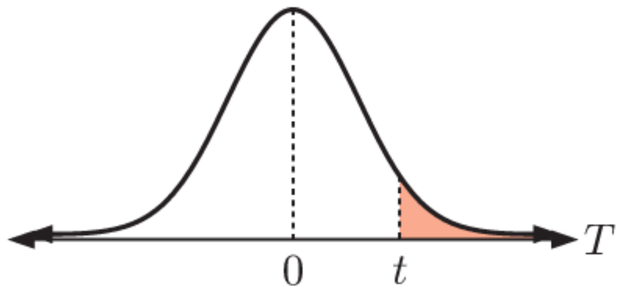
However the test statistic follows a t-distribution, so the p-values, critical values, and critical regions are interpreted the same way as for the t-test.

H_1: $\rho < 0$



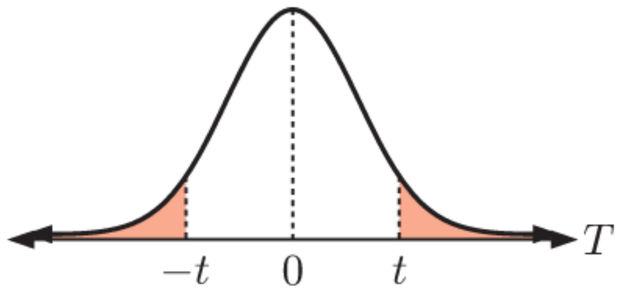
p-value $= \mathrm{P}(T \leqslant t)$

H_1: $\rho > 0$



p-value $= \mathrm{P}(T \geqslant t)$

H_1: $\rho \neq 0$



p-value $= 2 \times \mathrm{P}(T \geqslant |t|)$

In this course we use technology to conduct tests about ρ. You should be able to use your calculator to obtain test statistics and p-values, and interpret these numbers in context.



Example 11 **Self Tutor**

Consider the bivariate data set:

x	2	2	2	2	3	3	3	4	4	5
y	8	10	12	14	8	10	12	8	10	8

Conduct a hypothesis test at the 5% level of significance to determine whether the variables are negatively correlated.

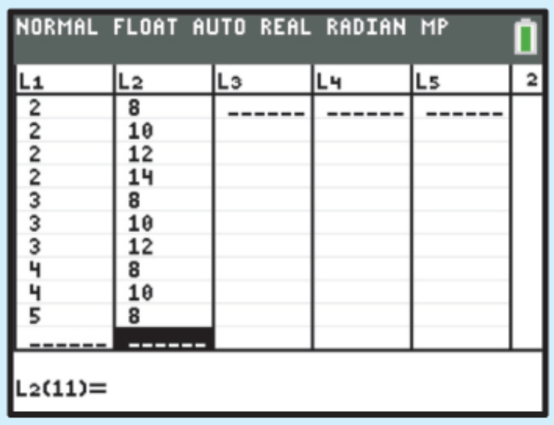
Step 1: Let ρ be the population product moment correlation coefficient between the variables. We use the hypotheses:

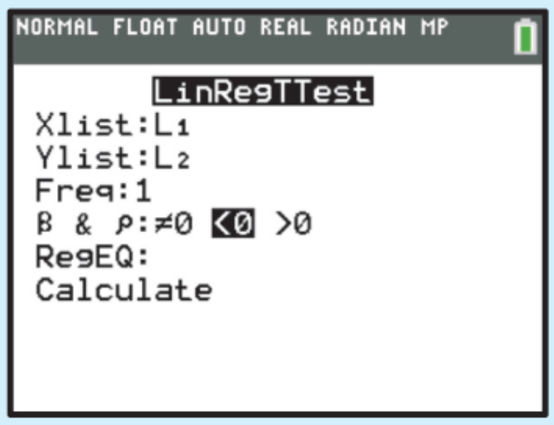
H_0: $\rho = 0$ {there is no correlation between the variables}

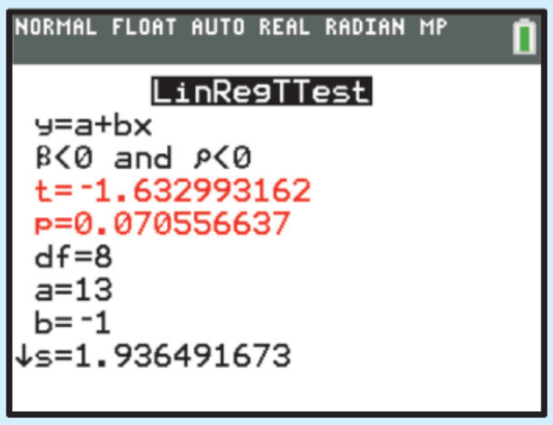
H_1: $\rho < 0$ {the variables are negatively correlated}

Step 2: The significance level is $\alpha = 0.05$.

Step 3:







The observed value of the test statistic ≈ -1.63.

Step 4: p-value ≈ 0.0706

Step 5: Since p-value $\geqslant 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the variables are not correlated.

EXERCISE 30I

1 Consider the bivariate data set:

x	1	2	3	4	5	6	7
y	4.1	6.2	4.5	9.8	7.7	5.8	8.8

A hypothesis test is to be conducted to determine whether the variables are positively correlated at the 5% level of significance.

a State the hypotheses to be tested.

b Use technology to calculate the test statistic and p-value.

c Determine the outcome of the hypothesis test.

2 The following table shows the *Mathematics mark* and *Physics mark* for a group of students:

Mathematics mark (x)	13	10	8	14	6	11	10	5	12	13
Physics mark (y)	21	15	14	20	12	16	9	10	17	12

Is there enough evidence to conclude that there is an association between a student's Mathematics mark and Physics mark at the: **a** 5% level **b** 1% level?

3 The *cholesterol level* in the bloodstream (x) and the *resting heart rate* (y) of 10 people are:

Cholesterol level (x)	5.32	5.54	5.45	5.06	6.13	5.00	4.90	6.00	6.70	4.75
Resting heart rate (y)	55	48	55	53	74	44	49	68	78	51

A researcher claims that there is a positive correlation between *cholesterol level* and *resting heart rate*. Based on this data, is the researcher's claim justified at the 1% level?

4 The table below shows the number of Sudoku puzzles (x) and the number of logic puzzles (y) solved in a three hour period by participants in a psychology experiment.

Number of Sudoku puzzles (x)	5	8	12	15	15	17	20	21	25	27
Number of logic puzzles (y)	3	11	9	6	15	13	25	15	13	20

Carry out a hypothesis test to determine whether the variables are linearly correlated at the 2% level.

5 Consider the data in the table below.

Weight of mother (x kg)	49	46	48	45	46	42	43	40	52	55	68
Birth weight of child (y kg)	3.8	3.1	2.5	3.0	3.2	2.8	3.1	2.9	3.4	2.9	3.9

The critical regions for a test of H_0: $\rho = 0$ against H_1: $\rho \neq 0$ for various significance levels are shown in the table alongside.

α	*Critical region*
0.1	$t \leqslant -1.83$ or $t \geqslant 1.83$
0.05	$t \leqslant -2.26$ or $t \geqslant 2.26$
0.02	$t \leqslant -2.82$ or $t \geqslant 2.82$
0.01	$t \leqslant -3.25$ or $t \geqslant 3.25$

a Use technology to calculate the value of the test statistic for this hypothesis test.

b Use the critical regions to determine the significance level(s) for which H_0 will be rejected.

c Check your answer to **b** using the p-value.

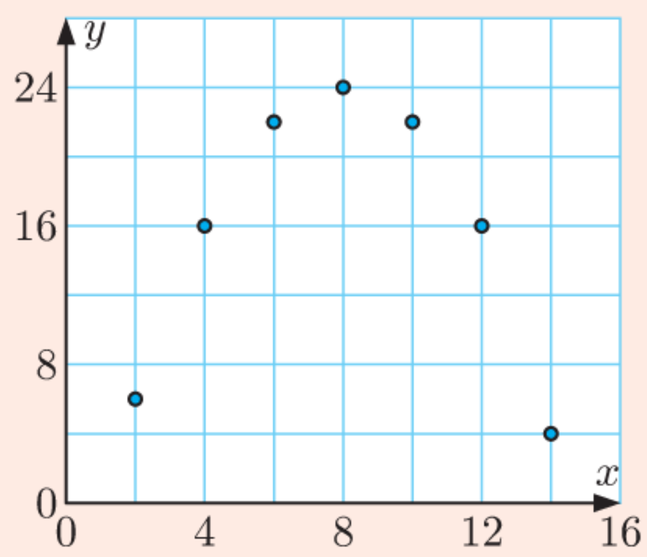
DISCUSSION

For the data set alongside, the sample product moment correlation coefficient $r = 0$.

x	2	4	6	8	10	12	14
y	6	16	22	24	22	16	6

When we construct a scatter diagram of y against x, there clearly *is* a relationship between the variables, but it is quadratic rather than linear.

Discuss the limitations of hypothesis tests for ρ when we want to establish the existence of *any* relationship between the variables.



J ERROR PROBABILITIES AND STATISTICAL POWER

In **Section B**, we defined the **significance level** α of a hypothesis test as the probability of making a Type I error.

$$\begin{aligned}\alpha &= \text{P(Type I error)} \\ &= \text{P(Reject } H_0 \mid H_0 \text{ true)}\end{aligned}$$

α and β are sometimes called the **Type I error rate** and the **Type II error rate** respectively.



In a similar way, we denote the probability of making a Type II error as β.

$$\begin{aligned}\beta &= \text{P(Type II error)} \\ &= \text{P(Retain } H_0 \mid H_0 \text{ false)}\end{aligned}$$

Suppose we conduct a hypothesis test on an $\alpha = 0.01$ level of significance and obtain p-value $= 0.021$.

We accept the null hypothesis H_0 since there is insufficient evidence to reject H_0. However, it is possible that H_0 is in fact not true, in which case we make a Type II error.

If the significance level was $\alpha = 0.05$ instead, then H_0 would have been rejected. β is therefore *dependent* on α.

Ideally, we want to minimise both α and β when conducting a hypothesis test. However, this is not possible since as α decreases, β increases. Thus as a compromise, we hold α constant and minimise β. This is one reason why we state the significance level α *before* conducting a hypothesis test.

We choose α rather than β because we know the distribution of the test statistic under H_0. By contrast, the alternative hypothesis H_1 involves an inequality, so we cannot know the distribution of the test statistic under H_1.

In this course, we only calculate β if the distribution under H_1 is given.

CALCULATING β FOR A Z-TEST

Suppose we conduct a Z-test with significance level α for the hypotheses: H_0: $\mu = \mu_0$
H_1: $\mu < \mu_0$

Under H_0, the test statistic $Z = \dfrac{\overline{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}}$ has the standard normal distribution $\text{N}(0, 1^2)$.

Under the alternative hypothesis H_1, μ is equal to some value μ_1 which is *less than* μ_0.

So under H_1, $\overline{X}_n \sim \text{N}\left(\mu_1, \left(\dfrac{\sigma}{\sqrt{n}}\right)^2\right)$

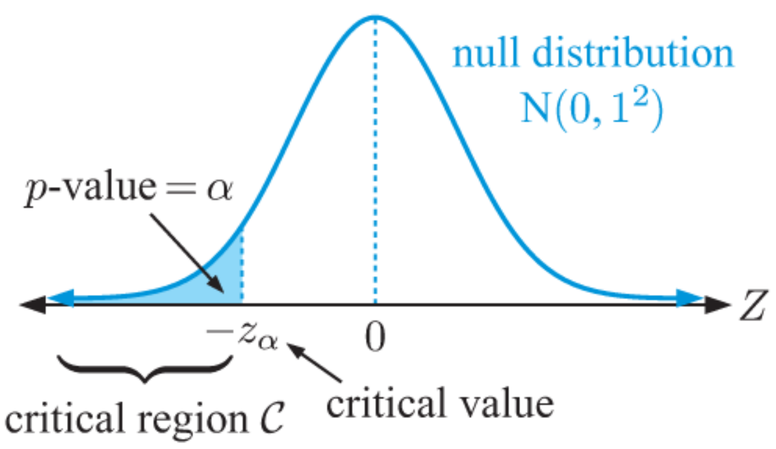
$$\therefore \quad \overline{X}_n - \mu_0 \sim \text{N}\left(\mu_1 - \mu_0, \left(\frac{\sigma}{\sqrt{n}}\right)^2\right)$$

$$\therefore \quad Z = \frac{\overline{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}} \sim \text{N}\left(\frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}}, 1^2\right)$$

Now H_0 is retained whenever the test statistic is not in the critical region $\mathcal{C}$.

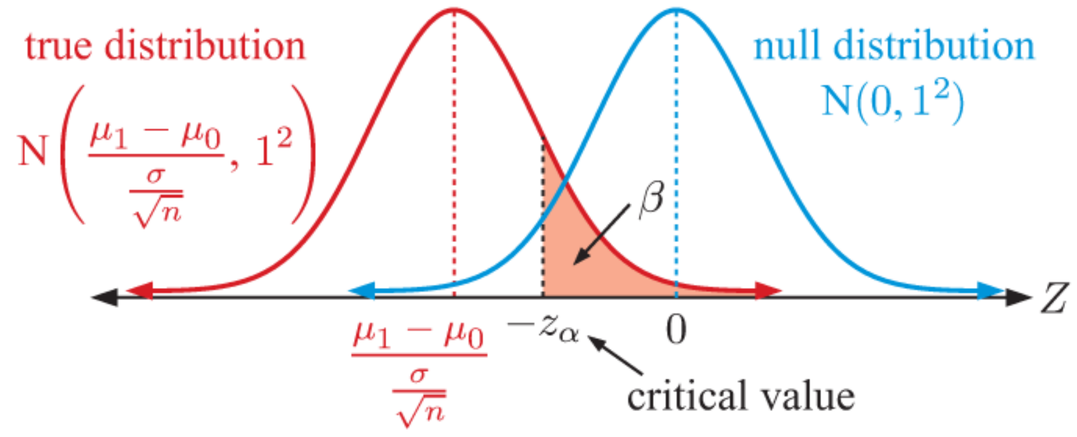
From **Section C**, for a Z-test with H_1: $\mu < \mu_0$:

$$\mathcal{C} = \{z \mid z \leqslant -z_\alpha\}$$



$$\beta = \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})$$

$$\therefore \quad \beta = \mathrm{P}\left(Z \notin \mathcal{C} \,\middle|\, Z \sim \mathrm{N}\left(\frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}}, 1^2 \right) \right)$$



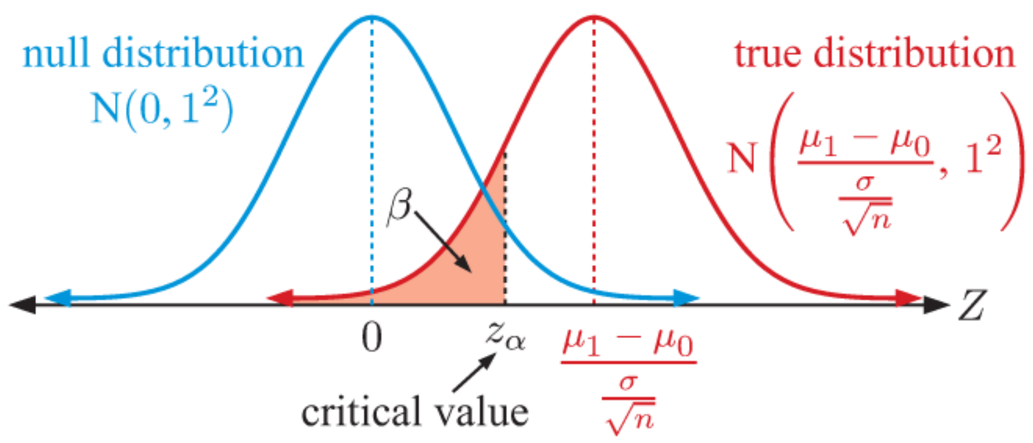
So, for $\boldsymbol{H_1}$: $\boldsymbol{\mu < \mu_0}$:

$$\beta = \mathrm{P}\left(Z > -z_\alpha \,\middle|\, Z \sim \mathrm{N}\left(\frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}}, 1^2 \right) \right)$$

We can apply similar arguments for the other alternative hypothesis forms:

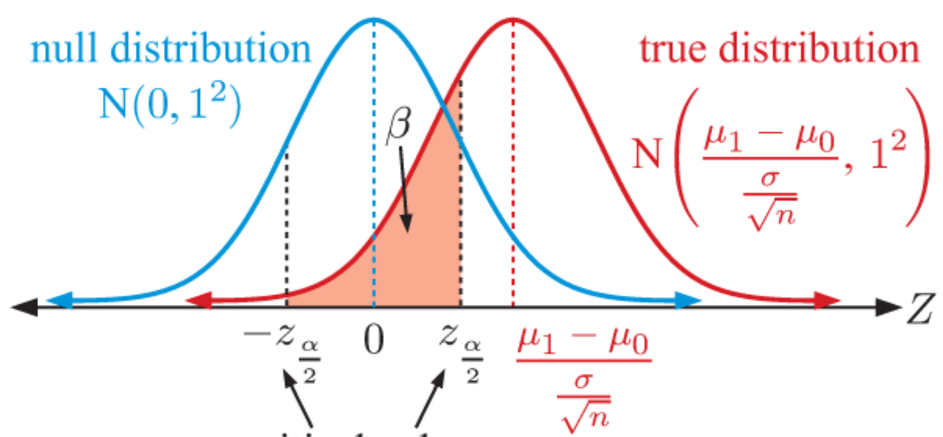
- For $\boldsymbol{H_1}$: $\boldsymbol{\mu > \mu_0}$:

$$\beta = \mathrm{P}\left(Z < z_\alpha \,\middle|\, Z \sim \mathrm{N}\left(\frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}}, 1^2 \right) \right)$$



- For $\boldsymbol{H_1}$: $\boldsymbol{\mu \neq \mu_0}$:

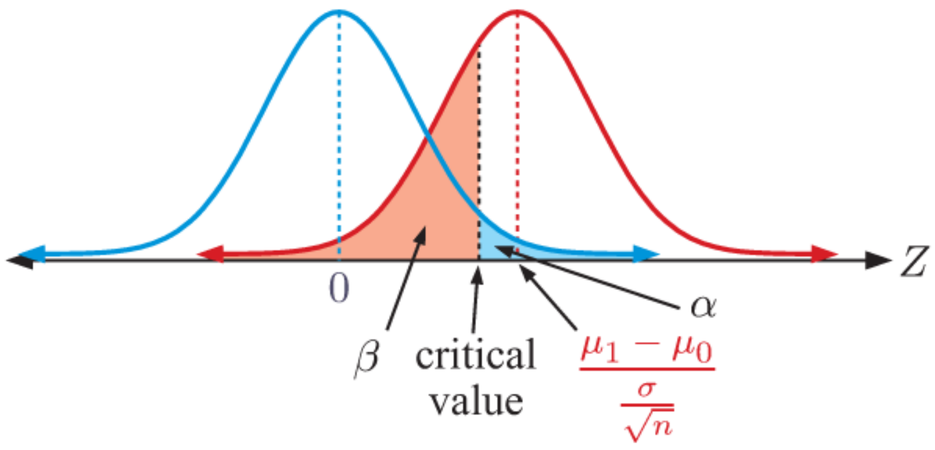
$$\beta = \mathrm{P}\left(-z_{\frac{\alpha}{2}} < Z < z_{\frac{\alpha}{2}} \,\middle|\, Z \sim \mathrm{N}\left(\frac{\mu_1 - \mu_0}{\frac{\sigma}{\sqrt{n}}}, 1^2 \right) \right)$$

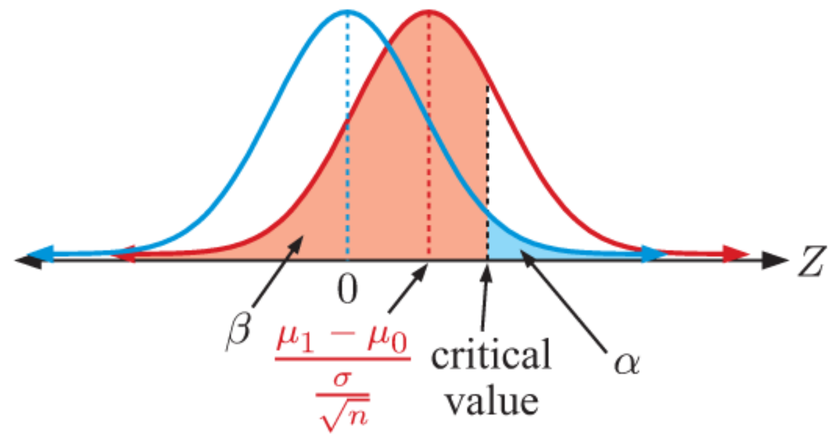


STATISTICAL POWER

The **power** of a hypothesis test is defined as $1 - \beta$, the probability of (correctly) rejecting the null hypothesis H_0 when H_0 is false.

The closer μ_1 is to μ_0, the larger the value of β, and hence the lower the power of the test.





Example 12 **Self Tutor**

Cans of chickpeas are labelled 400 g. It is known that the true weight of cans is normally distributed with standard deviation 10 g.

A statistician wishes to test whether the mean weight of a can is less than 400 g. He uses a sample of 12 cans, and a 2% level of significance.

a Find the probability of making:

i a Type I error **ii** a Type II error, given the true mean $\mu = 395$ g.

b Find the power of the test, given the true mean $\mu = 395$ g.

a **i** $\text{P(Type I error)} = \alpha = 0.02$

ii H_0: $\mu = 400$ g $\quad$ H_1: $\mu < 400$ g

H_0 is retained if

$Z > -z_{0.02} \approx -2.054$

The true distribution of Z is

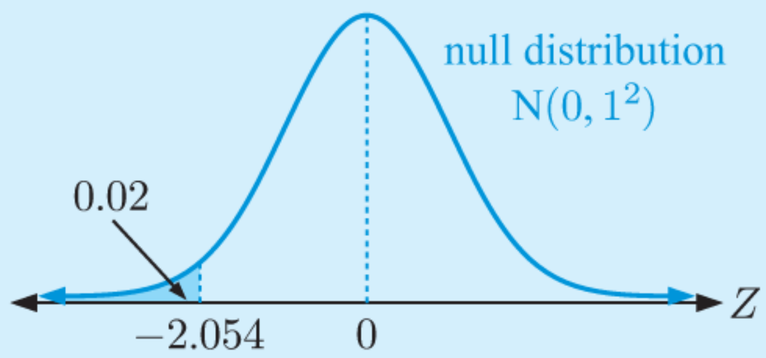
$$\text{N}\left(\frac{395 - 400}{\frac{10}{\sqrt{12}}}, 1^2\right) \equiv \text{N}(-\sqrt{3}, 1^2)$$

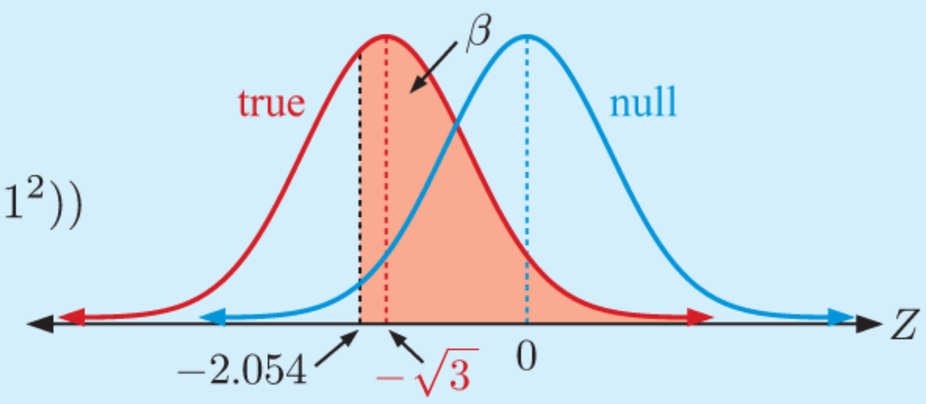
$\therefore \quad \beta = \text{P(Type II error)}$

$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$

$\approx \text{P}(Z > -2.054 \mid Z \sim \text{N}(-\sqrt{3}, 1^2))$

≈ 0.626





b Power $= 1 - \beta \approx 0.374$

EXERCISE 30J.1

1 In a population of adult trout, the length of fish X cm is normally distributed with known variance 6 cm^2, but the mean μ cm is unknown.

It is proposed to test the hypotheses H_0: $\mu = 27$ cm against H_1: $\mu > 27$ cm using a sample of size 9.

a Find the critical region for the test if the significance level is:

i 5% **ii** 1%

b Suppose the true mean $\mu = 29.2$.
Calculate P(Type II error) when P(Type I error) is:
i 0.05 **ii** 0.01

2 The weight of pumpkins from a very large crop is normally distributed with standard deviation 0.7 kg but unknown mean μ.
Using a random sample of 15 pumpkins from the crop, a statistician conducts a hypothesis test at the 5% level of significance.
Suppose that the true value of the mean is actually 6.4 kg. For each alternative hypothesis, calculate the probability β of a Type II error.

a H_1: $\mu > 6$ kg **b** H_1: $\mu \neq 6$ kg **c** H_1: $\mu < 6$ kg

3 The length of a beam produced in a manufacturing process is normally distributed with standard deviation 0.15 m. The manufacturer claims the mean length of such beams is 3.5 m.
A random sample of 20 beams is taken and their mean length $\overline{x}$ is calculated.

a State suitable hypotheses for a two-tailed Z-test.

b For a 1% level of significance, find the critical region for the test.

c The true mean length is 3.4 m. Calculate:
i the probability of making a Type II error **ii** the power of the test.

Example 13 **Self Tutor**

A sample of size 25 is taken from a normal population with unknown mean μ and known variance 36.

To test the hypotheses H_0: $\mu = 42$, H_1: $\mu > 42$, the decision rule is:

accept H_0 if the sample mean $\overline{x} < 43.5$

reject H_0 if the sample mean $\overline{x} \geqslant 43.5$.

a Find the probability of a Type I error α for the given decision rule.

b Suppose the true value of the mean is 44. Find the probability of a Type II error, β.

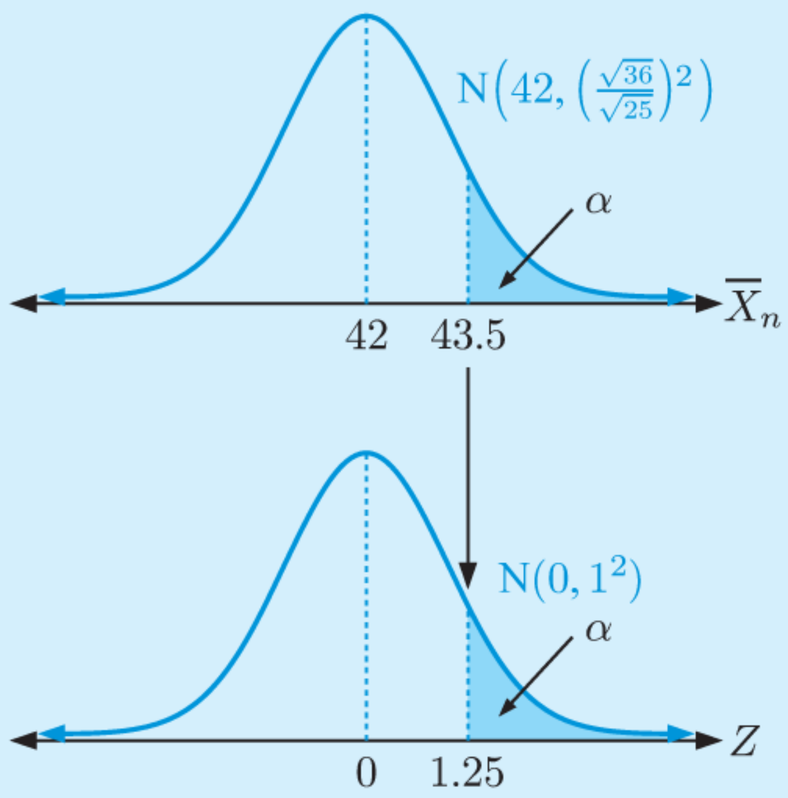
a Under H_0, the z-score of 43.5 is

$$\frac{43.5 - 42}{\frac{\sqrt{36}}{\sqrt{25}}} = 1.25$$

So, the decision rule can be equivalently written as:

accept H_0 if $z < 1.25$
reject H_0 if $z \geqslant 1.25$

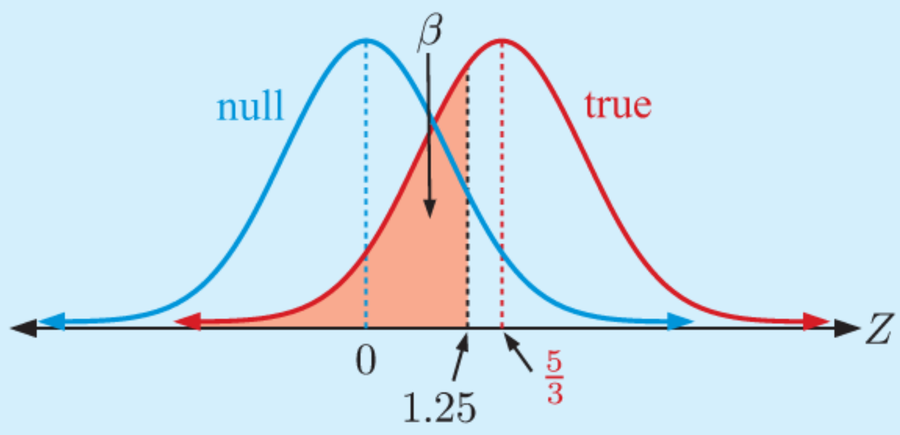
$\therefore \quad \alpha = \text{P(Type I error)}$
$= \text{P(Reject } H_0 \mid H_0 \text{ true)}$
$= \text{P}(Z \geqslant 1.25 \mid Z \sim \text{N}(0, 1^2))$
≈ 0.106



b The true distribution of Z is

$$\mathrm{N}\left(\frac{44-42}{\frac{\sqrt{36}}{\sqrt{25}}},\ 1^2\right) \equiv \mathrm{N}\left(\tfrac{5}{3},\ 1^2\right)$$

$$\begin{aligned}\therefore\quad \beta &= \mathrm{P}(\text{Type II error})\\ &= \mathrm{P}(\text{Retain } H_0 \mid H_0 \text{ false})\\ &= \mathrm{P}\left(Z < 1.25 \mid Z \sim \mathrm{N}\left(\tfrac{5}{3},\ 1^2\right)\right)\\ &\approx 0.338\end{aligned}$$



4 A sample of size 16 is taken from a normal population with unknown mean μ and known variance 64.
The sample is used to test the hypotheses H_0: $\mu = 150$, H_1: $\mu > 150$.
The decision rule for the test is: accept H_0 if the sample mean $\overline{x} < 155$
reject H_0 if $\overline{x} \geqslant 155$.

a Find $\alpha = \mathrm{P}(\text{Type I error})$.

b Suppose the true mean is $\mu = 159$.

i Find the probability of a Type II error, β.

ii Calculate the power of the test.

5 A machine fills each can of fizzy drink with volume Y cm^3, where Y is normally distributed with mean μ and standard deviation 2 cm^3.
The mean μ is believed to be 330 cm^3. In order to check this value, a random sample of 16 cans is selected.
The following hypotheses are tested using this sample: H_0: $\mu = 330$
H_1: $\mu \neq 330$.
H_0 will be rejected if $\overline{y} \leqslant 329$ or $\overline{y} \geqslant 331$. Otherwise, H_0 is accepted.

a Find the significance level for this test.

b The true mean $\mu = 328$ cm^3.

i Find the probability of a Type II error.

ii Calculate the power of the test.

6 Consider a Z-test of H_0: $\mu = 0$ against H_1: $\mu \neq 0$ with a sample of size $n = 4$ and significance level α for a population with standard deviation $\sigma = 1$.

a Suppose the true mean $\mu = 2$.

i Copy and complete this table.

ii Describe what happens to β as α decreases.

α	0.1	0.05	0.01	0.001
β				

b Suppose now we set $\alpha = 0.05$.

i Calculate β for each possible "true value" of μ in the table alongside.

ii Discuss the following statement:

"The further away the true value of the mean is from the null value, the easier it is to detect a genuine difference."

True value of μ	β
± 1	
± 2	
± 5	
± 10	

ERROR ANALYSIS WITH OTHER DISTRIBUTIONS

The definitions we have seen for α and β can be applied to other hypothesis tests. We now consider the calculation of error probabilities of hypothesis tests for:

- the mean of a Poisson population
- a population proportion.

Example 14 **Self Tutor**

A discrete random variable X has a Poisson distribution with unknown mean λ. We wish to test the hypothesis H_0: $\lambda = 2$ against H_1: $\lambda < 2$.

A random sample $\{x_1, ..., x_{12}\}$ of 12 independent values is taken from X.

The decision rule is: accept H_0 if $\sum_{i=1}^{12} x_i > 16$, otherwise reject it.

a Define the critical region for $t = \sum_{i=1}^{12} x_i$.

b Define a Type I error for this scenario, and calculate P(Type I error).

c The true mean of $\lambda = 1.5$.
Define a Type II error for this scenario, and calculate P(Type II error).

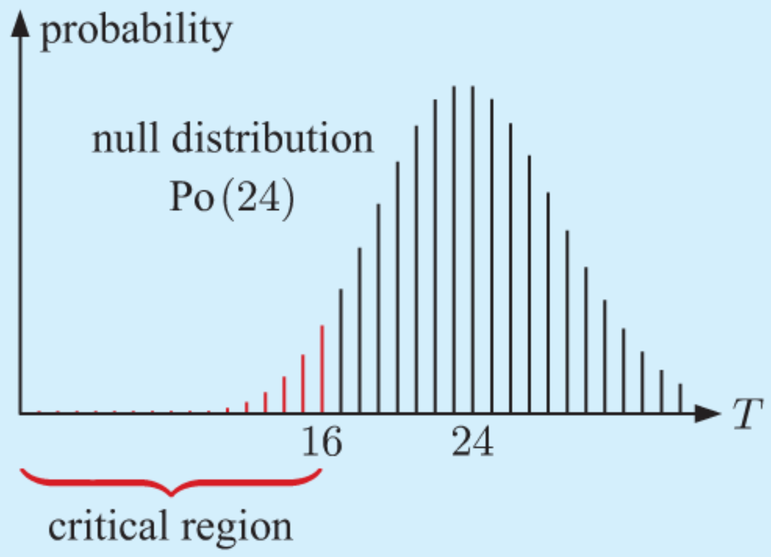
---

a Since X is a discrete random variable with values 0, 1, 2, ..., so is t.
$\therefore$ the critical region is $\{t \mid t \leqslant 16\}$.

b A Type I error is rejecting H_0 when H_0 is in fact true. This means deciding $\lambda < 2$ when in fact $\lambda = 2$ and $X \sim \text{Po}(2)$.

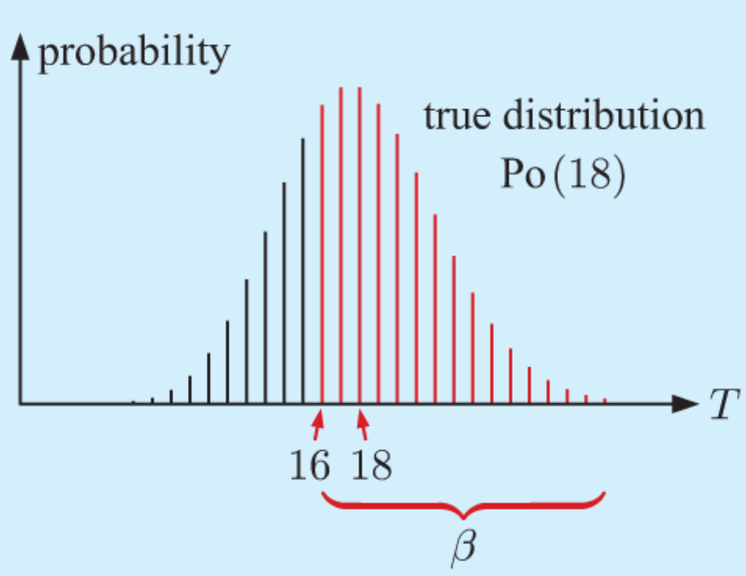
$$\begin{aligned}&\text{P(Type I error)}\\ &= \text{P(Reject } H_0 \mid H_0 \text{ true)}\\ &= \text{P}(T \leqslant 16 \mid T \sim \text{Po}(24))\\ &\approx 0.0563\end{aligned}$$





c A Type II error is accepting H_0 when H_0 is in fact false. This means accepting $\lambda = 2$ when in fact $\lambda = 1.5$.

$$\begin{aligned}&\text{P(Type II error)}\\ &= \text{P(Retain } H_0 \mid H_0 \text{ false)}\\ &= \text{P}(T > 16 \mid T \sim \text{Po}(18))\\ &\approx 0.625\end{aligned}$$



Example 15 **Self Tutor**

To test whether a coin is biased towards tails, the following decision rule is adopted:

Toss the coin 180 times. If 104 or more tails are obtained, accept that the coin is biased towards tails.

a Define the null hypothesis H_0 and alternative hypothesis H_1.

b Calculate P(Type I error) and state the level of significance for this test.

c The coin is actually biased, and the probability of obtaining a tail with each toss is 0.65. Find P(Type II error) and state the power of this test.

---

a Let p be the probability of obtaining tails in one coin toss.

H_0: $p = 0.5$ {the coin is fair}

H_1: $p > 0.5$ {the coin is biased towards tails}

b A Type I error means deciding the coin is biased when it is in fact fair.

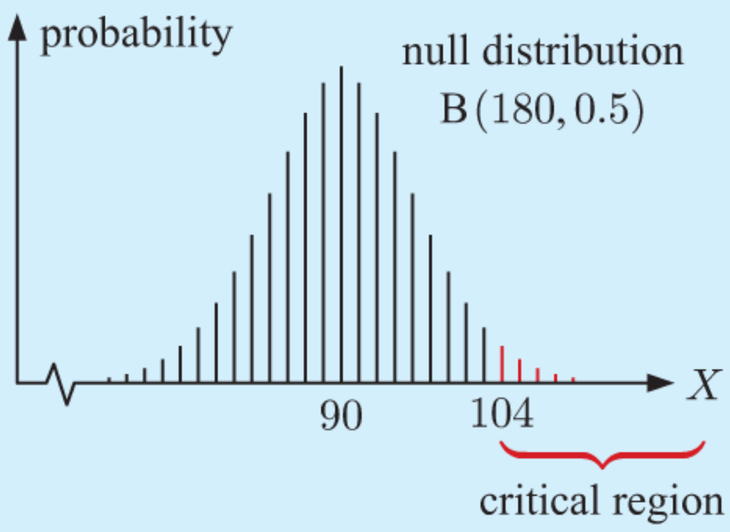
$\alpha = \text{P(Type I error)}$

$= \text{P(Reject } H_0 \mid H_0 \text{ true)}$

$= \text{P}(X \geqslant 104 \mid X \sim \text{B}(180, 0.5))$

≈ 0.0219

$\therefore$ the significance level ≈ 0.0219



c A Type II error means deciding the coin is fair when it is in fact biased.

If $p = 0.65$, then $X \sim \text{B}(180, 0.65)$.

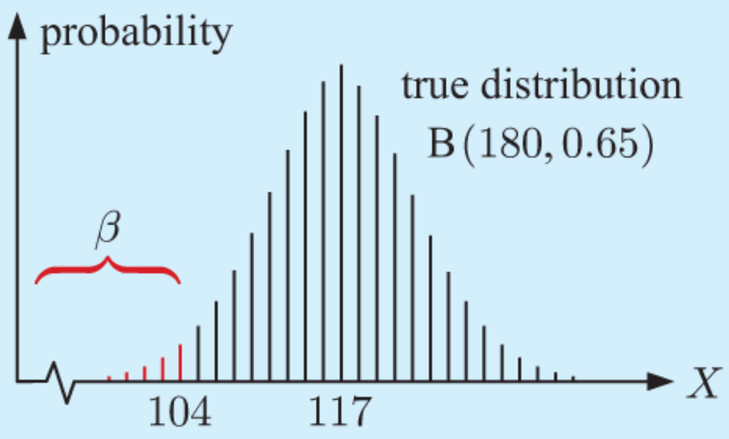
$\therefore \beta = \text{P(Type II error)}$

$= \text{P(Retain } H_0 \mid H_0 \text{ false)}$

$= \text{P}(X < 104 \mid X \sim \text{B}(180, 0.65))$

≈ 0.0184

$\therefore$ the power of the test $= 1 - \beta \approx 0.9816$



EXERCISE 30J.2

1 A tetrahedral die has faces marked 1, 2, 3, and 4. To test whether the die is biased towards rolling a 4, the following decision rule is adopted:

Roll the die 300 times. If at least 88 4s are rolled, we accept that the die is biased towards rolling a 4.

a Define the null hypothesis H_0 and the alternative hypothesis H_1.

b i Define a Type I error for this example.

ii Find P(Type I error) and state the level of significance for the test.

c The die is actually biased with P(rolling a 4) $= 0.32$. Find P(Type II error).

2 Eri has a coin which has probability p of giving a head when tossed. She believes that the coin is fair. However, Eri's friend, Mayuki, thinks that the coin is biased, and that $p > 0.5$.

To determine which of them is correct, Eri tosses the coin 12 times. Let X denote the number of heads obtained.

a State appropriate null and alternative hypotheses.

b Eri rejects the null hypothesis if $X \geqslant 10$.

i What name is given to the region $X \geqslant 10$?

ii Find P(Type I error) and state the significance level of this test.

c In fact, the coin is biased and $p = 0.6$.

i Find P(Type II error) and state the power of this test.

ii If Eri uses the decision rule from **b**, what type of error is Eri at risk of making?

3 A random variable X has a Poisson distribution with mean λ, where λ equals 3 or 4. To test the value of λ the following hypotheses are defined: H_0: $\lambda = 3$

H_1: $\lambda = 4$.

A random sample $\{x_1, x_2, ..., x_9\}$ of independent values is taken from X, with replacement.

If $\sum_{i=1}^{9} x_i \leqslant 37$, then H_0 is accepted, otherwise H_1 is accepted.

a Find the level of significance for this test. **b** Calculate the power of the test.

4 Consider the following hypotheses for the mean of a Poisson population: H_0: $\lambda = 2$

H_1: $\lambda > 2$.

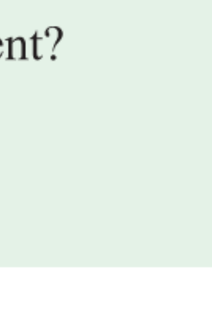
A sample of size $n = 5$ is taken and the hypotheses are tested with significance level $\alpha = 0.1$.

a Find the critical region for this test.

b The true population mean $\lambda = 2.5$. Calculate the power of the test.

SUMMARY — HYPOTHESIS TESTING FLOWCHART

Click on this icon to obtain a flowchart to help you remember which hypothesis test to use, and how to calculate and make decisions with the p-value.



REVIEW SET 30A

1 Buses serving route 033F are scheduled to arrive at stop 2 at 9:45 am each day. The bus company wants to determine whether the buses serving this route are arriving late to stop 2. Write down the set of hypotheses that the bus company should consider.

2 When a null hypothesis H_0 and an alternative hypothesis H_1 are tested at a 10% significance level, the p-value ≈ 0.0794.

a Interpret the p-value. **b** What does the significance level represent?

c Is there enough evidence to reject H_0 in this case?

d If H_0 is accepted when it is actually false, what type of error has been made?

3 Quickshave produces disposable razorblades. They claim that the mean number of shaves before a blade has to be thrown away is 13. A researcher wishing to test the claim asks 30 men to supply data on the number of shaves they get from one Quickshave blade. The sample mean found was 12.8.

Given the population standard deviation $\sigma = 1.6$, use a critical region to test the manufacturer's claim at a 2% level.

4 The house prices in a large town are normally distributed. A real estate agent claims that the mean house price is £438 000. To test the real estate agent's claim, 30 recently sold houses were randomly selected and the mean price $\overline{x}$ was calculated. The standard deviation for this sample was £23 500. At a 2% level of significance, what values of $\overline{x}$, to the nearest £500, would support the agent's claim?

5 Rosario owns an apricot orchard. Last year, the mean weight of his apricots was 90 grams. Rosario fears that severe droughts this year may have reduced the weight of his apricots. To address his concerns, Rosario randomly selected 20 apricots from his current harvest and recorded their weights in grams:

88	72	93	71	86	94	70	99	86	80
92	93	88	78	83	72	79	75	78	84

a Write down the hypotheses that Rosario should test.

b Which hypothesis test should Rosario use to test his concerns?

c Are Rosario's concerns justified at the 1% level? Use a critical region in your decision.

6 Joe and Ruben are friends who like to fish in their free time. Joe is very competitive and claims that he is a better fisherman than Ruben. This table summarises the data for the number of fish caught during their last 12 fishing trips:

	Mean	*Standard deviation*
Joe	$\overline{x}_1 = 11.0$	$s_1 \approx 3.34$
Ruben	$\overline{x}_2 = 10.25$	$s_2 \approx 2.26$

a Write down the hypotheses that Joe should use to test his claim.

b Use a two-sample t-test with a 5% level of significance to test Joe's claim.

7 Kelly recorded the time spent shopping, in minutes, by a sample of customers in two neighbouring supermarkets.

Supermarket A:	12	28	13	7	22	19	4	13	6	11		
Supermarket B:	14	35	32	21	14	8	2	16	24	27	19	42

a Should a two-sample or paired t-test be used to determine whether there is a significant difference between the supermarkets? Explain your answer.

b Conduct the appropriate test with a 10% level of significance.

8 E10 fuel is unleaded fuel that has been mixed with 10% ethanol. The fuel economy of E10 and regular unleaded was recorded for 10 different cars:

E10 (km L^{-1})	9.6	11.2	9.2	14.1	11.5	9.2	11.8	12.3	12.0	10.3
Regular unleaded (km L^{-1})	10.7	11.8	9.3	13.4	12.4	9.5	12.2	13.1	12.8	11.0

Use a paired t-test to test the claim that the fuel economy of E10 is just as good as regular unleaded, with a 5% level of significance.

9 A traditional training method for teaching people to type accurately has a success rate of 70%. A new set of exercises is claimed to have a higher success rate.

30 people were randomly selected to try the new exercises, and 26 people successfully learned how to type accurately.

On a 5% level of significance, is this sufficient evidence to conclude that the new exercises are better?

10 An aged care office has staffing concerns about the number of calls they receive. If the number of calls per 5 minute interval exceeds 2, then a new receptionist should be hired.

Over a particular 1 hour time interval, 29 calls were received.

Assuming that calls arrive in a Poisson distribution, conduct a hypothesis test with a 1% significance level to determine whether a new receptionist should be hired.

11 The bivariate data below compares the height (x) and weight (y) of 11 men.

Height (x cm)	164	167	173	176	177	178	180	180	181	184	192
Weight (y kg)	68	88	72	96	85	89	71	100	83	97	93

Conduct a hypothesis test with $\alpha = 0.05$ to determine whether the variables are linearly correlated.

12 A machine is used to fill packets with rice. The weight of the contents in each packet is X grams. X is normally distributed with mean μ grams and standard deviation 3.71 grams.

The packets are labelled as containing 500 g. To check this statement, a sample of 13 packets is selected, and the mean contents $\overline{x}$ is calculated.

The hypotheses for the test are H_0: $\mu = 500$ g and H_1: $\mu \neq 500$ g.

H_0 will be rejected if $\overline{x} \leqslant 498$ or $\overline{x} \geqslant 502$. Otherwise, H_0 will be accepted.

a State the critical region for the hypothesis test.

b Show that the significance level for the test is approximately 0.0519.

c Find the probability of a Type II error with this test given that the true value of μ is 498.4 grams.

13 An advertisement claimed that 20% of all households used Ongodo washing powder. A consumer group consulted a statistician to help test the advertisement's claim. The statistician recommended sampling 100 households at random. He said the advertisement's claim was only reasonable if at least 13 households in the sample use Ongodo.

a For this test, write down the:

i acceptance region **ii** critical region.

b Find the significance level of this test.

c The true proportion of households using Ongodo washing powder is 17.5%. Calculate the power of this test.

REVIEW SET 30B

1 A seafood company farms fish for a restaurant chain. The manager is concerned that the fish supplied are too small. As a consequence he considers the hypotheses:

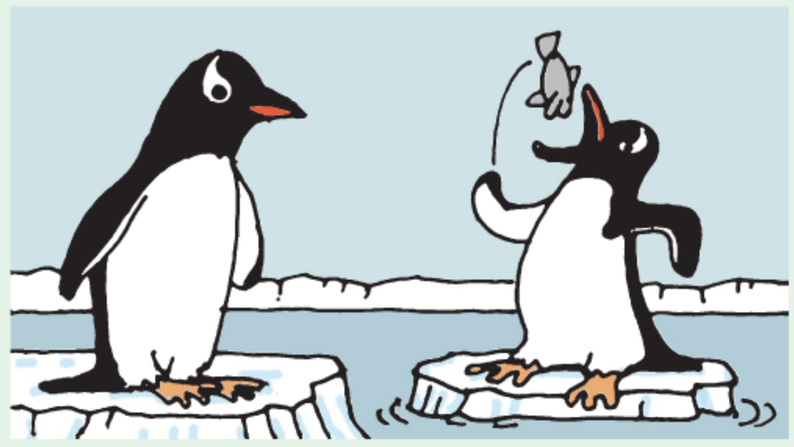
H_0: The seafood company is not supplying undersize fish.

H_1: The seafood company is supplying undersize fish.

What conclusion would result in: **a** a Type I error **b** a Type II error?

2 Yarni's resting pulse rate was 68 beats per minute for many years. However, with a sensible diet and exercise, she hopes to reduce this rate. After six months, Yarni's mean pulse rate is now 65 beats per minute, with standard deviation 1.732. These statistics were calculated from 42 measurements. Using the p-value, is there sufficient evidence at a 5% level, to conclude that Yarni's pulse rate has decreased?

3 In 2011, the mean weight of a gentoo penguin from a penguin colony was 7.82 kg, with standard deviation 1.83 kg. Exactly one year after these statistics were found, a sample of 48 penguins from the same colony were found to have mean weight 7.55 kg. Is there sufficient evidence, at a 5% level of significance, to suggest that the mean weight in 2012 differs from that in 2011?



4 As part of a routine health check of its employees, a company wants to check whether their systolic blood pressure is too high. High systolic blood pressure is generally diagnosed when a blood pressure test is more than 140 mm Hg.

A doctor measures the systolic blood pressure of a random sample of 35 company employees. She finds that the sample mean is 143.7 mm Hg with standard deviation 11.2 mm Hg.

Conduct a one-sample t-test with a 5% level of significance to determine whether the company's concerns are justified.

5 The average distance Arthur can hit a golf ball is 115 metres.

After spending time with a professional, Arthur measured the distance of 30 drives. The results in metres were as follows:

100	126	93	171	131	94	136	144	138	110
168	132	100	49	156	119	119	150	146	139
149	145	122	56	140	118	115	73	105	133

Is there sufficient evidence at the 5% level to claim that Arthur has improved?

6 To decide the credit limit of a prospective credit card holder, a bank gives points based on factors such as employment, income, home and car ownership, and general credit history.

The points totals of randomly selected people living in Maple Grove and Berkton are:

Maple Grove:	14	11	13	13	15	12	12	12	10	11	11	11	12	13
	14	13	11	12	14	14	14	13	15	14	11	10	11	16
	11	12	12	10	11	10	10	12	13	13	13	12		
Berkton:	11	10	12	14	11	12	14	11	11	14	13	13	14	13
	12	12	14	11	12	11	12	12	11	12	13	11	12	13
	12	12	10	10	12	11	11	9	14	10	13	13	10	12

Conduct a two-sample t-test with a 10% level of significance to determine whether there is a difference between the average points total of the two suburbs.

7 The following scores are the final IB examination results for students who took a revision course and students who did not:

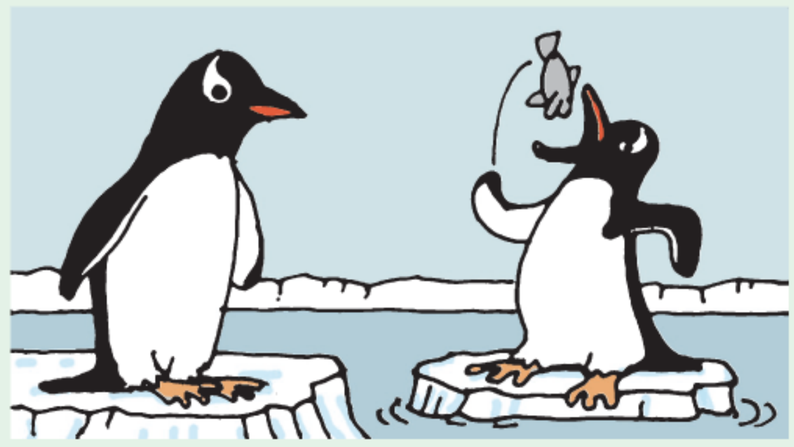
Revision course:	32	39	31	35	40	33	34	33	34	35	39	33	32	35
No revision course:	28	31	30	23	33	31	36	36	38	35				

a Find the mean and standard deviation of each sample.

b Conduct a two-sample t-test with a 10% level of significance to determine whether the revision course was effective.

2 Yarni's resting pulse rate was 68 beats per minute for many years. However, with a sensible diet and exercise, she hopes to reduce this rate. After six months, Yarni's mean pulse rate is now 65 beats per minute, with standard deviation 1.732. These statistics were calculated from 42 measurements. Using the p-value, is there sufficient evidence at a 5% level, to conclude that Yarni's pulse rate has decreased?

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A doctor measures the systolic blood pressure of a random sample of 35 company employees. She finds that the sample mean is 143.7 mm Hg with standard deviation 11.2 mm Hg.

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100	126	93	171	131	94	136	144	138	110
168	132	100	49	156	119	119	150	146	139
149	145	122	56	140	118	115	73	105	133

Is there sufficient evidence at the 5% level to claim that Arthur has improved?

6 To decide the credit limit of a prospective credit card holder, a bank gives points based on factors such as employment, income, home and car ownership, and general credit history.

The points totals of randomly selected people living in Maple Grove and Berkton are:

Maple Grove:	14	11	13	13	15	12	12	12	10	11	11	11	12	13
	14	13	11	12	14	14	14	13	15	14	11	10	11	16
	11	12	12	10	11	10	10	12	13	13	13	12		
Berkton:	11	10	12	14	11	12	14	11	11	14	13	13	14	13
	12	12	14	11	12	11	12	12	11	12	13	11	12	13
	12	12	10	10	12	11	11	9	14	10	13	13	10	12

Conduct a two-sample t-test with a 10% level of significance to determine whether there is a difference between the average points total of the two suburbs.

7 The following scores are the final IB examination results for students who took a revision course and students who did not:

Revision course:	32	39	31	35	40	33	34	33	34	35	39	33	32	35
No revision course:	28	31	30	23	33	31	36	36	38	35				

a Find the mean and standard deviation of each sample.

b Conduct a two-sample t-test with a 10% level of significance to determine whether the revision course was effective.

Chapter 31

χ^2 hypothesis tests

Contents:

A The χ^2 goodness of fit test
B Estimating distribution parameters in a goodness of fit test
C Critical regions and critical values
D The χ^2 test for independence



OPENING PROBLEM

A multiple choice quiz has 10 questions. Each question has 4 choices, only 1 of which is correct.

The quiz is given to 150 people, and the number of correct answers for the participants is summarised in the table alongside.

Number of correct answers	*Frequency*
0	7
1	34
2	52
3	36
4	12
5 or more	9

A statistician analysing the data assumes that each person answers all questions by randomly guessing.

Things to think about:

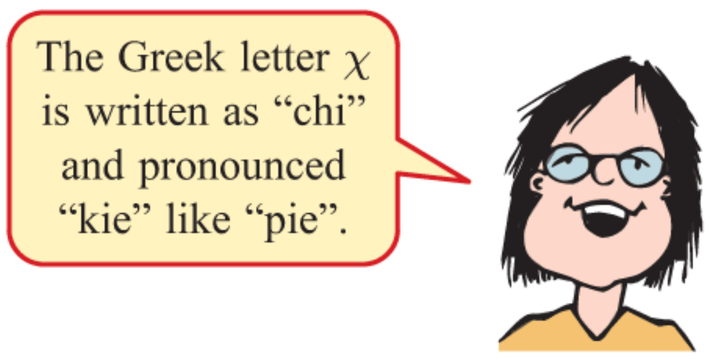
a Let X be the number of questions answered correctly by a person randomly guessing answers. What is the distribution of X?

b Do you think the statistician's model is appropriate? Explain your answer.

c How can we *measure* how appropriate the statistician's model is?

When we observe a variable in a population, we do not always know its distribution. If we *choose* a distribution to model the variable, we will want to know how well the distribution fits our observations.

In this Chapter we study χ^2 **(chi-squared)** tests to assess how appropriate a statistical model is.



A THE χ^2 GOODNESS OF FIT TEST

Suppose Rico rolls a die 60 times and obtains the rolls in the table. Since the relative frequencies or *proportions* of the outcomes are quite different, Rico claims that his die is *biased*.

Number	*Frequency*
1	20
2	10
3	5
4	8
5	7
6	10

Rico could use the binomial distribution to test a hypothesis about a *single* population proportion, as we did in **Chapter 30**.

For example, he could test the hypotheses H_0: $p = \frac{1}{6}$, H_1: $p \neq \frac{1}{6}$ where $p = \text{P(rolling a } 1)$.

However, these hypotheses do not take into account the probabilities of rolling the other numbers.

If X is the number rolled with Rico's die, then H_0 being true *might* mean that the die is fair.

x	1	2	3	4	5	6
$\text{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

fair

However, there are also infinitely many biased distributions for X for which H_0 is true. An example is shown opposite.

x	1	2	3	4	5	6
$\text{P}(X = x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$

biased

To test whether his die is biased, Rico needs a test for *multiple* proportions, or a **probability distribution**. He needs the χ^2 **goodness of fit test**.

THE HYPOTHESES

For Rico's die, let p_1 be the probability of rolling a 1, p_2 be the probability of rolling a 2, and so on.

Our null hypothesis H_0 is that the die is fair. If this is the case, then each number is *equally likely* to occur on each roll. We therefore have H_0: $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{6}$, ..., $p_6 = \frac{1}{6}$.

Our alternative hypothesis H_1 is that the die is *not* fair. If this is the case, then at least one outcome has a probability which is different from the others. We therefore have
H_1: at least one of $p_1, p_2, ..., p_6 \neq \frac{1}{6}$.

Consider a scenario with k categories. Let p_i be the population proportion of individuals in category i, where $p_1 + p_2 + ... + p_k = 1$.

The **hypotheses** in a χ^2 goodness of fit test have the form:

H_0: $p_1 = p_{01}$, $p_2 = p_{02}$, ..., and $p_k = p_{0k}$

H_1: at least one of $p_i \neq p_{0i}$

where p_{0i} is the population proportion of category i under the null hypothesis.

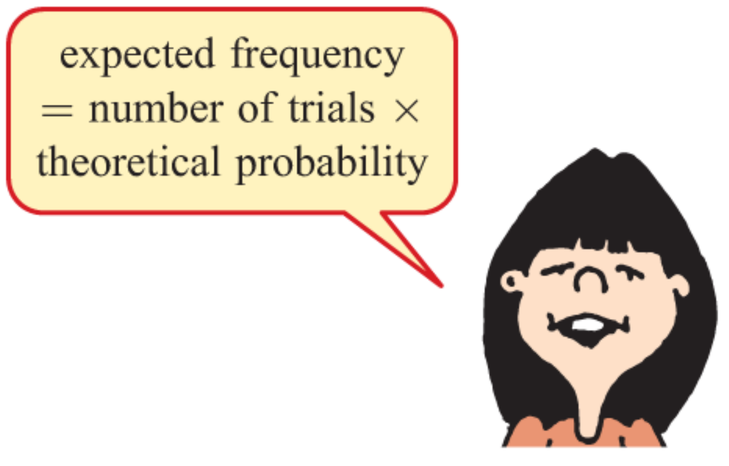
In Rico's experiment, $p_{01}, p_{02}, ..., p_{06}$ are all equal to $\frac{1}{6}$. However, in general the population proportions under H_0 do not have to all be the same.

THE TEST STATISTIC

In our study of probability we calculated the number of times we *expect* an event to occur given its theoretical probability.

For example, if Rico's die was fair then we would expect to see $60 \times \frac{1}{6} = 10$ of each number.

We are therefore interested in how the *observed* frequencies differ from their *expected* values.



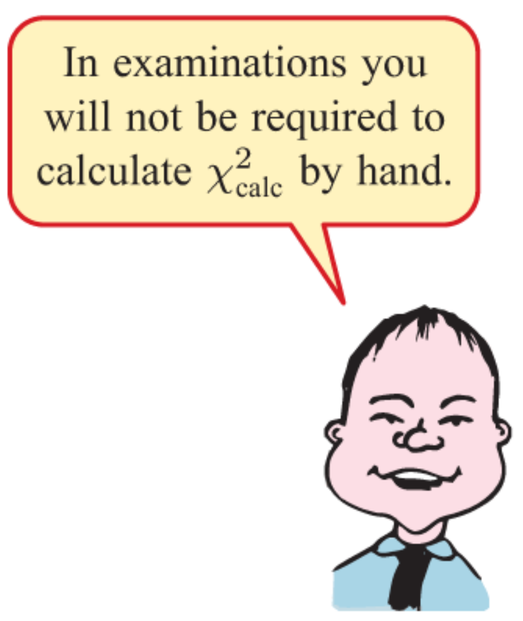
The **test statistic** for a χ^2 goodness of fit test is: $\chi^2_{\text{calc}} = \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$

where f_{obs} is an **observed** frequency

f_{exp} is an **expected** frequency.

For the die rolling example, we can calculate the test statistic χ^2_{calc} with the help of a table:

Number	f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$(f_{\text{obs}} - f_{\text{exp}})^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
1	20	10	10	100	10
2	10	10	0	0	0
3	5	10	−5	25	2.5
4	8	10	−2	4	0.4
5	7	10	−3	9	0.9
6	10	10	0	0	0
				Total	13.8



So, $\chi^2_{\text{calc}} = 13.8$.

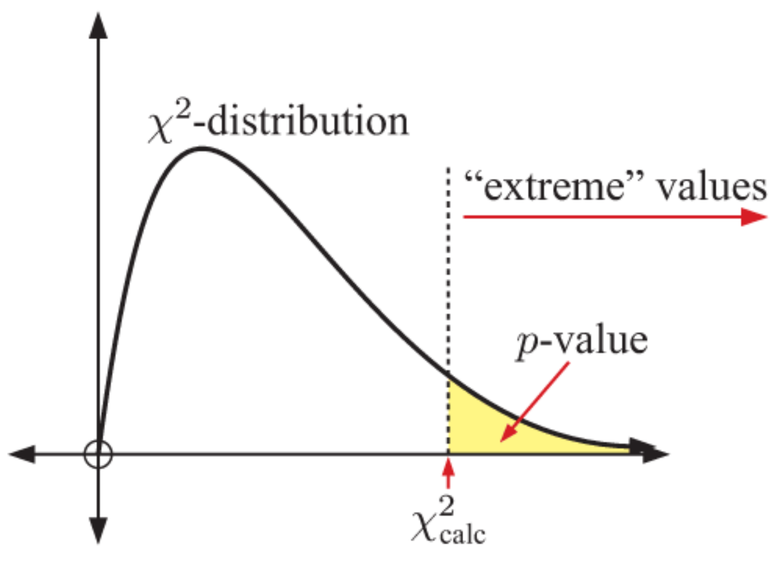
THE p-VALUE

In order to make a decision on whether or not to reject H_0 based on χ^2_{calc}, we need to calculate a ***p*-value** and compare it to the **significance level** α of the test.

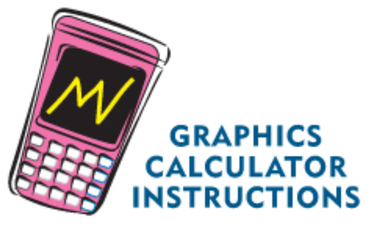
For a t-test, we calculated the p-value as the probability of observing a value that is as "extreme" as the test statistic. Depending on the form of the alternative hypothesis H_1, this would correspond to the upper tail, lower tail, or both tails of the t-distribution.

In the goodness of fit test, if H_0 was not true then the difference between the observed frequencies and expected frequencies would be large. Hence the value of χ^2_{calc} would also be large. So, for the χ^2 goodness of fit test:

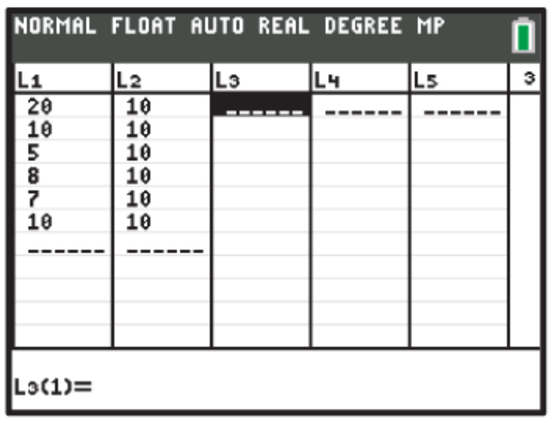
> ***p*-value** = probability of observing a value **greater than or equal to** χ^2_{calc}.

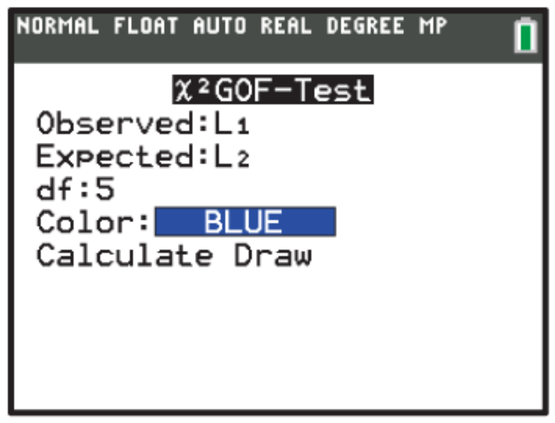


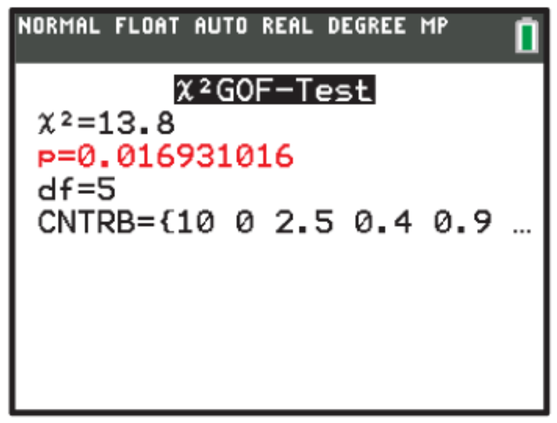
You can use your calculator to evaluate p-values.



For the die rolling example, we obtain:

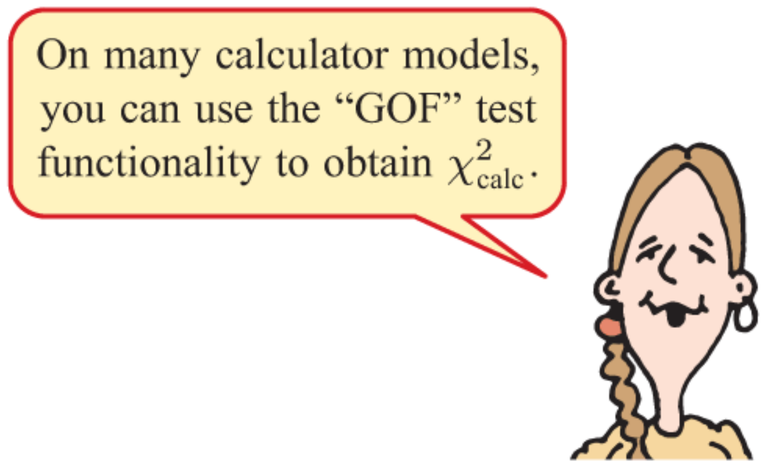






So, the p-value ≈ 0.0169. For a test of H_0 on a significance level of $\alpha = 0.05$, this is enough evidence to reject H_0 in favour of H_1.

We therefore conclude that Rico's die is biased.



DEGREES OF FREEDOM

In finding the p-value with your calculator, you should have noticed that you also need a value for "df". Like in the t-test, this stands for "**degrees of freedom**".

In a χ^2 goodness of fit test, the value of df is the number of values that are *free to vary*.

For example, consider the population proportions of 3 categories p_1, p_2, and p_3. Since they are proportions, they must add to 1. We can therefore write any one of the proportions in terms of the other two. For example, $p_1 = 1 - p_2 - p_3$.

So, only two of the proportions are *free to vary*.

For a χ^2 goodness of fit test, **df = number of categories − 1.**

SUMMARY OF THE χ^2 GOODNESS OF FIT TEST

Step 1: State the **null hypothesis** H_0 and the **alternative hypothesis** H_1.
Step 2: State the **significance level** α.
Step 3: Calculate the value of the **test statistic**: $\chi^2_{\text{calc}} = \sum \dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$.
Step 4: Use technology to calculate the ***p*-value**, using df = number of categories − 1.
Step 5: Reject H_0 if p-value $< \alpha$, otherwise accept H_0.
Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Notice the similarity between the steps of the goodness of fit test and all of the other hypothesis tests we have seen so far.

Example 1 **Self Tutor**

The table alongside shows the grades received by university students taking a second year Computer Science course.

Grade	*Frequency*
High Distinction	16
Distinction	21
Credit	21
Pass	59
Fail	34
Total	151

In the following semester, a new coordinator is appointed for the course. The new coordinator is concerned by the high number of High Distinctions and Distinctions awarded, and wants to determine if the course should be adjusted.

It is considered usual if 5% of students receive a High Distinction, 10% receive a Distinction, 15% receive a Credit, 40% receive a Pass, and the remaining 30% receive a Fail.

Conduct a χ^2 goodness of fit test to determine whether the course should be adjusted with a 5% level of significance.

Step 1: Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of students who receive a High Distinction, Distinction, Credit, Pass, and Fail respectively.

The hypotheses that should be tested are:

H_0: $p_1 = 0.05$, $p_2 = 0.1$, $p_3 = 0.15$, $p_4 = 0.4$, $p_5 = 0.3$
H_1: at least one of $p_1 \neq 0.05$, $p_2 \neq 0.1$, ..., or $p_5 \neq 0.3$.

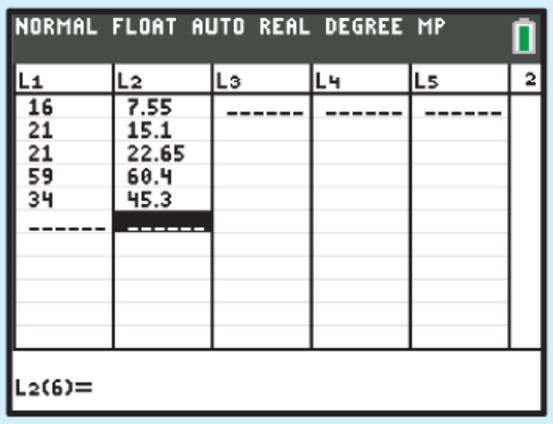
Step 2: The significance level is $\alpha = 0.05$.

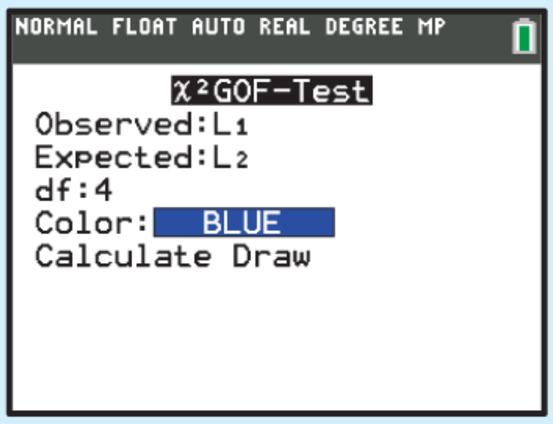
Step 3:

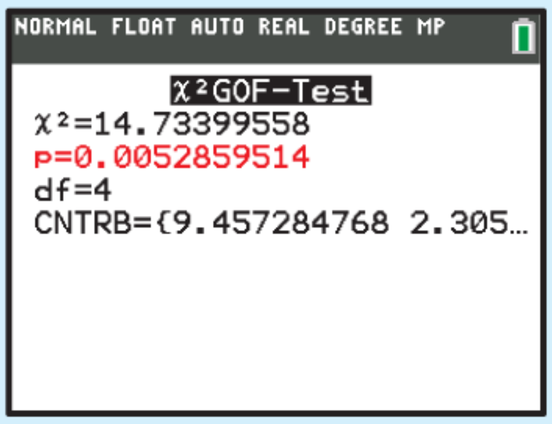
Grade	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
High Distinction	16	$151 \times 0.05 = 7.55$	≈ 9.4573
Distinction	21	$151 \times 0.1 = 15.1$	≈ 2.3053
Credit	21	$151 \times 0.15 = 22.65$	≈ 0.1202
Pass	59	$151 \times 0.4 = 60.4$	≈ 0.0325
Fail	34	$151 \times 0.3 = 45.3$	≈ 2.8188
		Total	≈ 14.734

So, $\chi^2_{\text{calc}} \approx 14.7$.

Step 4: $\text{df} = 5 - 1 = 4$







Using technology, p-value $\approx 0.005\,29$.

Step 5: Since p-value $< 0.05 = \alpha$, we have enough evidence to reject H_0 in favour of H_1 on a 5% level of significance.

Step 6: Since we have accepted H_1, we conclude that the course should be adjusted.

LIMITATIONS

In order for χ^2 to be distributed appropriately, the sample size n must be sufficiently large. Generally, n is sufficiently large if none of the expected frequencies is less than 5.

In cases where there are expected frequencies less than 5, we can **combine** similar categories to get more reliable results.

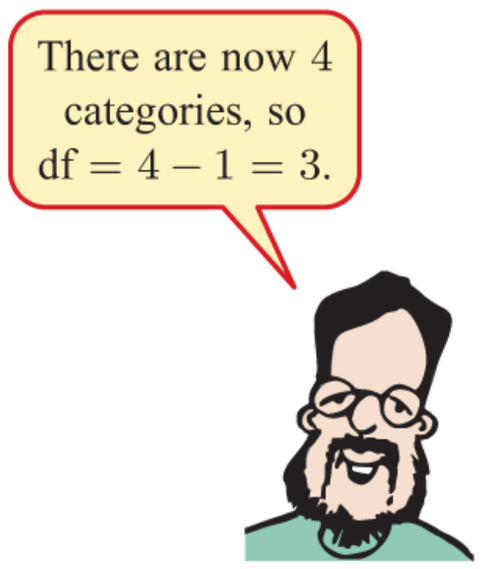
For example, when investigating the number of serious sports injuries treated by a hospital, the following expected frequencies are calculated. Since the expected frequencies of two age groups are less than 5, the data for some categories is combined.

Original table

Age group	*Expected frequency*
10 - 15	14.9
15 - 20	14.1
20 - 30	15.6
30 - 40	4.4
40 - 50	0.9
50+	0.2

Combined

Age group	*Expected frequency*
10 - 15	14.9
15 - 20	14.1
20 - 30	15.6
30+	$4.4 + 0.9 + 0.2 = 5.5$



EXERCISE 31A

1 When a coin was tossed 96 times, 54 heads were observed. A χ^2 goodness of fit test is to be conducted to determine if the coin is biased with a 5% level of significance.

a Write down the hypotheses that should be considered.

b How many tails were observed?

c Calculate the *expected* frequencies of heads and tails assuming the coin is fair.

d Use a table to help you calculate χ^2_{calc}.

e State the number of degrees of freedom.

f Use technology to find the p-value.

g Is there enough evidence to conclude that the coin is biased?

2 In the last election, 54% of voters voted for party A, 30% of voters voted for party B, and the rest voted for party C.

A polling agency conducted a survey asking 300 voters which party they are going to vote for in the upcoming election. The results are shown alongside.

Party	A	B	C
Voters	141	105	54

Conduct a χ^2 goodness of fit test with a 1% level of significance to determine whether there has been a change in the proportions of voters supporting each party since the last election.

3 Brian owns a chocolate café. He wants to start offering ice cream in addition to his chocolate menu items. He initially makes the same amount of chocolate, strawberry, vanilla, honeycomb, and choc-chip ice cream, assuming that the flavours will be equally liked.

The number of sales of each ice cream flavour are shown in the table alongside.

Flavour	*Sales*
chocolate	54
strawberry	48
vanilla	35
honeycomb	28
choc-chip	40
Total	205

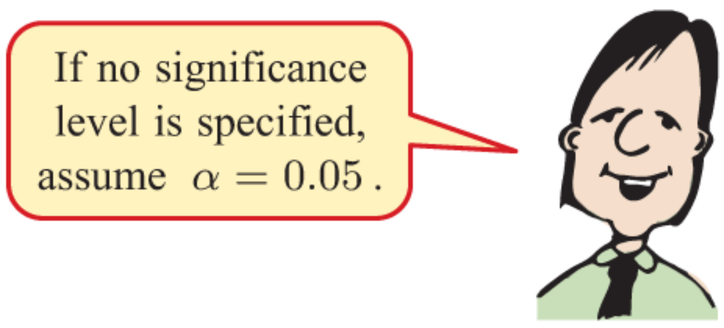
Conduct a χ^2 goodness of fit test with a 10% level of significance to determine whether Brian should change the amount of each ice cream flavour that he makes.

4 Of the people living in London in 2001, 71.2% identified as White, 12.1% as Asian/Asian British, 10.9% as Black/Black British, 3.2% as mixed ethnicity, and 2.6% as other ethnicities.

The table alongside shows the number of people recorded for each ethnic group in the 2011 UK Census.

Ethnic group	*Frequency*
White	4 887 435
Asian/Asian British	1 511 546
Black/Black British	1 088 640
Mixed	405 279
Other	281 041
Total	8 173 941

Conduct a χ^2 goodness of fit test to determine whether there was a significant change in London's demographics between 2001 and 2011.



5 In Australia, the NAPLAN tests are used to gauge the literacy and numeracy skills of students. The students are allocated into "bands" based on the score they obtain. The results for the Year 9 students at a particular school, and the national percentages for each band, are shown below.

Band	*School frequency*	*National percentage*
10	5	7.9%
9	9	16.7%
8	55	29.8%
7	53	29.7%
6	23	13.5%
5 and below	5	2.4%
Total	150	100%

a Use the national percentages to calculate the expected frequency for each band.

b Conduct a χ^2 goodness of fit test with a 1% significance level to determine whether there is a substantial difference between the school's results and the rest of the nation.

c Explain why you may wish to combine the "Band 6" and "Band 5 and below" categories.

d Combine the appropriate categories and repeat **b**. Comment on your results.

Example 2 **Self Tutor**

The data alongside shows the number of children born to 150 Indian women in a 5-year period in the 19th century. Test at a 5% level of significance, whether the data is binomial with parameters $n = 5$ and $p = 0.5$.

Number of children	*Number of women*
0	4
1	19
2	41
3	52
4	26
5	8

Step 1: The hypotheses are:

H_0: the data is from B(5, 0.5)

H_1: the data is not from B(5, 0.5).

We do not need to specify each individual probability in the hypothesis because they are implied by the distribution we are considering.



Step 2: The significance level is $\alpha = 0.05$.

Step 3: We calculate the probabilities for each value of x given $X \sim$ B(5, 0.5):

x	0	1	2	3	4	5
$P(X = x)$	0.031 25	0.156 25	0.3125	0.3125	0.156 25	0.031 25

This gives us the following expected frequency table:

Number of children (x)	f_{obs}	f_{exp}
0	4	$150 \times 0.031\,25 = 4.6875$
1	19	$150 \times 0.156\,25 = 23.4375$
2	41	$150 \times 0.3125 = 46.875$
3	52	$150 \times 0.3125 = 46.875$
4	26	$150 \times 0.156\,25 = 23.4375$
5	8	$150 \times 0.031\,25 = 4.6875$

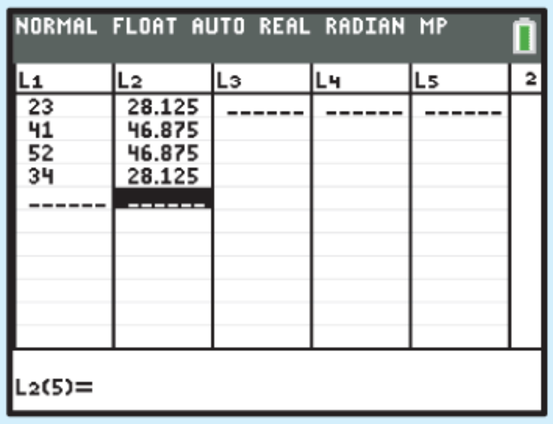
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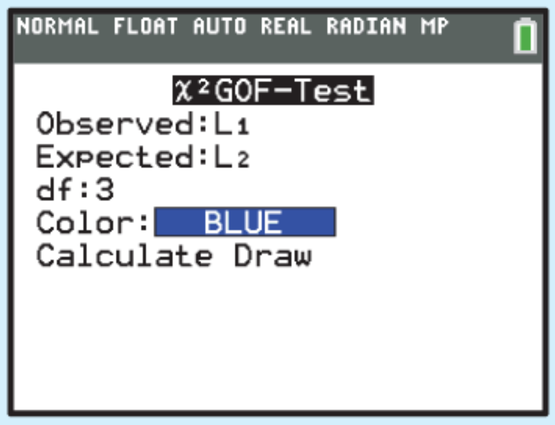
There are expected frequencies less than 5, so we combine "categories" appropriately:

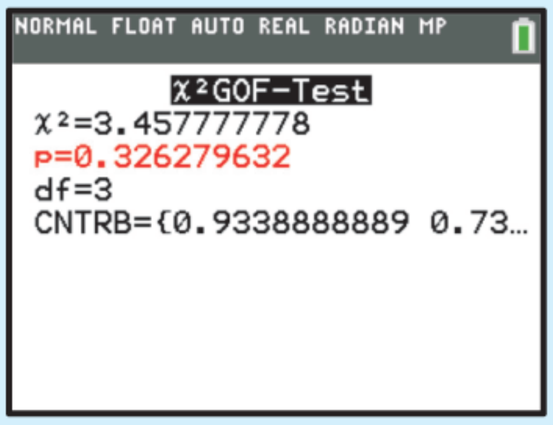
Number of children (x)	f_{obs}	f_{exp}	$\dfrac{(f_{obs} - f_{exp})^2}{f_{exp}}$
$\leqslant 1$	23	28.125	≈ 0.9339
2	41	46.875	≈ 0.7363
3	52	46.875	≈ 0.5603
$\geqslant 4$	34	28.125	≈ 1.2272
		Total	≈ 3.4578

So, $\chi^2_{calc} \approx 3.46$.

Step 4: $\text{df} = 4 - 1 = 3$







Using technology, p-value ≈ 0.326.

Step 5: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0. We therefore accept H_0.

Step 6: Since we have accepted H_0, we conclude that the data is from the binomial distribution B(5, 0.5).

6 Answer the **Opening Problem** on page **830**. Use a 10% level of significance in your hypothesis test.

7 It is claimed that the data in the table comes from a Poisson distribution with mean 5. Test this claim using a goodness of fit test at a 1% level of significance.

Score	$\leqslant 3$	4	5	6	$\geqslant 7$	*Total*
Frequency	6	9	10	7	4	36

ACTIVITY MENDEL'S DATA

Gregor Johann Mendel (1822 - 1884) is often credited as being "the father of modern genetics".

One of his most important contributions was the use of recessive and dominant genes to describe inherited traits.

During his research, he conducted experiments involving over 28 000 plants, the majority of which were pea plants. The table below shows the results of one such experiment.



Gregor Johann Mendel

Type of pea	*Frequency*
Yellow round seeds	315
Green round seeds	108
Yellow wrinkled seeds	101
Green wrinkled seeds	32
Total	556

According to Mendel's model, he expected the ratio

yellow round : green round : yellow wrinkled : green wrinkled $= 9 : 3 : 3 : 1$

What to do:

1 Calculate the expected frequency of each type of pea.

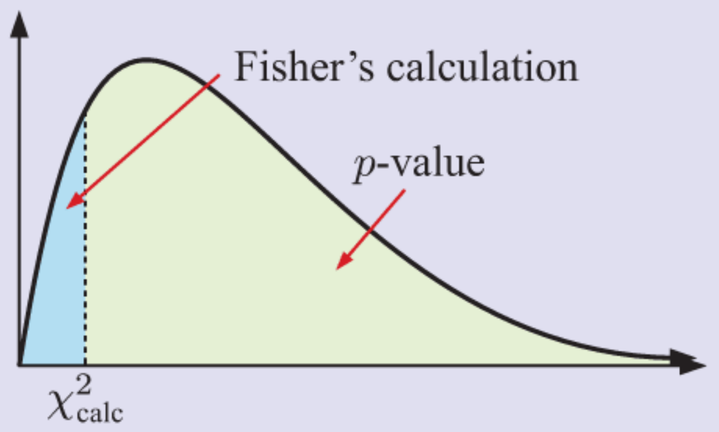
2 In the context of a χ^2 goodness of fit test, show that $\chi^2_{\text{calc}} \approx 0.470$ and p-value ≈ 0.925.

3 What can we conclude based on these results?

THEORY OF KNOWLEDGE

In 1936, **Sir Ronald Aylmer Fisher** analysed Mendel's experiments described in the previous Activity.

He considered the probability that Mendel's results were *consistent* with his expectations, and found that the probability of observing a test statistic *less* than $\chi^2_{\text{calc}} \approx 0.470$ was ≈ 0.075.



In fact, Fisher observed results like this for all of Mendel's experiments. By combining all of Mendel's data, Fisher found that the probability of getting data as good as Mendel's was about 4 in 100 000. Fisher concluded that Mendel had manipulated the data to obtain the results he desired.

1 Does Fisher's finding invalidate the importance of Mendel's contribution to biology and genetics?

Today, the manipulation of data and "data mining" is a major problem in research. Many academic publications only report findings which are "statistically significant", as these findings are more likely to yield more interesting results and lead to further research.

2 Will all claims in academic publications on statistical data necessarily be true? You might want to consider your findings in **Investigation 1** in **Chapter 30**.

3 Discuss the role that statistical interpretation plays in research ethics.

The MMR vaccine controversy was caused by a fraudulent paper published in 1998 which claimed a causal relationship between the MMR vaccine and autism in children. The paper has since been retracted after thorough investigation. However, because of the widespread misconceptions that it has caused, it has been cited as "perhaps the most damaging medical hoax of the last 100 years".

4 Research the details of the MMR vaccine controversy. In particular, consider how the authors collected and used the data cited in the original paper.

5 What other things should a statistician be mindful of when analysing data?

6 Who do you think was responsible for the damage caused by the MMR controversy? Was it the authors of the paper, the media, or the general public?

B ESTIMATING DISTRIBUTION PARAMETERS IN A GOODNESS OF FIT TEST

When the parameters of the distribution of interest are unknown, we must estimate them from the data.

For example, suppose in **Example 2** that we want to test whether the data is binomial with $n = 5$, but without specifying the probability of success p in advance.

In this case we estimate p using the property that the mean of $\mathrm{B}(n, p)$ is np.

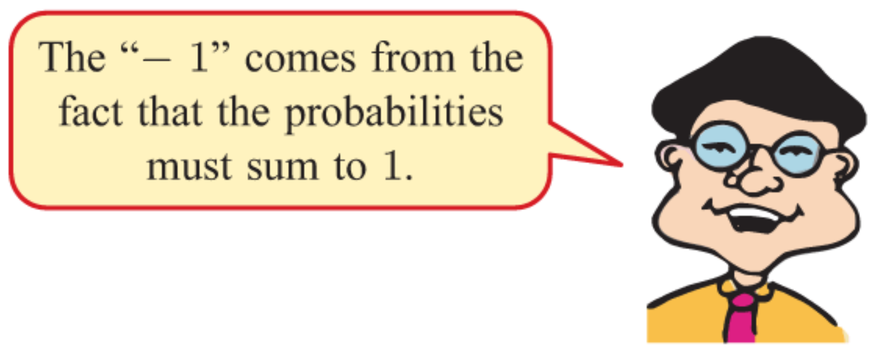
Now $\overline{x} \approx \dfrac{4(0) + 19(1) + ... + 8(5)}{150} \approx 2.6733$

$\therefore\ p \approx \dfrac{\overline{x}}{n} \approx \dfrac{2.6733}{5} \approx 0.5347$

However, since the parameter p is estimated, we would need to reduce the degrees of freedom by 1.

In general:

df = number of categories − number of estimated parameters − 1

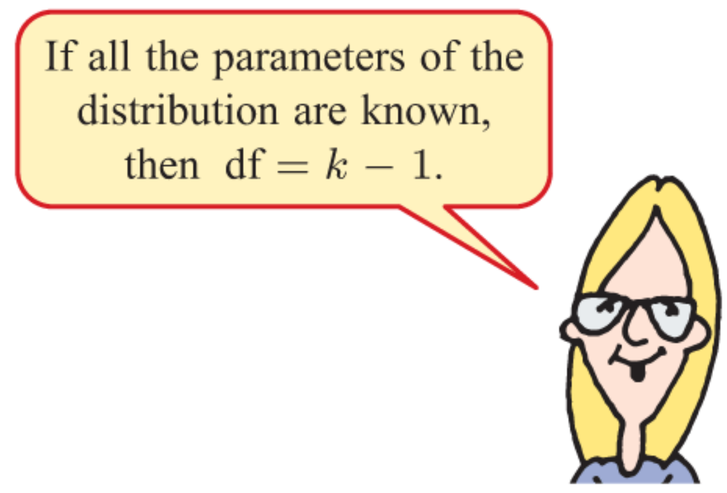


Alternatively, if we think of estimating a parameter as a "restriction" like the one on the sum of the probabilities:

df = number of categories − number of restrictions

For example, the table below gives the degrees of freedom of the goodness of fit test for various distributions when the data is sorted into k categories:

Distribution	*Estimated parameters*	df
Binomial $\mathrm{B}(n, p)$	$p \approx \dfrac{\overline{x}}{n}$	$k-2$
Poisson $\mathrm{Po}(\lambda)$	$\lambda \approx \overline{x}$	$k-2$
Normal $\mathrm{N}(\mu, \sigma^2)$	$\sigma^2 \approx s^2$	$k-2$
	$\mu \approx \overline{x},\quad \sigma^2 \approx s^2$	$k-3$



Example 3 **Self Tutor**

A manufacturer measures the volume of liquid in 100 bottles produced in their factory. The results are summarised in the frequency table alongside.

Use a χ^2 goodness of fit test at a 5% level of significance to determine whether a normal distribution is a suitable model for the data.

Volume (x mL)	*Number of bottles*
$266 \leqslant x < 272$	1
$272 \leqslant x < 274$	16
$274 \leqslant x < 276$	26
$276 \leqslant x < 278$	19
$278 \leqslant x < 280$	20
$280 \leqslant x < 286$	18

Step 1: H_0: the data is from a normal distribution.
H_1: the data is *not* from a normal distribution.

Step 2: The significance level is $\alpha = 0.05$.

Step 3: We estimate the mean μ and standard deviation σ of the normal distribution using the data alongside.

Using technology, $\mu \approx \overline{x} \approx 277.24$

$\sigma \approx s \approx 3.4027$

Volume (x mL)	*Midpoint*	*Frequency*
$266 \leqslant x < 272$	269	1
$272 \leqslant x < 274$	273	16
$274 \leqslant x < 276$	275	26
$276 \leqslant x < 278$	277	19
$278 \leqslant x < 280$	279	20
$280 \leqslant x < 286$	283	18
	Total	100

So, the proposed distribution is $X \sim \text{N}(277.24,\ 3.4027^2)$.

Using this distribution, the probabilities associated with each class interval are:



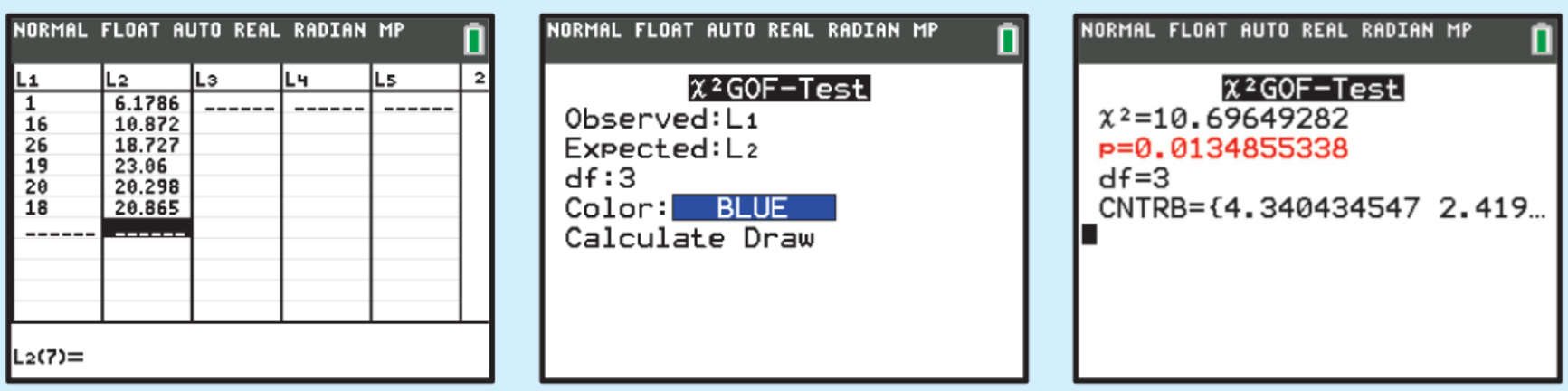
Volume (x mL)	*Probability*
$266 \leqslant x < 272$	≈ 0.0618
$272 \leqslant x < 274$	≈ 0.1087
$274 \leqslant x < 276$	≈ 0.1873
$276 \leqslant x < 278$	≈ 0.2306
$278 \leqslant x < 280$	≈ 0.2030
$280 \leqslant x < 286$	≈ 0.2086

This gives us the expected frequency table:

Volume (x mL)	f_{obs}	f_{exp}	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
$266 \leqslant x < 272$	1	≈ 6.18	≈ 4.3404
$272 \leqslant x < 274$	16	≈ 10.87	≈ 2.4192
$274 \leqslant x < 276$	26	≈ 18.73	≈ 2.8244
$276 \leqslant x < 278$	19	≈ 23.06	≈ 0.7147
$278 \leqslant x < 280$	20	≈ 20.30	≈ 0.0044
$280 \leqslant x < 286$	18	≈ 20.86	≈ 0.3933
		Total	≈ 10.6965

So, $\chi^2_{\text{calc}} \approx 10.7$.

Step 4: $\text{df} = 6 - 2 - 1 = 3$



Using technology, p-value ≈ 0.0135.

Step 5: Since p-value $< 0.05 = \alpha$, we have sufficient evidence to reject H_0 in favour of H_1.

Step 6: Since we have accepted H_1, we conclude that the data is *not* normally distributed.

EXERCISE 31B

1 The table shows the number of days last week the students at a Bangkok school used public transport.

Number of days	0	1	2	3	4	5	6	7
Number of students	9	16	22	38	57	62	41	26

Conduct a χ^2 goodness of fit test at the 5% level of significance to determine whether the data is binomially distributed.

2 The number of accidents reported to the local police station over a period of 104 weeks are recorded in the table:

Number of accidents	0	1	2	3
Number of weeks	52	22	20	10

a Use the data set above to find the mean number of accidents per week.

b Conduct a χ^2 goodness of fit test at the 5% level of significance to determine whether a Poisson distribution would be an appropriate model for this data set.

3 Consider the data in the frequency table.

x	*Frequency*
$0 \leqslant x < 2.5$	159
$2.5 \leqslant x < 5$	245
$5 \leqslant x < 7.5$	210
$7.5 \leqslant x < 10$	164
$10 \leqslant x < 12.5$	74
$12.5 \leqslant x < 15$	30
Total	882

a Estimate the population standard deviation σ.

b Conduct a χ^2 goodness of fit test to determine whether the data is from a normal distribution with mean 5.

4 The data below shows the time (in minutes) between calls at an information centre for 80 successive phone calls:

4.2	6.0	5.6	15.0	12.3	4.9	7.3	14.6	4.0	4.6
1.4	3.5	9.9	5.9	8.2	5.8	3.6	8.1	10.8	9.6
16.0	2.8	6.0	4.3	7.2	10.2	7.8	5.6	8.6	10.9
14.2	3.5	4.7	5.6	5.1	1.2	11.2	10.8	8.9	6.2
5.1	3.4	4.4	7.8	10.5	14.5	12.8	14.4	6.4	2.3
13.7	6.3	3.0	7.1	13.2	6.8	1.9	11.1	8.2	5.3
12.8	3.3	3.5	4.3	7.4	9.2	9.9	4.2	9.2	5.9
9.1	3.6	9.9	5.2	6.0	5.9	8.0	3.7	3.0	7.6

a Organise the data into the class intervals in the table.

b Draw a histogram to display the data.

c Describe the distribution of the data.

d Estimate the mean and standard deviation of the population.

e Conduct a χ^2 goodness of fit test at a 1% level of significance to determine whether the data is normally distributed. Comment on your conclusion.

Time (t min)	*Frequency*
$0 \leqslant t < 2.5$	
$2.5 \leqslant t < 5$	
$5 \leqslant t < 7.5$	
$7.5 \leqslant t < 10$	
$10 \leqslant t < 12.5$	
$12.5 \leqslant t < 15$	
$15 \leqslant t < 17.5$	
Total	80

C CRITICAL REGIONS AND CRITICAL VALUES

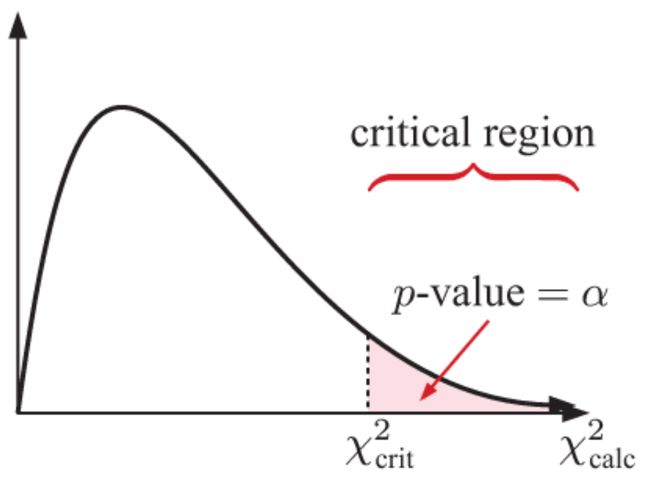
We have seen that for a hypothesis test with significance level α:

- The **critical region** of the test is the set of all values of the test statistic that results in rejecting the null hypothesis.
- The **critical value** of the test is the value in the critical region that has the largest p-value associated with it.

For a χ^2 goodness of fit test, we denote the critical value as $\boldsymbol{\chi^2_{crit}}$.

Since we use the upper tail of the null distribution in calculating the p-value, the critical region is the set of values $\geqslant \chi^2_{crit}$.

The inequality $\boldsymbol{\chi^2_{calc} \geqslant \chi^2_{crit}}$ is called the **rejection inequality**.



The critical value χ^2_{crit} depends on the degrees of freedom (df) and the significance level α of the test.

The table alongside shows the critical value for various values of df and α.

Click on the icon for a more detailed table of critical values.

CRITICAL VALUES

Degrees of freedom (df)	*Significance level* α		
	10%	5%	1%
1	2.71	3.84	6.63
2	4.61	5.99	9.21
3	6.25	7.81	11.34
4	7.78	9.49	13.28
5	9.24	11.07	15.09
6	10.64	12.59	16.81
7	12.02	14.07	18.48
8	13.36	15.51	20.09
9	14.68	16.92	21.67
10	15.99	18.31	23.21

χ^2 GOODNESS OF FIT TEST PROCEDURE WITH CRITICAL VALUES

Step 1: State the **null hypothesis** H_0 and the **alternative hypothesis** H_1.

Step 2: State the **significance level** α.

Step 3: Calculate the value of the **test statistic**: $\chi^2_{calc} = \sum \dfrac{(f_{obs} - f_{exp})^2}{f_{exp}}$.

Step 4: Find the **critical value** χ^2_{crit} given df and α.

Step 5: Reject H_0 if $\chi^2_{calc} \geqslant \chi^2_{crit}$, otherwise accept H_0.

Step 6: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

EXERCISE 31C

1 A goodness of fit test is to be used to test the following hypotheses:

H_0: $p_1 = \frac{1}{5}$, $p_2 = \frac{1}{5}$, ..., $p_5 = \frac{1}{5}$

H_1: at least one of $p_1, p_2, ..., p_5 \neq \frac{1}{5}$ with a 5% level of significance.

The value of the test statistic is $\chi^2_{\text{calc}} = 10.3$.

a Write down the critical value χ^2_{crit} for the test.

b Hence determine whether H_0 should be rejected.

2 Lucy bought a large packet of Chewy Chews lollies. She counted the number of lollies of each colour it contained. Her results are shown in the table.

Colour	*Frequency*
red	12
yellow	17
green	20
blue	16
Total	65

The manufacturer of Chewy Chews claims that they make equal quantities of each colour.

Lucy wants to test the manufacturer's claim with a goodness of fit test using a 10% level of significance.

a Write down the hypotheses that Lucy should test.

b Calculate χ^2_{calc}.

c Find the critical value χ^2_{crit} for the test.

d Does Lucy have enough evidence to reject the manufacturer's claim?

e Verify your answer by examining the p-value.

3 An internet service provider (ISP) conducted a survey asking their customers how satisfied they were with their service. They found that:

- 5% said "very satisfied"
- 20% said "dissatisfied"
- 25% said "satisfied"
- 9% said "very dissatisfied".
- 41% said "neutral"

To improve customer satisfaction, the ISP made several changes to the internet plans they offered. They then repeated the same survey 6 months later. The results are shown in the table opposite.

Response	*Frequency*
very satisfied	25
satisfied	78
neutral	77
dissatisfied	36
very dissatisfied	17
Total	233

Conduct a χ^2 goodness of fit test using a critical value and a 1% level of significance to determine whether the ISP's changes were effective.

4 The table alongside shows the number of people served at a tourist information centre per hour.

Number of people served	*Frequency*
0	14
1	38
2	49
3	36
4	14
5	4
6	4
$\geqslant 7$	1
Total	160

Conduct a χ^2 goodness of fit test using a critical value and a 10% level of significance to determine whether the data is Poisson with rate $\lambda = 2.2$.

5 Omni Transport has a fleet of 550 buses. The air pressure in every bus tyre is periodically checked. From the last test, the number of tyres per bus with incorrect pressure is summarised alongside.

Conduct a χ^2 goodness of fit test using a critical value and a 5% level of significance to determine whether the data is binomially distributed.

Number of tyres	*Frequency*
0	307
1	180
2	54
3	9
4	0

D THE χ^2 TEST FOR INDEPENDENCE

This table shows the results of a sample of 400 randomly selected adults classified according to *gender* and *regular exercise*.

We call this a 2×2 **contingency table**.

	Regular exercise	*No regular exercise*	*Sum*
Male	110	106	216
Female	98	86	184
Sum	208	192	400

We are interested in how the variables *gender* and *regular exercise* are related. The variables may be **dependent**, for example females may exercise more regularly than males. Alternatively, the variables may be **independent**, which means the gender of a person has no effect on whether they exercise regularly.

In this Section, we will apply what we have learned about the χ^2 goodness of fit test to determine whether two variables from the same sample are independent.

THE HYPOTHESES

We have previously seen that if two events A and B are independent, then

$$\text{P}(A \text{ and } B) = \text{P}(A \cap B) = \text{P}(A) \times \text{P}(B).$$

For example, if the variables *gender* and *regular exercise* are independent, then

$$\begin{aligned}\text{P}(\textit{male} \cap \textit{regular exercise}) &= \text{P}(\textit{male}) \times \text{P}(\textit{regular exercise}) \\ &= \tfrac{216}{400} \times \tfrac{208}{400}\end{aligned}$$

In a contingency table with r rows and c columns, there are $r \times c$ possible combinations, and hence $r \times c$ probabilities to consider.

Instead of writing all these probabilities down when we formulate the hypotheses, we can equivalently write:

H_0: the variables are independent

H_1: the variables are dependent

CALCULATING THE TEST STATISTIC

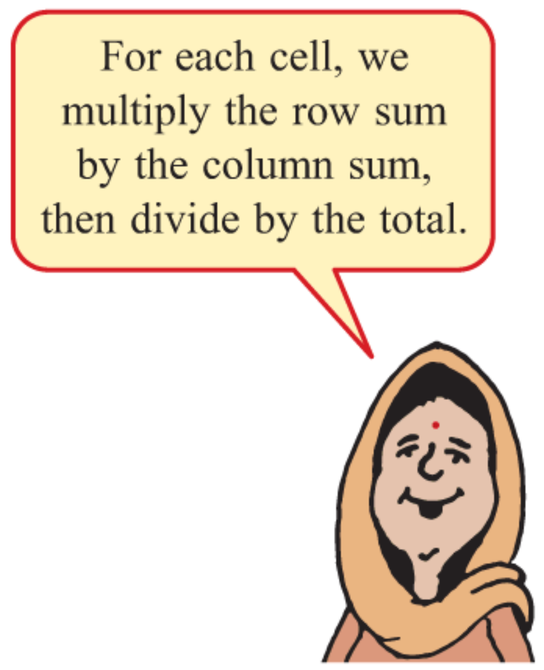
Having established a theoretical probability for each possible outcome, we can calculate **expected frequencies** and hence the test statistic χ^2_{calc}.

For example, in a sample of 400 adults, we would expect

$$400 \times \left(\frac{216}{400} \times \frac{208}{400}\right) = \frac{216 \times 208}{400} = 112.32 \text{ to be male and exercise regularly.}$$

Performing similar calculations for each possible outcome, we can complete an **expected frequency table**.

	Regular exercise	*No regular exercise*	*Sum*
Male	$\frac{216 \times 208}{400} = 112.32$	$\frac{216 \times 192}{400} = 103.68$	216
Female	$\frac{184 \times 208}{400} = 95.68$	$\frac{184 \times 192}{400} = 88.32$	184
Sum	208	192	400



Using the formula for χ^2_{calc} on page **831**,

$$\begin{aligned}\chi^2_{\text{calc}} &= \sum \frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}} \\ &= \frac{(110 - 112.32)^2}{112.32} + ... + \frac{(86 - 88.32)^2}{88.32} \\ &\approx 0.2170\end{aligned}$$

DEGREES OF FREEDOM

We have seen that **degrees of freedom (df)** is the number of values which are free to vary.

Consider the 2×2 contingency table alongside, with the sum values given.

	A_1	A_2	*Sum*
B_1			12
B_2			8
Sum	15	5	20

The value in the top left corner is free to vary. It can take many possible values, one of which is 9. However, once we set this value, the remaining values are *not* free to vary, as they are determined by the row and column sums.

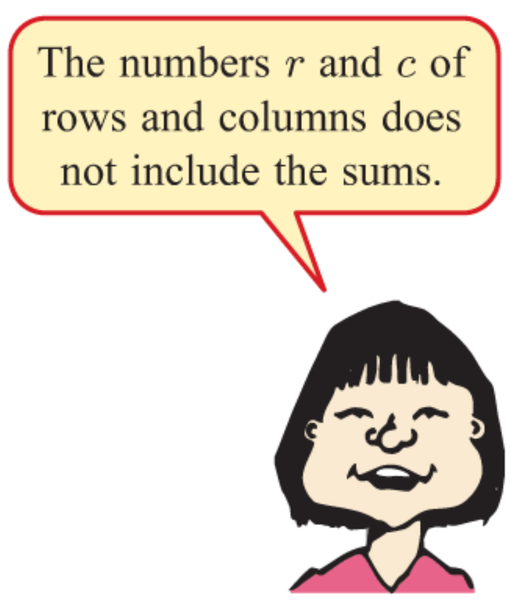
So, the number of degrees of freedom is 1, which is $(2-1) \times (2-1)$.

	A_1	A_2	*Sum*
B_1	9	3	12
B_2	6	2	8
Sum	15	5	20

In a 3×3 contingency table, we can choose $(3-1) \times (3-1) = 4$ values before the remaining values are not free to vary.

	C_1	C_2	C_3	*Sum*
D_1				12
D_2				8
D_3				13
Sum	13	9	11	33

	C_1	C_2	C_3	*Sum*
D_1	5	3	4	12
D_2	2	4	2	8
D_3	6	2	5	13
Sum	13	9	11	33



For a contingency table which has r rows and c columns,
$\mathbf{df} = (r-1)(c-1)$.

DISCUSSION

Discuss how the above formula for df is related to the formula we used in **Section B**:

$$\text{df} = \text{number of categories} - \text{number of estimated parameters} - 1.$$

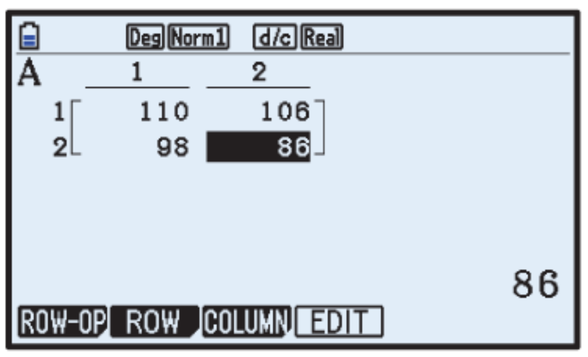
THE p-VALUE

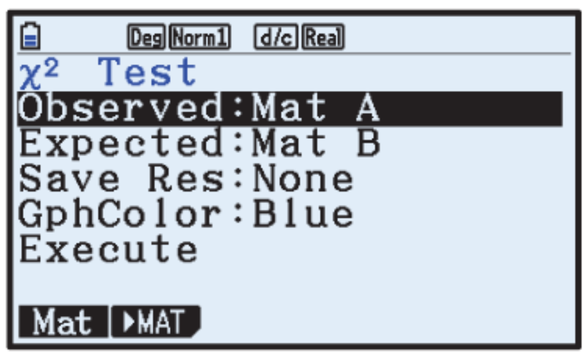
Like we did for the χ^2 goodness of fit test, we use technology to calculate χ^2_{calc} and the ***p*-value**.

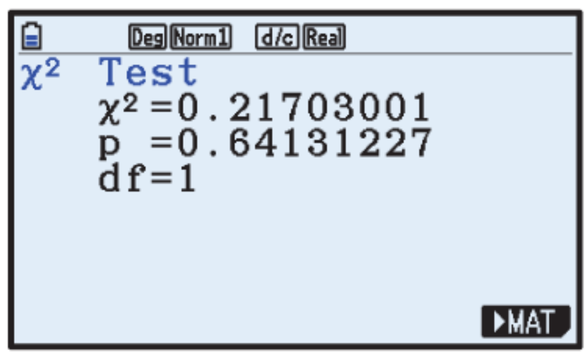
Your **graphics calculator** likely has a special function for the χ^2 test for independence.



GRAPHICS CALCULATOR INSTRUCTIONS







CRITICAL VALUES

We can use the same table of critical values on page **842** to find the critical value χ^2_{crit} of the test.

However, for a χ^2 test for independence, remember that the degrees of freedom $= (r-1)(c-1)$.

SUMMARY OF THE χ^2 TEST FOR INDEPENDENCE

Step 1: State the **null hypothesis** H_0 and the **alternative hypothesis** H_1, which have the form:
H_0: the variables are independent
H_1: the variables are dependent.

Step 2: State the **significance level** α.

Step 3: Calculate $\mathbf{df} = \boldsymbol{(r-1)(c-1)}$ where r and c are the number of rows and columns of the contingency table respectively.

Step 4: Construct the **expected frequency table** and calculate the value of the **test statistic**:
$\chi^2_{\text{calc}} = \sum \dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$.

Step 5: Use technology to calculate the ***p*-value** *or* find the **critical value** $\boldsymbol{\chi^2_{\text{crit}}}$ for the test.

Step 6: Reject H_0 if p-value $< \alpha$ *or* if $\chi^2_{\text{calc}} \geqslant \chi^2_{\text{crit}}$, otherwise accept H_0.

Step 7: Interpret your decision in the context of the problem. Write your conclusion in a sentence.

Example 4 **Self Tutor**

A survey was given to randomly chosen high school students from years 9 to 12 on possible changes to the school's canteen. The results are shown in this contingency table.

	Year group			
	9	10	11	12
Change	7	9	13	14
No change	14	12	9	7

At a 5% significance level, test whether a student's *canteen preference* depends on their *year group*.

Step 1: H_0: *year group* and *canteen preference* are independent.
H_1: *year group* and *canteen preference* are not independent.

Step 2: The significance level is $\alpha = 0.05$.

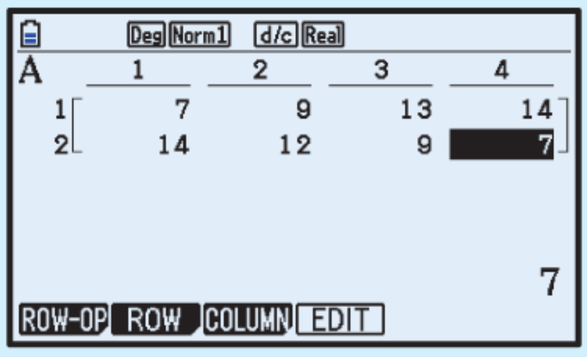
Step 3: df $= (2-1)(4-1) = 3$

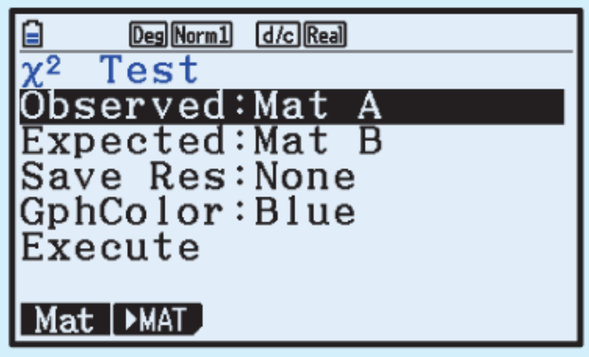
Step 4: The 2×4 contingency table is:

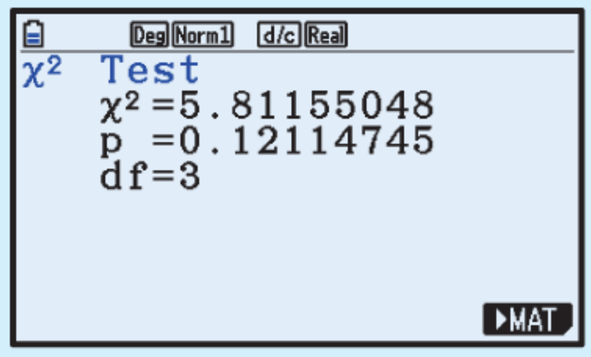
	Year group				
	9	10	11	12	*Sum*
C	7	9	13	14	43
C'	14	12	9	7	42
Sum	21	21	22	21	85

The expected frequency table is:

	Year group			
	9	10	11	12
C	10.6	10.6	11.1	10.6
C'	10.4	10.4	10.9	10.4







Using technology, $\chi^2_{\text{calc}} \approx 5.81$.

Step 5: From the screenshots above, p-value ≈ 0.121.

Step 6: Since p-value $> 0.05 = \alpha$, we do not have enough evidence to reject H_0 in favour of H_1 on a 5% significance level. We therefore accept H_0.

Step 7: We conclude that the variables *year group* and *canteen preference* are independent.

SUMMARY HYPOTHESIS TESTING FLOWCHART

Click on this icon to obtain a printable flow chart to help you remember which χ^2 hypothesis test to use.

HYPOTHESIS TESTING FLOWCHART

EXERCISE 31D.1

1 Construct an expected frequency table for the contingency table:

a

	Junior school	*Middle school*	*High school*	*Sum*
Plays sport	35	59	71	165
Does not play sport	23	27	35	85
Sum	58	86	106	250

b

	Drove to work	*Cycled to work*	*Public transport*	*Sum*
Male				44
Female				36
Sum	46	14	20	80

c

	Wore hat and sunscreen	*Wore hat or sunscreen*	*Wore neither*	*Sum*
Sunburnt	3	5	13	
Not sunburnt	36	17	1	
Sum				

2 Consider the contingency table:

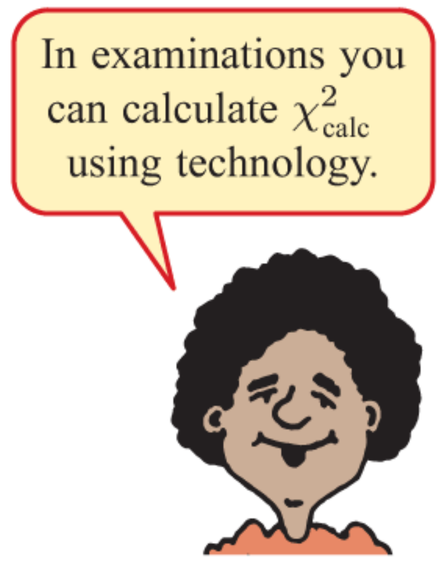
	Pass Maths test	*Fail Maths test*	*Sum*
Male	24	26	50
Female	36	14	50
Sum	60	40	100

a Construct the corresponding expected frequency table.

b Interpret the value in the top left corner of your expected frequency table.

c Calculate χ^2_{calc} by copying and completing this table:

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$(f_{\text{obs}} - f_{\text{exp}})^2$	$\dfrac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
24				
26				
36				
14				
			Total	



3 This contingency table shows the responses of a randomly chosen sample of adults regarding the person's *weight* and whether they have *diabetes*.

Conduct a χ^2 test for independence at a 5% level to determine whether there is a link between *weight* and suffering from *diabetes*.

	Weight		
	Light	*Medium*	*Heavy*
Diabetic	11	19	26
Non-diabetic	79	68	69

4 The table alongside shows the way in which a random sample of people intend to vote in the next election.

	Age of voter		
	18 to 35	36 to 59	60+
Party A	85	95	131
Party B	168	197	173

a For a 10% significance level, what is the critical value χ^2_{crit}?

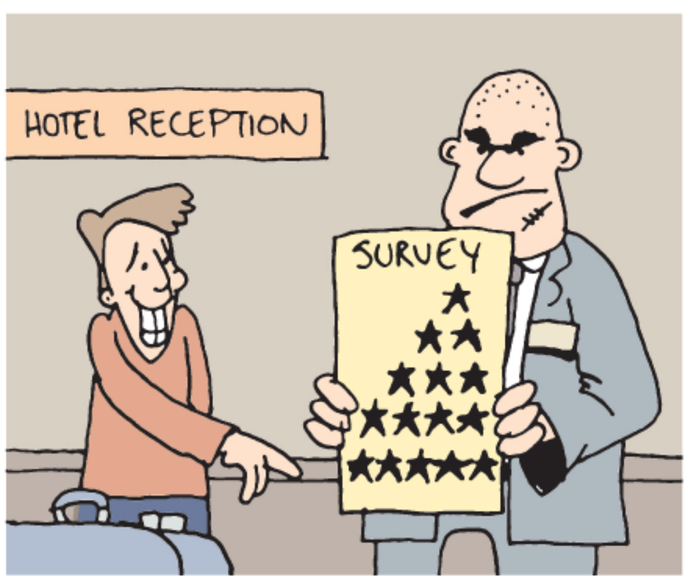
b Conduct a χ^2 test for independence at a 10% level to determine whether there is any association between the *age of a voter* and the *party they intend to vote for*.

5 The guests staying at a hotel are asked to provide their *reason for travelling*, and to *rate* the hotel on a scale from Poor to Excellent. The results are shown in the table:

		Rating			
		Poor	*Fair*	*Good*	*Excellent*
Reason for travelling	*Business*	27	25	20	8
	Holiday	9	17	24	30

a Show that, at a 5% significance level, the variables *reason for travelling* and *rating* are dependent.

b By examining the contingency table, describe how a guest's *rating* is affected by their *reason for travelling*.



6 Hockey player Julie wondered whether the position you played affected your likelihood of being injured. She asked a random sample of hockey players what position they played, and what injuries they had sustained in the last year.

Injury type \ *Position*	*Forward*	*Midfielder*	*Defender*	*Goalkeeper*
No injury	23	18	24	7
Mild injury	14	34	23	11
Serious injury	10	16	13	7

Test, at a 10% significance level, whether the variables *position* and *injury type* are independent.

7 Consider the contingency table alongside:

a Construct the expected frequency table.

b Are any of the expected frequencies less than 5?

c Combine the data so that none of the cells have an expected frequency less than 5.

d For the combined data, test at a 5% level whether there is a link between *age* and *owning a pet*.

Age \ *Owns a pet*	*Yes*	*No*
0 - 19	5	3
20 - 29	32	22
30 - 49	42	58
50+	39	34

8 The following table is a result of a major investigation considering *intelligence level* and *smoking habits*.

Smoking habits \ *Intelligence level*	*Low*	*Average*	*High*	*Very high*
Non smoker	279	386	96	2
Medium level smoker	123	201	58	5
Heavy smoker	100	147	64	2

a Construct the expected frequency table.

b Test at a 1% level whether there is a link between *intelligence level* and *smoking habits*.

c Combine appropriate columns so that none of the expected frequencies are less than 5.

d Perform this test again with the table in **c** at a 1% level. Is your conclusion the same as in **b**?

YATES' CONTINUITY CORRECTION (EXTENSION)

The χ^2 test for independence may be unreliable if the number of degrees of freedom is 1. This occurs when we have a 2×2 contingency table.

To improve the reliability of the test in this case, we can apply **Yates' continuity correction**. We use a modified formula to find χ^2_{calc} as follows:

If $\text{df} = 1$, we use $\boldsymbol{\chi^2_{\text{calc}} = \sum \dfrac{(|f_{\text{obs}} - f_{\text{exp}}| - 0.5)^2}{f_{\text{exp}}}}$

where $|f_{\text{obs}} - f_{\text{exp}}|$ is the **absolute value** or **modulus** of $f_{\text{obs}} - f_{\text{exp}}$.

When conducting the χ^2 test with Yates' continuity correction, we cannot use technology to calculate χ^2_{calc} and the p-value. We must therefore calculate χ^2_{calc} by hand and use the critical value to make a decision about H_0 instead.

Example 5 **Self Tutor**

80 people were surveyed about whether they enjoy surfing and skiing. The results are shown alongside.

Test, at a 1% level, whether there is an association between *enjoying surfing* and *enjoying skiing*.

Enjoy skiing? \ *Enjoy surfing?*	*Yes*	*No*
Yes	17	15
No	8	40

Step 1: H_0: The variables *enjoying surfing* and *enjoying skiing* are independent.
H_1: The variables *enjoying surfing* and *enjoying skiing* are not independent.

Step 2: The significance level is $\alpha = 0.01$.

Step 3: $\text{df} = (2-1)(2-1) = 1$

Step 4: The 2×2 contingency table is:

Enjoy skiing? \ *Enjoy surfing?*	*Yes*	*No*	*Sum*
Yes	17	15	32
No	8	40	48
Sum	25	55	80

The expected frequency table is:

Enjoy skiing? \ *Enjoy surfing?*	*Yes*	*No*
Yes	10	22
No	15	33

We find χ^2_{calc} using Yates' continuity correction:

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5$	$(\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5)^2$	$\dfrac{(\lvert f_{\text{obs}} - f_{\text{exp}} \rvert - 0.5)^2}{f_{\text{exp}}}$
17	10	7	6.5	42.25	4.225
15	22	−7	6.5	42.25	1.920
8	15	−7	6.5	42.25	2.817
40	33	7	6.5	42.25	1.280
				Total	10.242

So, $\chi^2_{\text{calc}} \approx 10.2$

Step 5: At a $\alpha = 0.01 = 1\%$ level with $\text{df} = 1$, the critical value $\chi^2_{\text{crit}} = 6.63$.

Step 6: Since $\chi^2_{\text{calc}} > \chi^2_{\text{crit}} = 6.63$, we have enough evidence to reject H_0 in favour of H_1 at a 1% level of significance.

Step 7: Since we have accepted H_1, we conclude that *enjoying surfing* and *enjoying skiing* are dependent.

EXERCISE 31D.2

1 Horace claims that he can predict the outcome of a coin toss. To test this, he tosses a coin 200 times, and tries to guess the outcome of each toss. The results are shown alongside.

Guess \ *Result*	*Heads*	*Tails*
Heads	54	50
Tails	41	55

a Construct the expected frequency table.

b Use Yates' continuity correction to find χ^2_{calc}.

c Test whether Horace's *guess* and the *result* are independent, at a 5% level with df $= 1$, $\chi^2_{\text{crit}} \approx 3.84$.

d Comment on the validity of Horace's claim.

2 The practical tests for a motorbike licence differ between France and Germany. An inquiry into the two systems yielded the following results for randomly selected candidates.

Country \ *Result*	*Pass*	*Fail*
France	56	29
Germany	176	48

A χ^2 test at a 10% significance level is used to investigate whether the result of a motorbike test is independent of the country where it took place.

a Construct the expected frequency table. **b** Write down the critical value χ^2_{crit}.

c Using Yates' continuity correction, find χ^2_{calc} for this data.

d What conclusion can be drawn from this χ^2 test?

REVIEW SET 31A

1 A particular χ^2 test has 6 degrees of freedom (df) and significance level $\alpha = 0.05$.

a State the critical value χ^2_{crit} for this test.

b The test statistic obtained for this test was $\chi^2_{\text{calc}} \approx 5.71$. Is there sufficient evidence to reject the null hypothesis?



2 When a menswear store opens, the sizes of its shirts are distributed as follows:

Size	Small	Medium	Large	X-Large	XX-Large
Percentage (%)	10	20	35	25	10

In the first week, the store sells these shirts:

Size	Small	Medium	Large	X-Large	XX-Large
Number sold	4	7	22	24	13

To determine whether the sales are consistent with the distribution of the sizes stocked by the store, a χ^2 goodness of fit test is performed at a 5% level of significance.

a Write down the:

i null hypothesis **ii** number of degrees of freedom.

b Calculate the p-value for the test.

c Decide whether the store should change the distribution of shirt sizes that it stocks.

3 The table below shows the responses to a survey about whether the city speed limit should be increased.

	Age of driver 18 to 30	31 to 54	55+
Increase	234	169	134
No increase	156	191	233

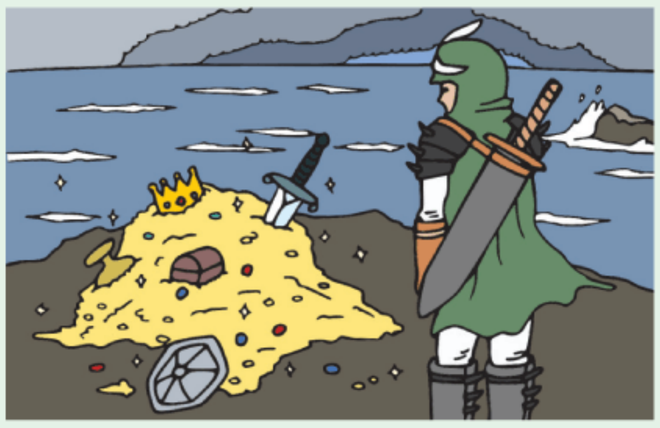
Conduct a χ^2 test for independence at a 10% level to determine whether there is any association between the *age of a driver* and their *opinion* on the speed limit.

4 A guide for a particular role playing game lists the following percentages for the rarity of items obtained when a "loot box" is opened.

Item rarity	*Percentage chance*
Super rare	5%
Rare	10%
Uncommon	25%
Common	60%

Emmanuel thinks that the guide is incorrect. To test his suspicions, he opens 250 loot boxes. His results are shown in the table below.

Item rarity	*Frequency*
Super rare	5
Rare	17
Uncommon	76
Common	152
Total	250



a Calculate the expected frequency for each item rarity.

b Conduct a χ^2 goodness of fit test with a 1% level of significance to determine whether Emmanuel's suspicions are justified.
Use a critical value to make your decision.

5 Red and blue biros are sold in packets of six. Each biro is either red or blue. The manufacturer claims that the number of red biros in a packet is binomially distributed. He collects 100 packets at random and obtains the information in the table.

a Calculate the mean number of red biros per packet.

b Hence estimate the probability that a randomly chosen biro is red.

c By calculating an appropriate χ^2 statistic, test the manufacturer's claim at a 10% significance level.

Number of red biros	*Number of packets*
0	1
1	3
2	9
3	17
4	31
5	28
6	11

6 A survey was conducted to determine whether the use of social media is linked to the amount of exercise a person gets. The table below summarises the results from a survey.
Test at a 5% level of significance whether the variables are related.

Exercise (x hours per week) \ *Social media* (y hours per week)	$y \leqslant 2$	$2 \leqslant y < 6$	$6 \leqslant y < 10$	$y \geqslant 10$
$x < 2$	5	13	11	21
$2 \leqslant x < 4$	16	22	18	14
$4 \leqslant x < 6$	20	26	16	9
$x \geqslant 6$	15	8	11	3

REVIEW SET 31B

1 A toy company claims to manufacture glass, agate, alabaster, and onyx marbles in the ratio $4 : 2 : 2 : 1$. The marbles are sold in bags of 50.

Aggie bought a bag of marbles and counted the number of each type. Her results are shown in the table.

Type	*Frequency*
glass	19
agate	16
alabaster	13
onyx	2

Test the manufacturer's claim using a χ^2 goodness of fit test with a 5% significance level.

2 In an experiment, a coin is tossed until a head appears. The number of tosses required is recorded. The results from 100 trials of the experiment are recorded alongside.

Number of tosses required	*Frequency*
1	46
2	20
3	12
4	8
5	5
6	3
7	4
$\geqslant 8$	2

a Let X be the number of tosses until a head appears in one trial of the experiment. Assuming the coin is fair:

i Explain why $P(X = x) = (0.5)^x$, $x = 1, 2, 3, ...$

ii Construct an expected frequency table for this distribution.

b By calculating an appropriate χ^2 statistic, test at a 5% significance level whether the distribution is a suitable model for the data.

3 In a study of students entering a university, a psychologist tested 2000 students. The psychologist wishes to determine whether the 2000 test scores come from a well-established normal distribution with mean 100 and variance 100.

a Copy and complete the table below, rounding expected frequencies to the nearest integer. Clearly explain how you obtain your answers.

Score (x)	*Observed frequencies*	*Expected frequencies*
$x < 70.5$	10	3
$70.5 \leqslant x < 80.5$	45	48
$80.5 \leqslant x < 90.5$	287	
$90.5 \leqslant x < 100.5$	641	
$100.5 \leqslant x < 110.5$	725	
$110.5 \leqslant x < 120.5$	250	253
$120.5 \leqslant x < 130.5$	40	38
$x \geqslant 130.5$	2	2

b Test the hypothesis at the 5% level of significance.

4 Consider the contingency table alongside. Test whether the variables P and Q are independent using a χ^2 test:

	Q_1	Q_2	Q_3	Q_4
P_1	19	23	27	39
P_2	11	20	27	35
P_3	26	39	21	30

a at a 5% level of significance

b at a 1% level of significance.

5 The frequency table alongside shows the mass m of newborn lambs in a field.
Conduct a χ^2 goodness of fit test at a 5% level of significance to determine whether the data is normally distributed.

Mass (m kg)	*Number of lambs*
$2 \leqslant m < 3$	6
$3 \leqslant m < 4$	11
$4 \leqslant m < 5$	37
$5 \leqslant m < 6$	18
$6 \leqslant m < 7$	4
$7 \leqslant m < 8$	3

6 The following table shows the results from an investigation considering *level of education* and *business success.*

		Education level			
		High school	*Graduate certificate*	*Undergraduate degree*	*Postgraduate*
Business success	*No success*	35	30	41	25
	Low success	28	41	26	29
	Success	35	24	41	56
	High success	52	38	63	72

At a 1% level with $\text{df} = 9$, the critical value is 21.67. Test at a 1% level whether there is a link between *education level* and *business success.*

ANSWERS

EXERCISE 1A

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
f $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
f $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** $x^{\frac{1}{2}}$ **b** $x^{\frac{3}{2}}$ **c** $x^{-\frac{1}{2}}$ **d** $x^{\frac{5}{2}}$ **e** $x^{-\frac{3}{2}}$

5 **a** ≈ 2.28 **b** ≈ 0.435 **c** ≈ 1.68 **d** ≈ 1.93
e ≈ 0.523

6 **a** $\sqrt[3]{5}$ **b** $\frac{1}{\sqrt{3}}$ **c** $9\sqrt{3}$ **d** $m\sqrt{m}$ **e** $x^3\sqrt{x}$

7 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 1B

1 **a** 1 **b** x **c** $x^{\frac{1}{2}}$ or $\sqrt{x}$

2 **a** $x^5 + 2x^4 + x^2$ **b** $2^{2x} + 2^x$ **c** $x + 1$
d $7^{2x} + 2(7^x)$ **e** $2(3^x) - 1$ **f** $x^2 + 2x + 3$
g $1 + 5(2^{-x})$ **h** $5^x + 1$ **i** $x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$
j $3^{2x} + 5(3^x) + 1$ **k** $2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5$ **l** $2^{3x} - 3(2^{2x}) - 1$

3 **a** $2^{2x} + 2^{x+1} - 3$ **b** $3^{2x} + 7(3^x) + 10$
c $5^{2x} - 6(5^x) + 8$ **d** $2^{2x} + 6(2^x) + 9$
e $3^{2x} - 2(3^x) + 1$ **f** $4^{2x} + 14(4^x) + 49$
g $x - 4$ **h** $2^{2x} - 9$ **i** $x - \dfrac{1}{x}$ **j** $x^2 + 4 + \dfrac{4}{x^2}$
k $7^{2x} - 2 + 7^{-2x}$ **l** $25 - 10(2^{-x}) + 2^{-2x}$

4 **a** $5^x(5^x + 1)$ **b** $10(3^n)$ **c** $7^n(1 + 7^{2n})$
d $5(5^n - 1)$ **e** $6(6^{n+1} - 1)$ **f** $16(4^n - 1)$
g $2^n(2^n - 8)$ **h** $5(2^{n-1})$ **i** $9(2^{2n-1})$

5 **a** $(3^x + 2)(3^x - 2)$ **b** $(2^x + 5)(2^x - 5)$
c $(4 + 3^x)(4 - 3^x)$ **d** $(5 + 2^x)(5 - 2^x)$
e $(3^x + 2^x)(3^x - 2^x)$ **f** $(2^x + 3)^2$
g $(3^x + 5)^2$ **h** $(2^x - 7)^2$ **i** $(5^x - 2)^2$

6 **a** $(2^x + 1)(2^x - 2)$ **b** $(3^x + 3)(3^x - 2)$
c $(2^x - 3)(2^x - 4)$ **d** $(2^x + 3)(2^x + 6)$
e $(2^x + 4)(2^x - 5)$ **f** $(3^x + 2)(3^x + 7)$
g $(3^x + 5)(3^x - 1)$ **h** $(5^x + 2)(5^x - 1)$
i $(7^x - 4)(7^x - 3)$

7 **a** 2^n **b** 10^a **c** 3^b **d** $\dfrac{1}{5^n}$ **e** 5^x
f $(\frac{3}{4})^a$ **g** $(\frac{8}{3})^k$ **h** 5 **i** 5^n

8 **a** $3^m + 1$ **b** $1 + 6^n$ **c** $4^n + 2^n$ **d** $4^x - 1$
e 6^n **f** 5^n **g** 4 **h** $2^n - 1$ **i** $\frac{1}{2}$

9 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 1C

1 **a** not exponential **b** exponential
c not exponential **d** exponential

2 **a** 1 **b** -1 **c** -2 **d** $-2\frac{1}{2}$ **e** $-2\frac{3}{4}$

3 **a** 15 **b** 135 **c** 5 **d** $\frac{5}{81}$ **e** $\frac{5}{3}$

4 **a** 32 **b** 2 **c** 4 **d** 1 **e** $\frac{1}{16}$

5 **a** $\frac{1}{5}$ **b** $\frac{1}{125}$ **c** 1 **d** 25 **e** 125

6 **a** 3 **b** 3.3 **c** ≈ 4.83 **d** ≈ 2.48 **e** ≈ 4.31

7 **a** yes **b** no **c** yes **d** yes **e** no **f** yes

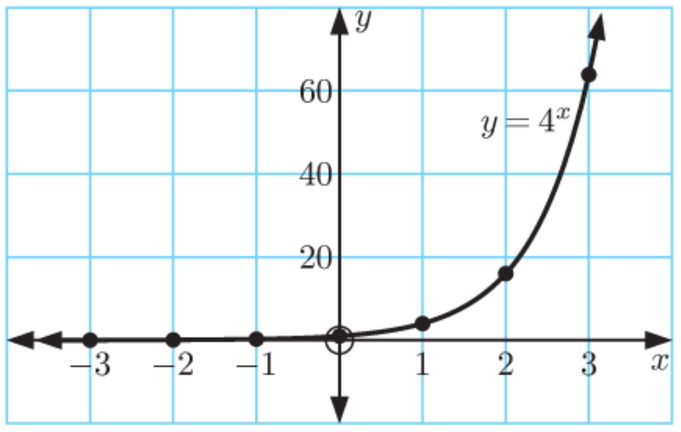
EXERCISE 1D

1 **a**

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

b **i** If x is increased by 1, the value of y is quadrupled.
ii If x is decreased by 1, the value of y is divided by 4.

c



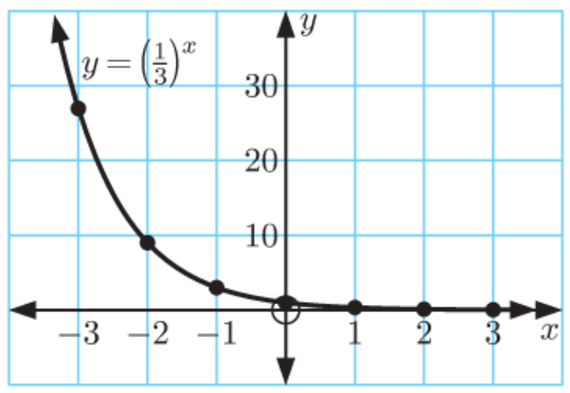
d **i** as $x \to \infty$, $y \to \infty$ **ii** as $x \to -\infty$, $y \to 0^+$

e $y = 0$

2 **a**

x	-3	-2	-1	0	1	2	3
y	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$

b



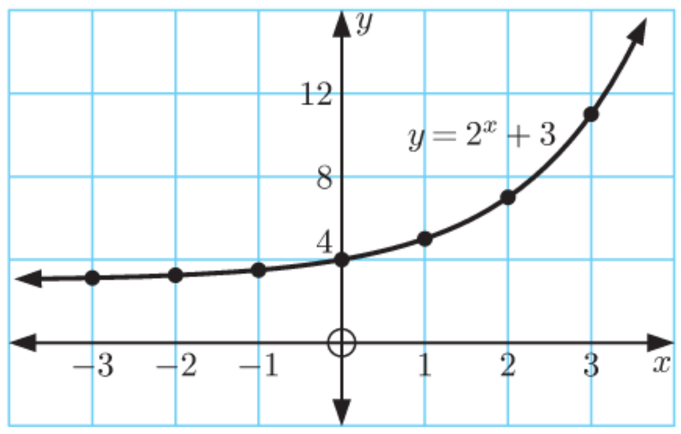
c decreasing

d **i** as $x \to \infty$, $y \to 0^+$ **ii** as $x \to -\infty$, $y \to \infty$

e $y = 0$

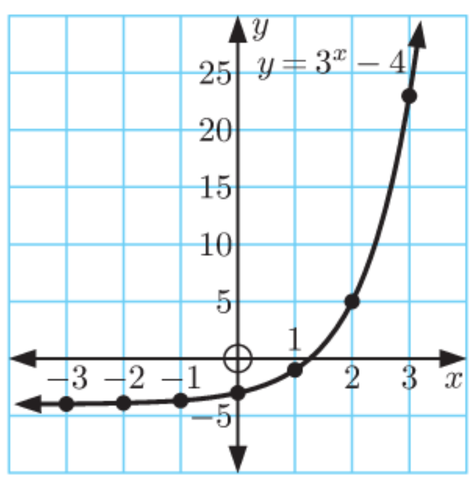
3 **a**

x	-3	-2	-1	0	1	2	3
y	$3\frac{1}{8}$	$3\frac{1}{4}$	$3\frac{1}{2}$	4	5	7	11



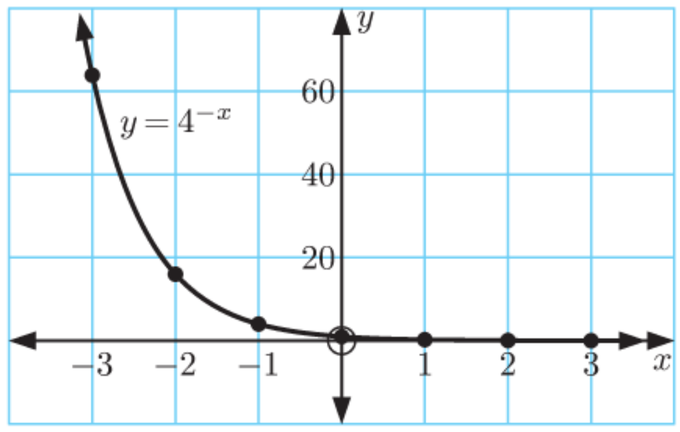
b

x	-3	-2	-1	0	1	2	3
y	$-3\frac{26}{27}$	$-3\frac{8}{9}$	$-3\frac{2}{3}$	-3	-1	5	23



c

x	-3	-2	-1	0	1	2	3
y	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$



d

x	-3	-2	-1	0	1	2	3
y	$\frac{5}{8}$	$\frac{5}{4}$	$\frac{5}{2}$	5	10	20	40

4 a i

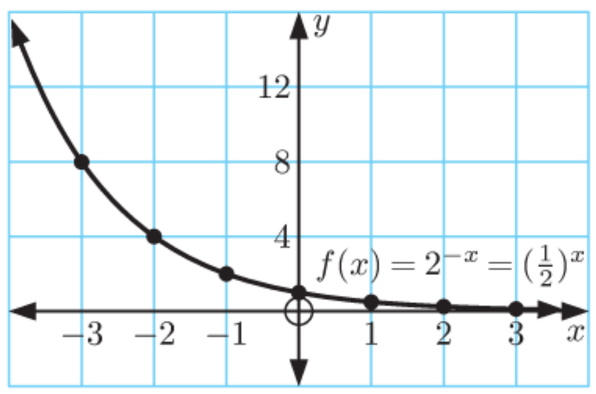
x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

ii

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

b $2^{-x} = (2^{-1})^x$
$= (\frac{1}{2})^x$

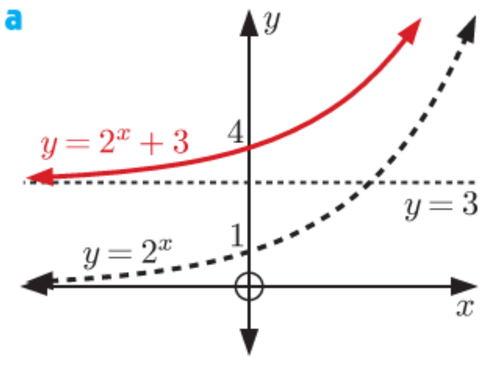
c



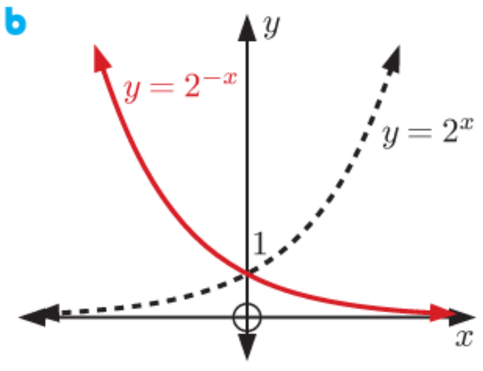
EXERCISE 1E

1 a C **b** B **c** E **d** A **e** D

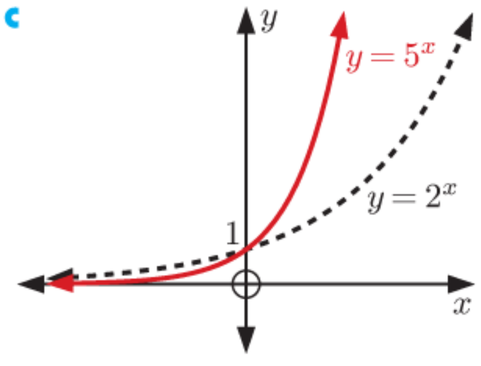
2 a



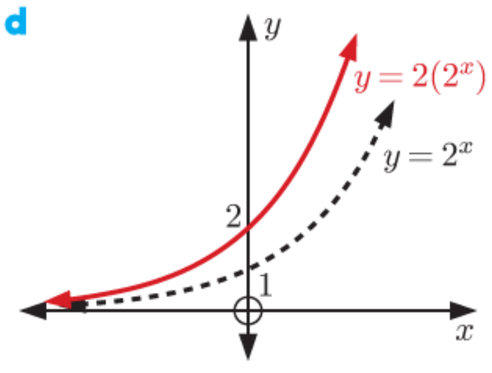
b



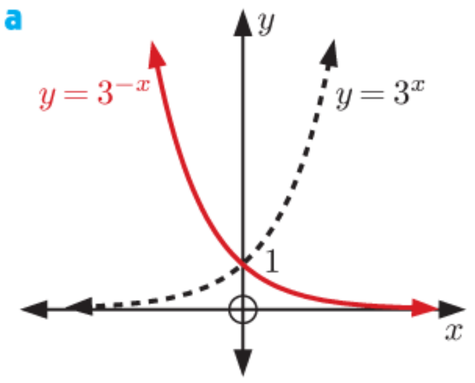
c



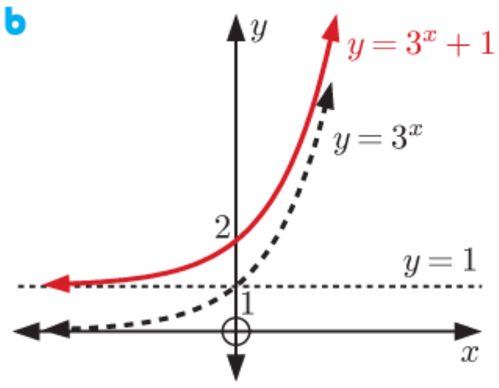
d



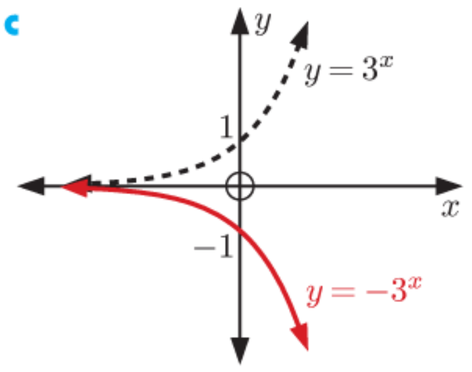
3 a



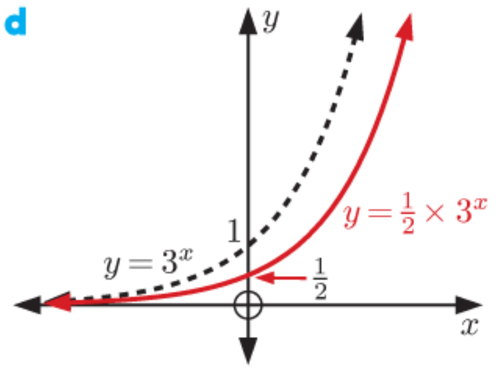
b



c



d

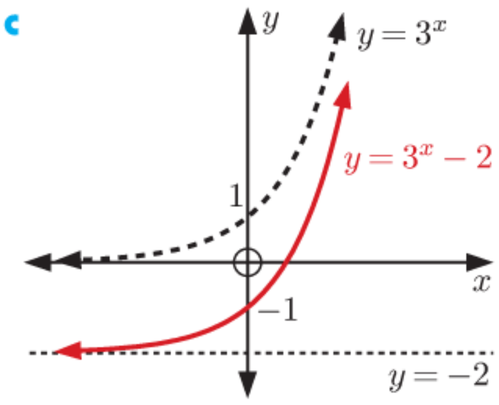


4 a $y = -1$ **b** $y = 4$ **c** $y = 1$ **d** $y = -5$

5 a 5 **b** -1 **c** 10 **d** $5\frac{1}{2}$

6 a i -1 **ii** 7 **iii** $-1\frac{8}{9}$ **b** $y = -2$

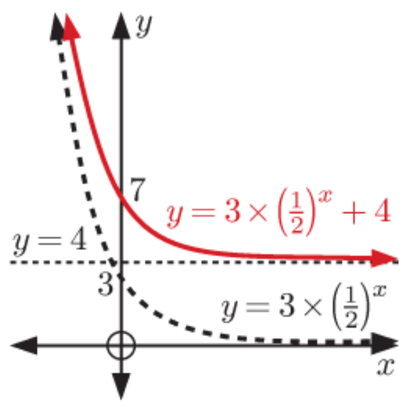
c



d Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -2\}$

7 a i 7 **ii** $4\frac{3}{4}$ **iii** 16 **b** $y = 4$

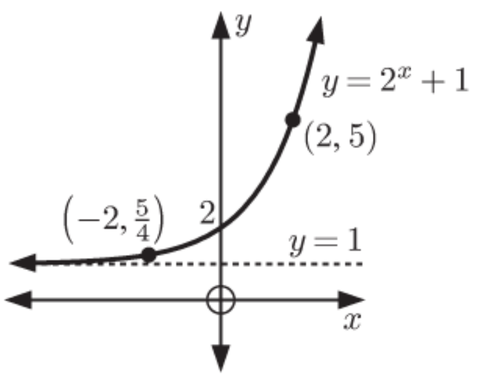
c



d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 4\}$

8 a i 2
ii $y = 1$
iii When $x = 2$, $y = 5$
When $x = -2$, $y = \frac{5}{4}$

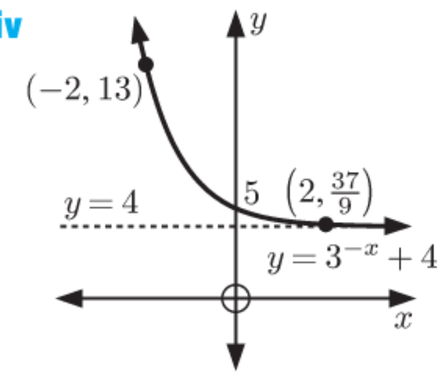
iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 1\}$

b i 5
ii $y = 4$
iii When $x = 2$, $y = \frac{37}{9}$
When $x = -2$, $y = 13$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 4\}$

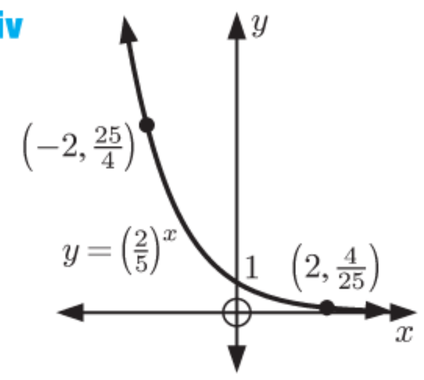
c **i** 1

ii $y = 0$

iii When $x = 2$, $y = \frac{4}{25}$

When $x = -2$, $y = \frac{25}{4}$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 0\}$

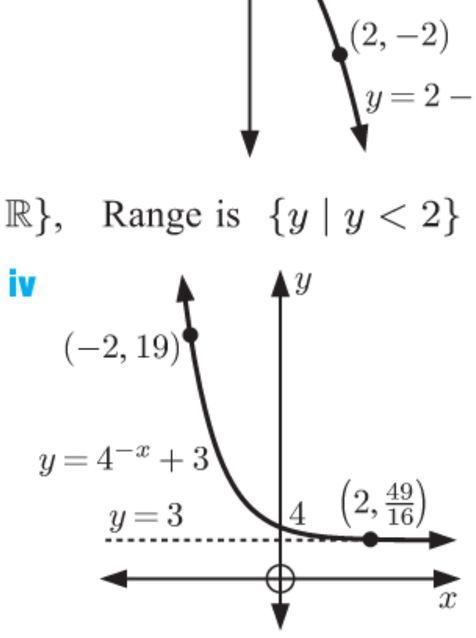
d **i** -2

ii $y = -3$

iii When $x = 2$, $y = -\frac{11}{4}$

When $x = -2$, $y = 1$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -3\}$

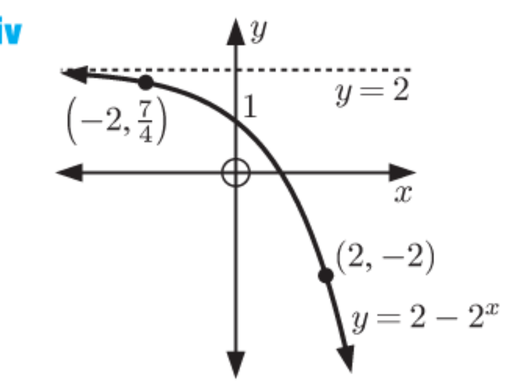
e **i** 1

ii $y = 2$

iii When $x = 2$, $y = -2$

When $x = -2$, $y = \frac{7}{4}$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 2\}$

f **i** 4

ii $y = 3$

iii When $x = 2$, $y = \frac{49}{16}$

When $x = -2$, $y = 19$

iv

v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 3\}$

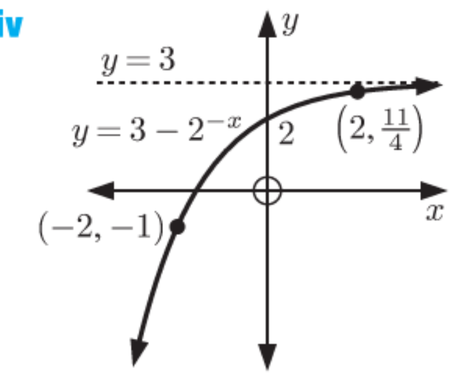
g **i** 2

ii $y = 3$

iii When $x = 2$, $y = \frac{11}{4}$

When $x = -2$, $y = -1$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 3\}$

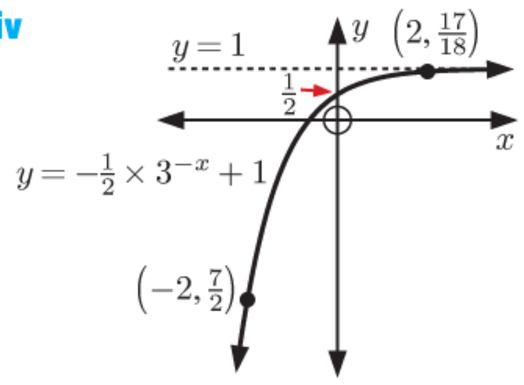
h **i** $\frac{1}{2}$

ii $y = 1$

iii When $x = 2$, $y = \frac{17}{18}$

When $x = -2$, $y = -\frac{7}{2}$

iv



v Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 1\}$

9 **a** $k = 5,\ c = -10$ **b** $y = 310$

10 **a** P(0, 2.5) **b** $a = 1.5$ **c** $y = 3.5$

11 **a** $y = 2^x + 3$ **b** $y = -6 \times 3^x + 5$

c $y = 4 \times 2^{-x} - 7$

12 **a** **i** k must be positive as the function is increasing and lies above the asymptote.

ii c must be negative as the horizontal asymptote is below the x-axis.

b $y = \frac{1}{2} \times 4^x - 6$ **c** $-\frac{11}{2}$ **d** $y = -6$

13 $f(x) = 8 \times 2^x + 2$ **14** $f(x) = -3 \times (\frac{3}{2})^{-x} + 4$

15 8 **16** **a** 3 **b** $y = -3$

EXERCISE 1F

1 **a** **i** $x \approx 1.6$ **ii** $x \approx -0.7$

2 **a** $x \approx 4.32$ **b** $x \approx 3.32$ **c** $x \approx 3.10$

d $x \approx 6.03$ **e** $x \approx 36.8$ **f** $x \approx 6.58$

3 **a** $x \approx 4.95$ **b** $x \approx 6.21$ **c** $x \approx 2.46$

d $x \approx 9.88$ **e** $x \approx 6.95$ **f** $x \approx 5.36$

g $x \approx 3.46$ **h** $x \approx 1.71$ **i** $x \approx 7.23$

4 **a** $a = \frac{1}{\sqrt{3}}$ **b** ≈ -1.02

5 **a** $k < 10$ **b** $k \geqslant 10$ **6** $x \approx 5.46$

EXERCISE 1G.1

1 **a** 3 m^2

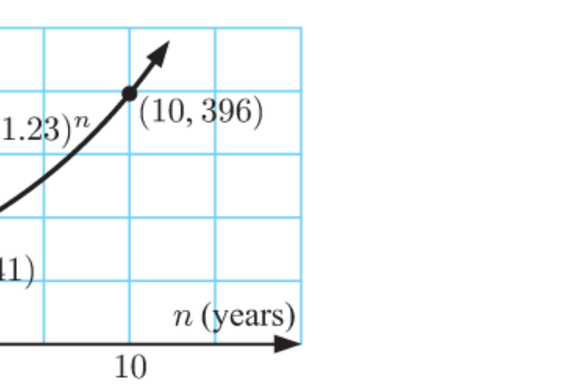
b 8%

c **i** ≈ 3.50 m^2

ii ≈ 6.48 m^2

iii ≈ 30.2 m^2

d



2 **a** 100 grams

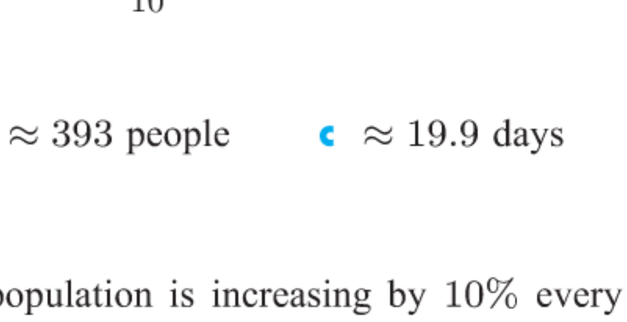
b The weight is increasing by 7% every hour.

c **i** ≈ 131 g

ii ≈ 197 g

iii ≈ 507 g

d

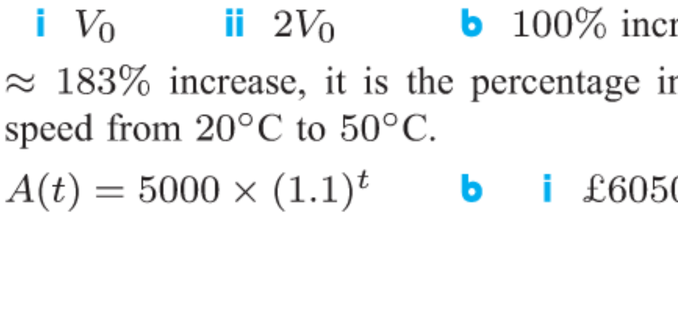


3 **a** $P_0 = 50$

b **i** ≈ 76 possums **ii** ≈ 141 possums

iii ≈ 396 possums

c



d ≈ 11.1 years

4 **a** 4 people **b** ≈ 393 people **c** ≈ 19.9 days

d $0 \leqslant t \leqslant 19.9$

5 **a** $B_0 = 200$

b $a = 1.1$, the bear population is increasing by 10% every year.

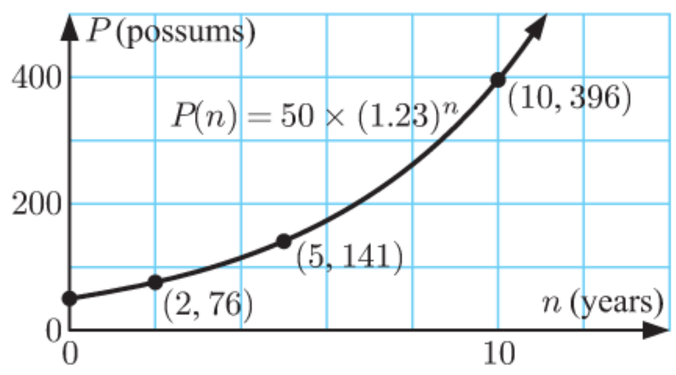
c ≈ 1350 bears **d** $\approx 159\%$ increase **e** ≈ 24.2 years

6 **a** **i** V_0 **ii** $2V_0$ **b** 100% increase

c $\approx 183\%$ increase, it is the percentage increase in reaction speed from 20°C to 50°C.

7 **a** $A(t) = 5000 \times (1.1)^t$ **b** **i** £6050 **ii** £8052.55

c



d ≈ 4.93 years

8 **a** $a = 1.075$

b $k = 360\,000$, the original value of the house was \$360 000.

c ≈ 5.86 years

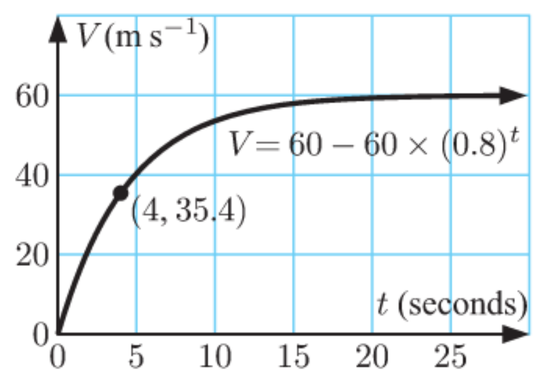
9 **a** $k = 160, \quad c = 40$ **b** ≈ 4.42 weeks

10 **a** When $t = 0, \quad V = c - k = 0$

$\therefore \quad c = k$

b $V = 60 - 60 \times (0.8)^t$ **c** $\approx 35.4 \text{ m s}^{-1}$

d

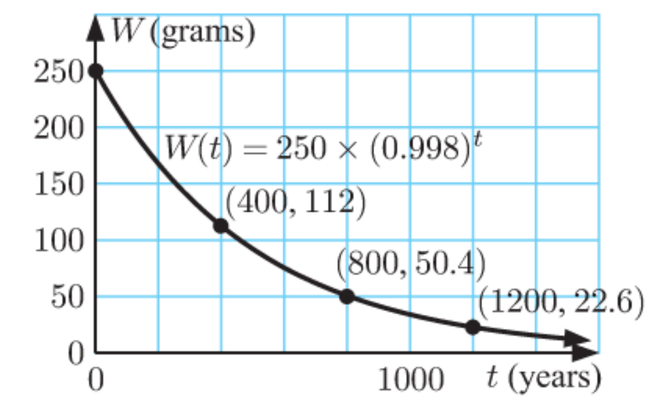


e The parachutist accelerates rapidly until he approaches his terminal velocity of 60 m s^{-1}.

EXERCISE 1G.2

1 **a** 250 g **b** **i** ≈ 112 g **ii** ≈ 50.4 g **iii** ≈ 22.6 g

c



d ≈ 346 years

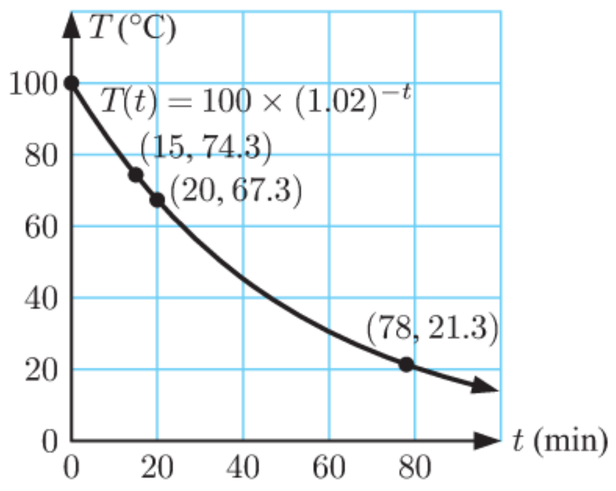
2 **a** 100°C

b **i** ≈ 74.3°C

ii ≈ 67.3°C

iii ≈ 21.3°C

c

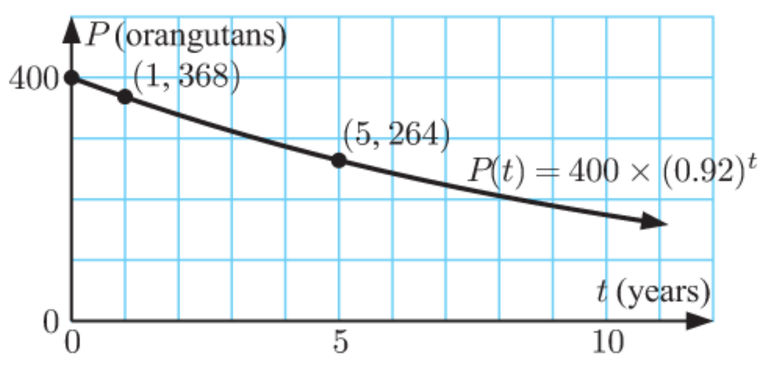


d ≈ 95.8 minutes

3 **a** $P(t) = 400 \times (0.92)^t$

b **i** 368 orangutans **ii** ≈ 264 orangutans

c



d ≈ 8.31 years, or ≈ 8 years 114 days

4 **a** As the intensity of light diminishes as the depth increases, we would expect that $0 < a < 1$.

b $a = 0.95$ **c** ≈ 2.77 units **d** ≈ 17.9 m

e $23.5 \leqslant d \leqslant 44.9$

5 **a** $a = 0.95$

b $k = 340$, the initial population of turtles was 340.

c ≈ 204 turtles

d No, the model is based on data collected since 2005. Before 2005 would be an extrapolation and therefore unreliable.

6 **a** $c = 1000$, the value of the car approaches a minimum "salvage value" of \$1000.

b $k = 23\,000$ **c** \$24 000 **d** ≈ 5 years

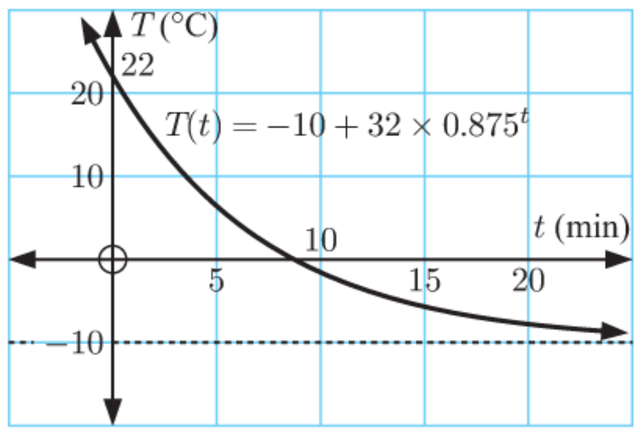
e No, the car depreciates by the same percentage each year *above* the salvage value of \$1000.

7 **a** $c = -10, \quad k = 32$

b -10°C, as the horizontal asymptote is $T = -10$ which suggests that the temperature in the freezer is -10°C.

c **i** 22°C **ii** ≈ 6.41°C **iii** ≈ -1.58°C

d



e ≈ 8.71 minutes

8 **a** 120 mg **b** ≈ 6.58 hours **c** ≈ 14.9 hours

9 **a** 800 g **b** ≈ 17.0 years **c** ≈ 68 years

10 **a** The initial weight of the isotope is 10 mg.

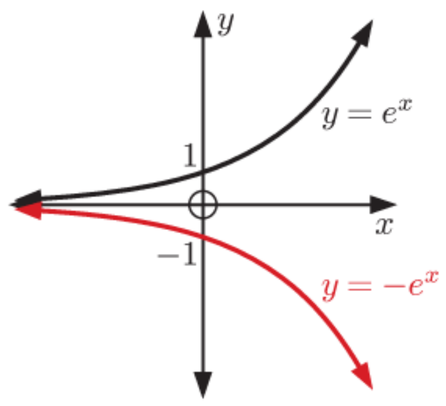
b $a \approx 0.7937$; each day the isotope's weight is decreasing by $\approx 20.63\%$.

c ≈ 6.30 mg **d** **i** ≈ 5.21 days **ii** ≈ 9.00 days

11 ≈ 2.52 hours **12** $\approx 18\,900$ years

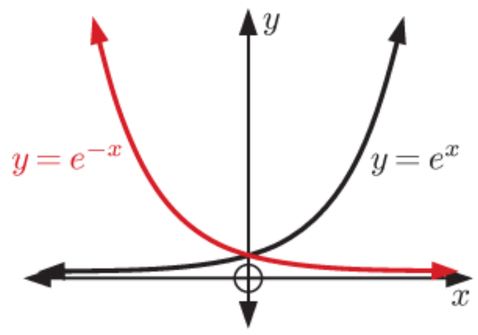
EXERCISE 1H

1



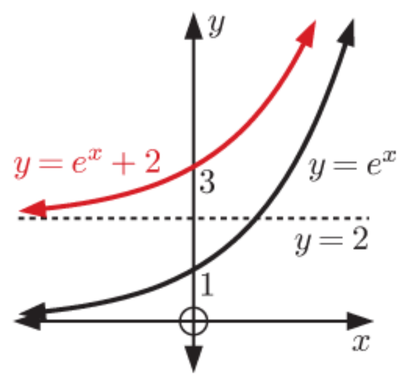
One is the other reflected in the x-axis.

2

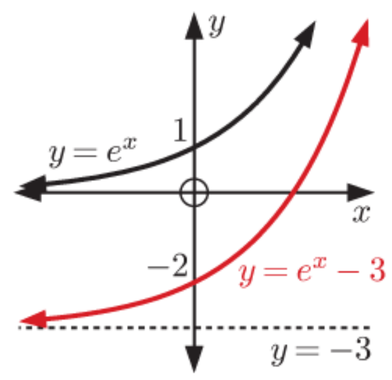


One is the other reflected in the y-axis.

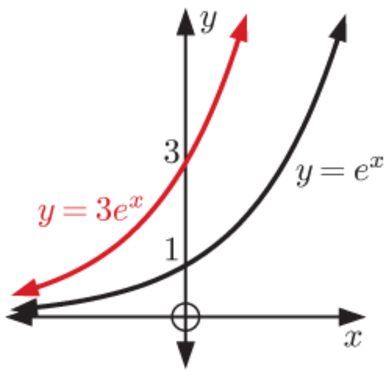
3 **a**



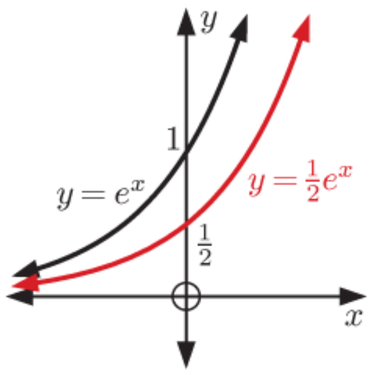
b



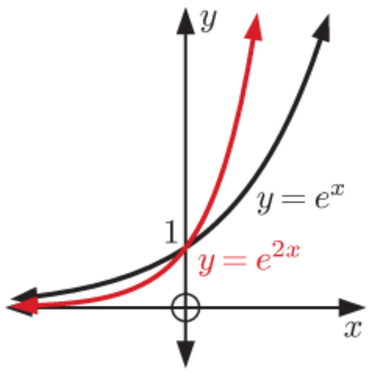
c



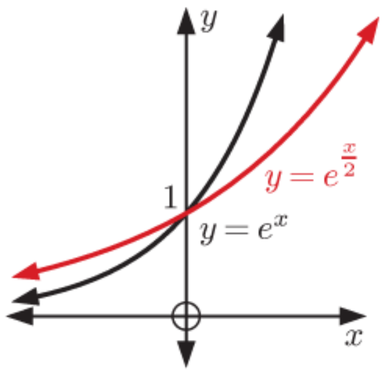
d



e



f



4 $k + c$

5 **a** ≈ 7.39 **b** ≈ 20.1 **c** ≈ 2.01 **d** ≈ 1.65
e ≈ 0.368

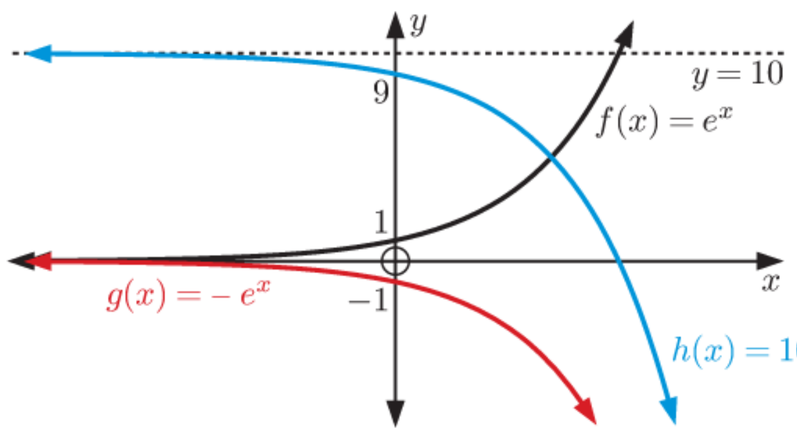
6 **a** $e^{\frac{1}{2}}$ **b** $e^{-\frac{1}{2}}$ **c** e^{-2} **d** $e^{\frac{3}{2}}$

7 **a** ≈ 10.074 **b** $\approx 0.099\,261$ **c** ≈ 125.09
d $\approx 0.007\,994\,5$ **e** ≈ 41.914 **f** ≈ 42.429
g ≈ 3540.3 **h** $\approx 0.006\,342\,4$

8 **a** $e^{2x} + 2e^x + 1$ **b** $1 - e^{2x}$ **c** $1 - 3e^x$

9 **a** $e^x(e^x + 1)$ **b** $(e^x + 4)(e^x - 4)$ **c** $(e^x - 6)(e^x - 2)$

10 **a**

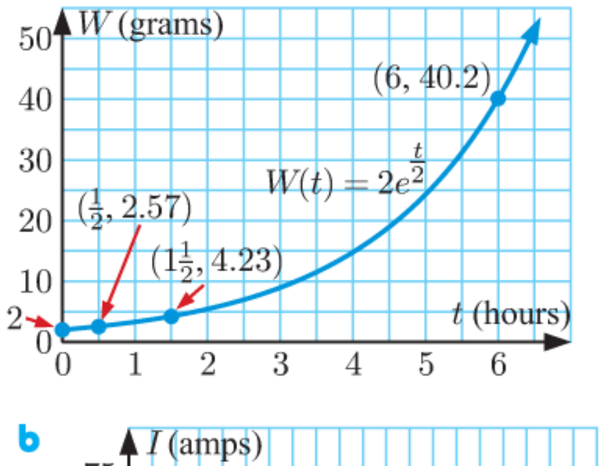


b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y < 0\}$
Range of h is $\{y \mid y < 10\}$

c For f: as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to 0^+$
For g: as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 0^-$
For h: as $x \to \infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 10^-$

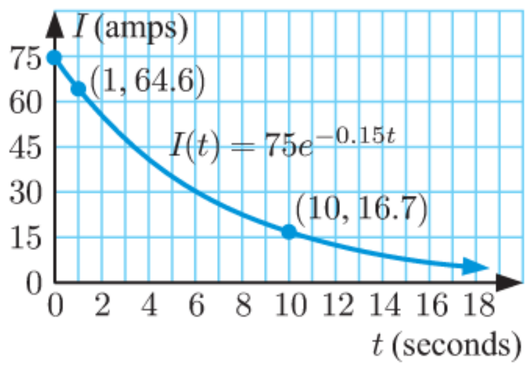
11 **a** **i** 2 g
ii ≈ 2.57 g
iii ≈ 4.23 g
iv ≈ 40.2 g

b



12 **a** **i** ≈ 64.6 amps
ii ≈ 16.7 amps
c ≈ 28.8 seconds

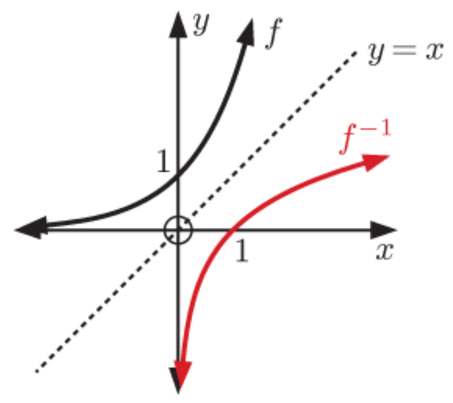
b



13 **a** $A = 7e^{-0.5t} + 3$
b The side as the tank never completely empties.
c ≈ 5.58 kL **d** ≈ 3.89 min

14 **a** decreasing **b** **i** 3900 m s^{-1} **ii** ≈ 2600 m s^{-1}
c ≈ 11.8 s

15 **a** $f^{-1}(x) = \log_e x$

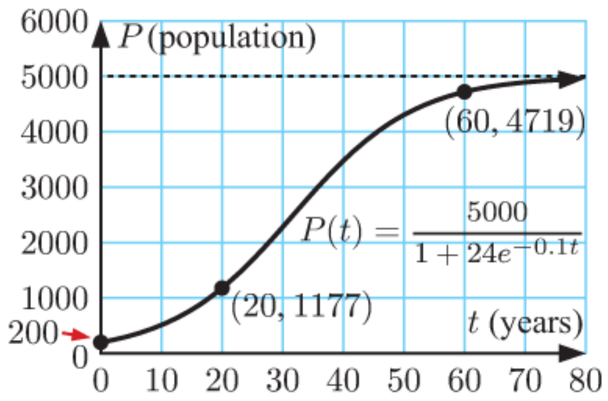


b Domain of f^{-1} is $\{x \mid x > 0\}$
Range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$

16 $e^1 \approx \sum_{i=0}^{19} \frac{1}{i!} 1^i \approx 2.718\,281\,828$

EXERCISE 11

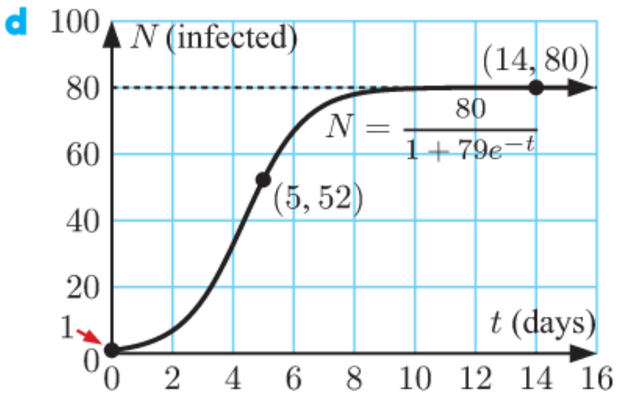
1 **a** 200 people
b **i** ≈ 1177 people **ii** ≈ 4719 people
c 5000 people
d



e ≈ 35.8 years

2 **a** $C = 79$
b as $t \to \infty$, $N \to 80^-$
c **i** ≈ 52 people
ii ≈ 80 people

d



3 **a** $L = 162$ **b** **i** ≈ 83.3 cm **ii** ≈ 139 cm
c 162 cm **d** ≈ 135 cm

4 **a** $N \approx \dfrac{290}{1 + 144e^{-0.819t}}$ **b** ≈ 198 people
c 290 people **d** after ≈ 7.04 hours ($\approx 3{:}03$ pm)

REVIEW SET 1A

1 **a** $2^{\frac{1}{4}}$ **b** $2^{\frac{7}{2}}$ **c** $2^{-\frac{2}{3}}$ **2** **a** 4 **b** $\frac{1}{9}$ **c** $\frac{1}{3}$

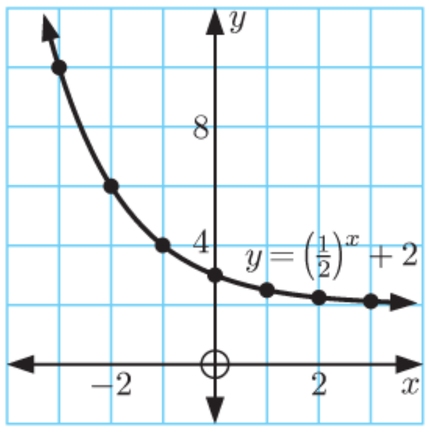
3 **a** $1 + e^{2x}$ **b** $2^{2x} + 10(2^x) + 25$ **c** $x - 49$

4 **a** yes **b** no

5 **a**

x	-3	-2	-1	0	1	2	3
y	10	6	4	3	$2\frac{1}{2}$	$2\frac{1}{4}$	$2\frac{1}{8}$

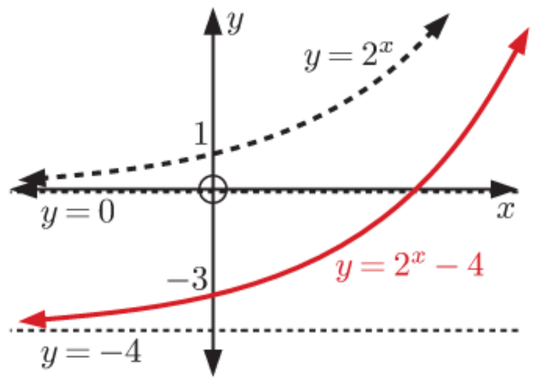
b



c **i** as $x \to \infty$, $y \to 2^+$ **ii** as $x \to -\infty$, $y \to \infty$

d $y = 2$

6

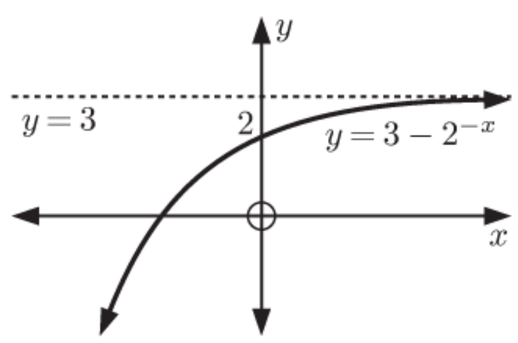


7 **a**

x	-2	-1	0	1	2
y	-1	1	2	$2\frac{1}{2}$	$2\frac{3}{4}$

b as $x \to \infty$, $y \to 3^-$
as $x \to -\infty$, $y \to -\infty$

c



d $y = 3$

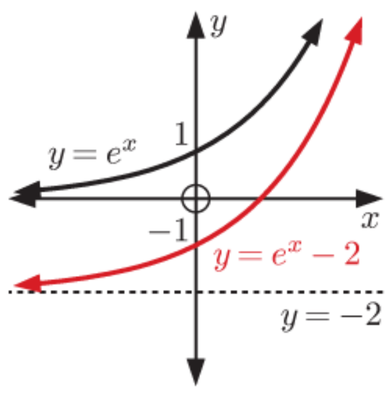
8 **a** $k = 8$, $a = \frac{4}{3}$ **b** $y = 0$ **c** $y = \frac{128}{9}$

9 **a** $x \approx 4.29$ **b** $x \approx 3.56$ **c** $x \approx 86.3$

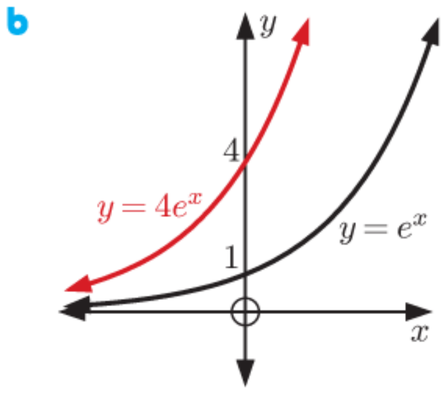
10 **a** 80 seals **b** ≈ 4.96 years **c** $\approx 74.9\%$ increase

11 **a** ≈ 54.6 **b** ≈ 0.135 **c** ≈ 331 **d** ≈ 3.19

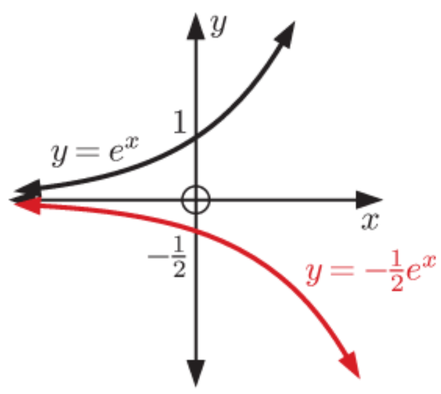
12 **a**



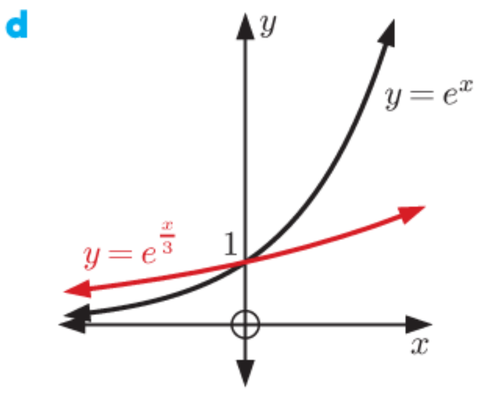
b



c



d



13 **a** $k = 700$, $c = 100$

b No, the computer depreciates by 40% each year above the computer's *minimum* price, but not overall.

c £251.20

d $V = 100$, the computer will never be worth less than £100.

14 **a** 1500 g **b** ≈ 98.7 years **c** ≈ 197 years

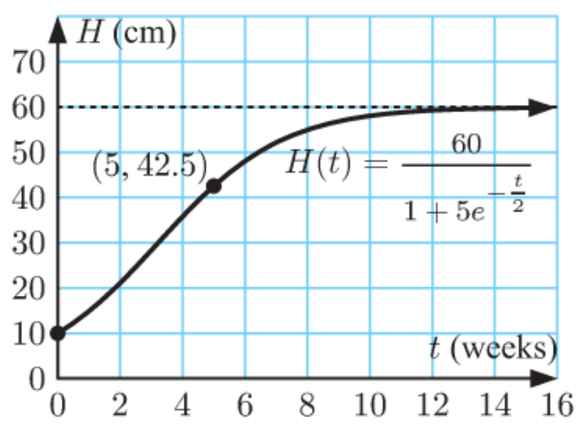
15 **a** $T = 60e^{-0.1t} + 20$ **b** $\approx 42.1°C$ **c** ≈ 17.9 min

16 **a** $L = 60$

b ≈ 42.5 cm

d ≈ 6.44 weeks after planting

c



REVIEW SET 1B

1 **a** $x^{\frac{1}{3}}$ **b** $x^{\frac{7}{2}}$ **c** $x^{-\frac{5}{2}}$

2 **a** ≈ 3.95 **b** ≈ 0.517 **c** ≈ 3.16

3 **a** $9 - 6e^x + e^{2x}$ **b** $x - 2 - x^{-1}$ **c** $2^x + 1$

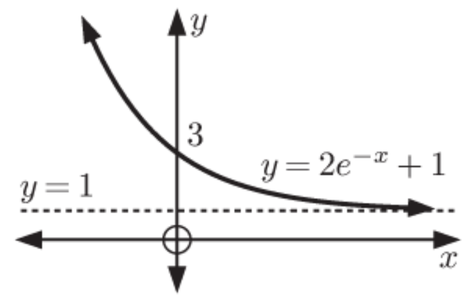
4 **a** $8(3^x)$ **b** $(2^x - 4)(2^x + 3)$ **c** $(e^x + 5)(e^x - 3)$

5 **a**

x	-2	-1	0	1	2
y	15.8	6.44	3	1.74	1.27

b as $x \to \infty$, $y \to 1^+$
as $x \to -\infty$, $y \to \infty$

c



d $y = 1$

6 **a** clock: £525, vase: £428

b clock: $V(t) = 500 \times (1.05)^t$
vase: $V(t) = 400 \times (1.07)^t$

c clock $\approx$ £1039.46, vase $\approx$ £1103.61 $\therefore$ the vase

d $500 \times (1.05)^t = 400 \times (1.07)^t$ and solve for t;
$t \approx 11.8$ years

7 **a** **i** 81 **ii** $\frac{1}{3}$ **b** $k = 9$

8 **a** $y = 6 \times 2^{-x} + 4$ **b** $y = -3 \times (\frac{1}{2})^x + 5$

9 **a** $y = 10 \times 5^{-x} - 2$ **b** $y = 48$

10 **a** $x \approx 2.45$ **b** $x \approx 2.67$ **c** $x \approx 2.79$

11 **a** $k > -2$ **b** $k \leqslant -2$

12 **a** $a = 1.15$

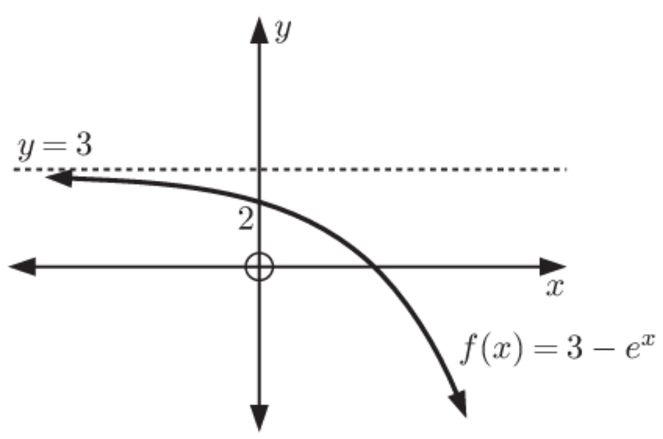
b $k = 800$, the initial population was 800 birds.

c ≈ 1610 birds

d No, the population would be unrealistically large. The birds would eventually reach a limiting population.

13 ≈ 152 g **14** **a** **C** **b** **E** **c** **A** **d** **B** **e** **D**

15 **a**



b Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 3\}$

c as $x \to \infty$, $f(x) \to -\infty$; as $x \to -\infty$, $f(x) \to 3^-$

16 **a** $L = \dfrac{20e - 20}{e - 2} \approx 47.8$, $C = \dfrac{e}{e - 2} \approx 3.78$

b ≈ 26 adult females **c** ≈ 29.6 months

EXERCISE 2A

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$
g $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$

2 **a** n **b** $a + 2$ **c** $1 - m$ **d** $a - b$

3 **a** $100 < 237 < 1000$
$\therefore \log 100 < \log 237 < \log 1000$
$\therefore 2 < \log 237 < 3$ **b** ≈ 2.37

4 **a** $-1 < \log 0.6 < 0$ **b** ≈ -0.22

5 **a** ≈ 1.88 **b** ≈ 2.06 **c** ≈ 0.48 **d** ≈ 2.92
e ≈ -0.40 **f** ≈ 3.51 **g** ≈ -2.10 **h** does not exist

6 **a** $x > 1$ **b** $x = 1$ **c** $0 < x < 1$ **d** $x \leqslant 0$

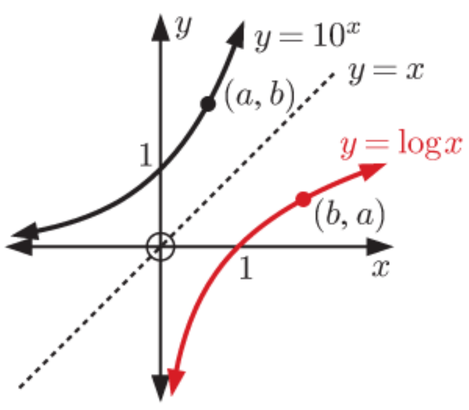
7 **a** $\approx 10^{0.7782}$ **b** $\approx 10^{1.7782}$ **c** $\approx 10^{3.7782}$
d $\approx 10^{-0.2218}$ **e** $\approx 10^{-2.2218}$ **f** $\approx 10^{1.1761}$
g $\approx 10^{3.1761}$ **h** $\approx 10^{0.1761}$ **i** $\approx 10^{-0.8239}$
j $\approx 10^{-3.8239}$

8 **a** **i** ≈ 0.477 **ii** ≈ 2.477
b $\log 300 = \log(3 \times 10^2) = \log(10^{\log 3} \times 10^2) = \ldots.$

9 **a** **i** ≈ 0.699 **ii** ≈ -1.301
b $\log(0.05) = \log(5 \times 10^{-2}) = \log(10^{\log 5} \times 10^{-2}) = \ldots.$

10 **a** $x = 100$ **b** $x = 10$ **c** $x = 1$
d $x = \frac{1}{10}$ **e** $x = \sqrt{10}$ **f** $x = \frac{1}{\sqrt{10}}$
g $x = 10\,000$ **h** $x = 0.000\,01$ **i** $x \approx 6.84$
j $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

11 **a** $b = 10^a$
c When $x = b$, $y = \log b = a$ {from **b**}
$\therefore$ (b, a) lies on the graph of $y = \log x$.
d They are inverse functions.
e



f 1 **g** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

EXERCISE 2B

1 **a** $\log 16$ **b** $\log 20$ **c** $\log 8$ **d** $\log\left(\frac{p}{m}\right)$
e $\log 4$ **f** $\log 2$ **g** 3 **h** $\log 25$
i $\log 24$ **j** 1 **k** 0 **l** $\log 28$

2 **a** $\log 700$ **b** $\log\left(\frac{2}{5}\right)$ **c** $\log 30$
d $\log\left(\frac{1}{20}\right)$ **e** $\log 200$ **f** $\log(0.005)$
g $\log(10^t \times w)$ **h** $\log\left(\frac{2}{5}\right)$ **i** $\log 20$

3 **a** $\log 96$ **b** $\log 72$ **c** $\log 8$ **d** $\log\left(\frac{25}{8}\right)$
e $\log 6$ **f** $\log\left(\frac{1}{2}\right)$ **g** 1 **h** $\log 25$ **i** 1

4 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

5 For example, for **a**, $\log 9 = \log(3^2) = 2 \log 3$.

7 **a** $p + q$ **b** $2q + r$ **c** $2p + 3q$ **d** $r + \frac{1}{2}q - p$
e $r - 5p$ **f** $p - 2q$

8 **a** $x + z$ **b** $z + 2y$ **c** $x + z - y$ **d** $2x + \frac{1}{2}y$
e $3y - \frac{1}{2}z$ **f** $2z + \frac{1}{2}y - 3x$

9 **a** 0.86 **b** 2.15 **c** 1.075

10 $\log 384$ **11** $1 + \log 12$

EXERCISE 2C

1 **a** 2 **b** 4 **c** $\frac{3}{2}$ **d** 0 **e** -1
f $\frac{1}{3}$ **g** -2 **h** $-\frac{1}{2}$

2 **a** 3 **b** 9 **c** $\frac{1}{5}$ **d** $\frac{1}{4}$ **e** a
f $1 + a$ **g** $a + b$ **h** ab

3 **a** $e^3 \approx 20.0855$, $e^4 \approx 54.5982$ **b** 3 and 4
c $\ln 40 \approx 3.689$

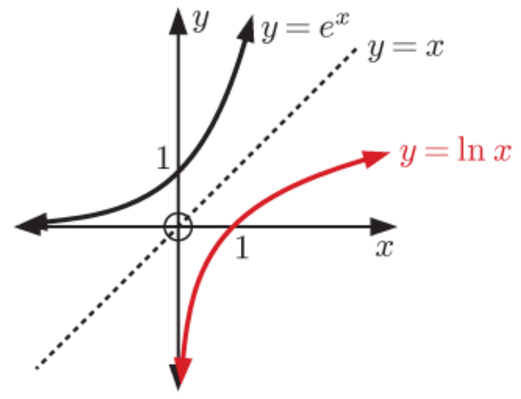
4 **a** ≈ 2.485 **b** ≈ 4.220 **c** ≈ 0.336
d ≈ -0.357 **e** ≈ 6.215

5 x does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.

6 **a** $\approx e^{1.7918}$ **b** $\approx e^{4.0943}$ **c** $\approx e^{8.6995}$
d $\approx e^{-0.5108}$ **e** $\approx e^{-5.1160}$ **f** $\approx e^{2.7081}$
g $\approx e^{7.3132}$ **h** $\approx e^{0.4055}$ **i** $\approx e^{-1.8971}$
j $\approx e^{-8.8049}$

7 **a** $x \approx 20.1$ **b** $x = e$ **c** $x = 1$
d $x \approx 0.368$ **e** $x \approx 0.006\,74$ **f** $x \approx 2.30$
g $x \approx 8.54$ **h** $x \approx 0.0370$

8 **a** **i** x **ii** x
b They are inverses of each other.
c



d Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

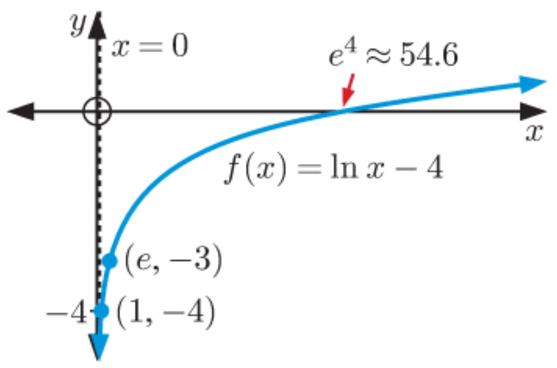
9 **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$
e $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln(4e)$ **h** $\ln\left(\frac{6}{e}\right)$
i $\ln 20$ **j** $\ln(4e^2)$ **k** $\ln\left(\frac{20}{e^2}\right)$ **l** $\ln 1 = 0$

10 **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$
e $\ln 6$ **f** $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$
i $\ln 16$ **j** $\ln(16e^2)$ **k** $\ln\left(\frac{3}{e}\right)$ **l** $\ln\left(\frac{\sqrt{e}}{8}\right)$

11 For example, for **a**, $\ln 27 = \ln(3^3) = 3 \ln 3$.

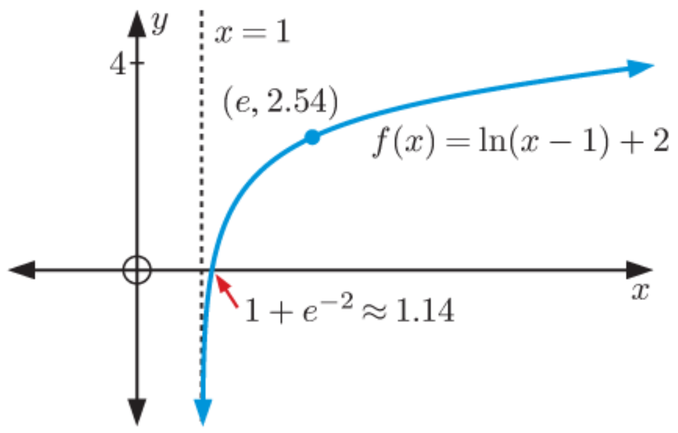
EXERCISE 2D

1 **a** **i** A translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.
ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$
iii $f(1) = -4$, $f(e) = -3$ **iv** e^4
v



vi $f^{-1}(x) = e^{x+4}$

b **i** A translation through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
ii Domain is $\{x \mid x > 1\}$, Range is $\{y \mid y \in \mathbb{R}\}$
iii $f(1)$ is undefined, $f(e) \approx 2.54$ **iv** $1 + e^{-2}$
v



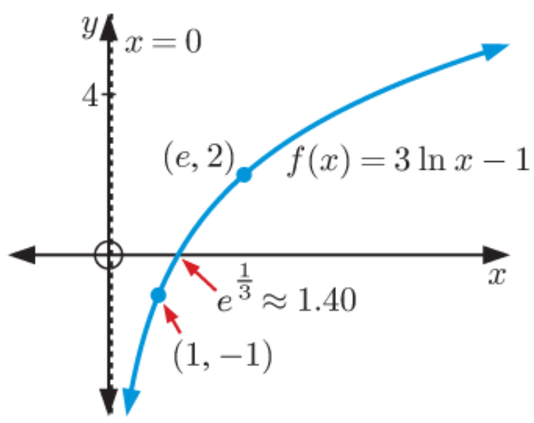
vi $f^{-1}(x) = e^{x-2} + 1$

c **i** A vertical stretch with scale factor 3, then a translation through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii $f(1) = -1$, $f(e) = 2$ **iv** $e^{\frac{1}{3}}$

v



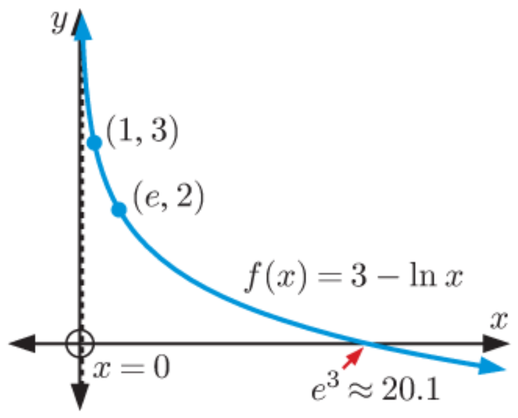
vi $f^{-1}(x) = e^{\frac{x+1}{3}}$

d **i** A reflection in the x-axis, then a translation through $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii $f(1) = 3$, $f(e) = 2$ **iv** e^3

v

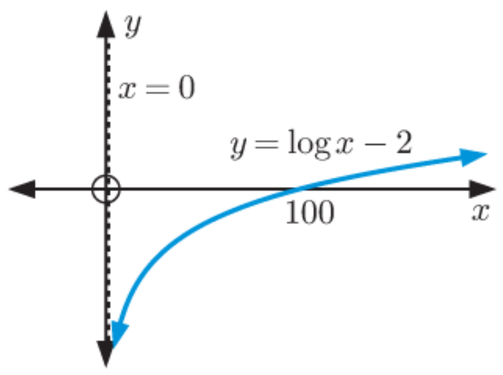


vi $f^{-1}(x) = e^{3-x}$

2 **a** **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept 100, no y-intercept

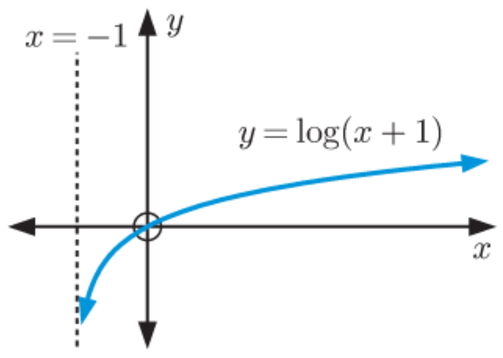
iii



b **i** Domain is $\{x \mid x > -1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = -1$, x and y-intercepts 0

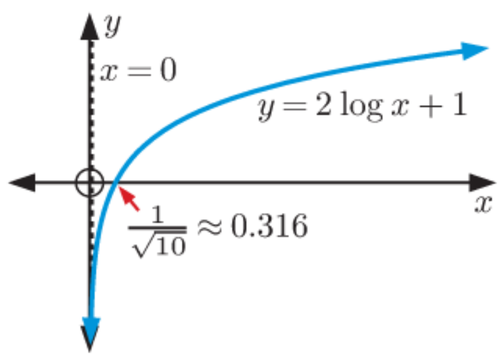
iii



c **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept $\frac{1}{\sqrt{10}}$, no y-intercept

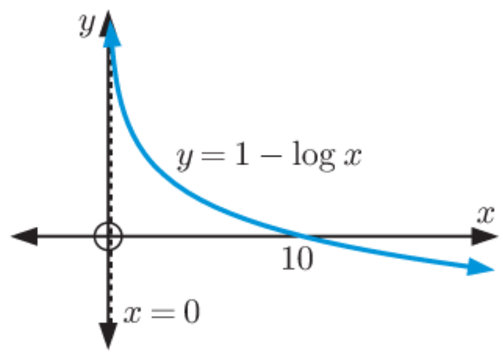
iii



d **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

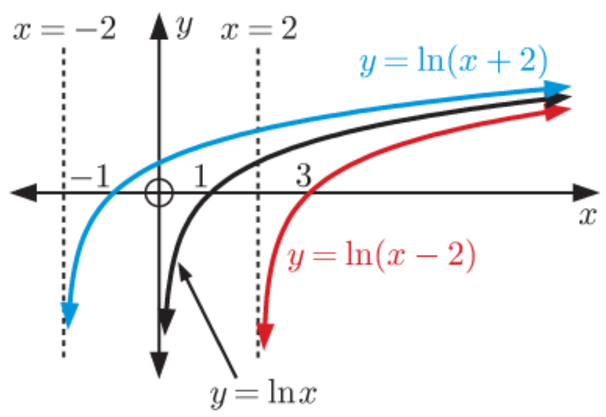
ii vertical asymptote is $x = 0$, x-intercept 10, no y-intercept

iii



3 **a** A is $y = \ln x$ as its x-intercept is 1.

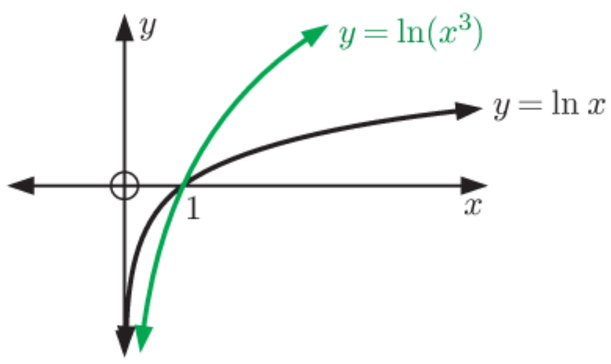
b



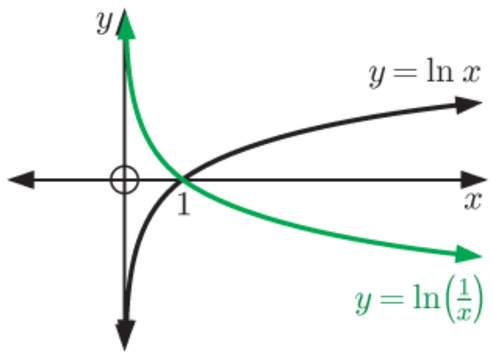
c $y = \ln x$ has vertical asymptote $x = 0$
$y = \ln(x - 2)$ has vertical asymptote $x = 2$
$y = \ln(x + 2)$ has vertical asymptote $x = -2$

4 $y = \ln(x^2) = 2 \ln x$, so she is correct.
This is because the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

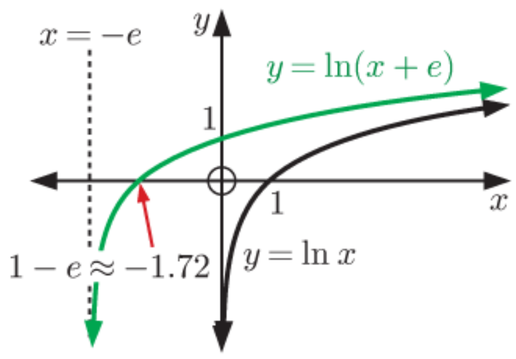
5 **a**



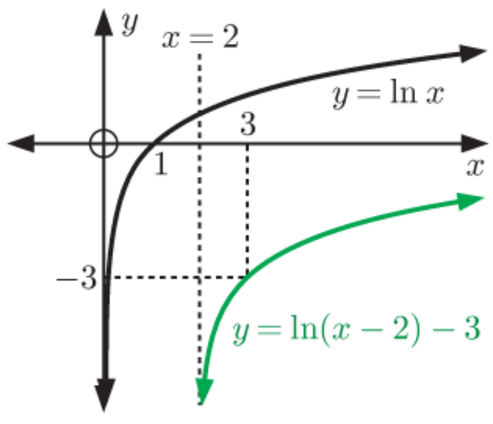
b



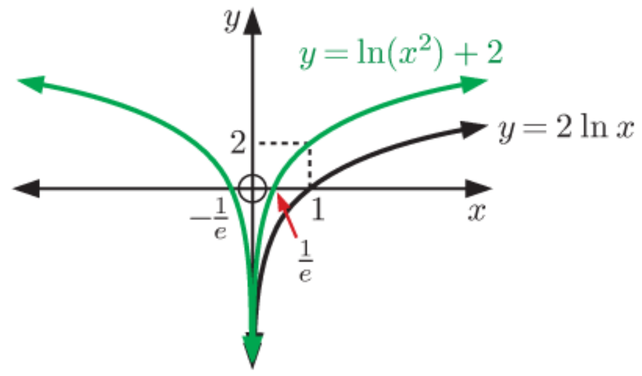
c



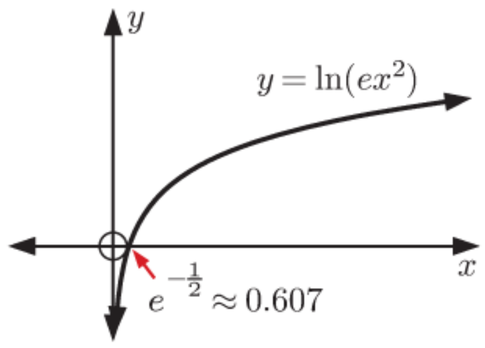
d



e



6 **a** $y = 1 + 2\ln x$ **b**



7 **a** $b = \log 50\,000$ **b** $a = \log 12\,500$

8 $f^{-1}(x) = \frac{1}{2}\ln x$

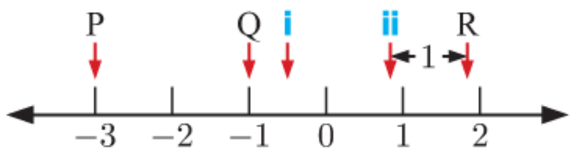
a $(f^{-1} \circ g)(x) = \frac{1}{2}\ln(2x-1)$

b $(g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{2}\right)$

EXERCISE 2E

1 **a** $\mathrm{P} = \frac{1}{1000}$, $\mathrm{Q} = \frac{1}{10}$

b

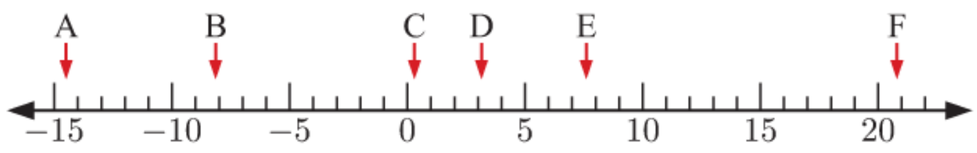


c No, since $\log 0$ is undefined.

2 **a**

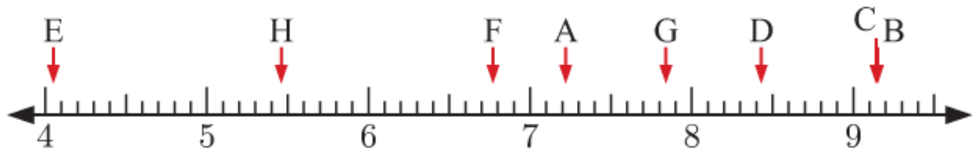
	Process	*Energy* (J)	*Logarithm*
A	average kinetic energy of a human red blood cell	3×10^{-15}	≈ -14.5
B	energy needed to raise a grain of sand by 1 mm	7×10^{-9}	≈ -8.15
C	kinetic energy when a 200 g apple falls 1 m	2×10^{0}	≈ 0.301
D	solar energy received by 1 m^2 of the Earth's surface directly under the Sun in 1 second	1.4×10^{3}	≈ 3.15
E	energy from the combustion of 1 m^3 of natural gas	4×10^{7}	≈ 7.60
F	total world energy consumption in 2016	6×10^{20}	≈ 20.8

b



3 **a**

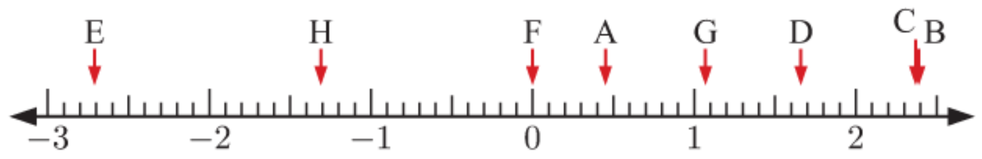
	Country	*Population*	*Logarithm*
A	Cambodia	16 482 646	≈ 7.22
B	China	1 420 062 022	≈ 9.15
C	India	1 368 737 513	≈ 9.14
D	Indonesia	269 536 482	≈ 8.43
E	Nauru	11 260	≈ 4.05
F	Singapore	5 836 496	≈ 6.77
G	Thailand	69 306 160	≈ 7.84
H	Vanuatu	288 017	≈ 5.46



b

	Country	*Population*	*Population relative to Singapore*	*Logarithm*
A	Cambodia	16 482 646	≈ 2.82	≈ 0.451
B	China	1 420 062 022	≈ 243	≈ 2.39
C	India	1 368 737 513	≈ 235	≈ 2.37
D	Indonesia	269 536 482	≈ 46.2	≈ 1.66
E	Nauru	11 260	$\approx 0.001\,93$	≈ -2.71
F	Singapore	5 836 496	1	0
G	Thailand	69 306 160	≈ 11.9	≈ 1.07
H	Vanuatu	288 017	≈ 0.0493	≈ -1.31

c



d Our scale in **c** is obtained by translating the scale in **b** ≈ 6.77 units left. This occurs since $\log\left(\frac{P}{P_S}\right) = \log P - \log P_S$.

4 **a** **i** If $M = 0$, $I = I_0$, the earthquake intensity is equal to the reference intensity.

ii If $M = 1$, $I = 10I_0$, the earthquake intensity is 10 times greater than the reference intensity.

b The magnitude of an earthquake follows a logarithmic scale. A magnitude 6 earthquake is $10^3 = 1000$ times more intense than a magnitude 3 earthquake.

c **i** 4 **ii** ≈ 5.70 **d** ≈ 5.01 times more intense

5 **a** The concentrations of H_3O^+ are extremely small numbers that would otherwise be impossible to compare.

b **i** ≈ 3.63 **ii** 10^{-7} mol L^{-1}

c **i** vinegar **ii** ≈ 4.30 **iii** ≈ 79.2 times greater

d Yes, if the concentration of H_3O^+ is greater than 1 mol L^{-1}, then the pH will be negative.

6 **a** ≈ 5.99 **b** $\approx 2.82 \times 10^{12}$ joules

7 **a** 261.6 Hz **b** ≈ 1.58 octaves

c **i** ≈ 2090 Hz **ii** ≈ 131 Hz **d** $\approx 1.06 : 1$

8 **b** ≈ 10.6 mm

REVIEW SET 2A

1 **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a + b + 1$

2 **a** ≈ 1.431 **b** ≈ -0.237 **c** ≈ 2.602 **d** ≈ 3.689

3 **a** $\approx 10^{1.5051}$ **b** $\approx 10^{-2.8861}$ **c** $\approx 10^{-4.0475}$

4 **a** $\ln 144$ **b** $\ln\left(\frac{3}{2}\right)$ **c** $\ln\left(\frac{25}{e}\right)$ **d** $\ln 3$

5 **a** $\log 144$ **b** $\log\left(\frac{16}{9}\right)$ **c** $\log 500$

6 **a** $2A + 2B$ **b** $A + 3B$ **c** $3A + \frac{1}{2}B$

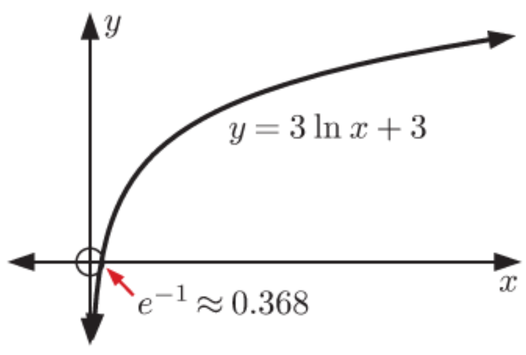
d $\frac{1}{2}(A + B)$ **e** $4B - 2A$ **f** $3A - 2B$

7 **a** $x = 10^{-3} = \frac{1}{1000}$ **b** $x = e^5$

8 **a** A vertical stretch with scale factor 3, then a translation through $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

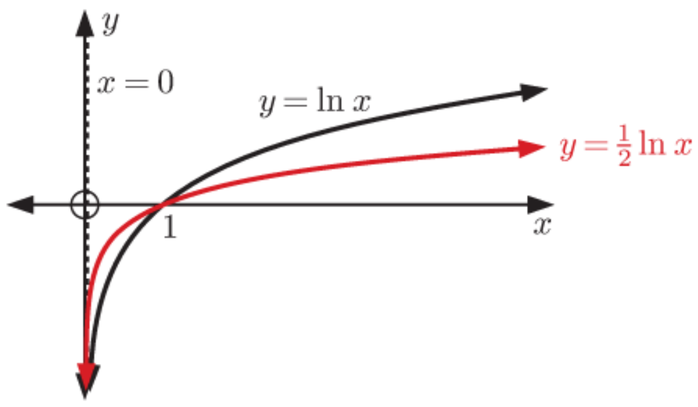
b $f(1) = 3$ **c** x-intercept e^{-1}, no y-intercept

d

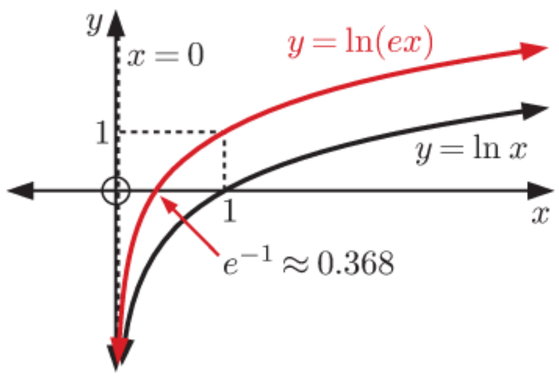


e $f^{-1}(x) = e^{\frac{x-3}{3}}$

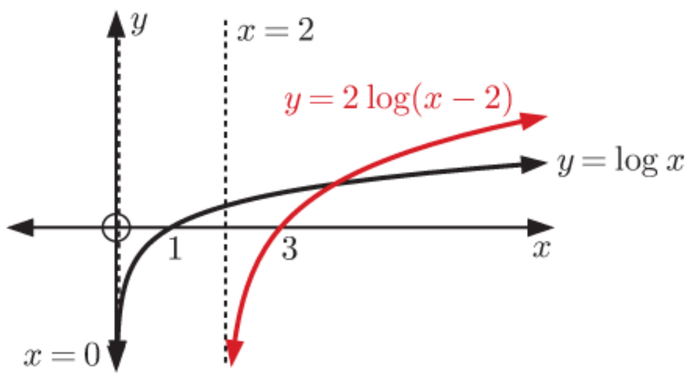
9 **a**



b



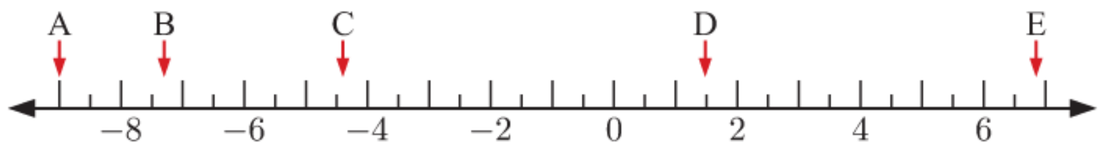
c



10 e and 10 **11** **a** 100 **b** (100, 2)

12

	Electromagnetic radiation type	*Wavelength* (m)	*Logarithm*
A	X-rays	0.000 000 001	-9
B	Ultraviolet	0.000 000 05	≈ -7.30
C	Infrared	0.000 04	≈ -4.40
D	High frequency microwaves	30	≈ 1.48
E	Super low frequency radio waves	7 000 000	≈ 6.85



REVIEW SET 2B

1 **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a+b$

2 **a** $\frac{3}{2}$ **b** -3 **c** $2x$ **d** $1-x$

3 **a** $\frac{2}{3}$ **b** $\frac{6}{5}$ **c** 8

4 **a** x^4 **b** 5 **c** $\frac{1}{2}$ **d** $3x$ **e** $-x$ **f** $-\log x$

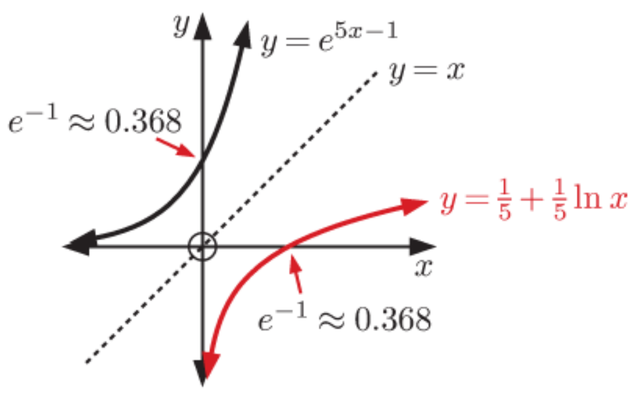
5 **a** $\approx e^{2.9957}$ **b** $\approx e^{8.0064}$ **c** $\approx e^{-2.5903}$

6 **a** $\ln 3$ **b** $\ln 4$ **c** $\ln 125$

7 $x = 1$ **8** $a = \ln 2$, $b = 3$

9 **a** $f^{-1}(x) = \frac{1}{5} + \frac{1}{5}\ln x$

b



10 **a** 100 times **b** **i** $\approx$ 73.0 dB **ii** $\approx$ 3.25 times

11 **a** **i** $\approx$ 13.5 years **ii** $\approx$ 31.2 years
b $\approx$ 52.8 years longer **c** $\approx$ 2.28%

EXERCISE 3A

1 **a** $\pm\frac{1}{2}$ cm **b** $\pm\frac{1}{2}$ mL **c** $\pm$50 mL **d** $\pm$250 g
e $\pm$0.05°C

2 **a** 26.5 mm to 27.5 mm **b** 38.25 cm to 38.35 cm
c 4.75 m to 4.85 m **d** 1.45 kg to 1.55 kg
e 24.5 g to 25.5 g **f** 3.745 kg to 3.755 kg

3 $36.35°\text{C} < T < 36.45°\text{C}$

4 **a** 1.055 km **b** 9.715 km **c** 10.05 km
For distances from 1 km up to 9.99 km, the watch displays the correct distance accurate to $\pm$0.005 km, but for distances from 10 km up to 99.9 km, the accuracy reduces to $\pm$0.05 km.

5 **a** 6.4 m, as it is the most different from the other measurements.
b 6.05 m, as it is in the range of values for a measurement of 6.0 m or 6.1 m.
c 10 cm

6 **a** $2.35\text{ m} < l < 2.45\text{ m}$ **b** $2.35n\text{ m} < L < 2.45n\text{ m}$

7 $4\text{ s} < t < 6\text{ s}$ **8** 248 ± 2 cm **9** $788\text{ cm} < l < 792\text{ cm}$

10 **a** 55.25 cm^2 **b** 41.25 cm^2

11 1092.25 ± 34 cm^2 **12** 36.125 ± 4.25 cm^2

13 196.5 ± 52.125 cm^3 **14** $\approx 1523.90 \pm 21.79$ cm^3

15 $\approx 1197.73 \pm 275.28$ cm^3 **16** 339.95 ± 7.74 cm^3

17 $27.6° < \theta < 34.4°$

18 **a** $3.60\text{ cm}^2 < A < 4.04\text{ cm}^2$
b $14.62\text{ cm} < P < 14.89\text{ cm}$

19 volume $= \frac{4}{3}\pi r^3$, surface area $= 4\pi r^2$
The rounding will have more effect on the volume, as the error is multiplied through 3 times rather than twice.

20 **a** 36.69 ± 1.15 cm^3 **b** 73.48 ± 1.54 cm^2

EXERCISE 3B

1 **a** €2460, $\approx 0.180\%$ **b** 467 people, $\approx 1.48\%$
c \$1890, $\approx 0.413\%$ **d** 189 cars, $\approx 6.72\%$

2 **a** 1.238 kg, $\approx 19.8\%$ **b** 2.4 m, $\approx 2.46\%$
c 3.8 L, $\approx 16.0\%$ **d** 22 hours, $\approx 30.6\%$

3 **a** 99.91 m^2 **b** 100 m^2 **c** 0.09 m^2, $\approx 0.0901\%$

4 **a** 3254.224 cm^3 **b** 3240 cm^3
c 14.224 cm^3, $\approx 0.437\%$

5 **a** **i** 0.5 cm, $\approx 0.290\%$ **ii** 0.5 cm, $\approx 9.09\%$
b The estimate of her height is more accurate since the maximum percentage error is much lower.

6 **a** 72 m^2 **b** \$6120 **c** 77.08 m^2 **d** $\approx 6.59\%$
e no **f** \$7650

7 **a** 5 m^2 **b** ≈ 1.7 m **c** 5.1 m^2 **d** 2%

8 **a** 65.25 km h^{-1} **b** 4.75 km h^{-1}, $\approx 7.28\%$

9 **a** $\approx 0.840\%$ **b** $\approx 1.29\%$

10 **b** **Hint:** Let the measure have the value 1.
c $\approx 1.50 \times 10^{-4}\,\%$

11 **a** ≈ 3.22 m^2 **b** $3.0375\text{ m}^2 < A < 3.4075\text{ m}^2$
c $\approx 6.01\%$

12 **a** ≈ 251 cm^3 **b** $173\text{ cm}^3 < V < 350\text{ cm}^3$ **c** $\approx 45.1\%$

13 **a** ≈ 4.40 hours ($\approx$ 4 h 24 min 5 s) **b** $\approx 0.287\%$

REVIEW SET 3A

1 **a** accurate to $\pm\frac{1}{2}$ cm **b** 35.5 cm to 36.5 cm
c $1260.25\text{ cm}^2 < A < 1332.25\text{ cm}^2$

2 $48\text{ cm} < P < 52\text{ cm}$ **3** $43.1\text{ m}^2 < A < 52.9\text{ m}^2$

4 **a** \$590, $\approx 22.8\%$ **b** 0.109 cm, $\approx 0.417\%$
c 386 people, $\approx 8.80\%$

5 $\approx 0.0811\%$

6 **a** $\approx 6.16\ \text{m}^2$ **b** $\approx 7.07\ \text{m}^2$ **c** $\approx 0.911\ \text{m}^2$, $\approx 14.8\%$

REVIEW SET 3B

1 **a** $14.85\ \text{s} < t < 14.95\ \text{s}$ **b** $6.66\ \text{m s}^{-1} < s < 6.77\ \text{m s}^{-1}$

2 $267.5\ \text{cm}^2 < A < 355.5\ \text{cm}^2$ **3** $57.9\ \text{km} < D < 61.9\ \text{km}$

4 **a** ≈ 175 cm **b** ≈ 2.87 cm, $\approx 1.61\%$

5 **a** **i** 2 m **ii** 2.24 m **iii** 2.236 m
b 2.24 m and 2.236 m

6 **a** $\approx 4.51\%$ **b** $\approx 1.32\%$ **c** $\approx 0.0507\%$
d $\approx 0.0402\%$ **e** $\approx 8.49 \times 10^{-6}\ \%$

7 **a** $\approx 38.5\ \text{cm}^2$ **b** $37.4\ \text{cm}^2 < A < 39.6\ \text{cm}^2$
c $\approx 2.92\%$

EXERCISE 4A

1 **a** \$232 **b** \$13 920 **c** \$1920

2 **a** £282.60 **b** £10 173.60 **c** £673.60

3 **a** \$1036.80 **b** \$2610 **4** €1903.28

5 **b** \$27 509.01
c **i** \$200 **ii** \$186.21
The balance of the loan is less in month 6 which means the interest paid will also be less.
d **i** \$4.02 **ii** \$6497.40 (\$6498 using technology)
e The monthly repayment was rounded up, so every month the payments have reduced the balance by a little extra.

6 **a** \$148.64 **b** \$729.28

7 **a** £490.61 **b** £15 598.73

8 **a** \$8500 **b** **i** \$518.58 **ii** \$1871.60 **iii** \$5492.57

9 **a** 6.30% p.a. **b** \$395.65

10 **a** **i** \$789.19 **ii** \$512.92 **iii** \$395.92
b The 3 year loan charges the least interest of \$3410.84 as more is paid off each month and therefore less interest is charged.

11 **a** €386.90 **b** €5214 **c** €10 169.13
d Ally pays more interest in the first $2\frac{1}{2}$ years than in the second $2\frac{1}{2}$ years.

12 **a** \$1827.33 **b** \$188 559.20 **c** \$162 745.03
d **i** \$3165.28 **ii** \$159 196.40 **iii** \$29 362.80

13 **a** \$2678.37 **b** \$418 703.37
c **i** \$2483.49
ii 4 years earlier; we assume the new rate is fixed for the remainder of the loan period.

EXERCISE 4B

1 **a** 25 years 4 months **b** \$3693.84

2 **a** 12 years 7 months **b** 3 years 1 month longer

3 **a** £5614.06
b No, he can only afford to spend £5614.06 per month. Otherwise his money will run out before he turns 84.

4 **a** \$1 094 748.09
b **i** \$8600.27 **ii** 11 years 11 months

5 **a** £11 512.29 **b** £394 007.62 **c** £1312.64

6 **a** $\$4500 \times 12 \times 20 = \$1\,080\,000$
b Maggie will earn interest on the money in the annuity account as she makes her regular withdrawals.
c \$618 117.53

7 The money will last forever.

8 **a** 7.19% **b** **i** 2 years 10 months **ii** €679.24

9 **a** \$5121.03 **b** \$322 605.07 **c** \$6708.44

10 **a** €3163.24 **b** €413 500.41
c **i** €3378.64 **ii** 9 years 3 months

REVIEW SET 4A

1 **a** \$455.43 **b** \$27 325.80 **c** \$4325.80

2 **a** €157.24 **b** €1086.93

3 **a** \$2884.74
b Total repayments $= \$2884.74 \times 12 \times 25 = \$865\,422$
Total interest charged $= \$865\,422 - \$410\,000 = \$455\,422$

4 **a** 8 years 7 months **b** \$2996.23

5 **a** \$799 813.28 **b** \$314 877.35

6 **a** €7861.43 **b** 14 years 3 months **c** €727 698.90
d **i** 9 years 7 months **ii** €7645.37

REVIEW SET 4B

1 **a** \$279.08 **b** \$1395.84

2 **a** **i** 11 742.52 pesos **ii** 8286.45 pesos
b The 4 year loan charges the least interest of 63 640.96 pesos as more is paid off each month and therefore less interest is charged.

3 **a** \$17 500 **b** \$1260.97 **c** \$9347.67
d **i** \$1269.19 **ii** \$65.76

4 **a** An annuity fund is an investment where an individual makes a lump-sum deposit, and then makes regular *withdrawals* from the account. We have previously considered compound interest investments that make regular *deposits* into an account.
b Diane is technically correct, but she will be able to withdraw more than £2000 per month since the money in the fund will earn interest.
c £3167.02

5 **a** €2467.29 **b** €448.52

6 **a** 4.90% p.a. **b** 6 years 7 months

EXERCISE 5A

1 **a** We have assumed that the cyclist travelled at a constant speed of $30\ \text{km h}^{-1}$ the entire time. This is not realistic, the cyclist will travel at different speeds uphill, downhill, and on flat ground.
b 60 km

2 **a** Briony constructed her model by finding the equation of the line through $(0,\ 8)$ and $(12,\ 23)$. Briony has assumed that the laptop will charge at a constant rate, and indefinitely. These assumptions are not realistic, but they may be satisfactory for this problem.
b $0 \leqslant C \leqslant 100, \quad t \geqslant 0$
Briony's model suggests that it is possible to have charge greater than 100%, which is not possible.
c 73.6 minutes
d **i** The laptop likely charges at a faster rate earlier on, then at a slower rate as it approaches a full charge.
ii Yes, 73.6 minutes was a useful estimate.

3 **a** $t = \frac{3}{20}d$ seconds **b** 75 seconds
c Rick will take a longer time to run 500 m than our prediction. He will not be able to run 500 m at the same pace that he runs 100 m.

4 **a** **C**, it is usually coldest at dawn, and warmest in the afternoon. Daily temperature should be roughly periodic.
b $\approx 28°\text{C}$

5 **a** **B**, most light globes last for 200 hours, then the number of working globes quickly decreases.

b ≈ 95 light globes

6 **a** Darren has assumed that the Earth is perfectly spherical, and that the lighthouse is perpendicular to the Earth's surface. These are reasonable assumptions.

b $D \approx 22.6$ km

7 **a** $h = (\frac{2}{15}x + 3)$ m; we have assumed that the power line is hanging perfectly straight and that the poles are directly opposite.

b $0 \leqslant x \leqslant 15$ **c** 4 m

d Yes, if the poles are 3 m and 5 m high, we expect the height of the power line above the middle of the road to be 4 m.

e Yes, provided he knows the width and height of the truck, he should be able to reasonably decide whether or not the truck can fit.

8 **a** **i** ≈ 71.8 seconds **ii** ≈ 51.7 seconds

We have assumed that there is no current, and that Antonio can swim/jog at a constant speed, in all situations.

b Swimming to C then jogging to B appears to be quicker.

c Swim to the point 16.2 m from C and jog 83.8 m to B, total time ≈ 49.7 seconds.

9 $4\frac{8}{19}$ hours (≈ 4 h 25 min) **10** $9\frac{3}{8}$ minutes (≈ 9 min 23 s)

11 15 hours **12** 6 hours **13** $1\frac{11}{13}$ hours (≈ 1 h 51 min)

EXERCISE 5B

1 **a** $C = 2.5x$

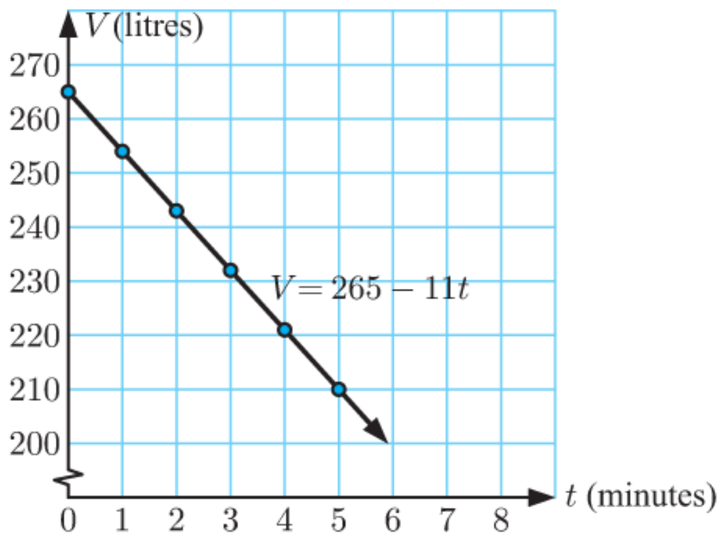
b The model fits the data points exactly.
$\therefore$ the model is exact.

c yes, \$30

2 **a**

Time (t minutes)	0	1	2	3	4	5
Volume (V litres)	265	254	243	232	221	210

b



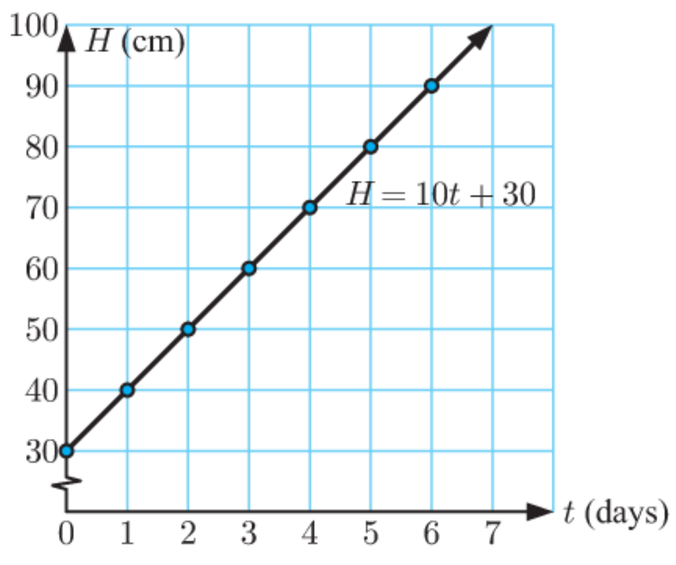
c $V = 265 - 11t$

d **i** 100 litres **ii** $\frac{265}{11} \approx 24.1$ minutes

3 **a**

t (days)	0	1	2	3	4	5	6
H (cm)	30	40	50	60	70	80	90

b



c It does not make sense to extend the model for $t < 0$ since the number of days after planting cannot be negative. It does not make sense to extend the model indefinitely for $t > 6$ since the plant will not continue to grow at this rate forever. However, we will assume that it does for the purposes of this model. $H(t)$ has domain $\{t \mid t \geqslant 0\}$.

d $H = 10t + 30$ **e** 7 days

4 **a** The points do not lie exactly on the line.
$\therefore$ the model is approximate.

b ≈ 160 pieces of litter; this estimate is likely to be inaccurate as it is an extrapolation. There would probably not be this much litter for Jack to pick up.

5 **a** 54.85 s, while the actual time was 57.0 s.
The values differ since the model is not exact. Some times will be higher or lower than what is predicted.

b The model would not be accurate for years much earlier than 1957, as the model would begin to predict unrealistically large times. The model should still be useful in estimating times a few years prior to 1957.

c 2005

d It is not suitable to extrapolate future times. Improvements in time no longer follow a linear trend.

EXERCISE 5C.1

1 **a** base rate: gradient $= 2$; the first 25 kL of water costs \$2 per kL.
higher rate: gradient $= 4$; each kL used above 25 kL costs \$4 per kL.

b \$160 **c** 15 kL

2 **a** 10 GB **b** \$5 per GB **c** \$45 **d** 18 GB

3 **a** The car was initially travelling at 10 m s^{-1}, then braked for 6 seconds to a complete stop. The car stopped for 5 seconds before accelerating for 4 seconds to a speed of 7 m s^{-1} which it maintained for the remaining 5 seconds.

b We have assumed that the car accelerates and decelerates at a constant rate. These assumptions are reasonable.

c **i** 5 m s^{-1} **ii** 30 m **d** 79 m

4 **a**



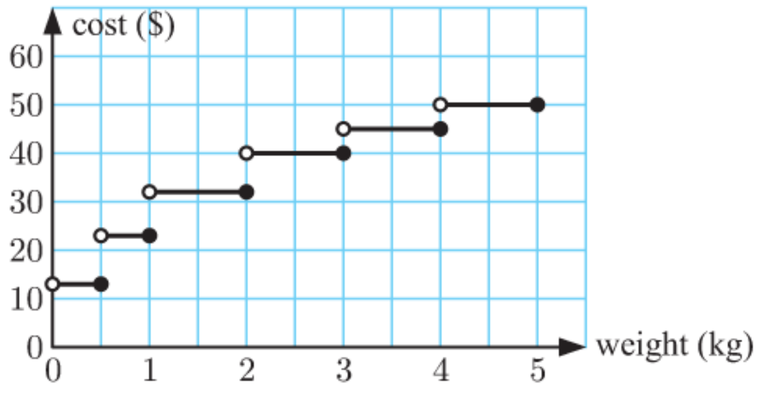
b \$45

c 17 kg

5 **a** **i** \$75 **ii** \$95 **b** **i** 15 min **ii** 60 min

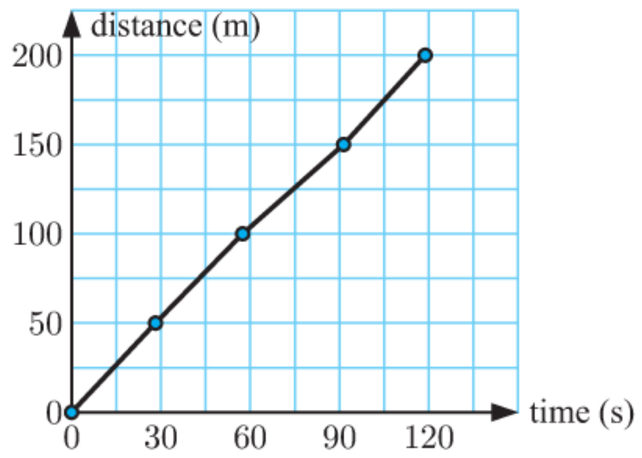
c \$155, assuming that the pattern shown in the graph continues.

6 **a**



b **i** \$23 **ii** \$32 **iii** \$45 **c** \$22

7 **a**



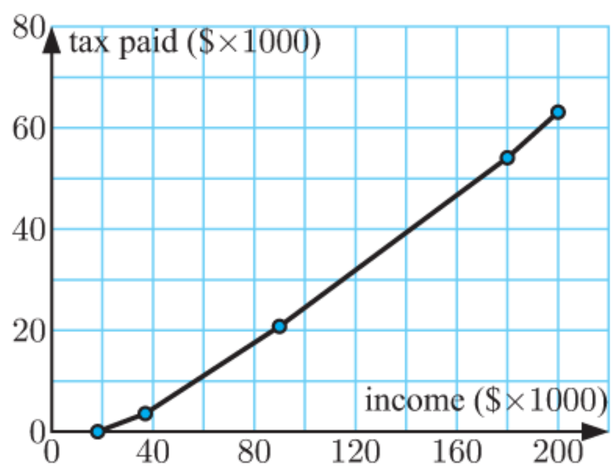
b We assume Laura swims at a constant rate in each 50 m leg.

c ≈ 104 m

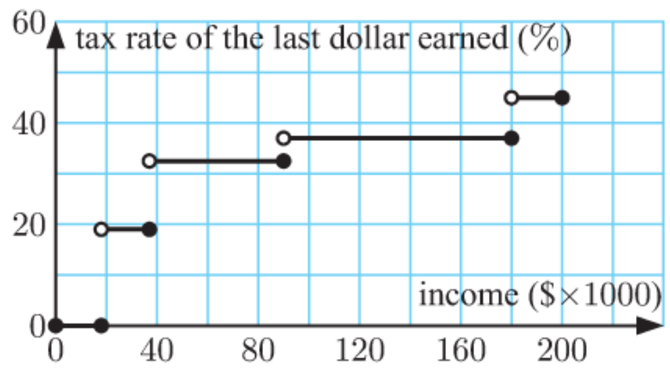
8 **a** £448.80 **b** **i** £9628.80 **ii** £43 543.20

c **i** £50 118.15 **ii** $\approx 32.6\%$

9 **a**



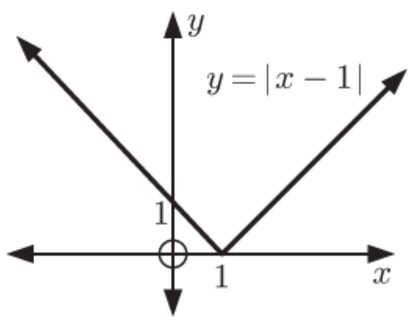
b



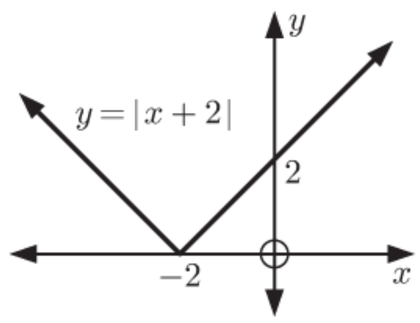
c **i** \$1862 **ii** \$7244.50 **iii** \$23 072.50

EXERCISE 5C.2

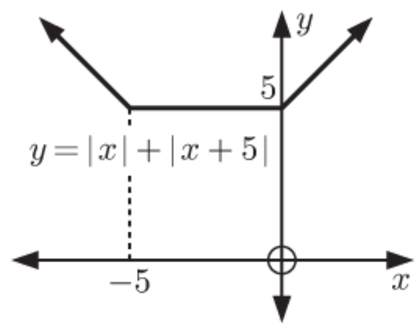
1 **a**



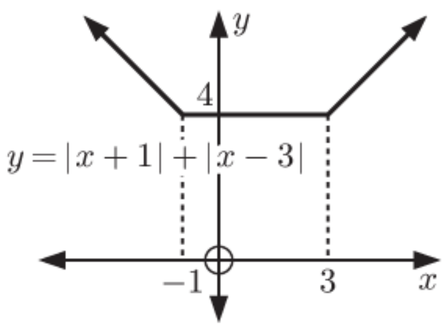
b



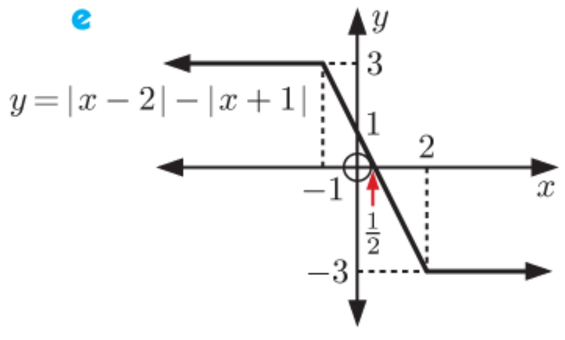
c



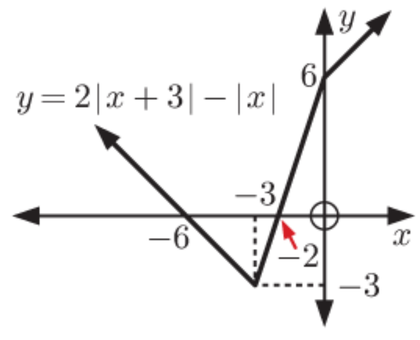
d



e



f



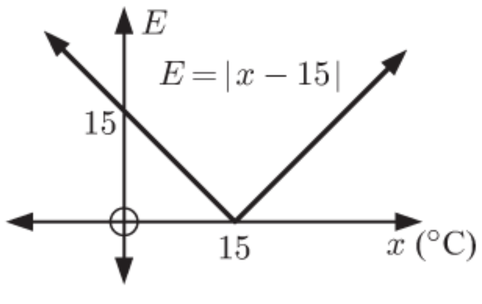
2 **a** $E = |x - 15|$

b **i** $E = 3$

ii $E = 0$

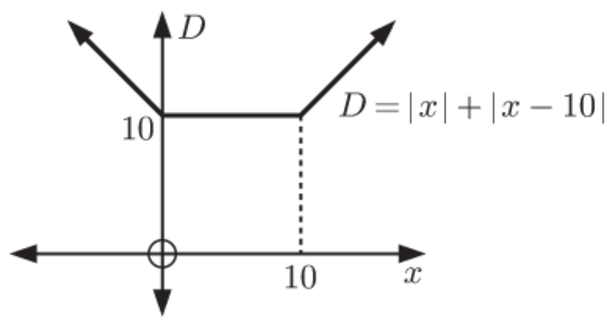
iii $E = 2$

c



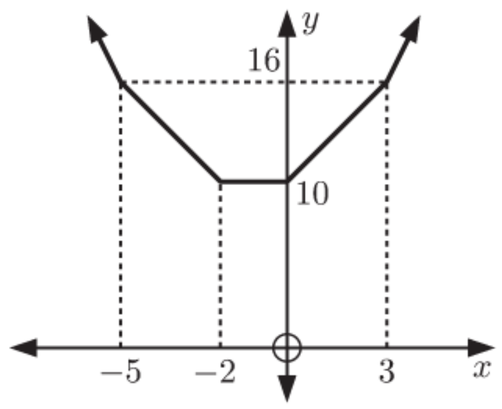
3 **a** $D = |x| + |x - 10|$ **b** **i** $D = 10$ **ii** $D = 16$

c



d $0 \leqslant x \leqslant 10, \quad D = 10$

4 **a**



b **i** The total distance from x to the four beaches O, P, Q, and R is $|x+5| + |x+2| + |x| + |x-3|$ km.

$\therefore$ the average distance to any beach is

$\frac{1}{4}(|x+5| + |x+2| + |x| + |x-3|)$ km.

ii Anywhere between O and Q inclusive, which corresponds to the minimum of our graph in **a**.

iii We minimise

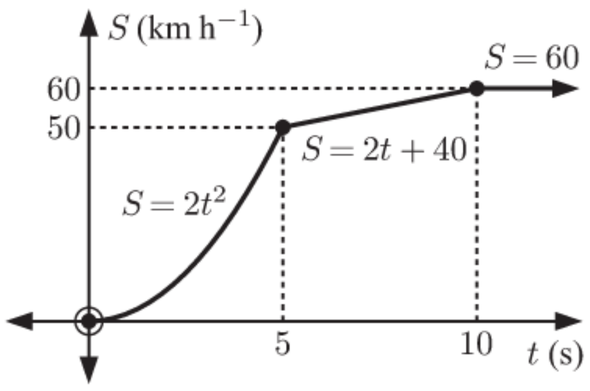
$y = \frac{1}{5}(|x+5| + |x+2| + |x| + |x-3| + |x-7|)$.

The ambulance service should be located at O.

EXERCISE 5C.3

1 **a** 30°C **b** 10 minutes **c** 25° C **d** 30 minutes

2 **a**



b 60 km h^{-1}

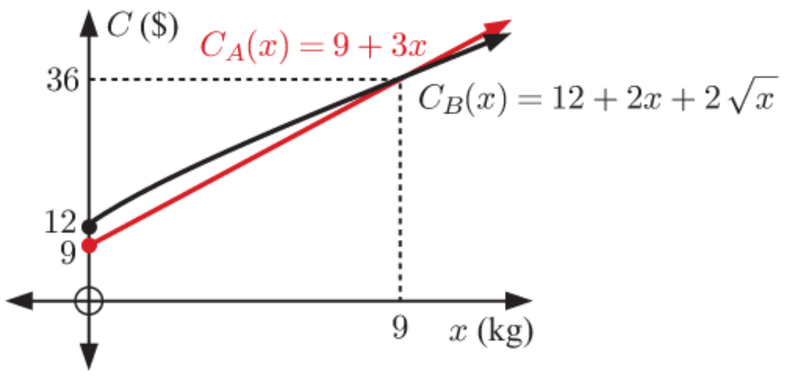
c **i** 18 km h^{-1}

ii 56 km h^{-1}

iii 60 km h^{-1}

d 4 seconds

3 **a**



b $C = \begin{cases} 9 + 3x, & 0 \leqslant x < 9 \\ 12 + 2x + 2\sqrt{x}, & x \geqslant 9 \end{cases}$

c **i** \$27 **ii** \$36

4 **a** $c = 45, \quad d = 50, \quad e = -105$

b 45 m **c** 20 m **d** ≈ 2.45 s, 4 s, and 6 s

EXERCISE 5D

1 **a** $y = \frac{1}{2}x + 1$

2 **a** $a + b + c = 7$
$4a + 2b + c = 10$
$9a + 3b + c = 11$

b $a = -1, \quad b = 6, \quad c = 2$

c $y = -x^2 + 6x + 2$

3 **a** $y = x^2 + 3x - 6$ **b** $y = -2x^2 + 4x + 9$

4 We find that $a = 0, \quad b = -2, \quad$ and $\quad c = 5$. So the three points lie on the line $y = -2x + 5$, but not on a quadratic.

5 **a** $y = -3x^2 + 10x$ **b** $y = x^3 - 4x^2 + 5x$

c $y = 2x + \dfrac{6}{x} - 3$

6 $a = 0.2,\ b = 0.5,\ c = 1$

7 **a** $h = \dfrac{36}{r} - r$

b **Hint:** Make h the subject in the equation of the surface area of a cylinder.

c 72π cm^2

d $h = -5$ cm, which is not reasonable, as height cannot be negative.

e 0 cm $< r <$ 6 cm

8 **a** $y = \frac{1}{125}x^2 - \frac{4}{5}x + 50$ **b** 37.2 m **c** $0 \leqslant x \leqslant 100$

9 **a** $d = 5.209$ **b** $a = 0.003,\ b = -0.117,\ c = 0.921$

c 0.829 m

REVIEW SET 5A

1 **a** $D = \frac{5}{2}t$ m **b** 1500 m

c The actual distance will be less than our prediction. Ben will not be able to kayak at the same speed for 10 minutes as he can for 40 seconds.

2 **a** $1\frac{1}{2}$ days

b The assumption is not reasonable, the job will take more time than our prediction in **a**.

3 **a** **A** **b** $\approx 40°$C

4 **a** gradient $= -5$, A-intercept $= 160$

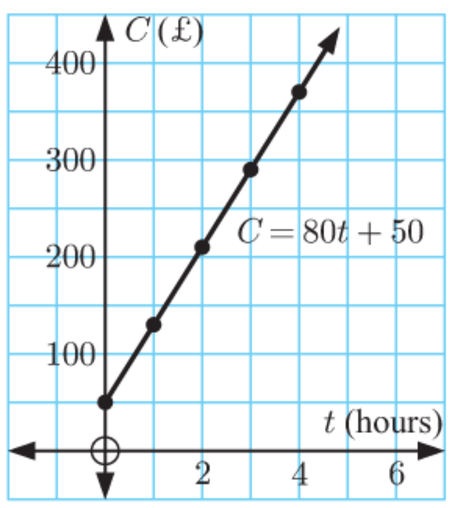
The barrel initially contained 160 L of oil and is leaking at a rate of 5 L per minute.

b $A = 160 - 5t$ **c** 85 litres **d** $0 \leqslant t \leqslant 32$

5 **a**

Time (t hours)	0	1	2	3	4
Cost (£C)	50	130	210	290	370

b

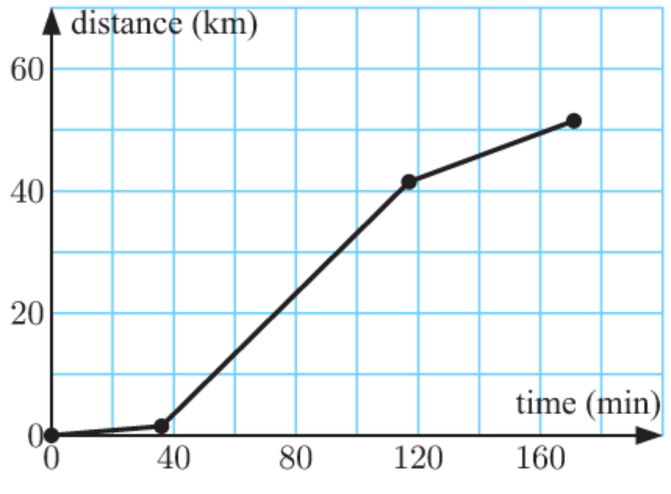


c $C = 80t + 50$

d £530

6 **a** **i** \$25 **ii** \$35 **b** 1 hour

7 **a**



b We have assumed that Alana travels at a constant speed during each leg of the race.

c $\approx$ 13 km

8 **a** $L = |x| + |x - 2|$ **b** **i** $L = 2$ **ii** $L = 4$

c $0 \leqslant x \leqslant 2,\ L = 2$ **d** The house should be built at G.

9 $y = \frac{1}{2}x^2 + 3x - 4$

10 **a** $y = 3x + \dfrac{4}{x}$ **b** $y = 2x^3 - 4x^2 + 3x$

11 **a** $a = \frac{7}{100},\ b = \frac{6}{5}$ **b** 99 m

c No, braking distance is dependent on the car and the reaction time of the driver.

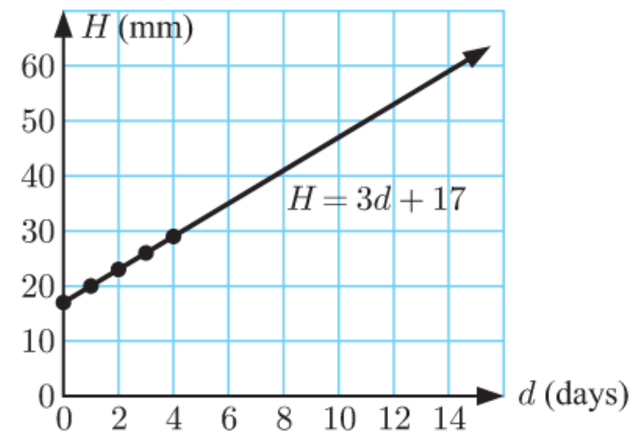
REVIEW SET 5B

1 $1\frac{8}{13}$ hours ($\approx$ 1 h 37 min), assuming that Todd and Sophie can work together at the same rate that they could individually.

2 **a**

d (days)	0	1	2	3	4
H (mm)	17	20	23	26	29

b



c $H = 3d + 17$ **d** 53 mm **e** every 21 days (3 weeks)

3 **a** Yes, these are reasonable assumptions. Not all of the wood in a tree is usable, and assuming that the trees are cylindrical makes it easier to perform calculations.

b $V = 0.8 \times \pi(0.225)^2 h$ m^3 or $V = \dfrac{81\pi h}{2000}$ m^3

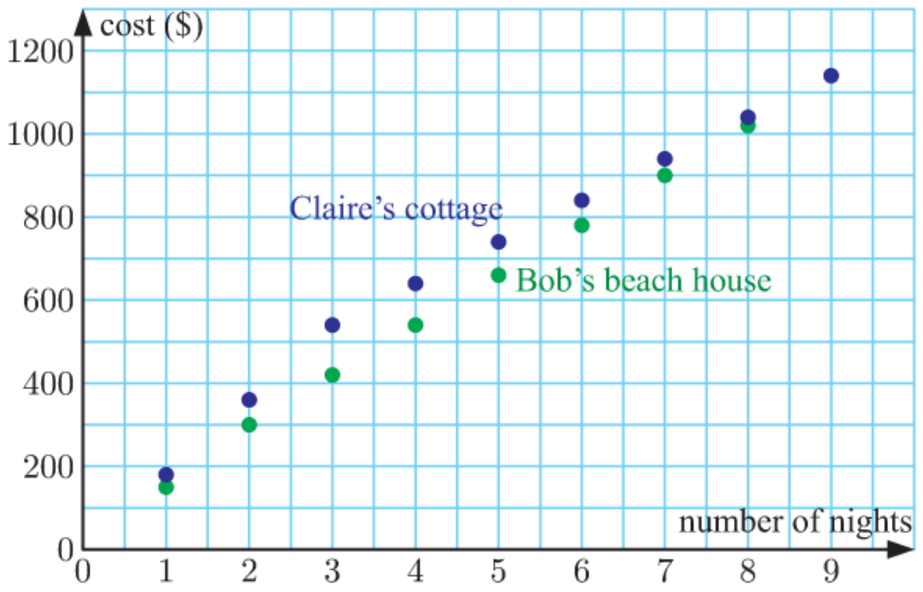
c ≈ 1.91 m^3

4 **a** The points do not lie exactly on the line.

$\therefore$ the model is approximate.

b 2040. This is an extrapolation, so this prediction is not likely to be reasonable.

5 **a**

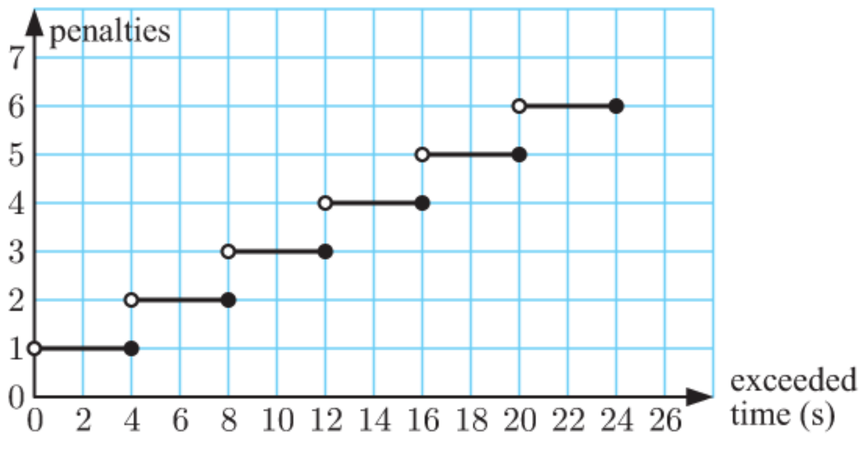


b **i** \$540 **ii** \$640 **c** 9 nights

d Bob's Beach House will be cheaper by \$20.

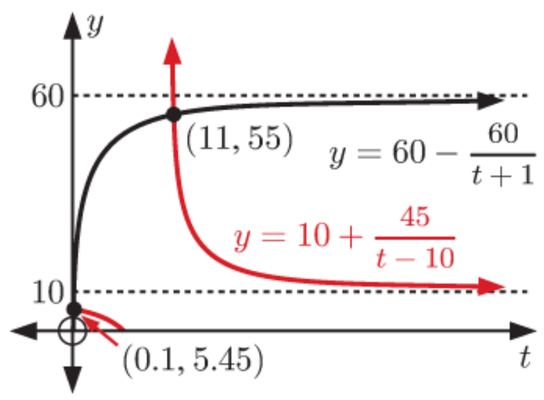
6 **a** piecewise linear model **b** $\approx 32.7\%$ **c** €36 841.48

7 **a**



b **i** 1 time penalty **ii** 2 time penalties

iii 0 time penalties **iv** 4 time penalties

8 **a** $k = 60$ **b** 

c $T = 11$ s
d 60 m s^{-1}
e **i** 50 m s^{-1} **ii** 19 m s^{-1}

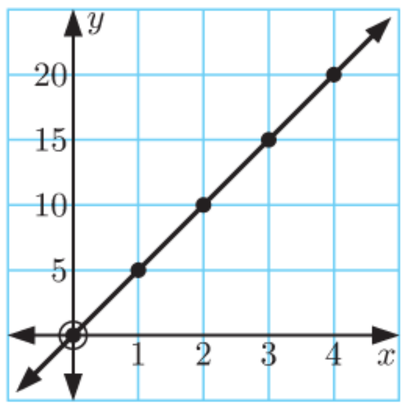
9 **a** $y = -\frac{1}{80}x^2 + x + \frac{1}{4}$ **b** 20.25 m **c** yes

10 **a** $y = x + 2\sqrt{x}$ **b** $y = -x^2 + 5x + \frac{8}{x}$

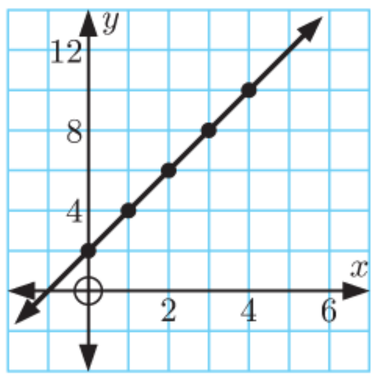
11 **a** If $x = 0$, the metal cannot be folded into a tray, and hence $V = 0$. $\therefore$ $d = 0$
b $a = 4$, $b = -120$, $c = 900$ **d** $0 < x < 15$

EXERCISE 6A

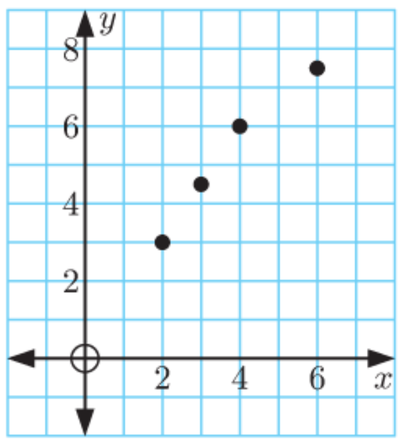
1 **D**

2 **a** **i** 

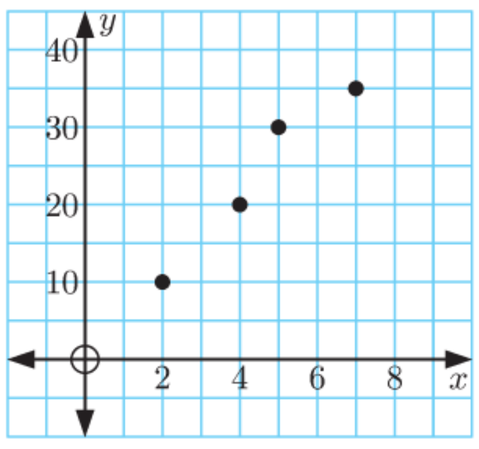
ii $y \propto x$, $k = 5$

b **i** 

ii y is not directly proportional to x.

c **i** 

ii y is not directly proportional to x.

d **i** 

ii y is not directly proportional to x.

3 **a** **i** €6 **ii** €12 **iii** €18 **iv** €24
b The graph of C against t is not a straight line through the origin.
$\therefore$ C and t are not directly proportional.

4 **a** y is doubled **b** x is multiplied by 6
c y is divided by 3 **d** x is increased by 20%
e y is decreased by 70% **f** x is increased by $\frac{10}{k}$

5 When the temperature in °C is 0, the temperature in °F is 32, so the graph of these variables does not pass through the origin.
$\therefore$ these variables are not directly proportional.

6 **a** directly proportional **b** not directly proportional
c directly proportional **d** not directly proportional
e not directly proportional

7 **a** $y = 72$ **b** $x = \frac{35}{4}$

8 **a** $k = 1.8$, the heater uses 1.8 kWh of energy each hour.
b 2.7 kWh **c** 3 h 20 min

9 **a** $k = 20$, 1 cm on the graph represents an actual length of 20 m.
b 5.5 cm **c** 300 m

10 **a** 366 kg m s^{-1} **b** ≈ 3.44 m s^{-1}

11 **a** **i** The graph of C against t has C-intercept 30, so does not pass through the origin.
$\therefore$ C and t are not directly proportional.
ii The graph of $(C - 30)$ against t passes through the origin and is a straight line.
$\therefore$ $(C - 30)$ and t are directly proportional.
b £330

EXERCISE 6B

1 **a** $A \propto r^2$, $k = \pi$ **b** $V \propto r^3$, $k = \frac{4}{3}\pi$
c $T \propto n^4$, $k = \frac{3}{4}$

2 **a** y is multiplied by 8 **b** y is divided by 1000
c y is increased by 72.8% **d** x is multiplied by $\sqrt[3]{2.5}$
e x is decreased by $\approx 11.2\%$ **f** x is increased by $\approx 26.0\%$

3 **a** $M = 90$ **b** $t = 6\sqrt{3} \approx 10.4$

4 **a** $V = 1920$ **b** $y = 3\sqrt[3]{6} \approx 5.45$

5 2500 g **6** **a** ≈ 11.9 mL **b** ≈ 5.45 cm

7 **a** $\approx 15.8\%$ increase **b** $\approx 26.0\%$ increase

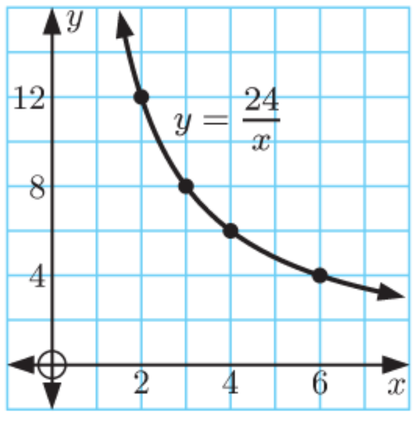
8 **a** $E \propto m$ **b** $E \propto v^2$ **c** E decreases by 19%
d Stopping distance $d \propto E$
and $E \propto v^2$
$\therefore$ $d \propto v^2$

9 $y \propto z^{\frac{1}{4}}$

EXERCISE 6C

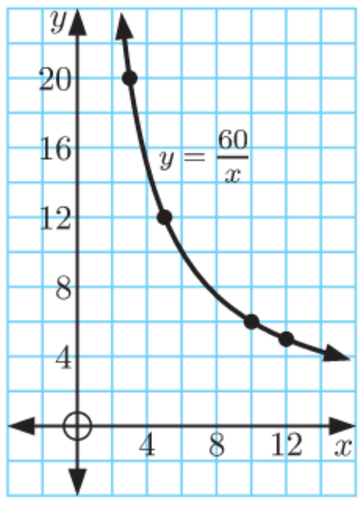
1 **B**

2 **a** $xy = 24$ for each point.
$\therefore$ x and y are inversely proportional, $y = \frac{24}{x}$.



b $xy = 20$ when $x = 1$, 2, or 5, and $xy = 24$ when $x = 4$.
$\therefore$ x and y are not inversely proportional.

c $xy = 60$ for each point.
$\therefore$ x and y are inversely proportional, $y = \dfrac{60}{x}$.

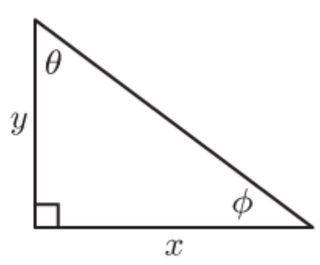


3 **a** No, $\theta = 90 - \phi$. There is no constant k such that $\theta \times \phi = k$.

b $\tan\theta = \dfrac{x}{y}$ and $\tan\phi = \dfrac{y}{x}$

$\therefore$ $\tan\theta = \dfrac{1}{\tan\phi}$

$\therefore$ $\tan\theta$ and $\tan\phi$ are inversely proportional.



4 **a** y is halved **b** y is multiplied by 7
c y is multiplied by $\frac{5}{9}$ **d** y is decreased by $\approx 23.1\%$

5 **a** $C = 5$ **b** $t = 4.5$ **6** 10 hours

7 **a** $3.75\ \text{m s}^{-2}$ **b** 0.75 kg

8 **a** If the *share price* is multiplied by a constant, then the *number of shares* Wendy can buy will be divided by the same constant.
$\therefore$ *number of shares* is inversely proportional to *share price*.

b **i** $\frac{125}{122}n$ shares (rounded down) **ii** $33\frac{1}{3}\%$

EXERCISE 6D

1 **a** y is divided by 8 **b** y is multiplied by $\frac{125}{27}$
c x is divided by 4

2 **a** $y = 3$ **b** $x = \frac{24}{5}$ **3** **a** $M = 216$ **b** $c = 48$

4 **a** $V = \pi r^2 h$

$\therefore$ $h = \dfrac{V}{\pi} \times \dfrac{1}{r^2}$

$\therefore$ $h \propto \dfrac{1}{r^2}$ {since V and π are constants}

b ≈ 10.4 cm **c** ≈ 2.79 cm
d The cans would otherwise be too small or too large to be practical for use.

5 **a** $\approx 24.9\%$ decrease **b** $\approx 30.7\%$ decrease

6 $z \propto x^2$ (or $x \propto \sqrt{z}$)

EXERCISE 6E

1 **a** $H = \dfrac{d^2}{16}$ **b** 16 m

2 **a** $m = \dfrac{l^3}{20\,000}$ **b** 6.25 kg **c** ≈ 27.1 cm

3 **a** $xy = 10$ for all 3 points, so it is reasonable to assume that y varies inversely with x.

b $y = \dfrac{10}{x}$ **c** $y = \frac{5}{4}$

4 **a** $x^2y = 5$ for all values in the table **b** $x = \sqrt{10}$

$\therefore$ $y = \dfrac{5}{x^2}$

so $k = 5$

5 **a** **i** $R = \dfrac{v^2}{200}$

ii When $v = 20$, $R = \dfrac{20^2}{200} = 2 \neq 4$

When $v = 30$, $R = \dfrac{30^2}{200} = 4.5 \neq 13.5$

When $v = 40$, $R = \dfrac{40^2}{200} = 8 \neq 32$

$\therefore$ this model is incorrect.

b When $v = 10$ and $R = 0.5$, $\dfrac{R}{v^3} = \dfrac{0.5}{10^3} = \dfrac{1}{2000}$

When $v = 20$ and $R = 4$, $\dfrac{R}{v^3} = \dfrac{4}{20^3} = \dfrac{1}{2000}$

When $v = 30$ and $R = 13.5$, $\dfrac{R}{v^3} = \dfrac{13.5}{30^3} = \dfrac{1}{2000}$

When $v = 40$ and $R = 32$, $\dfrac{R}{v^3} = \dfrac{32}{40^3} = \dfrac{1}{2000}$

$\dfrac{R}{v^3} = \frac{1}{2000}$ for all data points, so $R = \frac{1}{2000}v^3$.

c 62.5 units **d** A reduction of $\approx 7.17\%$.

EXERCISE 6F

1 **a** $y \approx 0.602x^4$ **b** $y \approx \dfrac{799}{x^3}$

2 **a** Inverse variation, as the points appear to lie on a curve which is asymptotic to both axes.

b $P \approx \dfrac{451}{x^2}$ **c** $P \approx 28.2$

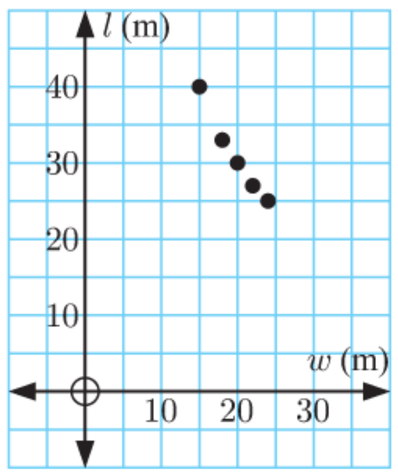
3 **a** Direct variation, as the graph of C against t should pass through the origin, and we expect the charge to increase as time increases.

b $C \approx 1.25t$ **c** $\approx 70\%$

d $0 \leqslant t \leqslant 80$, as the battery charge must lie between 0 and 100%.

4 **a** Inverse variation, as $lw \approx 600$ for all l and w.

b



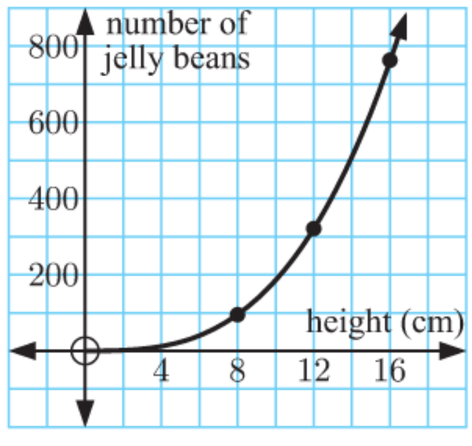
Yes, the points appear to lie on a curve which is asymptotic to both axes, as expected for inverse variation.

c $l \approx \dfrac{602}{w}$ **d** ≈ 26.2 m

e $15 \leqslant w \leqslant 24.5$, so the width of the block is practical for building a house, and the block is no wider than it is long.

5 **a** $R \approx 0.0197s^{2.29}$ **b** ≈ 3.84 m **c** $\approx 10.2\ \text{m s}^{-1}$

6 **a**



b no

c $V = \pi r^2 h = \dfrac{\pi h^3}{4}$ (since $r = \dfrac{h}{2}$)

$\therefore\ V \propto h^3$ $\{\frac{\pi}{4}$ is constant$\}$

The volume, and hence number of jelly beans, is proportional to the cube of the jar height.

d number of jelly beans, $N \approx 0.183h^3$ **e** 1464 jelly beans

7 **a** $F \approx \dfrac{5.63}{d^2}$ **b** ≈ 35.2 N **c** ≈ 0.0931 m

8 **a** $P \approx 2.93 \times T^{0.0521}$

Since multiplying the values of P does *not* result in the values of t being multiplied by the same number, then P is *not* directly proportional to T.

b **i**

Temperature (T K)	278.15	283.15	288.15	293.15
Pressure ($P \times 10^5$ Pa)	3.22	3.28	3.33	3.39

Temperature (T K)	298.15	303.15	308.15
Pressure ($P \times 10^5$ Pa)	3.45	3.51	3.57

ii $P \approx 0.0112T$

iii Yes, our power model suggests that $P \propto T$, where T is measured in kelvin.

REVIEW SET 6A

1 **C** **2** **a** A is multiplied by 4 **b** t is increased by 5%

3 ≈ 106 N

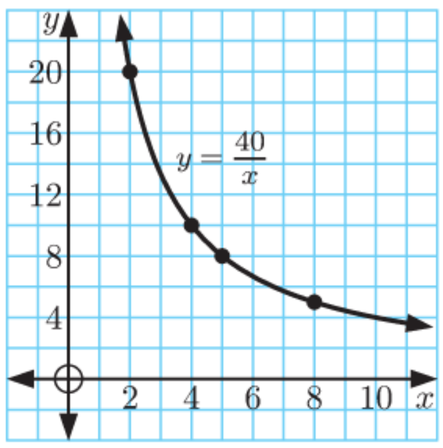
4 **a** $y \propto x^2,\ k = 5$ **b** $P \propto n^4,\ k = \frac{2}{3}$

c $V \propto a^3,\ k = \frac{\sqrt{5}}{4}$

5 **a** $y = \frac{15}{32}x^2$ **b** **i** $y = 7.5$ **ii** $x = 8\sqrt{5} \approx 17.9$

6 **a** $xy = 40$ for each point.

$\therefore\ x$ and y are inversely proportional, $y = \dfrac{40}{x}$.



b $xy = 60$ when $x = 3$, 5, or 10, and $xy = 64$ when $x = 8$. $\therefore\ x$ and y are not inversely proportional.

7 625 THz **8** **a** ≈ 0.516 ohms **b** ≈ 0.321 cm

9 **a** $y = \frac{2}{3}x^2$ **b** $y = 80\frac{2}{3}$

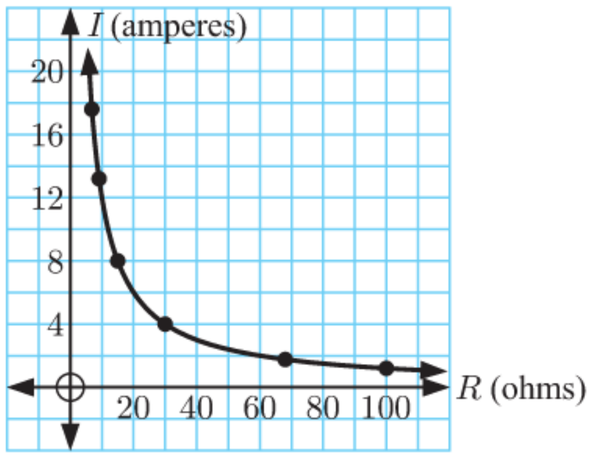
10 **a** **Hint:** Write the base lengths l in terms of h.

b $V = \frac{2}{5}h^3$ **c** $\frac{2}{5} \times 4^3 = 25.6$ ✓ $\frac{2}{5} \times 6^3 = 86.4$ ✓

d **i** $V = 204.8$ **ii** $h = 5$

11 **a** $y \approx 1.50x^3$ **b** $y \approx \dfrac{500}{x^4}$

12 **a**

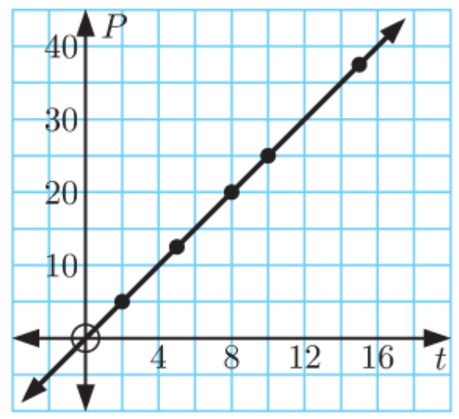


b Inverse variation, as $RI \approx 120$ for each point. The points appear to lie on a curve which is asymptotic to both axes.

c $I \approx \dfrac{120}{R}$ **d** ≈ 0.48 amperes

REVIEW SET 6B

1 **a**



b The graph of P against t is a straight line which passes through the origin.

c $P = 2.5t$

2 **a** 2.9 N **b** 1.8 A **3** **B**

4 **a** $A = \pi r^2$, π is a constant

$\therefore\ A \propto r^2$

b 80.625 kg **c** ≈ 23.1 cm

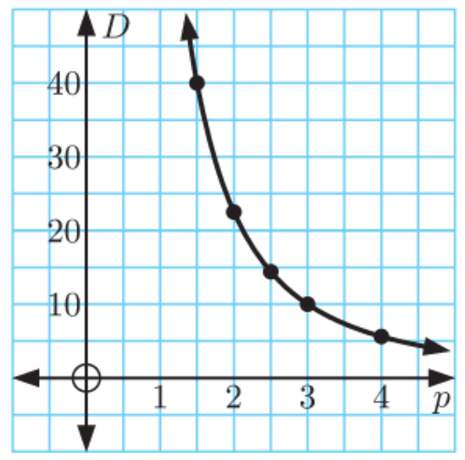
5 **a** $y = \dfrac{3456}{x^3}$ **b** **i** $y = 54$ **ii** $x = -12$

6 6.75 days **7** decreases by $\approx 8.71\%$

8 **a** $y = \dfrac{28}{x}$ **b** $y = 280$

9 **a** $P = \dfrac{262}{42\,875}v^3$ **b** **i** ≈ 557 W **ii** ≈ 32.3 km h^{-1}

10 **a**



b We see that $Dp^2 = 90$ for each point.

The points appear to lie on a curve which is asymptotic to both axes, so the model $D = \dfrac{k}{p^2}$ seems appropriate.

c $k = 90$ **d** $D = 3.6$

11 **a** Direct variation, as the graph of y against x appears to pass through the origin and is increasing.

b $y \approx 0.753x^2$

12 **a**

d (m)	1	5	10	15	20
I (W m^{-2})	63.7	2.55	0.637	0.283	0.159
$I \times d$	63.7	12.75	6.37	4.245	3.18

b $I \approx \dfrac{63.7}{d^2}$ **c** $\approx 49.0\%$ decrease

EXERCISE 7A

1 **a** weak, positive, linear correlation, with no outliers

b strong, negative, linear correlation, with one outlier

c no correlation

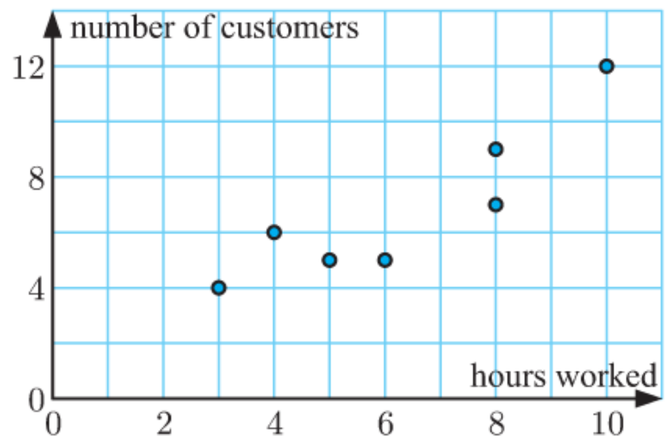
d strong, negative, non-linear correlation, with one outlier

e moderate, positive, linear correlation, with no outliers

f weak, positive, non-linear correlation, with no outliers

2 **a** *Hours worked* is the explanatory variable.
Number of customers is the response variable.

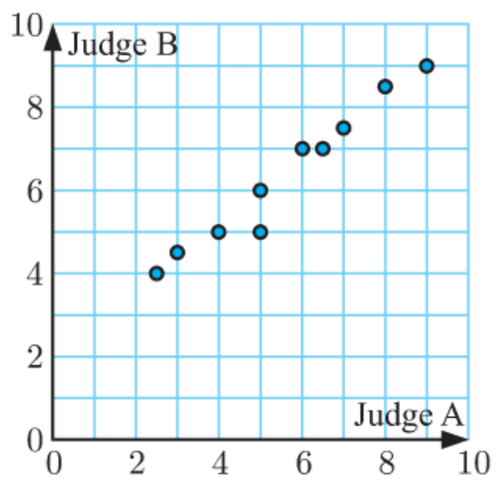
b



c **i** Monday and Friday **ii** Wednesday and Sunday

d The more hours that Tiffany works, the more customers she is likely to have.

3 **a**

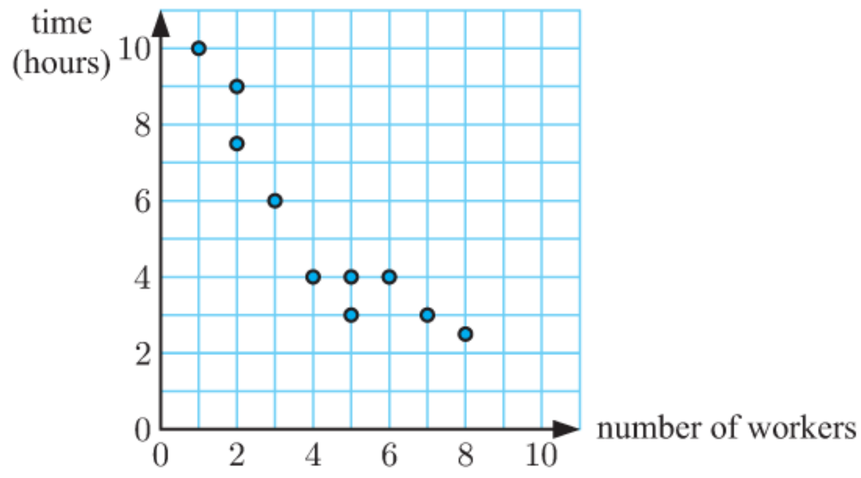


b There appears to be **strong, positive**, **linear** correlation between Judge A's scores and Judge B's scores. This means that as Judge A's scores increase, Judge B's scores **increase**.

c No, the scores are related to the quality of the ice skaters' performances.

4 **a** **i** job G **ii** job C

b



c There is a strong, negative, non-linear correlation between *number of workers* and *time*.

5 **a** **D** **b** **A** **c** **B** **d** **C**

6 **a** There is a moderate, positive, linear correlation between *hours of study* and *marks obtained*.

b The test is out of 50 marks, so the outlier (> 50) appears to be an error. It should be discarded.

c Yes, this is a causal relationship as spending more time studying for the test is likely to cause a higher mark.

7 **a** Not causal, dependent on genetics and/or age.

b Not causal, dependent on the size/location of the fire.

c Causal, an increase in advertising is likely to cause an increase in sales.

d Causal, the childrens' adult height is determined by the genetics they receive from their parents to a great extent.

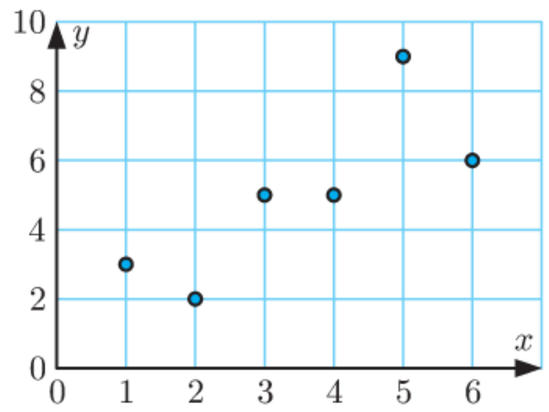
e Not causal, dependent on population of town.

EXERCISE 7B

1 weak, positive correlation

2 **a** **B** **b** **A** **c** **D** **d** **C** **e** **E**

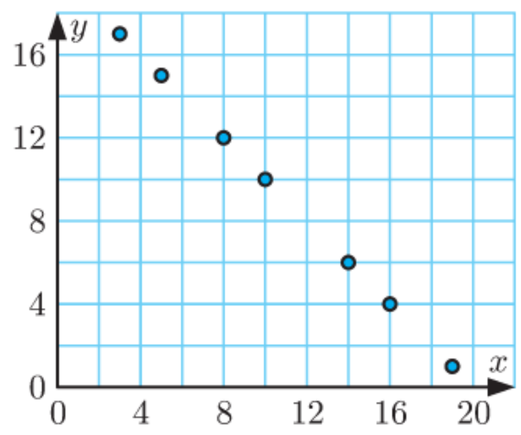
3 **a** **i**



ii $r \approx 0.786$

iii moderate, positive correlation

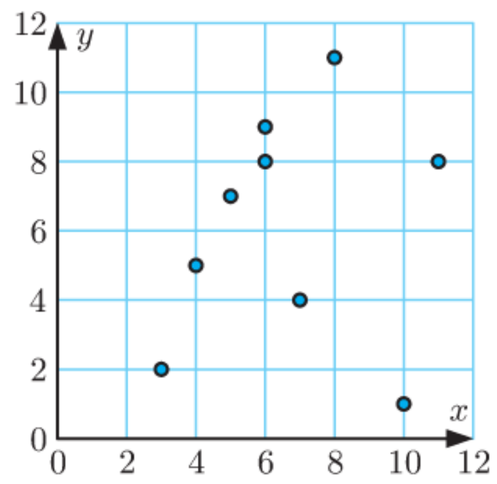
b **i**



ii $r = -1$

iii perfect, negative correlation

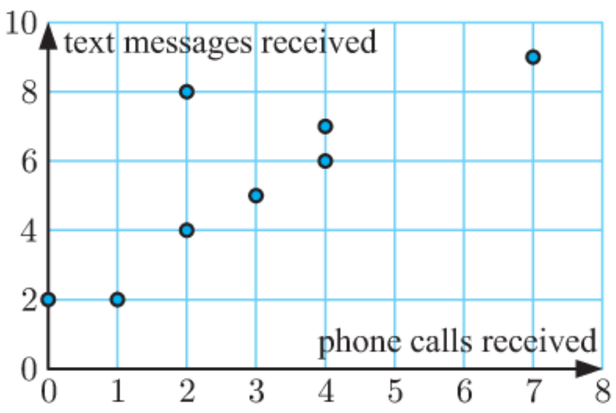
c **i**



ii $r \approx 0.146$

iii very weak, positive correlation

4 **a**



b $r \approx 0.816$

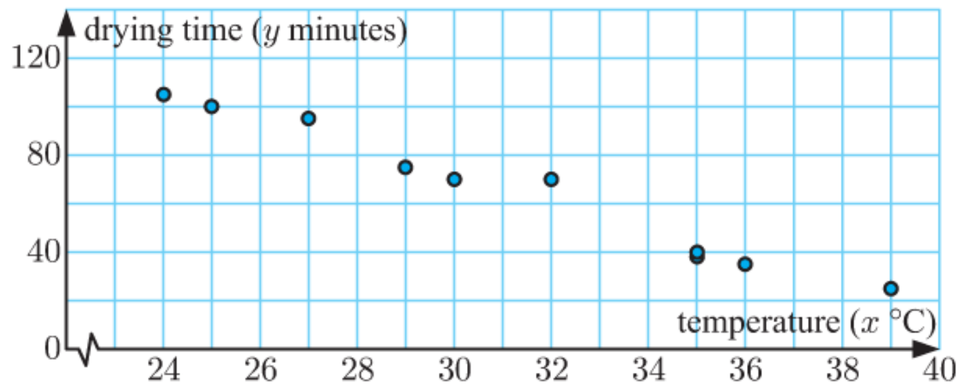
c moderate, positive correlation

d Those students who receive several phone calls are also likely to receive several text messages and vice versa.

5 **a** $r \approx 0.917$

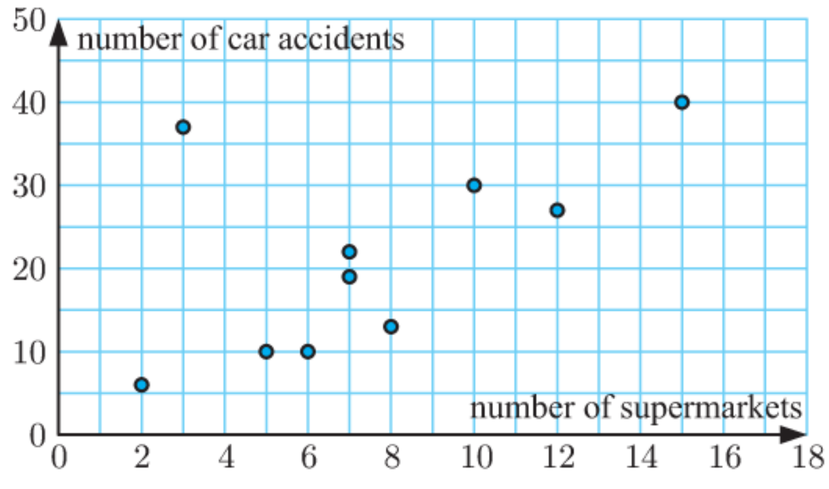
b strong, positive correlation. In general, the higher the young athlete's age, the further they can throw a discus.

6 **a**



b $r \approx -0.987$ **c** very strong, negative correlation

7 **a**



b $r \approx 0.572$

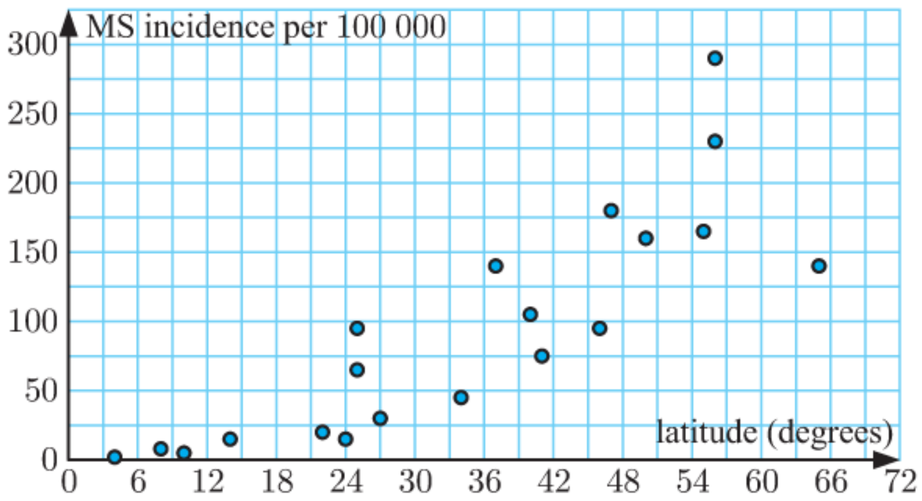
c The point $(3, 37)$, which represents 37 car accidents in a town with 3 supermarkets, is an outlier.

d **i** $r \approx 0.928$ **ii** strong, positive correlation

iii Removing the outlier had a very significant effect on the value of r.

e No, it is not a causal relationship. Both variables depend on the number of people in each town, not on each other.

8 **a**



b $r \approx 0.849$ **c** moderate, positive correlation

d The incidence of MS is higher near the poles.

EXERCISE 7C

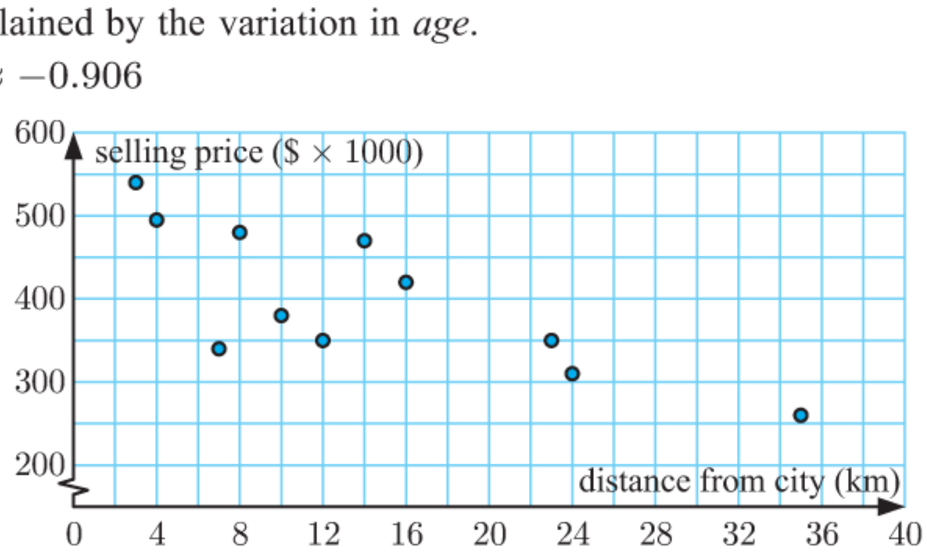
1 57.8% of the variation in the *number of visitors* can be explained by the variation in *maximum temperature*.

2 $r^2 \approx 0.598$. 59.8% of the variation in the amount of *money lost* can be explained by the variation in *time spent gambling*.

3 $r^2 \approx 0.133$. 13.3% of the variation in *heart rate* can be explained by the variation in *age*.

4 $r \approx -0.906$

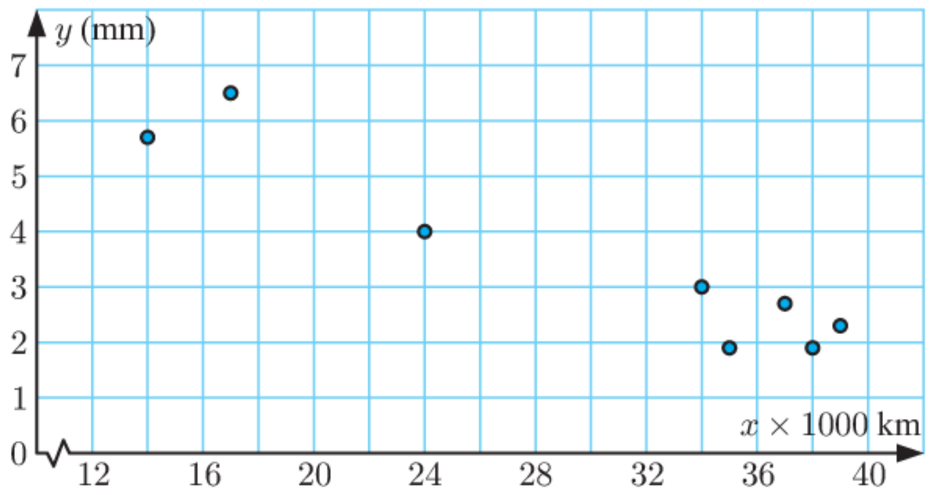
5 **a**



b $r^2 \approx 0.561$ **c** $\approx 56.1\%$

d The age of the house, the size of the land.

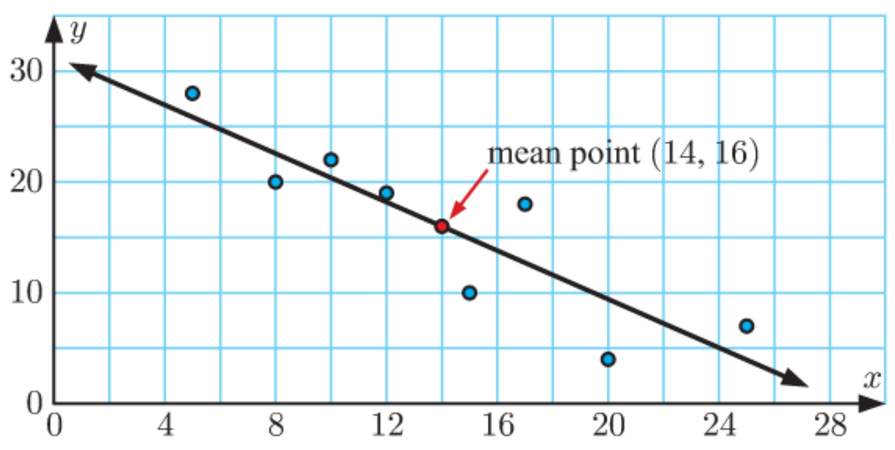
6 **a**



b $r^2 \approx 0.904$. 90.4% of the variation in *tread depth* can be explained by the variation in the *distance travelled*.

EXERCISE 7D

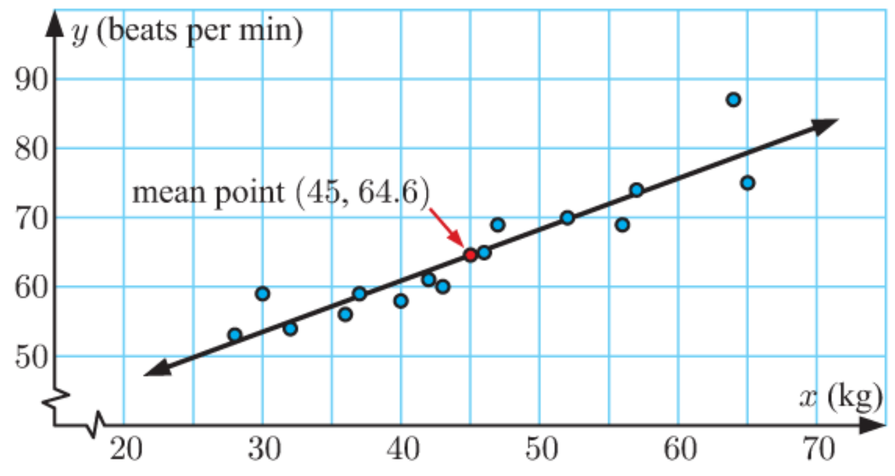
1 **a, f**



b negatively correlated **c** $r \approx -0.881$

d strong, negative correlation **e** $(14, 16)$ **g** $y \approx 7$

2 **a, e**



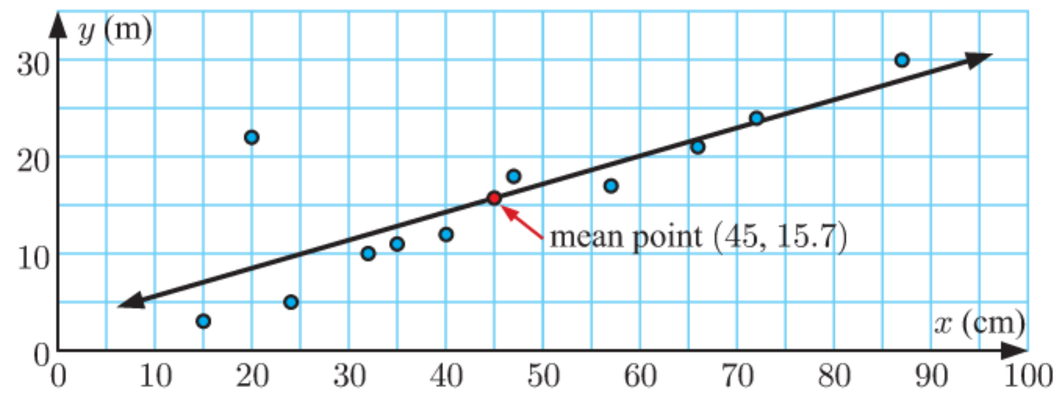
b $r \approx 0.929$

c There is a strong, positive correlation between *weight* and *pulse rate*.

d $(45, 64.6)$

f 68 beats per minute. This is an interpolation, so the estimate is reliable.

3 **a, e**



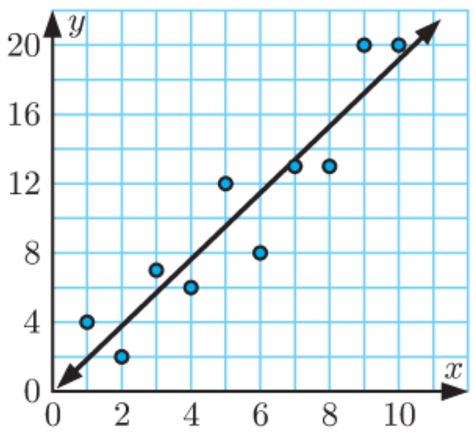
b $(20, 22)$ **c** very tall and thin **d** $(45, 15.7)$

f ≈ 37 m. This is an extrapolation, so the prediction may not be reliable.

g ≈ 25 cm. This is an interpolation, so the estimate is reliable.

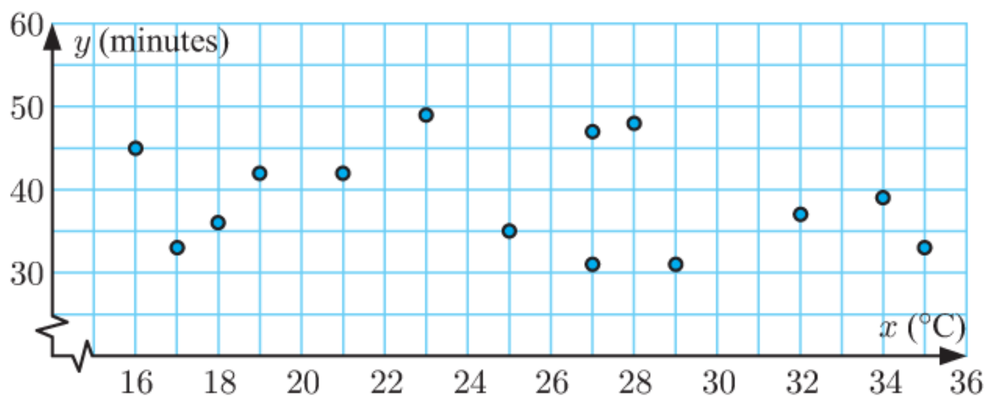
EXERCISE 7E

1 **a, c**



b $y \approx 1.92x - 0.0667$

2 **a**



b $r \approx -0.219$, $r^2 \approx 0.0481$

c There is a very weak, negative correlation between *temperature* and *time*.

d No, as there is almost no correlation and only about 4.81% of the variation in y is explained by the variation in x.

3 **a** $r \approx -0.924$, $r^2 \approx 0.853$

b There is a strong, negative, linear correlation between the *petrol price* and the *number of customers*.

c $y \approx -4.27x + 489$

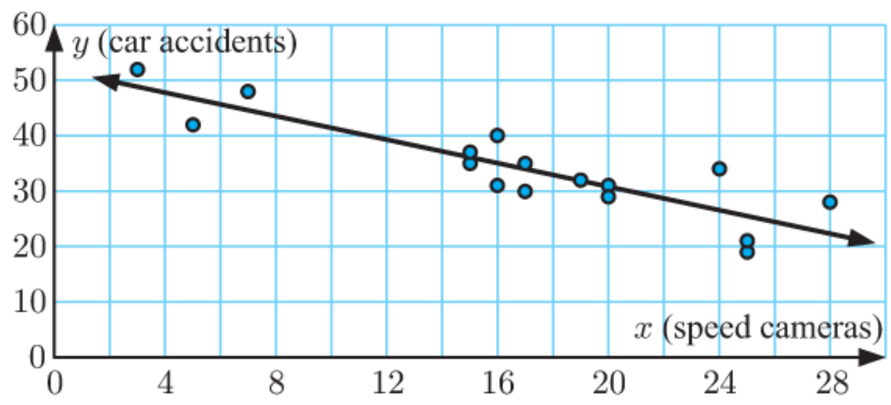
d Approximately 85.3% of the variation in the *number of customers* can be explained by the variation in the *petrol price*. So, the linear model fits the data quite well.

e ≈ -4.27; this indicates that for every cent per litre the petrol price increases by, the number of customers will decrease by approximately 4.27.

f ≈ -5.10 customers **g** ≈ 105.3 cents per litre

h In **f**, it is impossible to have a negative number of customers. This extrapolation is not valid.
In **g**, this is an interpolation, so this estimate is likely to be reliable.

4 **a**



b $r \approx -0.878$, $r^2 \approx 0.771$

c There is a strong, negative, linear correlation between *number of speed cameras* and *number of car accidents*.

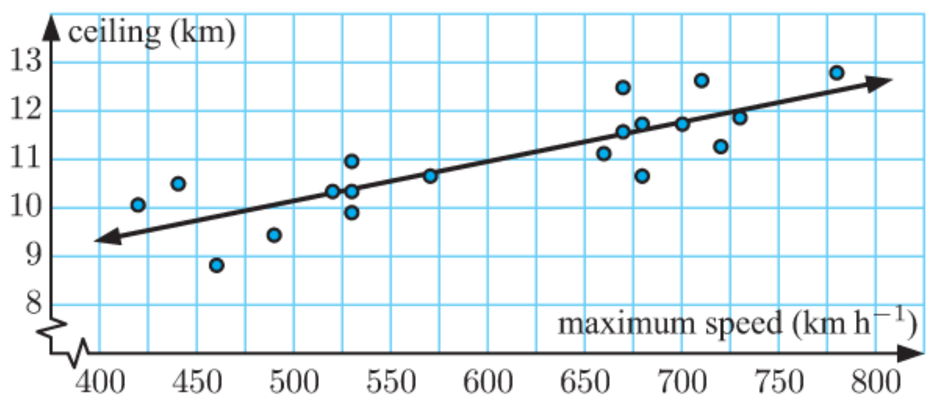
d $y \approx -1.06x + 52.0$

e Approximately 77.1% of the variation in the *number of car accidents* can be explained by the variation in the *number of speed cameras*. So, the linear model fits the data moderately well.

f gradient: ≈ -1.06; this indicates that for every additional speed camera, the number of car accidents per week decreases by an average of 1.06.
y-intercept: ≈ 52.0; this indicates that if there were no speed cameras in a city, an average of 52.0 car accidents would occur each week.

g ≈ 41.4 car accidents

5 **a**, **d**



b $r \approx 0.840$, $r^2 \approx 0.706$

c moderate, positive, linear correlation

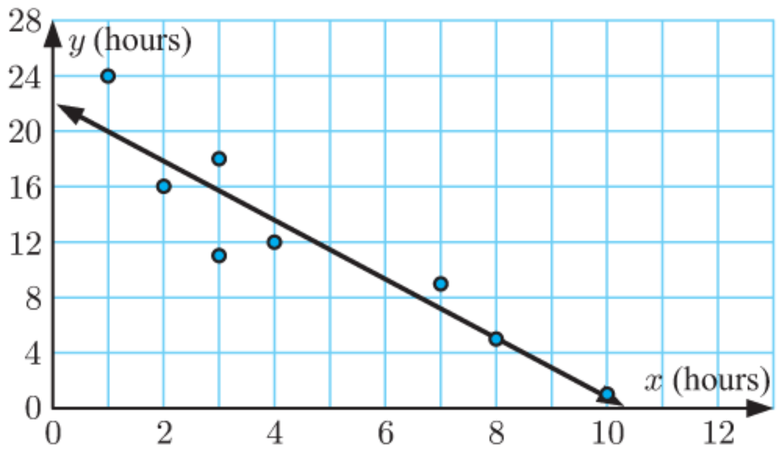
d $y \approx 0.008\,12x + 6.09$

e Approximately 70.6% of the variation in the *ceiling* can be explained by the variation in the *maximum speed*. So, the linear model fits the data moderately well.

f $\approx 0.008\,12$; this indicates that for each additional km h^{-1}, the ceiling increases by an average of 0.008 12 km or 8.12 m.

g ≈ 11.0 km **h** ≈ 605 km h^{-1}

6 **a**, **d**



b $r \approx -0.927$, $r^2 \approx 0.859$

c There is a strong, negative, linear correlation between *time exercising* and *time watching television*.

d $y \approx -2.13x + 22.1$

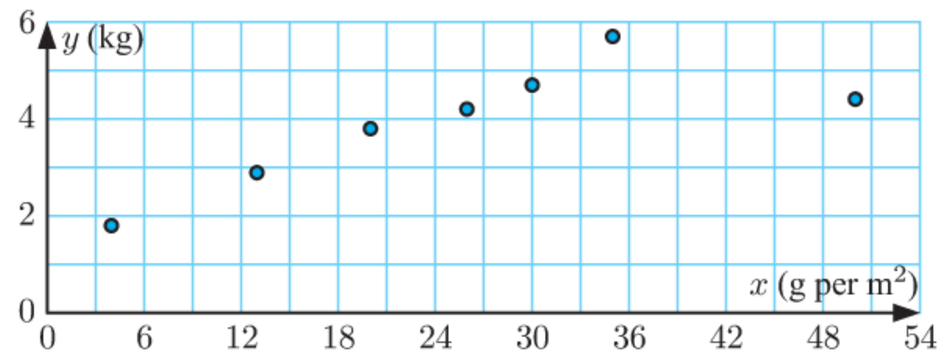
e Approximately 85.9% of the variation in the *time watching television* can be explained by the variation in the *time exercising*. So, the linear model fits the data quite well.

f gradient: ≈ -2.13; this indicates that for each additional hour a child exercises each week, the number of hours they spend watching television each week decreases by 2.13.
y-intercept: ≈ 22.1; this indicates that for children who do not spend time exercising, they would watch television for an average of 22.1 hours per week.

g **i** 9 hours per week **ii** ≈ 7.22 hours per week

iii This particular child spent more time watching television than predicted.

7 **a**



(50, 4.4) is the outlier.

b **i** reduces the strength of the correlation

ii decreases the gradient of the regression line

c **i** $r \approx 0.798$, $r^2 \approx 0.637$ **ii** $r \approx 0.993$, $r^2 \approx 0.985$

d **i** $y \approx 0.0672x + 2.22$ **ii** $y \approx 0.119x + 1.32$

e The one which excludes the outlier, as this will be more accurate for an interpolation.

f Too much fertiliser often kills the plants. In this case, the outlier should be kept when analysing the data as it is a valid data value. If the outlier is a recording error caused by bad measurement or recording skills, it should be removed before analysing data.

EXERCISE 7F

1 **a** parallel forms **b** test-retest **c** test-retest

2 **a**
- the same treadmill should be used
- each measurement should be taken at approximately the same time of day (if they are taken on different days)
- each volunteer should wear the same footwear for each measurement
- Jo should ask each volunteer to keep their eating, sleeping, and exercise routines the same throughout the duration of the experiment.

b test-retest reliability; familiarity with the test will not affect the performance of the volunteers.

3 **a** $r \approx -0.687$

b The *ATP rank* of each player is being used to predict the *number of grand slams* won, the latter of which is the criterion variable.

c There is only a weak, negative correlation between the variables. So, the predictor variable *ATP rank* has little criterion validity.

4 **a** $r \approx 0.999$

b The post office worker wants to know if the set of scales is measuring *accurately*.

c criterion validity; the post office worker wants to know if the *measured weight* accurately predicts the *standard weight*.

d By comparing each measured weight with the corresponding standard weight, we can see there is a systematic error in the measured weight (between 400 mg to 700 mg over the expected standard weight). This is not detectable if only Pearson's product-moment correlation is considered.

5 **a** $r \approx 0.508$

b The same set of cards is used, but they are placed randomly each time. So, each repetition of the test is different.

c The correlation between the *1st game* and *2nd game* times is weak. So, the test is not very reliable.

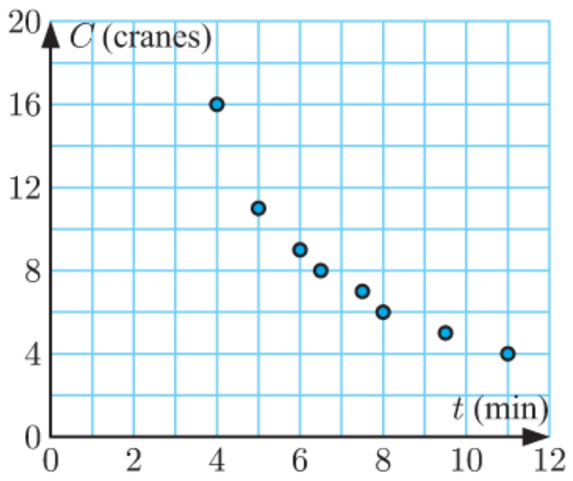
d The cards are likely to be in different positions in the 2nd game so remembering where the cards are in the first game will not give the child an advantage.

EXERCISE 7G

1 **a** B **b** C **c** A **2** **a** A **b** A

3 **a** negative **b** no

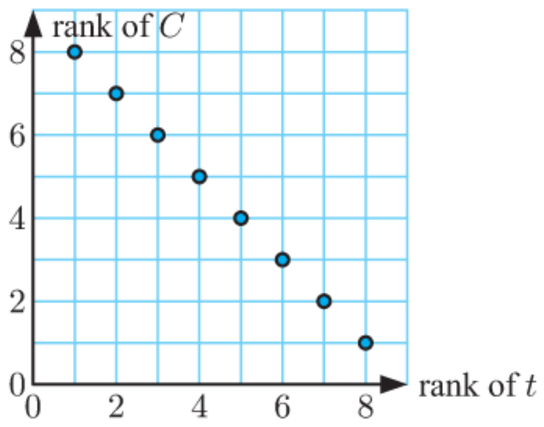
4 **a**



b $r_p \approx -0.921$

c

t (min)	4	5	6	6.5	7.5	8	9.5	11
rank of t	1	2	3	4	5	6	7	8
C	16	11	9	8	7	6	5	4
rank of C	8	7	6	5	4	3	2	1

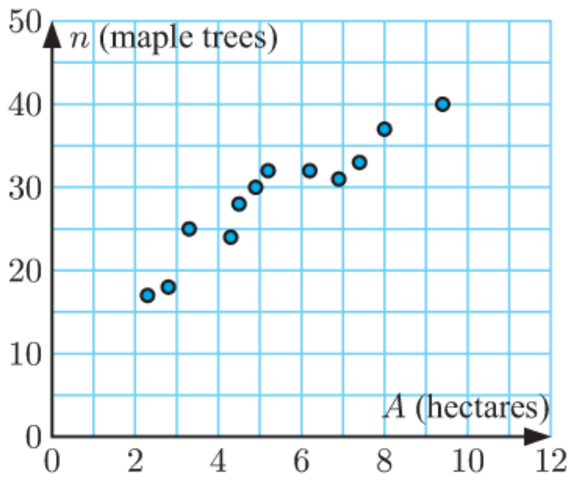


d $r_s = -1$

e The scatter diagram of the raw data shows a non-linear trend. Spearman's rank correlation coefficient is therefore more appropriate.

f strong, negative, linear correlation

5 **a**

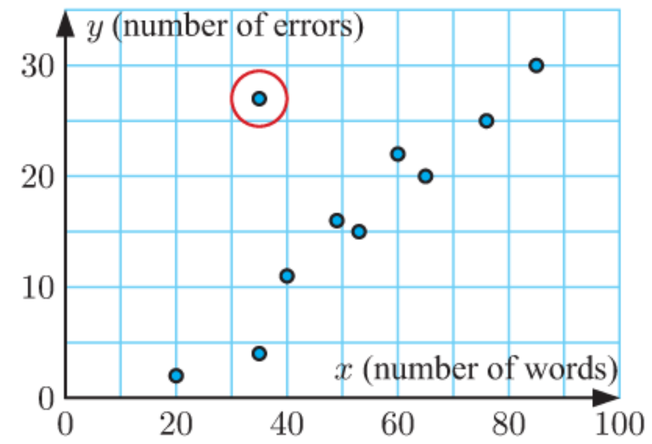


b There is a positive, linear correlation in the scatter diagram of the raw data. We expect to see the same in the rank scatter diagram. r_p and r_s will be very similar.

c $r_p \approx 0.943$, $r_s \approx 0.970$

6 **a** $r_p \approx 0.762$, $r_s \approx 0.681$

b



c $r_p \approx 0.976$, $r_s \approx 0.967$ **d** r_s

7 **a**

Longest jump (y m)	5.29	5.22	4.64	4.62	4.58
Placing	1	2	3	4	5

Longest jump (y m)	4.38	4.31	4.28	3.94	3.89
Placing	6	7	8	9	10

b *longest jump*

c **i** The variable *placing* has the values 1 to 10 which act as a ranking.

ii The longest jump *always* decreases as the placing increases.

REVIEW SET 7A

1 **a** strong, positive, linear correlation, with no outliers

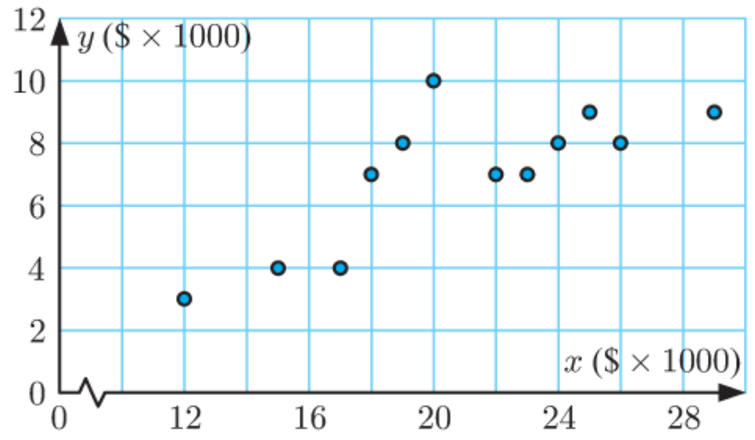
b weak, negative, linear correlation, with one outlier

c strong, negative, non-linear correlation, with no outliers

2 **a** The correlation between water bills and electricity bills is likely to be positive, as a household with a high water bill is also likely to have a high electricity bill, and vice versa.

b No, there is not a causal relationship. Both variables mainly depend on the number of occupants in each house.

3 **a**

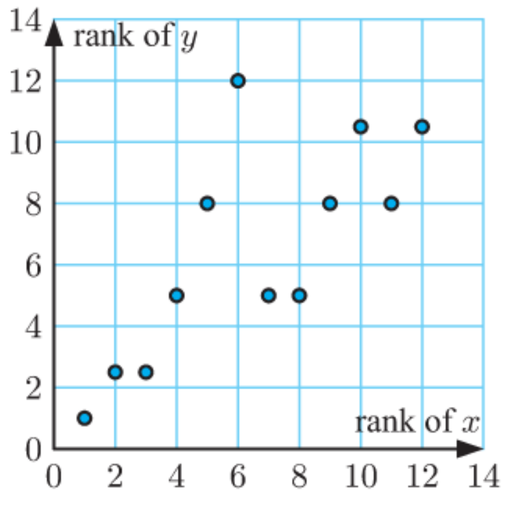


b $r_p \approx 0.776$

c

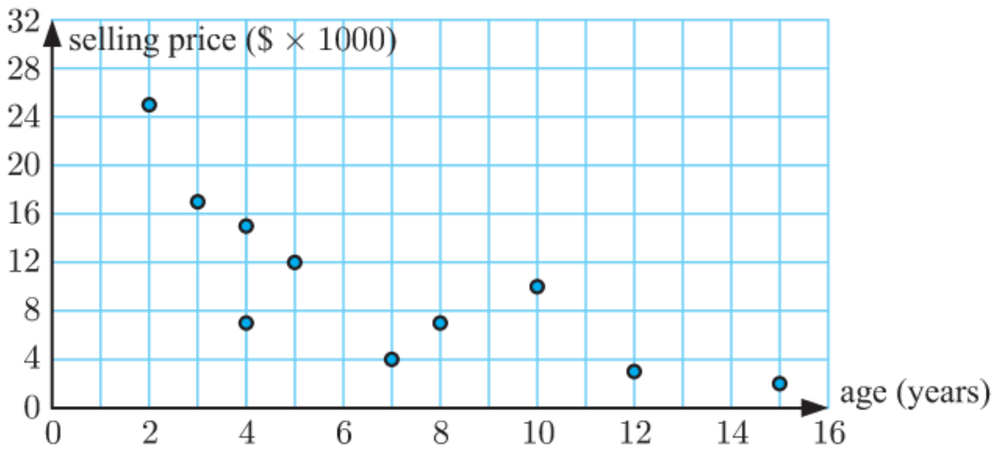
Ticket sales ($\$x \times 1000$)	25	22	15	19	12	17	24	20	18	23	29	26
rank of x	10	7	2	5	1	3	9	6	4	8	12	11

Beverage sales ($\$y \times 1000$)	9	7	4	8	3	4	8	10	7	7	9	8
rank of y	10.5	5	2.5	8	1	2.5	8	12	5	5	10.5	8



d $r_s \approx 0.744$ **e** moderate, positive correlation

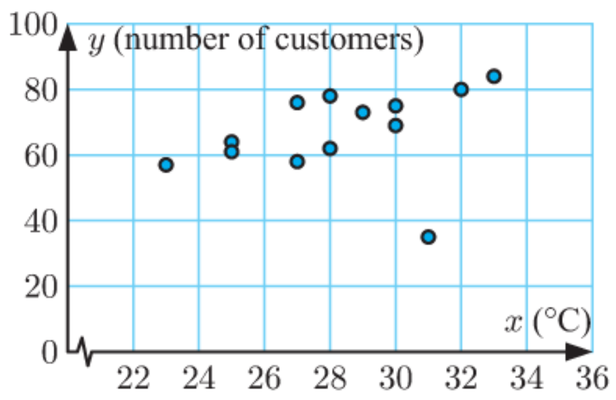
4 **a**



b moderate, negative correlation **c** $r^2 \approx 0.608$

d $\approx 60.8\%$ **e** mileage, condition, and features

5 **a**



b $r_p \approx 0.314$

c $r_s \approx 0.511$

d (31, 35)

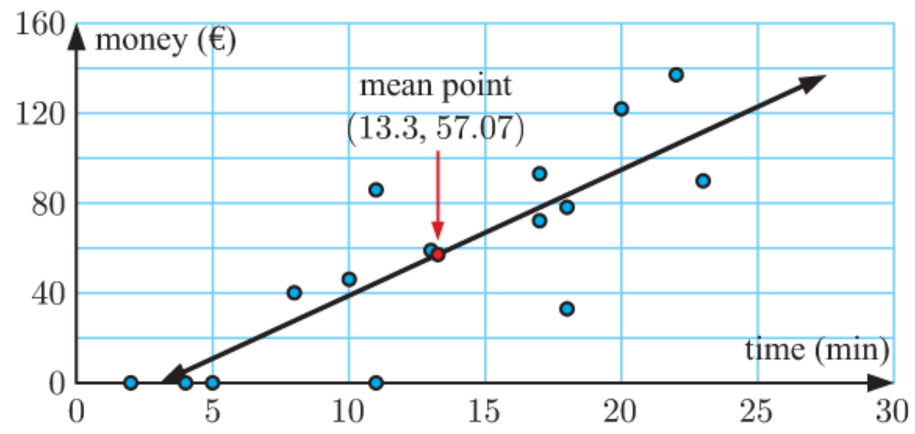
e $r_p \approx 0.801$, $r_s \approx 0.764$

f r_p

g moderate, positive, linear correlation

6 **a** mean time ≈ 13.3 min, mean spending $\approx$ €57.07

b



c There is a moderate, positive, linear correlation between *time in the store* and *money spent*.

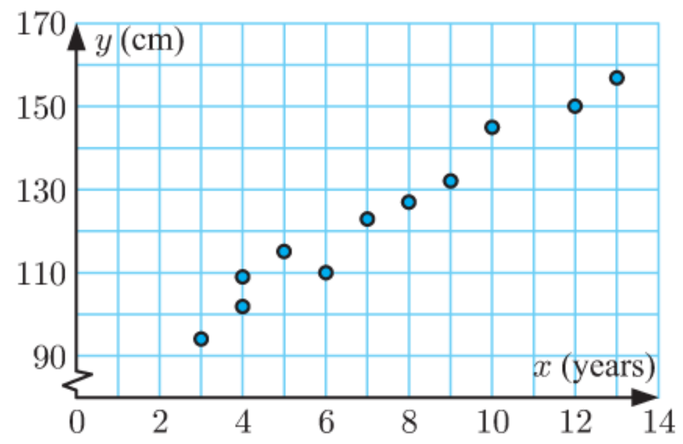
7 test-retest reliability; each titration should be identical.

8 **a** $r \approx 0.836$

b The *short response* marks are being used to predict the *mid-year exam* results, the latter of which is the criterion variable.

c There is a moderate, positive correlation between the variables. So, the predictor variable *short response* has moderate criterion validity.

9 **a**

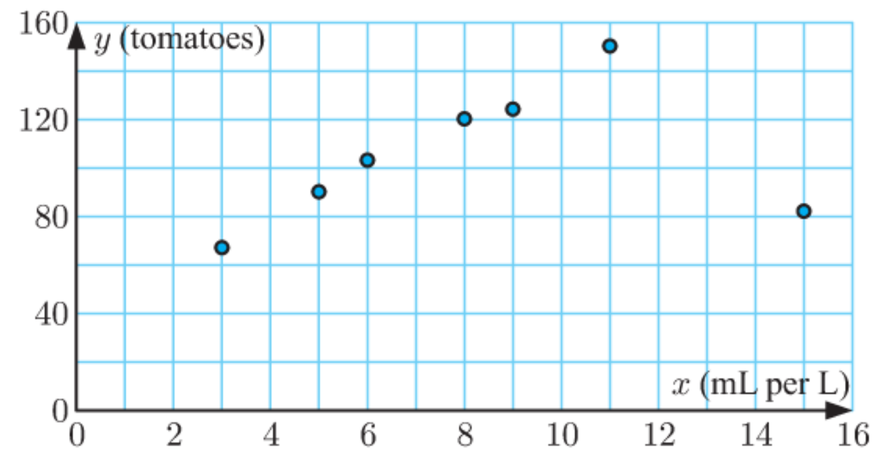


b $y \approx 5.98x + 80.0$

c ≈ 5.98; this indicates that each year, a child grows taller by an average of 5.98 cm.

d ≈ 110 cm **e** 10 years old

10 **a**



b $r^2 \approx 0.115$; approximately 11.5% of the variation in *yield of tomatoes* can be explained by the variation in *spray concentration*.

c Yes, (15, 82) is an outlier.

d $r^2 \approx 0.989$. Yes it is now reasonable to draw a linear model.

e $y \approx 9.93x + 39.5$

f gradient: ≈ 9.93; this indicates that for every additional mL per L the spray concentration increases by, the yield of tomatoes per bush increases on average by 9.93.
y-intercept: ≈ 39.5; this indicates that if the tomato bushes are not sprayed, the average yield per bush is approximately 39.5 tomatoes.

g **i** ≈ 109 tomatoes per bush **ii** ≈ 16.2 mL per L

h In **g i**, this is an interpolation, so this estimate is likely to be reliable. In **g ii**, this is an extrapolation, so this estimate may not be reliable.

REVIEW SET 7B

1 **a** Negative correlation. As prices increase, the number of tickets sold is likely to decrease.
Causal. Less people will be able to afford tickets as the prices increase.

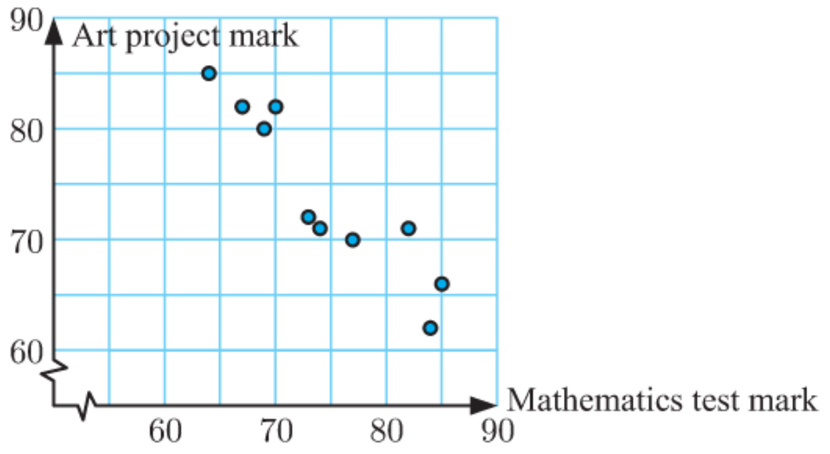
b Positive correlation. As ice cream sales increase, the number of shark attacks is likely to increase.
Not causal. Both of these variables are dependent on the number of people at the beach.

2 **a** moderate, positive correlation

b $r^2 \approx 0.619$; approximately 61.9% of the variation in *energy* can be explained by the variation in *fat content*.

3 **a** **C** **b** **A** **c** **B**

4 **a**

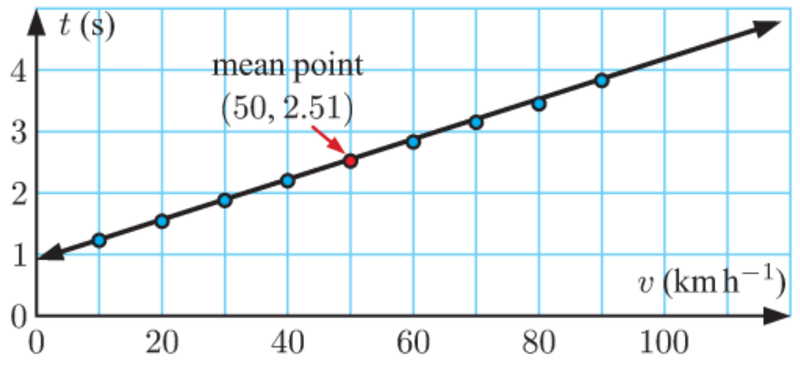


b There is a strong, negative, linear correlation between the Mathematics and Art marks.

c $r \approx -0.930$

5 **a** (50, 2.51)

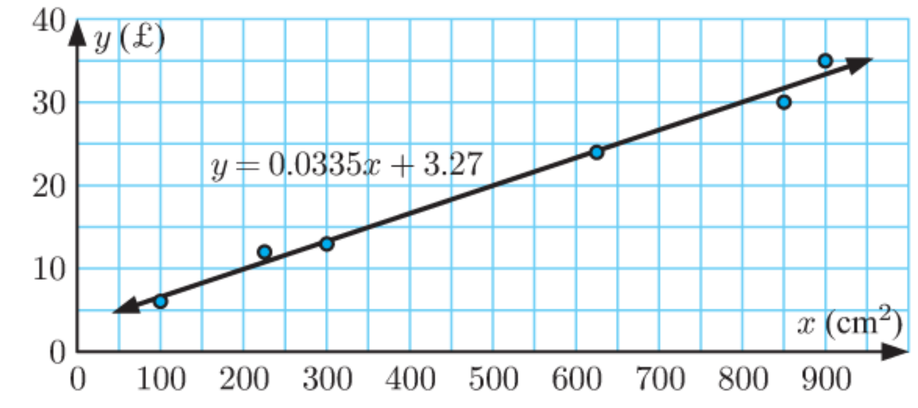
b



c **i** ≈ 2.7 seconds **ii** ≈ 4.4 seconds

d The estimate in **c i**, since it is an interpolation.

6 **a, d**

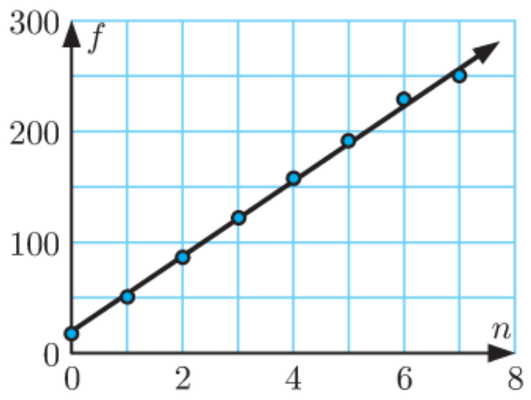


b $r \approx 0.994$, $r^2 \approx 0.988$

c There is a very strong, positive, linear correlation between *area* and *price*.

e $\approx$ £43.42, this is an extrapolation, so it may be unreliable.

7 **a, e**



There is a very strong, positive, linear correlation between *number of waterings* and *flowers produced*.

b $f \approx 34.0n + 19.3$

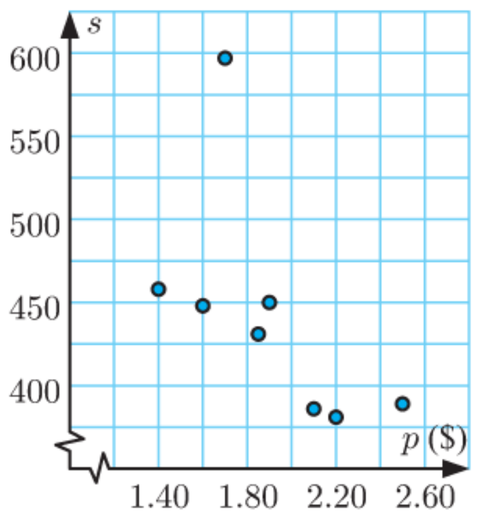
c $r^2 \approx 0.998$; approximately 99.8% of the variation in the number of *flowers produced* can be explained by the variation in the *number of waterings*. So, the linear model fits the data extremely well.

d Yes, plants need water to grow, so it is expected that an increase in watering will result in an increase in flowers.

f **i** 104 flowers $(n = 2.5)$, 359 flowers $(n = 10)$

ii $n = 2.5$ is reliable, as it is an interpolation.
$n = 10$ is unreliable as it is an extrapolation and over-watering could be a problem.

8 **a**



b Yes, the point (1.70, 597) is an outlier. It should not be deleted as there is no evidence that it is a recording error.

c $s \approx -116p + 665$

d ≈ -116; this indicates that with every additional dollar the price increases by, the number of sales decreases by 116.

e $r^2 \approx 0.346$; approximately 34.6% of the variation in *sales* can be explained by the variation in *price*.

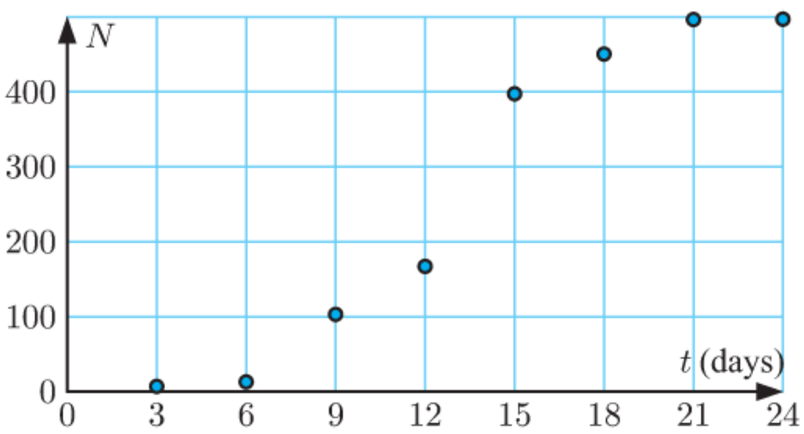
f No, the prediction would not be accurate, as it is an extrapolation.

9 **a** $r \approx 0.874$ **b** parallel forms

c There is a strong, positive correlation between the *Week 1* and *Week 2* results. So, Ms Wang's tests are very reliable.

d No; it does not consider the students' oral skills, grammar skills, sentence structure, and so on. Also, it is only relevant to Chinese. No other languages are considered.

10 **a**



b $r_p \approx 0.963$ **c** $r_s = 1$

d Spearman's rank correlation; the scatter diagram shows a clear non-linear trend in the data.

EXERCISE 8A

1 **a** $P = 2.5 - \frac{3}{4}\ln t$ **b** $L = 2 + \frac{2}{3}\ln k$

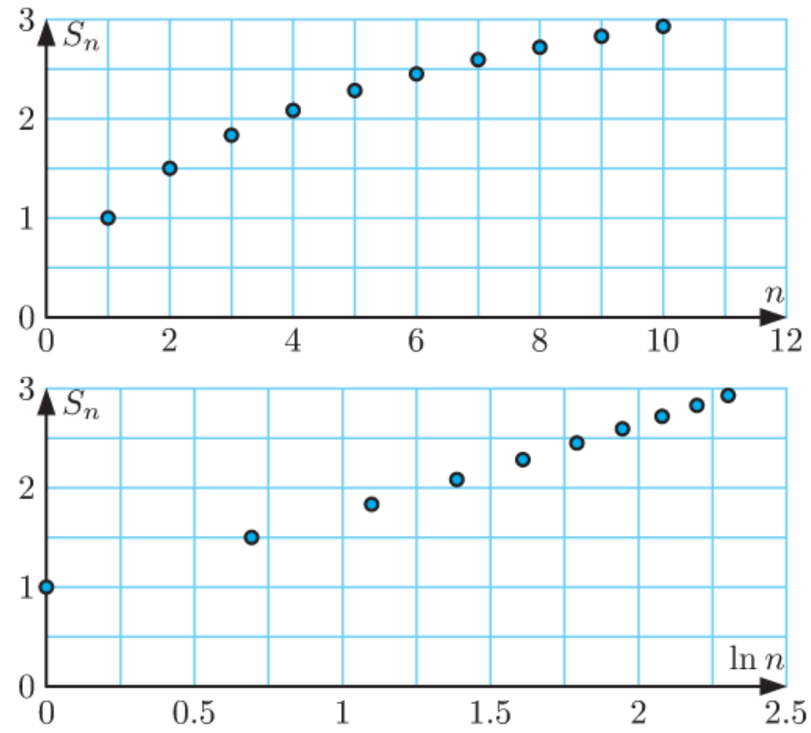
c $A \approx 1.37 + 0.526 \ln r$

2 **a** equivalent;

$$\begin{aligned} y &= p + q\ln(rx) \\ &= p + q(\ln r + \ln x) \\ &= \underbrace{(p + q\ln r)}_{=a} + \underbrace{q}_{=b}\ln x \end{aligned}$$

b not equivalent; the laws of logarithms cannot be used to write $\ln(x - \gamma)$ in the form $\ln x$ + constant.

3 **a**



b The graph of S_n against $\ln n$ appears to be linear.

c $S_n \approx 0.940 + 0.849 \ln n$

d No; as $n \to \infty$, $\ln n \to \infty$

EXERCISE 8B

1 **a** $y \approx 7.39 \times 1.82^x$ **b** $G \approx 0.607 \times 4.48^t$

c $Q \approx 1100 \times 0.174^n$

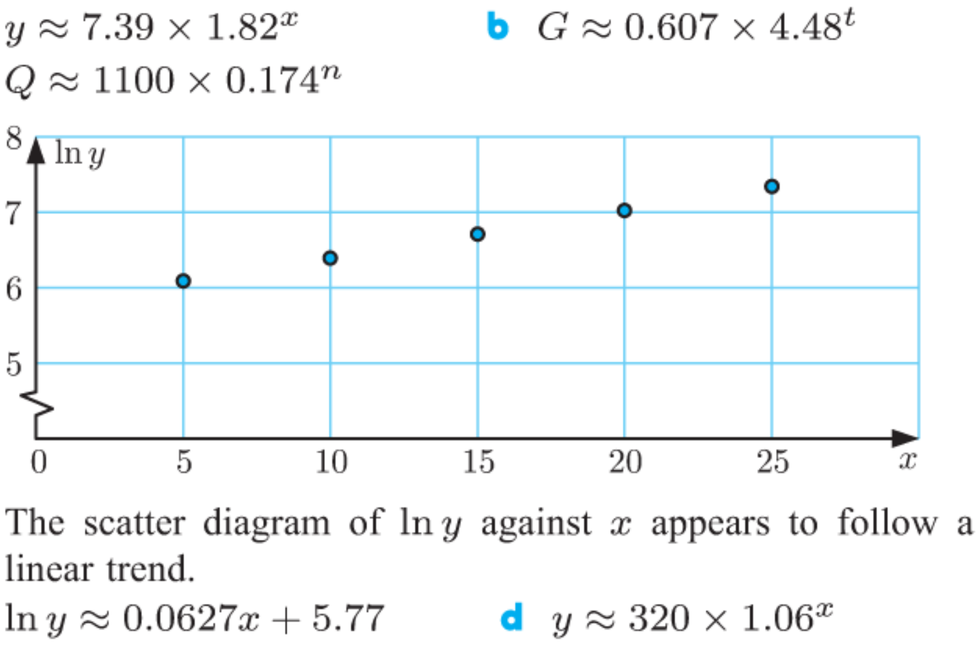
2 **a**

b The scatter diagram of $\ln y$ against x appears to follow a linear trend.

c $\ln y \approx 0.0627x + 5.77$ **d** $y \approx 320 \times 1.06^x$

e **i** ≈ 529 **ii** ≈ 2874

3 **a**



b The scatter diagram of $\ln V$ against t appears to be linear.

c $\ln V \approx -0.105t + 5.70$ **d** $V \approx 300 \times 0.900^t$

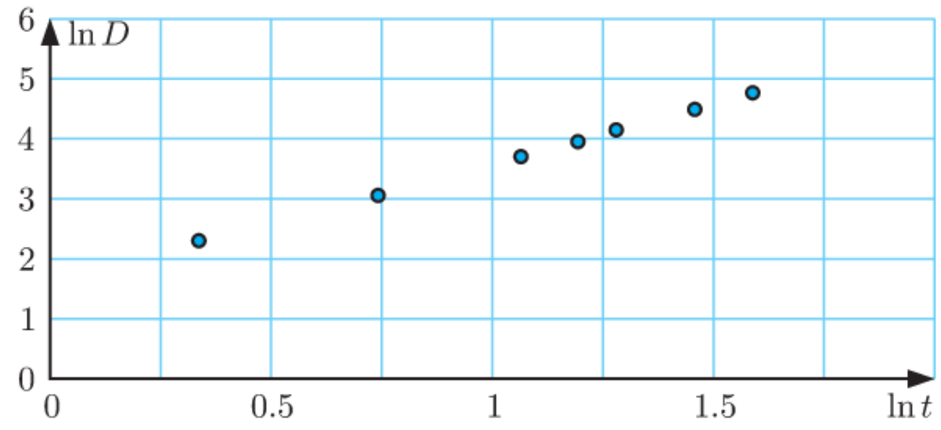
e ≈ 97.9 minutes

EXERCISE 8C

1 **a** $y \approx 2.72 \times x^{0.25}$ **b** $R \approx 0.135 \times p^3$

c $T \approx 66.7 \times k^{-0.4}$

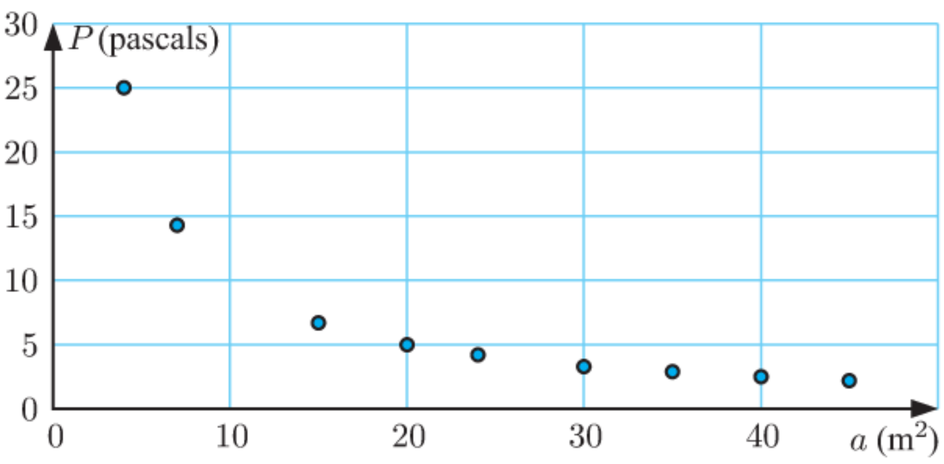
2 **a**



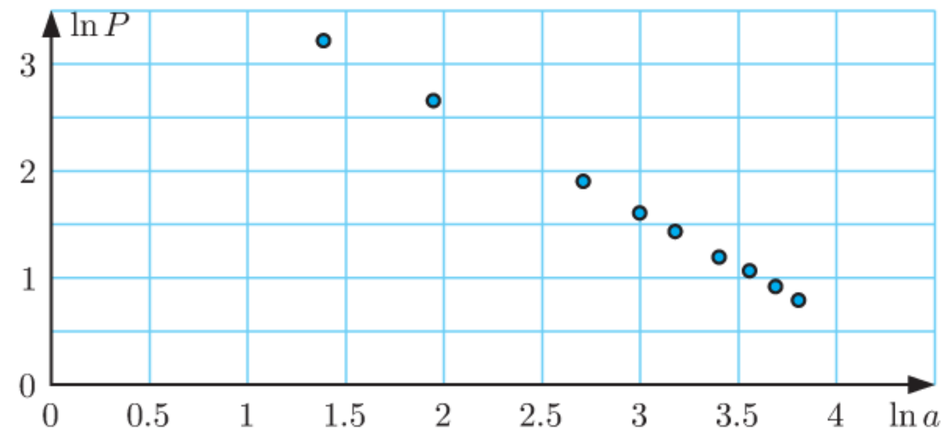
b The scatter diagram of $\ln D$ against $\ln t$ appears to be linear.

c $\ln D \approx 1.97 \ln t + 1.62$ **d** $D \approx 5.04 \times t^{1.97}$

3 **a**



b



The scatter diagram of $\ln P$ against $\ln a$ appears to be linear, so a power model is appropriate.

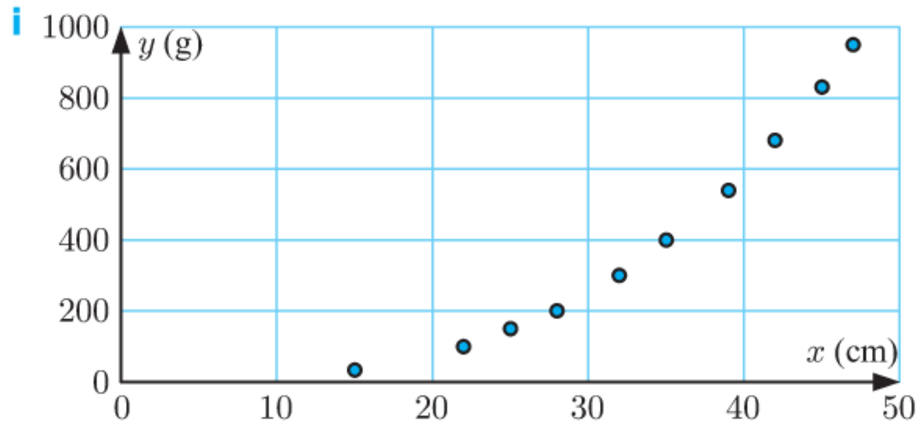
c $\ln P \approx -1.00 \ln a + 4.61$ **d** $P \approx 100 \times a^{-1.00}$

e $P \approx 10.0$ pascals

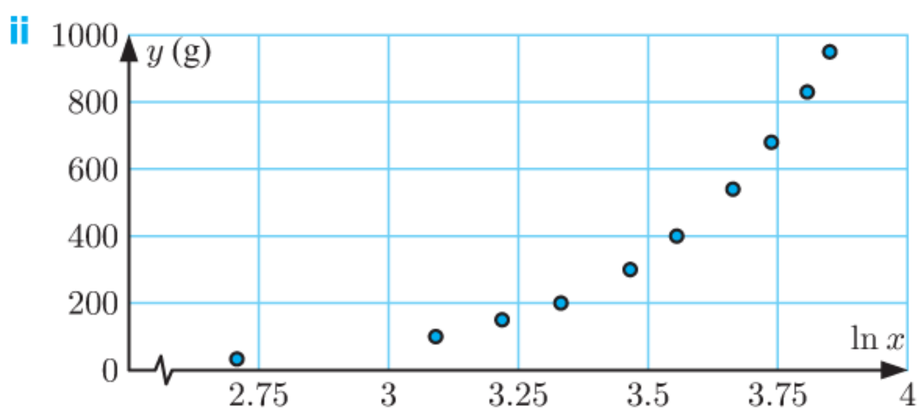
The estimate is reliable as we are interpolating.

EXERCISE 8D

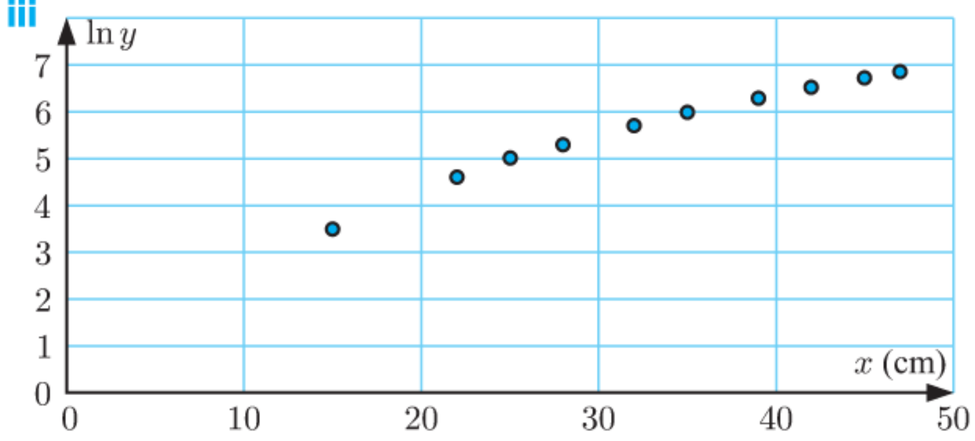
1 **a** **i**



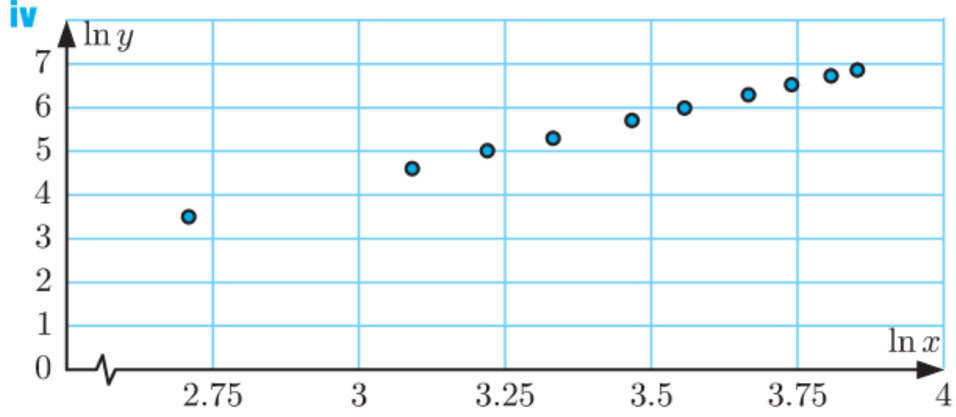
ii



iii



iv

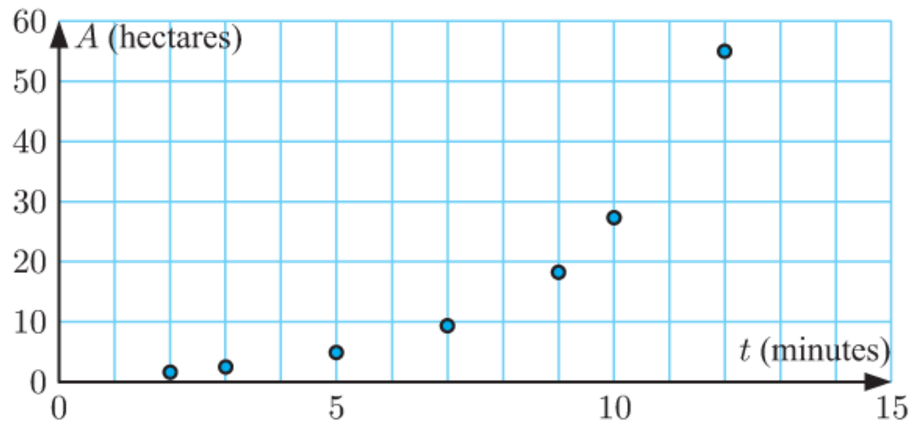


b The scatter diagram of $\ln y$ against $\ln x$ appears to be linear.

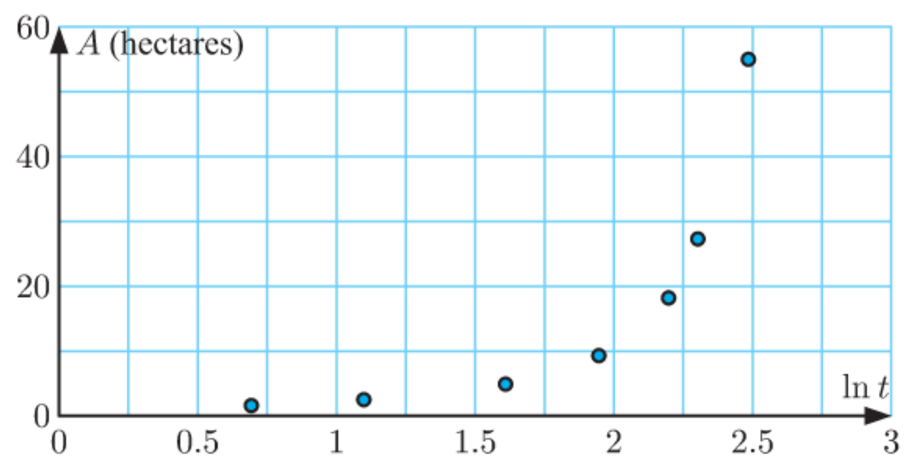
c $\ln y \approx 2.94 \ln x - 4.48, \quad r^2 \approx 0.9998$

d $y \approx 0.0113 \times x^{2.94}$

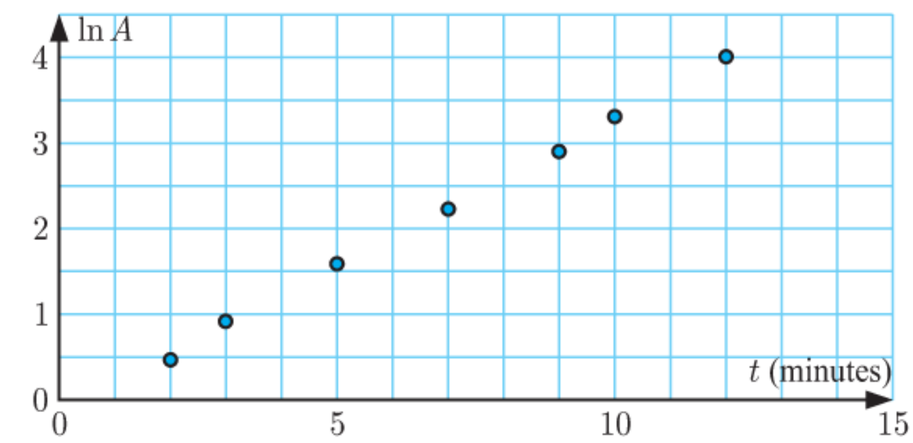
2 **a** **i**



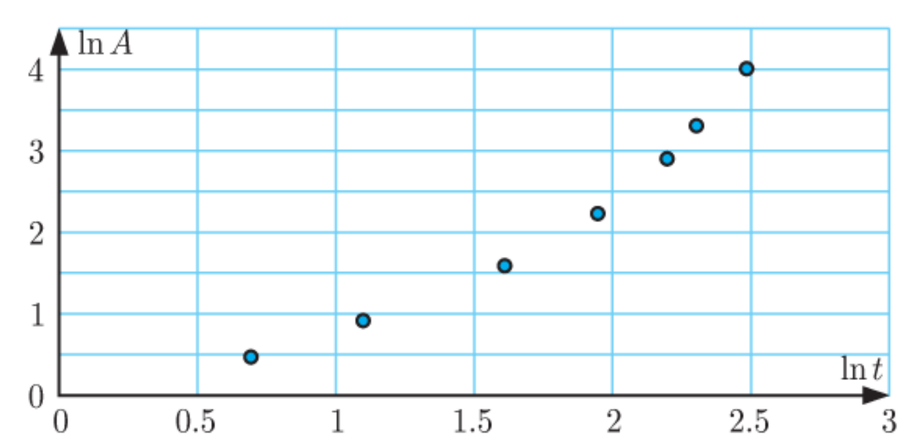
ii



iii



iv

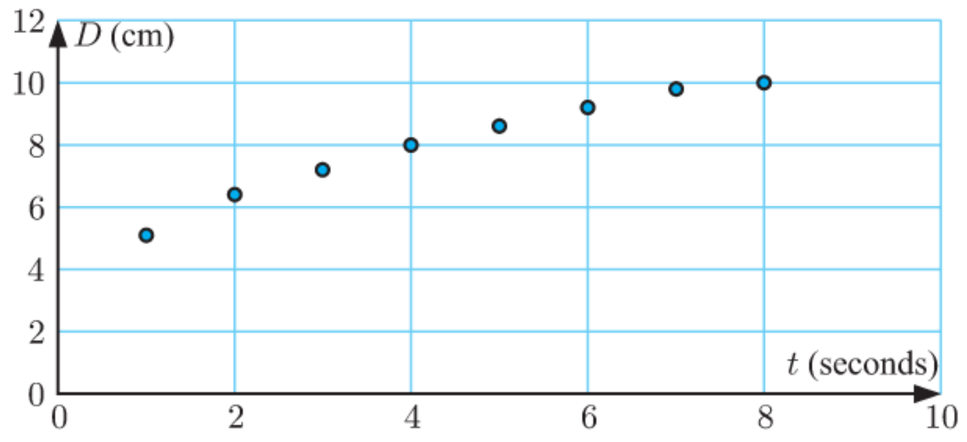


b The scatter diagram of $\ln A$ against t follows a linear trend.

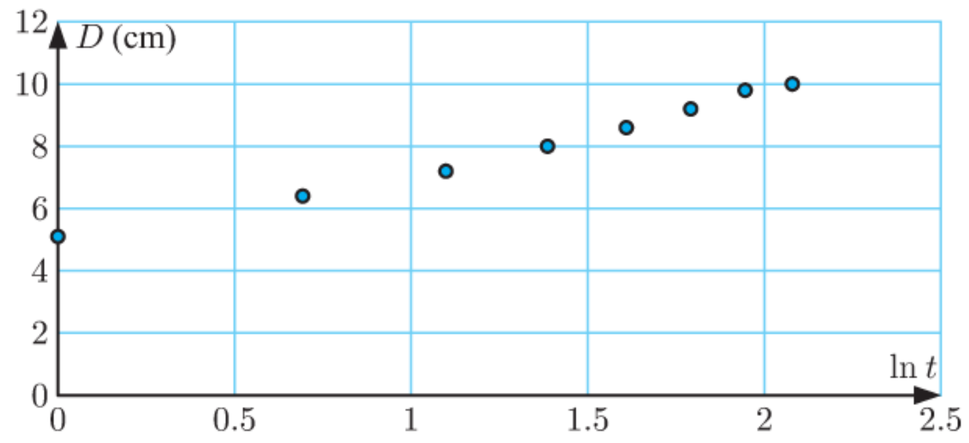
c $\ln A \approx 0.347t - 0.178, \quad r^2 \approx 0.999$

d $A \approx 0.837 \times 1.42^t$

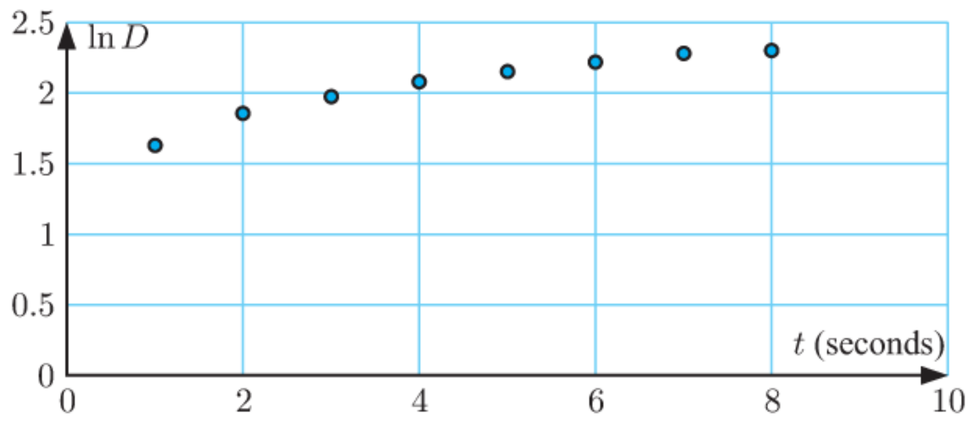
3 **a** **i**



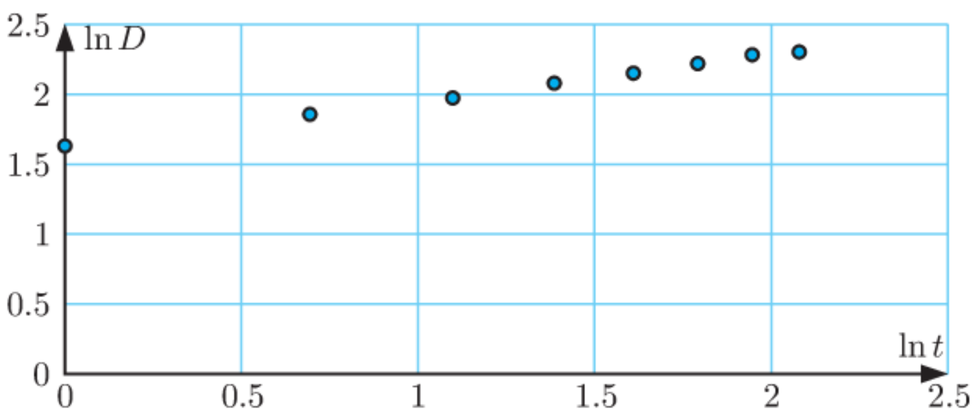
ii



iii



iv

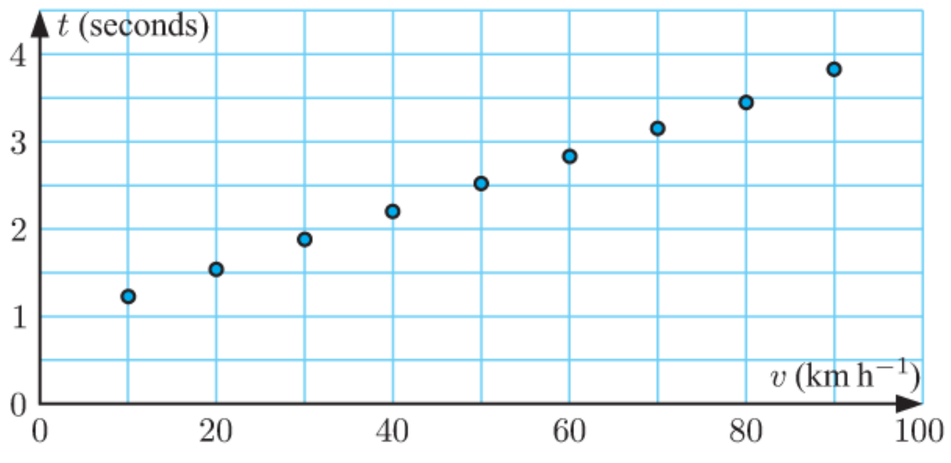


b The scatter diagram of $\ln D$ against $\ln t$ appears to be linear, so a power model is appropriate.

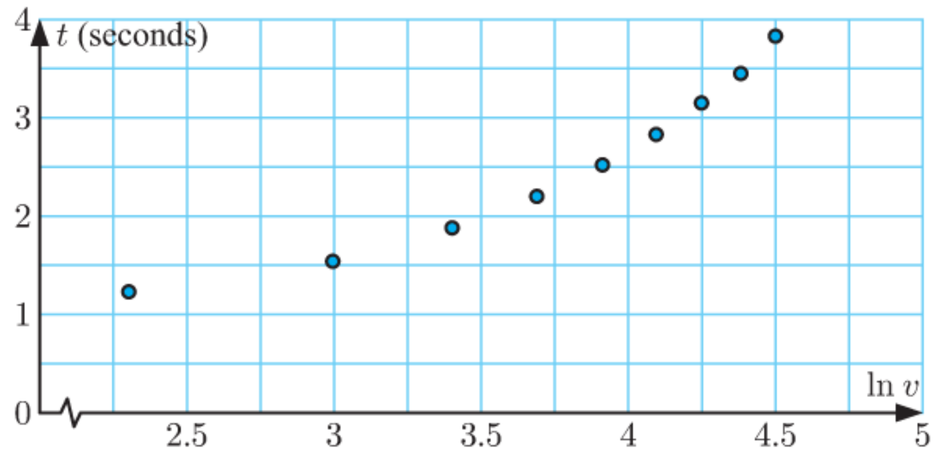
c $D \approx 5.08 \times t^{0.330}$

d $D \approx 7.67$ cm, and since this is an interpolation, this estimate is likely to be reliable.

4 **a** **i**

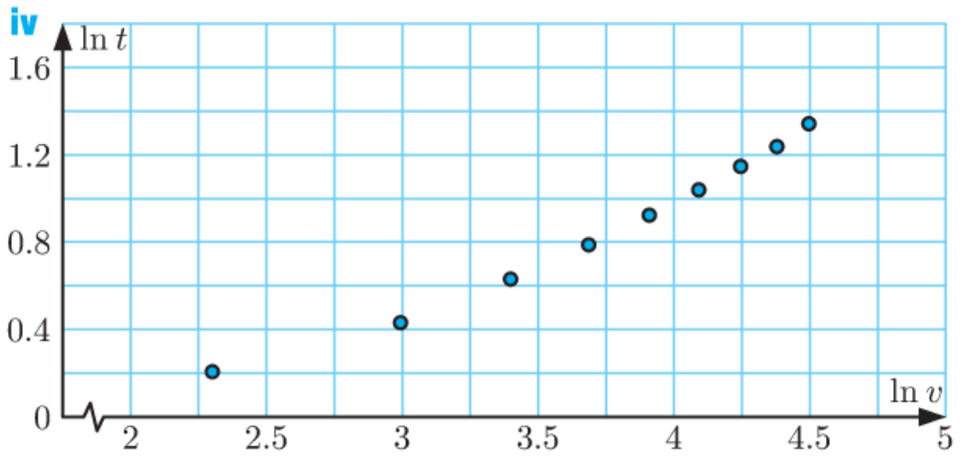


ii



iii

iv



b The scatter diagram of t against v appears to be linear, so a linear model is appropriate.

c $t \approx 0.0322v + 0.906$ **d** $t \approx 4.44$ seconds

5 **a** $V \approx 250 \times 1.03^t$ **b** $V \approx$ £522 000

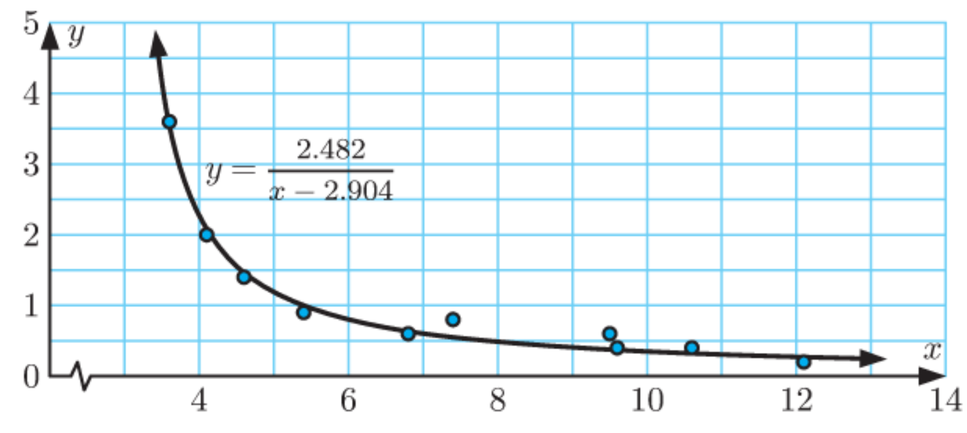
6 **a** $a \approx 0.997, \quad k \approx 1.50$ **b** appropriate, as $r \approx 1$

c As $a \approx 1$ and $k \approx \frac{3}{2}$, $\quad T \propto r^{\frac{3}{2}}$

$\therefore \quad T^2 \propto r^3$

EXERCISE 8E

1 **a**



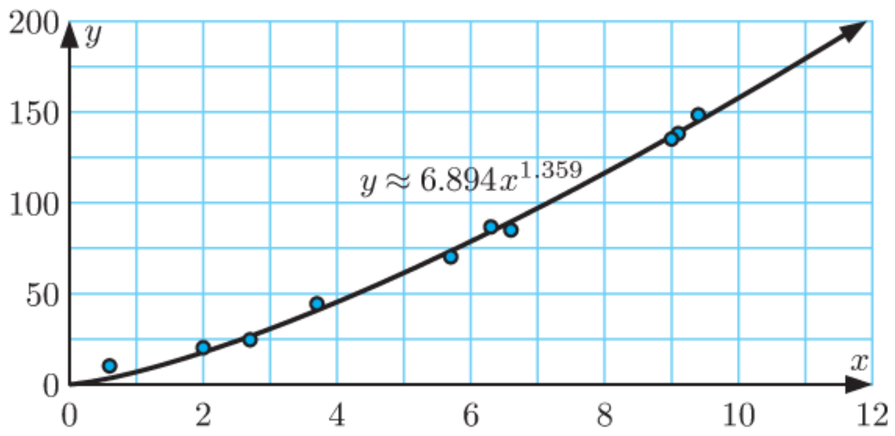
b

x	y	*Residual*
9.6	0.4	0.0293
7.4	0.8	0.248
3.6	3.6	0.0339
4.6	1.4	−0.0634
5.4	0.9	−0.0944
10.6	0.4	0.0775
12.1	0.2	−0.0699
4.1	2	−0.0753
6.8	0.6	−0.0371
9.5	0.6	0.224

c $SS_{\text{res}} \approx 0.144$

d SS_{res} is small (less than 1), so the model fits the data very well.

2 **a**



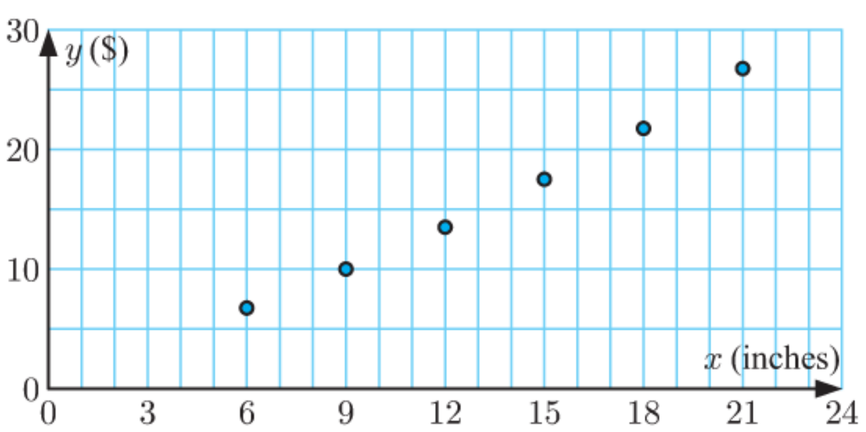
b

x	y	*Residual*
2.7	24.6	−1.99
3.7	44.4	3.6
5.7	70.2	−3.2
9.1	138.1	−0.514
2	20.2	2.52
9	135.1	−1.45
9.4	148.4	3.54
6.6	85	−4.58
6.3	86.7	2.6
0.6	10.2	6.76

c $SS_{\text{res}} \approx 122$

d $y \approx 12.5x^{1.03}$, the power of x is close to $n = 1.359$, but 12.5 is very different from $a \approx 6.894$.
$SS_{\text{res}} \approx 1160$, so this model is less appropriate.

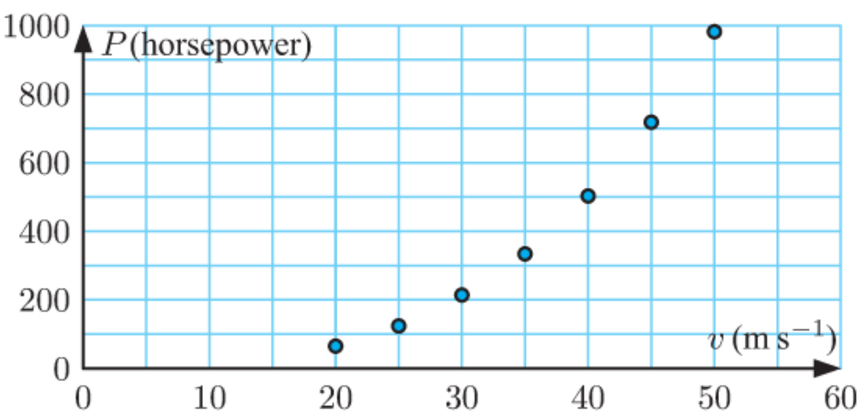
3 **a**



b $y \approx 0.0233x^2 + 0.697x + 1.78$ **c** $SS_{\text{res}} \approx 0.0254$

d SS_{res} is very small which suggests that the model is a good fit for the data. Also, we can assume that
price $\propto$ area of each pizza
$\propto$ (radius)^2
$\propto$ (diameter)^2

4 **a**

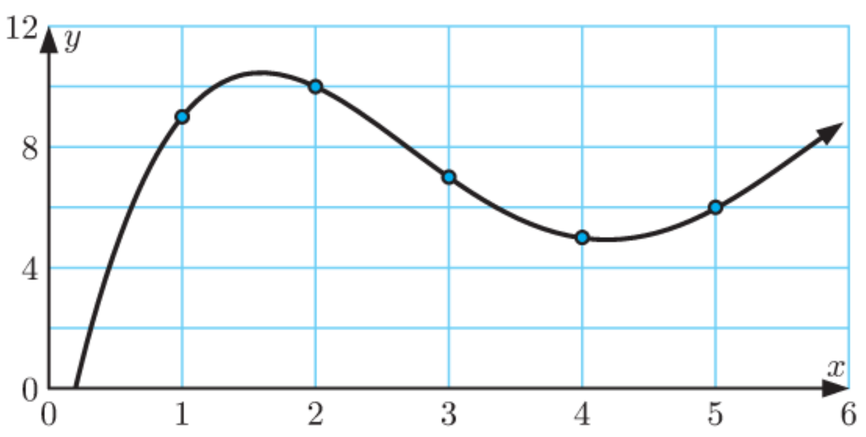


b $P \approx 0.007\,56v^3 + 0.0390v^2 - 1.62v + 21.8$

c $SS_{\text{res}} \approx 14.3$

5 **a** $y \approx -0.125x^4 + 2.083x^3 - 11.375x^2 + 22.417x - 4$

b



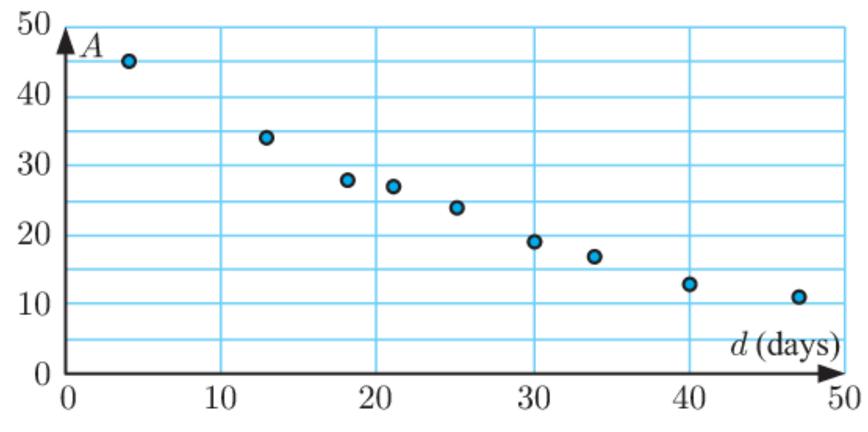
c $SS_{\text{res}} = 0$; the fitted model passes through each data point.

REVIEW SET 8A

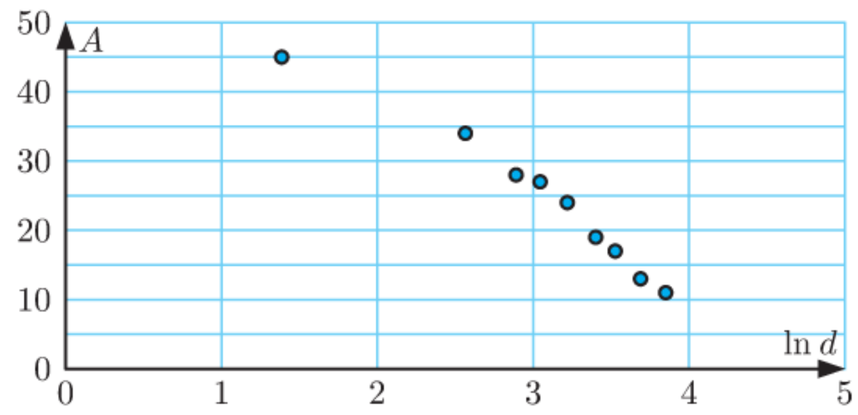
1 **a** $y \approx 20.1x^{0.5}$ **b** $P \approx 148 \times 0.435^n$

c $T = 60 - 2\ln n$

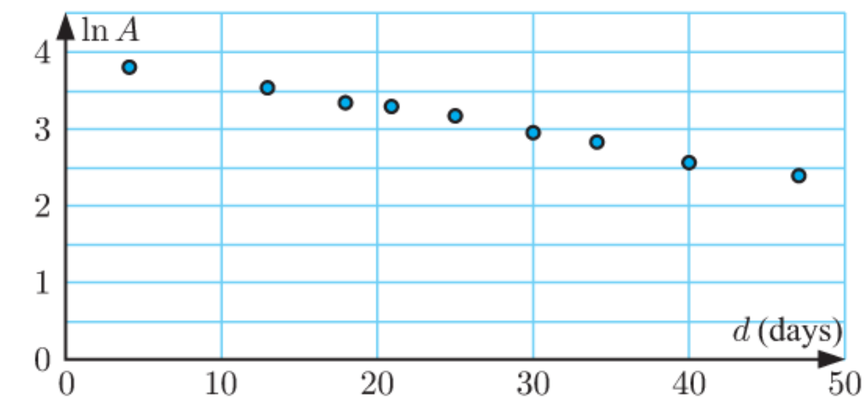
2 **a** **i**



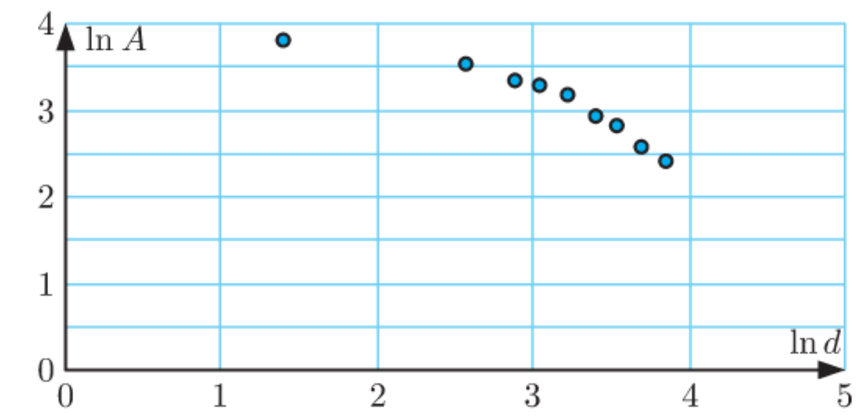
ii



iii



iv

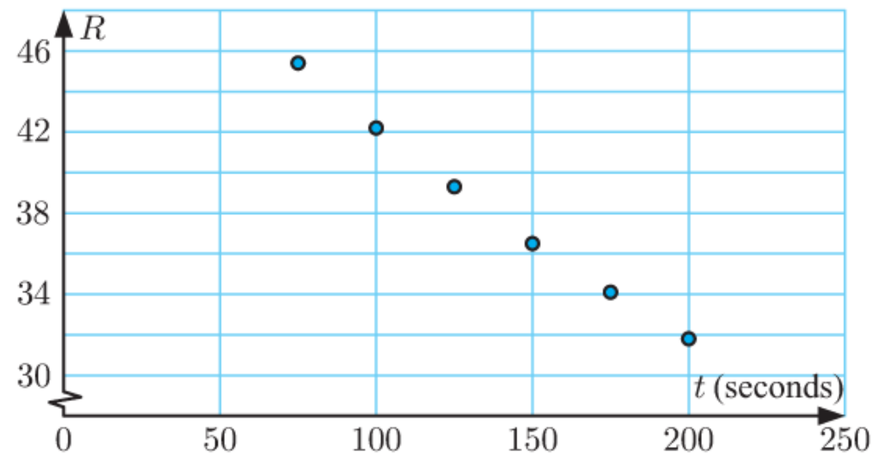


b The scatter diagram of $\ln A$ against d appears to be linear.

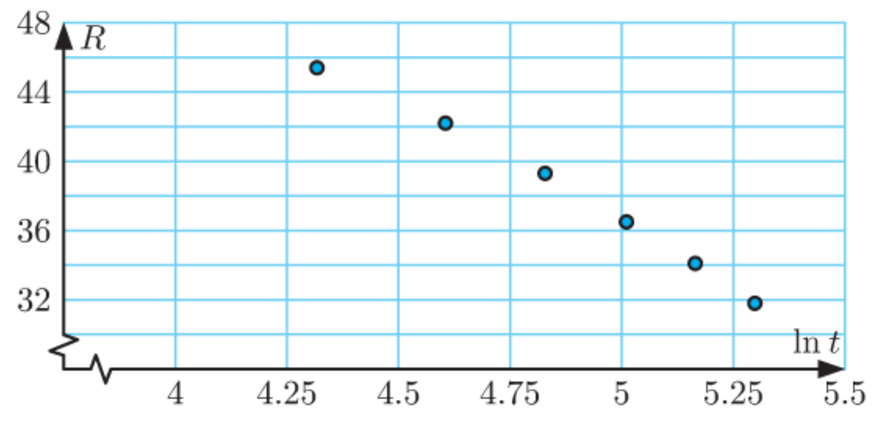
c $\ln A \approx -0.0336d + 3.9631$

d $A \approx 52.6 \times 0.967^d$ **e** ≈ 53

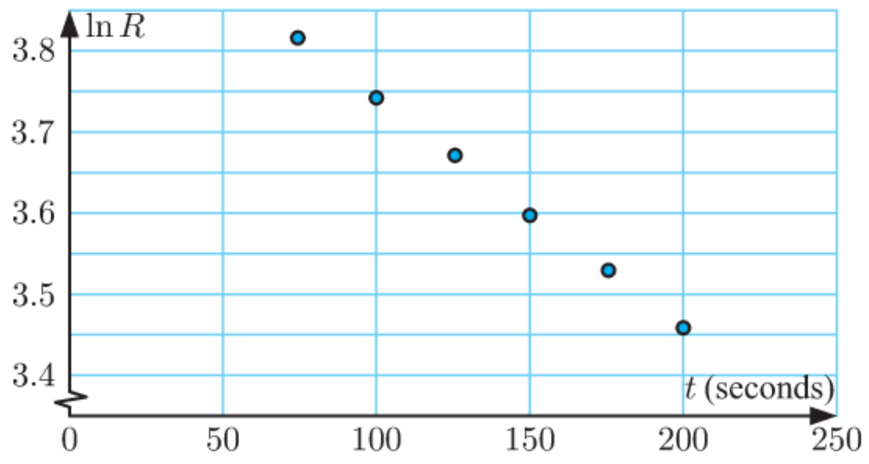
3 **a** **i**



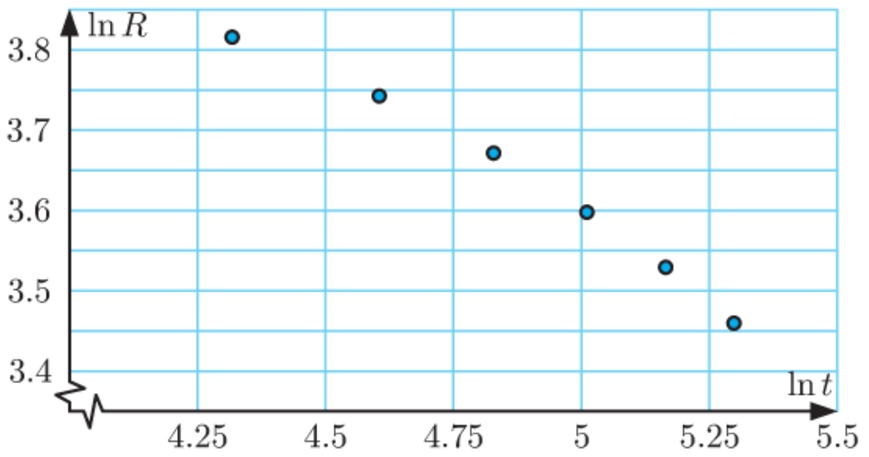
ii



iii



iv

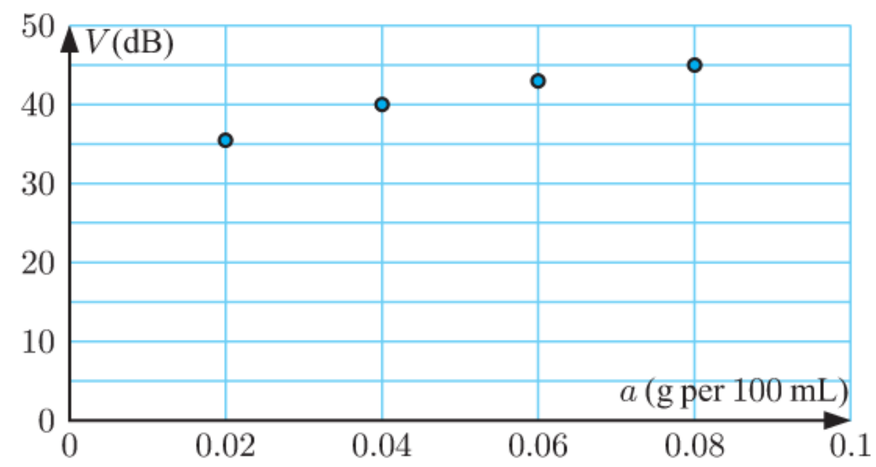


b The scatter diagram of $\ln R$ against t appears to be linear ($r^2 \approx 0.999$), so an exponential model is most appropriate.

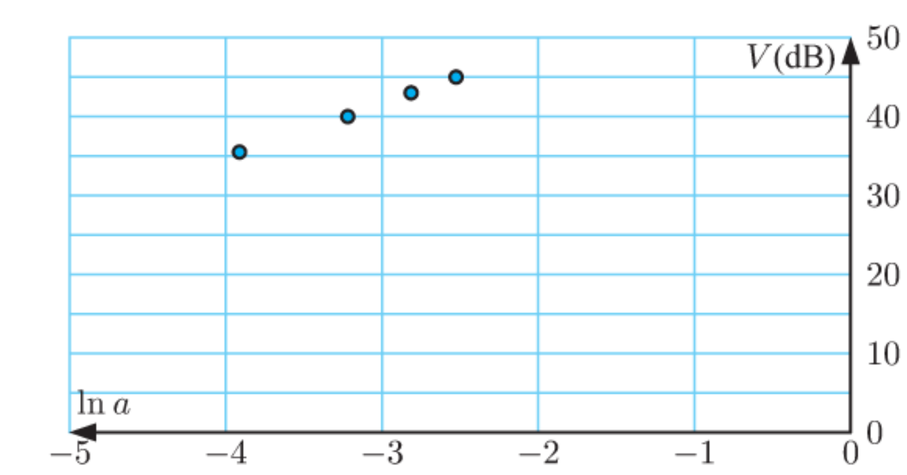
c $R \approx 56.1 \times 0.997^t$

d $R \approx 35.6$, and since we are interpolating, this estimate is likely to be reliable.

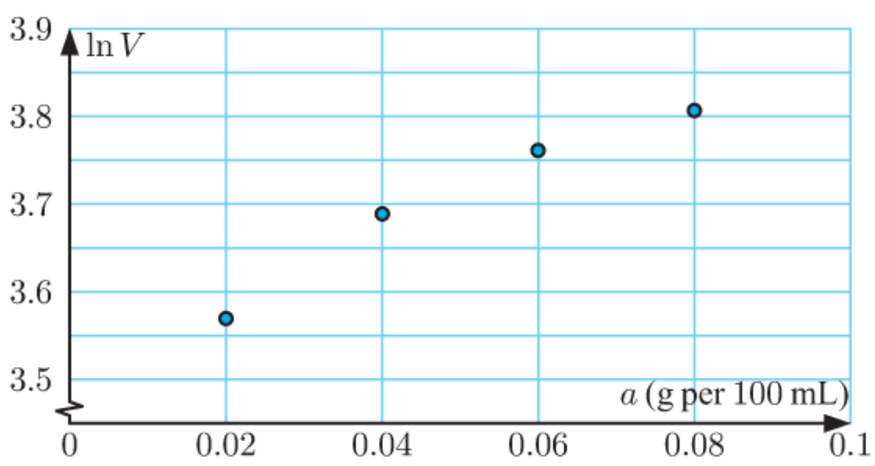
4 a i



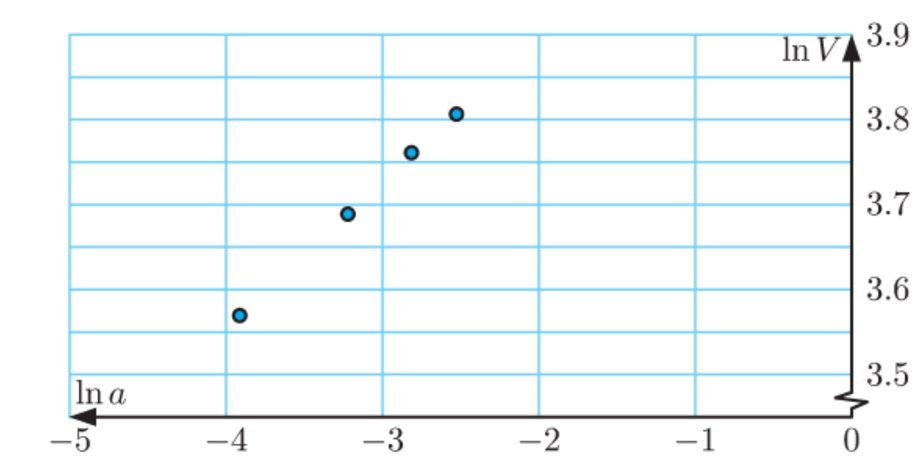
ii



iii



iv



b The scatter diagram of $\ln V$ against $\ln A$ appears to be linear ($r^2 \approx 0.999$), so a power model is most appropriate.

c $V \approx 69.6 \times a^{0.172}$ **d** **i** ≈ 50.2 dB **ii** ≈ 41.6 dB

5 a $a \approx 127m^{-1.01}$ (power model); the scatter diagram of $\ln a$ against $\ln m$ appears to be linear ($r^2 \approx 0.999$), so a power model is most appropriate.

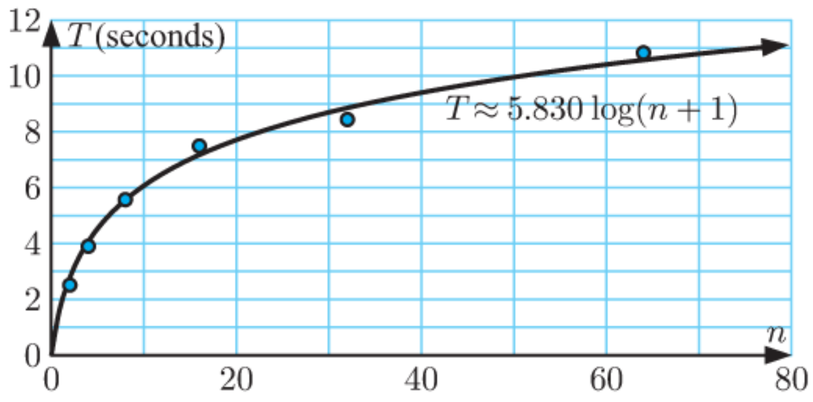
b From Newton's 2nd law: $F = ma$

$$\therefore \quad 125 = ma$$

$$\therefore \quad a = \frac{125}{m} = 125m^{-1}$$

The coefficients for the power model in **a** are quite close to the ones expected above.

6 a



b

n	T	*residual*
2	2.51	−0.272
4	3.9	−0.175
8	5.57	0.006 77
16	7.49	0.316
32	8.44	−0.413
64	10.83	0.261

c $SS_{\text{res}} \approx 0.443$

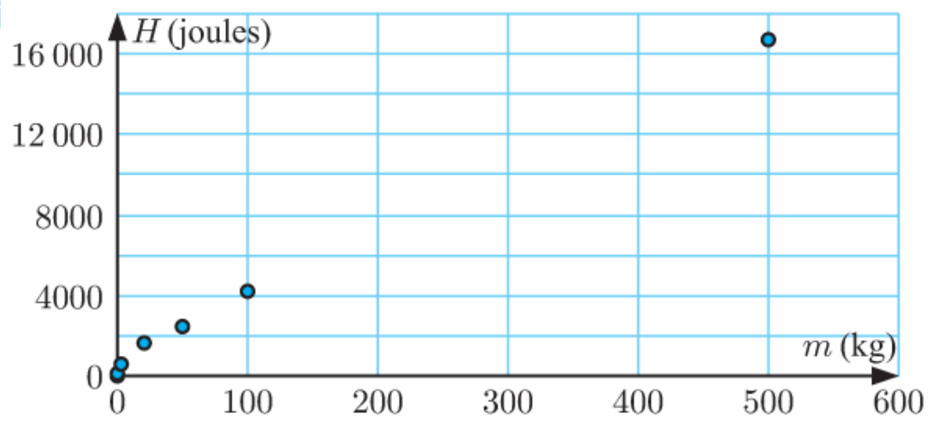
d The value of SS_{res} is small (less than 1), so the model fits the data very well.

REVIEW SET 8B

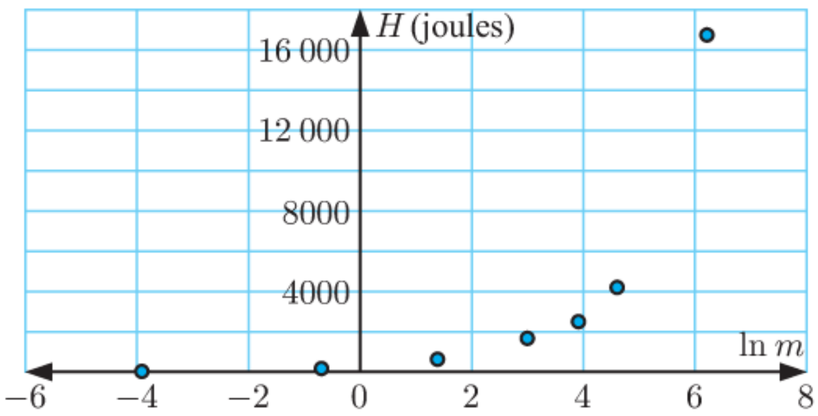
1 a $N \approx 0.607 \times 1.396^x$

b $K = \dfrac{e^2}{\sqrt{d}}$ $\quad (K \approx 7.39 \times d^{-0.5})$

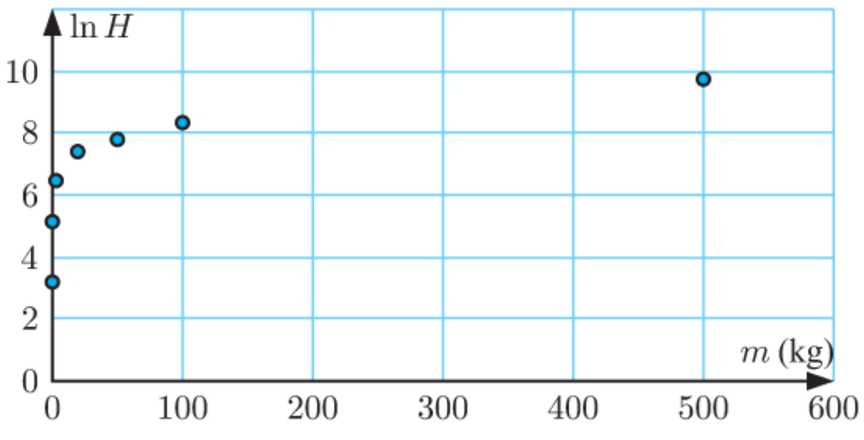
2 a i



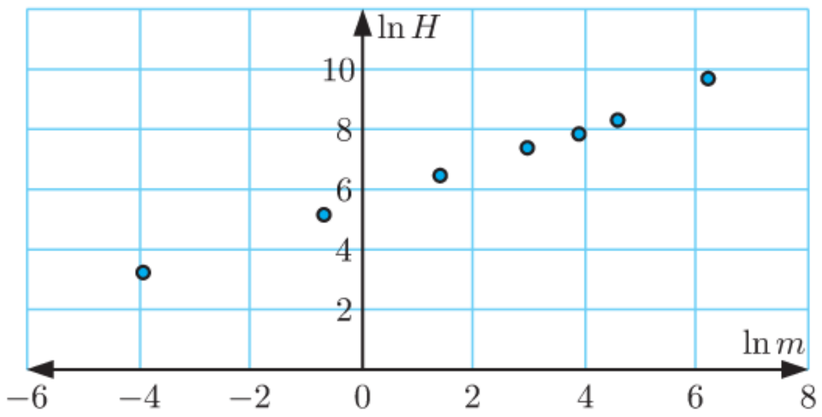
ii



iii



iv



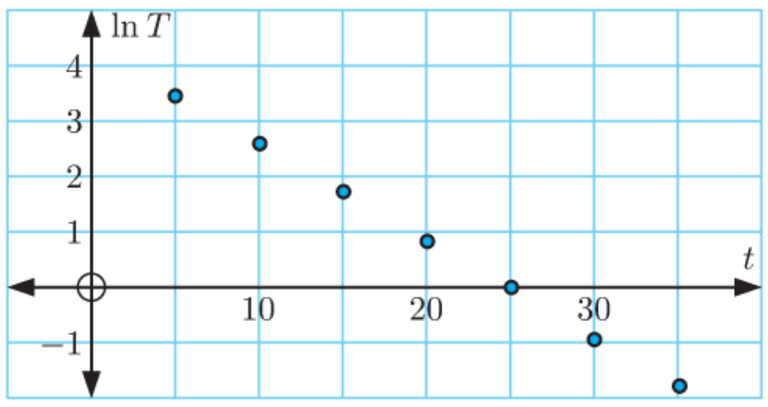
b The scatter diagram for $\ln H$ against $\ln m$ appears to be linear.

c $\ln H \approx 0.624 \ln m + 5.58, \quad r^2 \approx 0.995$

d $H \approx 265 \times m^{0.624}$

e $H \approx 53\,900$ joules. Not likely to be accurate, as we are extrapolating a long way beyond the upper pole.

3 **a**



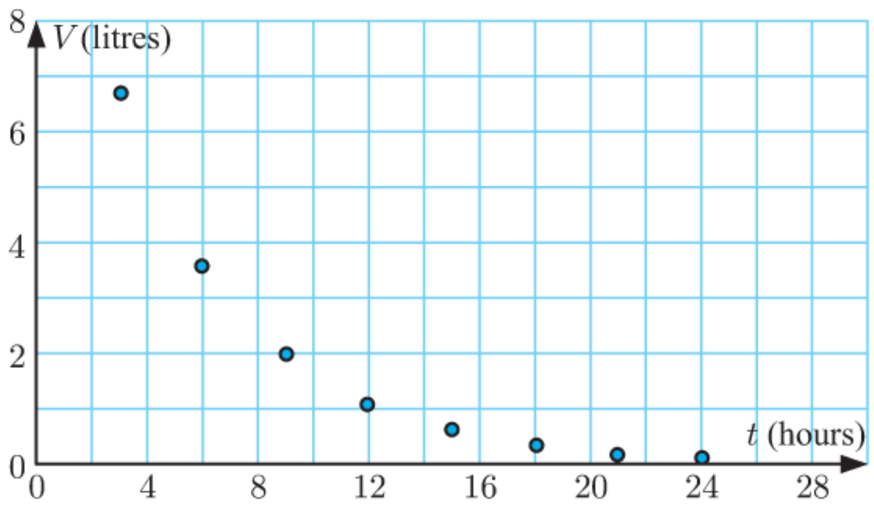
b Yes; the scatter diagram of $\ln T$ against t appears to be linear.

c $\approx 101°\text{C}$ **d** $\approx 78.4°\text{C}$

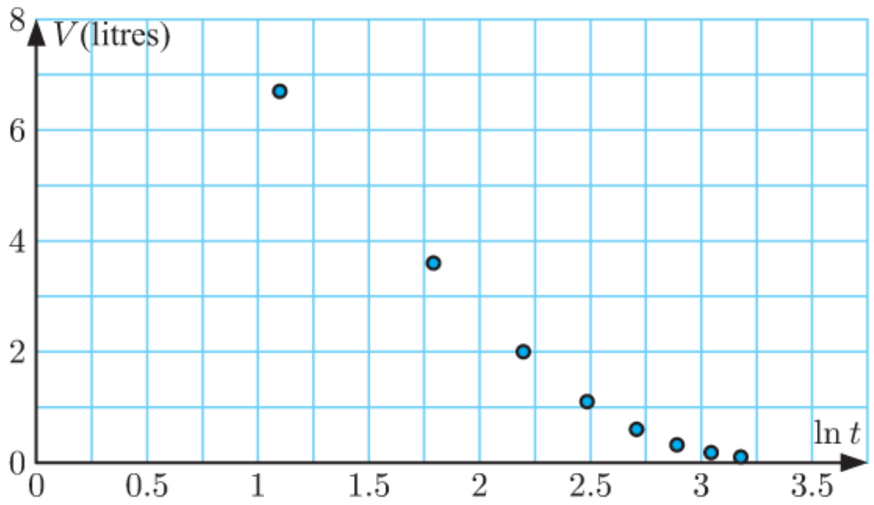
4 **a** The scatter diagram of b against $\ln P$ appears to be linear.

b $b \approx 9.03 - 0.441nP$ **c** ≈ 31.8 torr

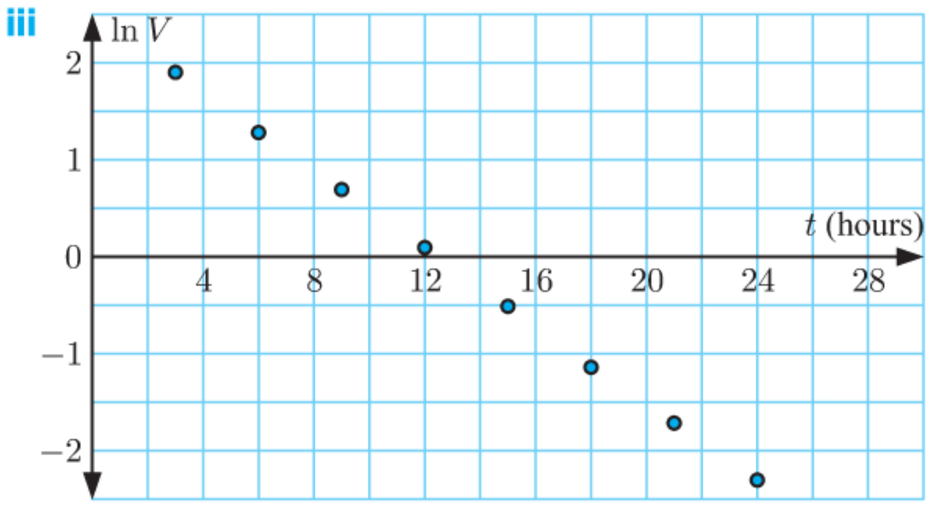
5 **a** **i**



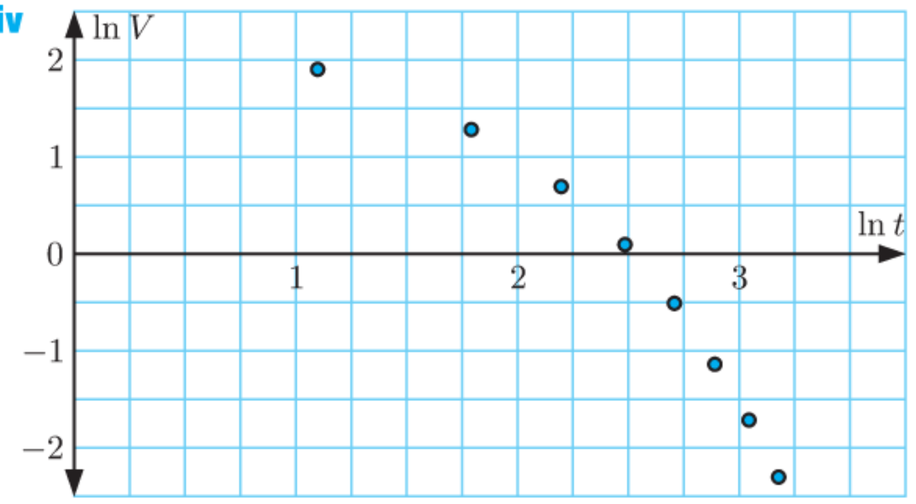
ii



iii



iv



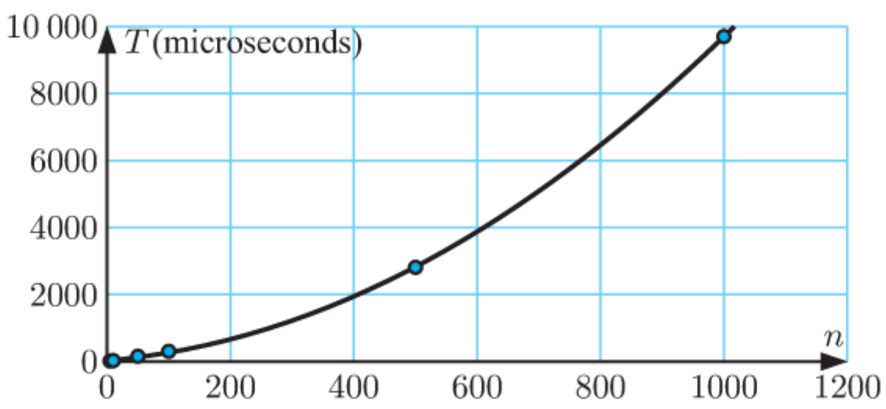
b $V \approx 12.1 \times 0.818^t$ **c** ≈ 4.45 litres **d** ≈ 10.5 litres

6 **a** $F \approx 138\,000 \times d^{-2.00}$, the graph of $\ln F$ against $\ln d$ appears to be linear.

b **i** ≈ 162 newtons **ii** ≈ 342 newtons

iii $\approx 0.005\,39$ newtons

7 **a**



b $T \approx 0.008\,15n^2 + 1.51n + 35.8$

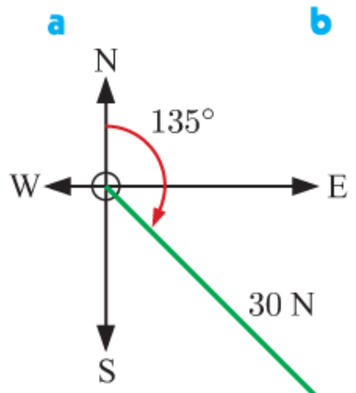
c $SS_{\text{res}} \approx 3550$ **d** ≈ 0.830 seconds

e The prediction in **d** is not very reliable because:

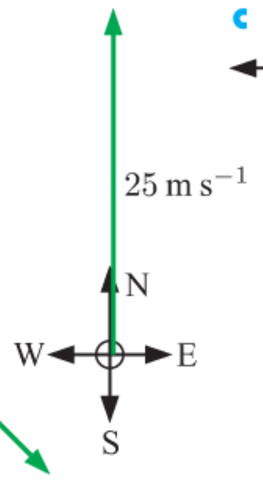
- it is an extrapolation far beyond the upper pole
- the value of SS_{res} is very large, suggesting a poor fit.

EXERCISE 9A.1

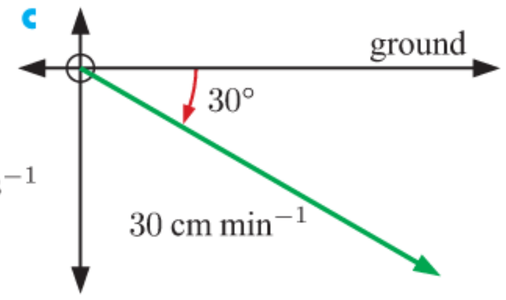
1 **a**



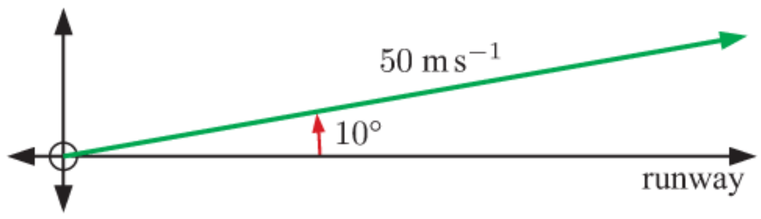
b



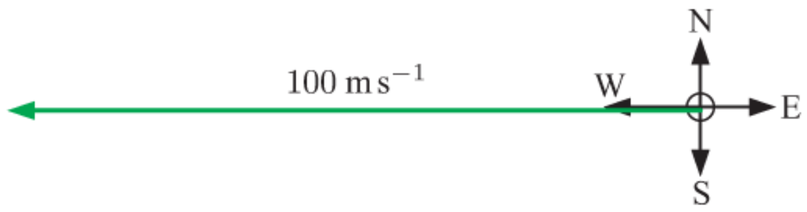
c



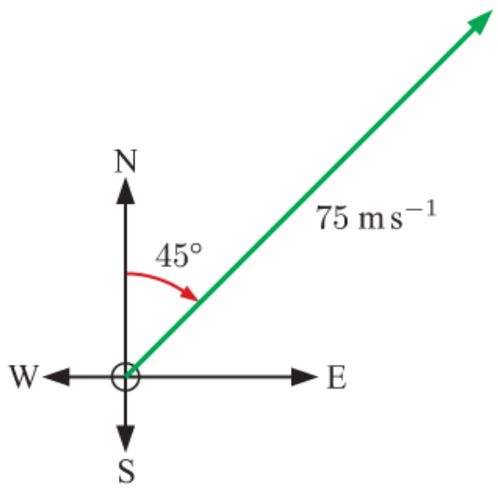
d



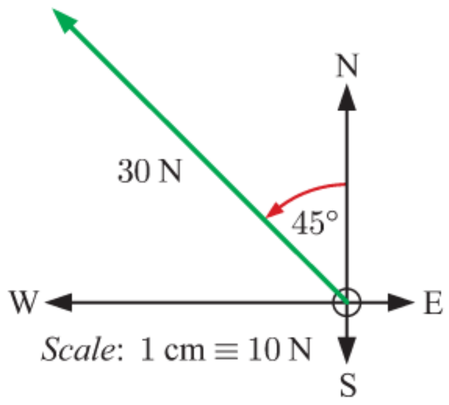
2 **a**



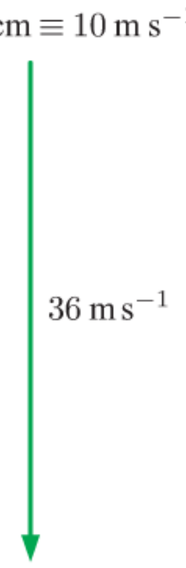
b



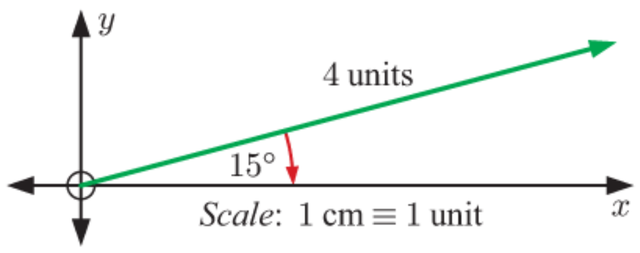
3 **a**



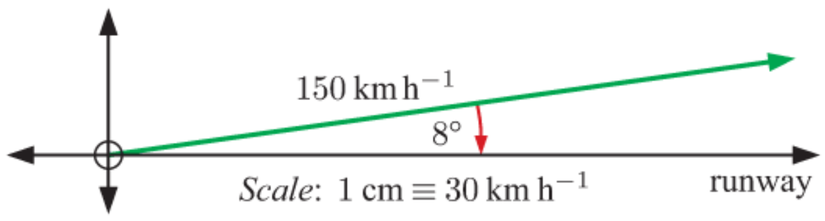
b



c



d



EXERCISE 9A.2

1 **a** p, q, s, t **b** p, q, r, t **c** p and r, q and t
d q, t **e** p and q, p and t

2 **a** true **b** true **c** false **d** false **e** true **f** false

3 **a** **i** $\overrightarrow{BC}$ **ii** $\overrightarrow{ED}$

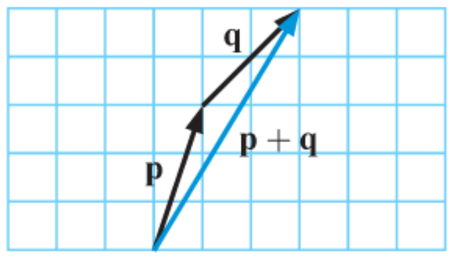
b **i** $\overrightarrow{FE}$, $\overrightarrow{BC}$

ii $\overrightarrow{DE}$, $\overrightarrow{EF}$, $\overrightarrow{FE}$, $\overrightarrow{FA}$, $\overrightarrow{AF}$, $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{DC}$

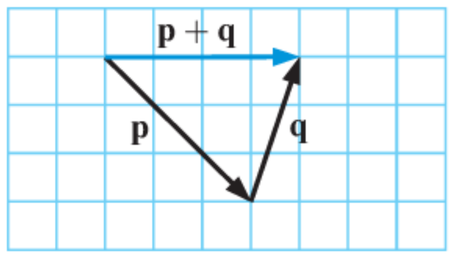
c $\overrightarrow{FC}$ (or $\overrightarrow{CF}$)

EXERCISE 9B.1

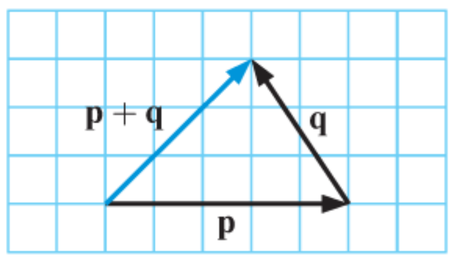
1 **a**



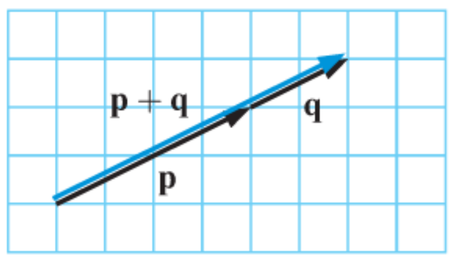
b



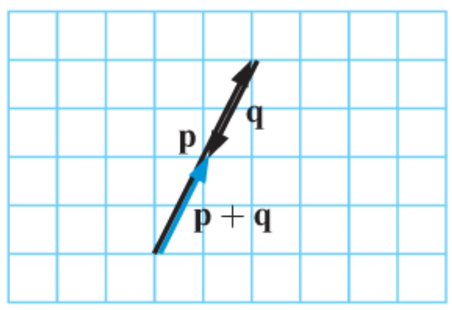
c



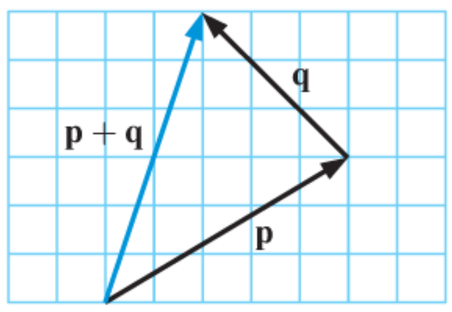
d



e

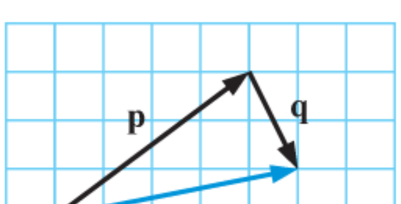


f

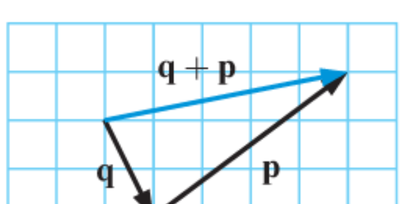


2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{BD}$ **c** 0 **d** $\overrightarrow{AD}$ **e** $\overrightarrow{AD}$ **f** 0

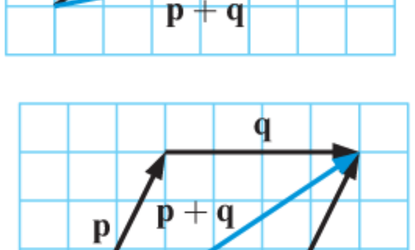
3 **a** **i**



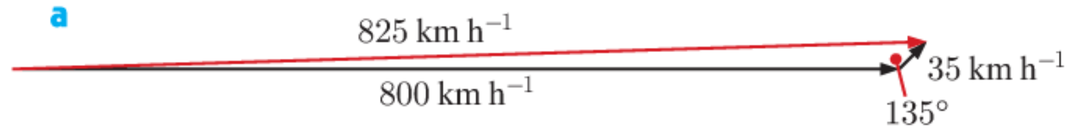
ii



b yes



5 **a**



Scale: 1 cm ≡ 125 km h^{-1}

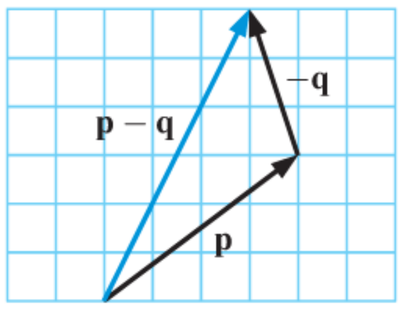
b We use vector addition.

c ≈ 825 km h^{-1}, 88° east of north

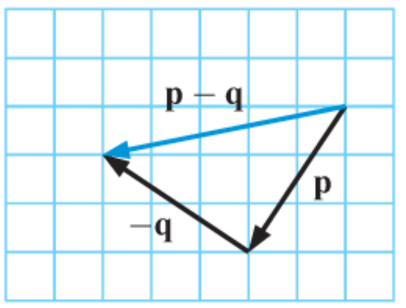
6 **a, b** ≈ 70.0 N, in the direction ≈ 357°

EXERCISE 9B.2

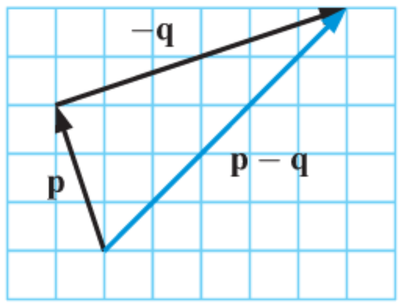
1 **a**



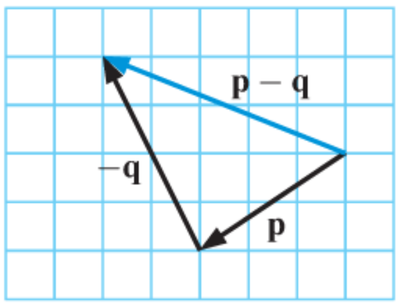
b



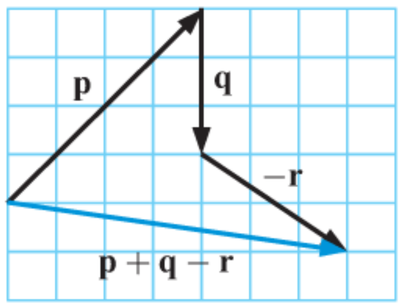
c



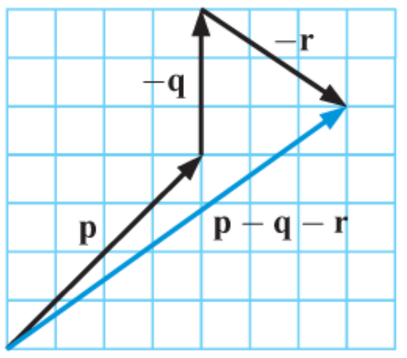
d



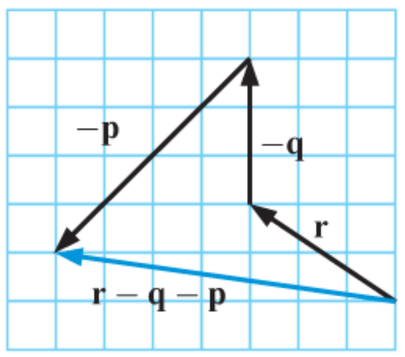
2 **a**



b



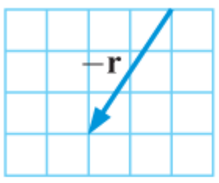
c



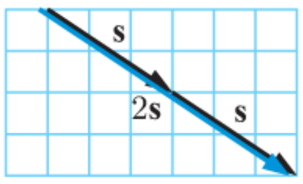
3 **a** $\overrightarrow{AB}$ **b** $\overrightarrow{AB}$ **c** 0 **d** $\overrightarrow{AD}$ **e** 0 **f** $\overrightarrow{AD}$

EXERCISE 9B.3

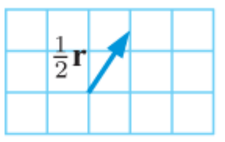
1 **a**



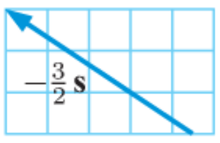
b



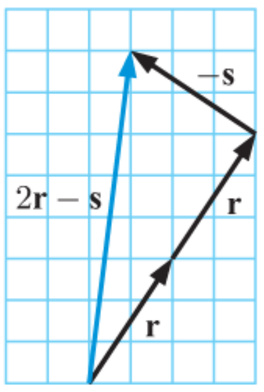
c



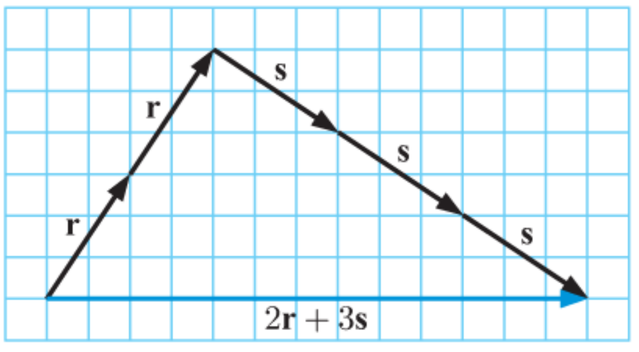
d



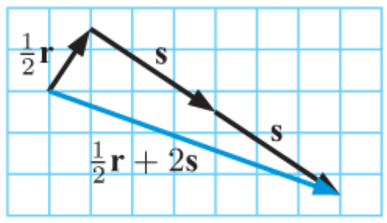
e



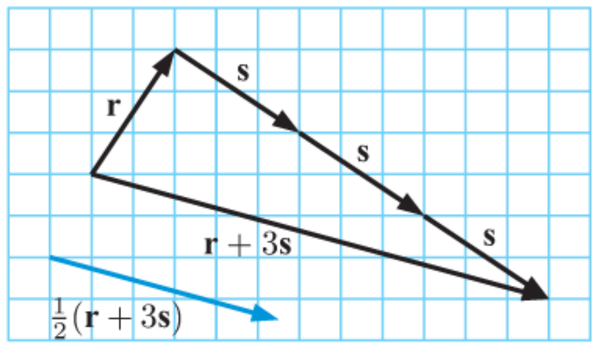
f



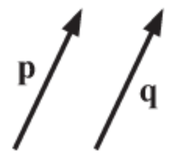
g



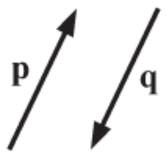
h



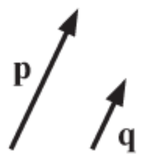
2 **a**



b

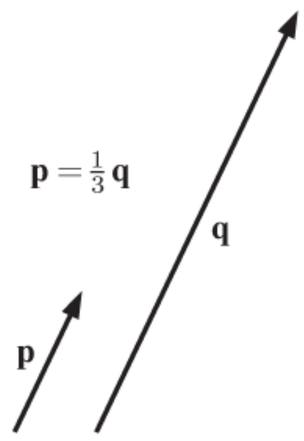


c

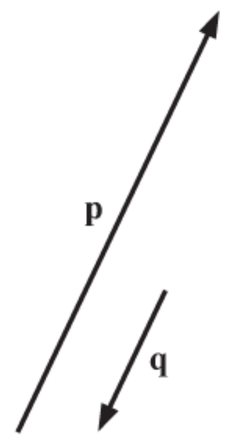


$\mathbf{p} = 2\mathbf{q}$

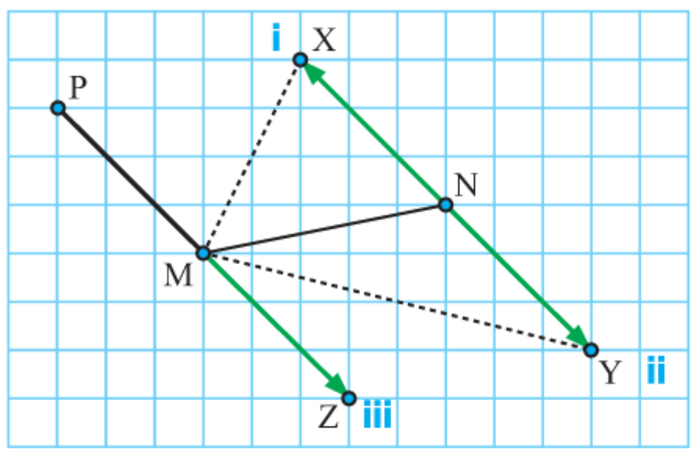
d



e



3 **a**



b a parallelogram

4 **a** $-\mathbf{p}$ **b** $\mathbf{p} + \mathbf{q}$ **c** $\frac{1}{2}(\mathbf{p} + \mathbf{q})$ **d** $\frac{1}{2}(\mathbf{q} - \mathbf{p})$

5 **a** $\mathbf{b}$ **b** $2\mathbf{b}$ **c** $\mathbf{b} - \mathbf{a}$ **d** $\mathbf{b} - \mathbf{a}$

6 **a** $\overrightarrow{BD} = \frac{1}{2}\mathbf{a}$ **b** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **c** $\overrightarrow{BA} = -\mathbf{b} + \mathbf{a}$

d $\overrightarrow{OD} = \mathbf{b} + \frac{1}{2}\mathbf{a}$ **e** $\overrightarrow{AD} = \mathbf{b} - \frac{1}{2}\mathbf{a}$ **f** $\overrightarrow{DA} = \frac{1}{2}\mathbf{a} - \mathbf{b}$

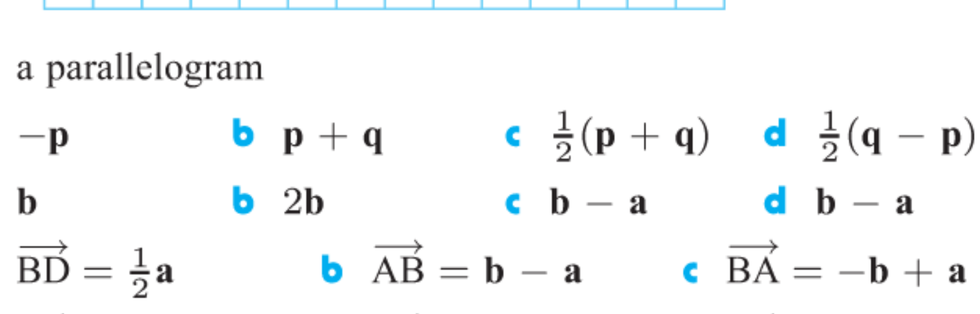
EXERCISE 9C

1 **a** $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $7\mathbf{i} + 3\mathbf{j}$ **b** $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$, $-6\mathbf{i}$

c $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $2\mathbf{i} - 5\mathbf{j}$ **d** $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $6\mathbf{j}$

e $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$, $-6\mathbf{i} + 3\mathbf{j}$ **f** $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$, $-5\mathbf{i} - 5\mathbf{j}$

2 **a** $3\mathbf{i} + 4\mathbf{j}$ **b** $2\mathbf{i}$ **c** $2\mathbf{i} - 5\mathbf{j}$ **d** $-\mathbf{i} - 3\mathbf{j}$



3 **a** **i** $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$, $-4\mathbf{i} - \mathbf{j}$ **ii** $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $-\mathbf{i} - 5\mathbf{j}$

iii $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $2\mathbf{i}$ **iv** $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $3\mathbf{i} - 4\mathbf{j}$

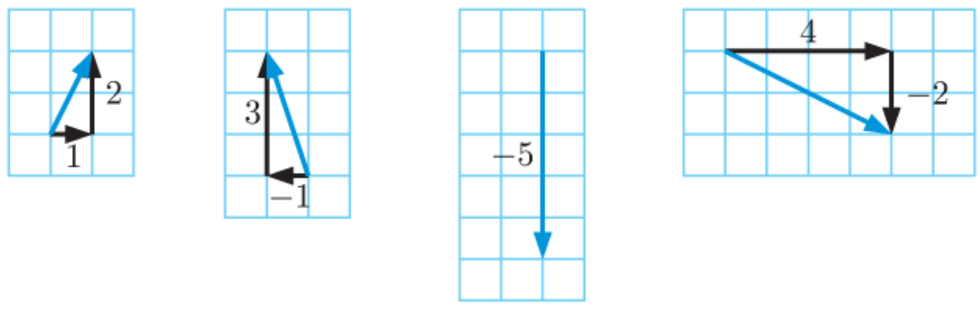
v $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$, $-3\mathbf{i} + 4\mathbf{j}$ **vi** $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $3\mathbf{i} + 5\mathbf{j}$

vii $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $5\mathbf{i} - 3\mathbf{j}$ **viii** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $2\mathbf{i} + \mathbf{j}$

ix $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$, $-4\mathbf{i} - \mathbf{j}$

b $\overrightarrow{BA} = \overrightarrow{ED}$, $\overrightarrow{AB} = \overrightarrow{DE}$, $\overrightarrow{AD} = \overrightarrow{BE}$, $\overrightarrow{DA} = \overrightarrow{EB}$

4 **a** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$



5 $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6 **a** $a = 3,\ b = 5$ **b** $a = 4,\ b = 6$ **c** $x = 4,\ y = 1$

EXERCISE 9D

1 **a** 5 units **b** 5 units **c** 2 units

d $\sqrt{8}$ units **e** 3 units

2 **a** $\sqrt{2}$ units **b** 13 units **c** $\sqrt{17}$ units

d 3 units **e** $|k|$ units

3 **a** unit vector **b** unit vector **c** not a unit vector

d unit vector **e** not a unit vector

4 **a** $k = \pm 1$ **b** $k = \pm 1$ **c** $k = 0$

d $k = \pm\frac{1}{\sqrt{2}}$ **e** $k = \pm\frac{\sqrt{3}}{2}$

5 $p = \pm 3$

EXERCISE 9E

1 **a** $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ **c** $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

e $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$ **f** $\begin{pmatrix} -5 \\ -3 \end{pmatrix}$ **g** $\begin{pmatrix} -6 \\ 4 \end{pmatrix}$ **h** $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$

2 **a** $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$ **c** $\begin{pmatrix} -8 \\ -1 \end{pmatrix}$ **d** $\begin{pmatrix} -6 \\ 9 \end{pmatrix}$

e $\begin{pmatrix} 0 \\ -5 \end{pmatrix}$ **f** $\begin{pmatrix} 6 \\ -9 \end{pmatrix}$

3 **a** $\mathbf{a} + \mathbf{0} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + 0 \\ a_2 + 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathbf{a}$

b $\mathbf{a} - \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_1 \\ a_2 - a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

4 **a** $\begin{pmatrix} -3 \\ -15 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 \\ 14 \end{pmatrix}$ **d** $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$

e $\begin{pmatrix} \frac{5}{2} \\ \frac{11}{2} \end{pmatrix}$ **f** $\begin{pmatrix} -7 \\ 7 \end{pmatrix}$ **g** $\begin{pmatrix} 5 \\ 11 \end{pmatrix}$ **h** $\begin{pmatrix} 3 \\ \frac{17}{3} \end{pmatrix}$

5 **a** **i** $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$ **ii** $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$ **iii** $\begin{pmatrix} 8 \\ -1 \end{pmatrix}$

b The vector expressions are equal, as each consists of the sum of 2 **p**s and 3 **q**s. Each expression is equal to $2\mathbf{p} + 3\mathbf{q}$.

6 **a** $k = 6$ **b** $k = -\frac{7}{5}$

7 **a** $\sqrt{13}$ units **b** $\sqrt{17}$ units **c** $5\sqrt{2}$ units

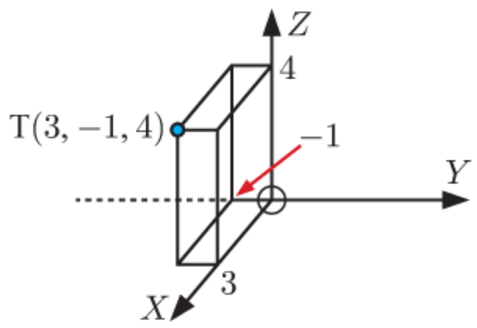
d $\sqrt{10}$ units **e** $\sqrt{29}$ units **f** $2\sqrt{13}$ units

g 11 units **h** $\sqrt{29}$ units

8 **a** **i** $\sqrt{10}$ units **ii** $2\sqrt{10}$ units **iii** $2\sqrt{10}$ units
iv $3\sqrt{10}$ units **v** $3\sqrt{10}$ units

EXERCISE 9F

1 **a**



b $\overrightarrow{OT} = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

c OT $= \sqrt{26}$ units

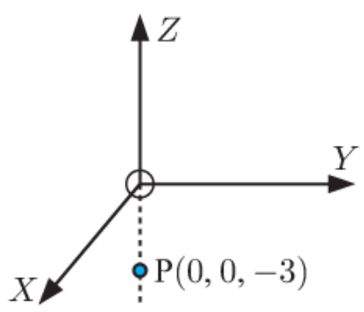
2 **a** $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = 2\mathbf{i} + 4\mathbf{k}$

b $\overrightarrow{OP} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

c $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = -2\mathbf{j}$

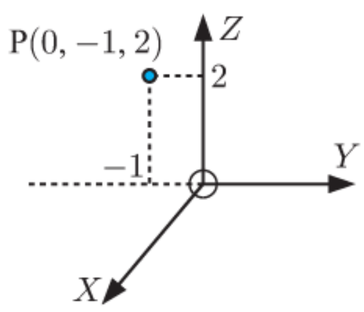
3 **a** 3 units **b** 5 units **c** $\sqrt{21}$ units **d** 3 units

4 **a**



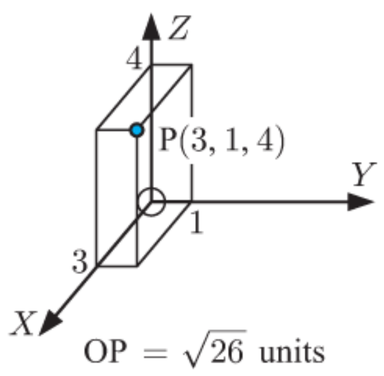
OP $= 3$ units

b



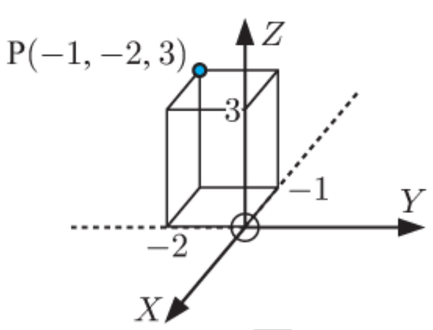
OP $= \sqrt{5}$ units

c



OP $= \sqrt{26}$ units

d



OP $= \sqrt{14}$ units

5 **a** $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$, $|\mathbf{a}| = \sqrt{21}$ units

b $\mathbf{b} = \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$, $|\mathbf{b}| = 5$ units

c $\mathbf{c} = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, $|\mathbf{c}| = \sqrt{13}$ units

6 **a** $a = 5,\ b = 6,\ c = -6$ **b** $a = 4,\ b = 2,\ c = 1$
c $a = 2,\ b = \frac{16}{5},\ c = \frac{26}{5}$

7 **a**, **c**, and **d** are unit vectors.

8 **a** $k = \pm\frac{\sqrt{11}}{4}$ **b** $k = \pm\frac{2}{3}$ **c** $k = \pm\frac{1}{\sqrt{3}}$
d $k = \pm\sqrt{14}$

9 **a** $m = \pm 3$ **b** $m = \pm\sqrt{11}$

10 $5 \leqslant \left| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right| \leqslant \sqrt{34}$

EXERCISE 9G

1 **a** $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$ **c** $\begin{pmatrix} 1 \\ 4 \\ -9 \end{pmatrix}$
d $\begin{pmatrix} -1 \\ \frac{3}{2} \\ -\frac{7}{2} \end{pmatrix}$ **e** $\begin{pmatrix} 1 \\ -4 \\ 7 \end{pmatrix}$ **f** $\begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

2 **a** $\sqrt{10}$ **b** $\sqrt{6}$ **c** $2\sqrt{10}$ **d** $2\sqrt{10}$
e $-3\sqrt{6}$ **f** $3\sqrt{6}$ **g** $3\sqrt{2}$ **h** $\sqrt{14}$

3 **a** $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ -8 \\ 8 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ -10 \\ 3 \end{pmatrix}$ **d** $\begin{pmatrix} -14 \\ 24 \\ 9 \end{pmatrix}$
e $\sqrt{38}$ **f** $\sqrt{3}$ **g** $\begin{pmatrix} \sqrt{11} \\ -3\sqrt{11} \\ 2\sqrt{11} \end{pmatrix}$ **h** $\begin{pmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{pmatrix}$

4 **a** $a = \frac{1}{3},\ b = 2,\ c = 1$ **b** $a = 1,\ b = -1,\ c = 2$
c $a = 4,\ b = -1,\ c = -19$
d $a = 0,\ b = -1,\ c = -1$ *or* $a = -\frac{1}{2},\ b = 0,\ c = -1$

EXERCISE 9H

1 **a** $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$

2 **a** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ **c** $\sqrt{17}$ units

3 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

b $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$ **c** $\overrightarrow{BC} = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

4 **a** $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 6 \\ -5 \end{pmatrix}$

5 **a** M(1, 4) **b** $\overrightarrow{CA} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$, $\overrightarrow{CM} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

6 **a** $\begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}$ **d** $\begin{pmatrix} -5 \\ -4 \\ -2 \end{pmatrix}$

7 $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$, AB $= \sqrt{29}$ units

8 **a** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -1 \\ -3 \end{pmatrix}$, $\overrightarrow{BA} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$

b $|\overrightarrow{AB}| = \sqrt{26}$ units, $|\overrightarrow{BA}| = \sqrt{26}$ units

9 **a** $\overrightarrow{NM} = \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$ **b** $\overrightarrow{MN} = \begin{pmatrix} -5 \\ 4 \\ 1 \end{pmatrix}$

c MN $= \sqrt{42}$ units

10 **a** $\overrightarrow{AB} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ **b** $5\sqrt{2}$ units

11 **a** $\sqrt{13}$ units **b** $\sqrt{14}$ units **c** 3 units

12 $\overrightarrow{AC} = -\mathbf{i} - 2\mathbf{k}$

13 **a** $\begin{pmatrix} -1 \\ 5 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 4 \\ -2 \end{pmatrix}$ **c** $\begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$

14 **a** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ k-3 \end{pmatrix}$, $|\overrightarrow{AB}| = \sqrt{16 + (k-3)^2}$ units

b $k = 0$ or 6

15 **a** B(4, 2) **b** C(2, 2) **16** **a** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **b** Q(3, 3)

17 **a** $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ -1 \end{pmatrix}$ **c** D$(-1, -2)$

18 B$(0, 3, 5)$, radius $= \sqrt{3}$ units

19 **a** $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix}$, AB $= \sqrt{17}$ units

b $\overrightarrow{AC} = \begin{pmatrix} -2 \\ -1 \\ -5 \end{pmatrix}$, AC $= \sqrt{30}$ units

c $\overrightarrow{CB} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$, CB $= \sqrt{35}$ units

d ABC is scalene, and not right angled.

21 **a** $|\overrightarrow{AB}| = \sqrt{158}$ units, $|\overrightarrow{BC}| = \sqrt{129}$ units, $|\overrightarrow{AC}| = \sqrt{29}$ units, and $29 + 129 = 158$

b area ≈ 30.6 units^2

22 **a** $r = 2,\ s = 4,\ t = -7$ **b** $r = -4,\ s = 0,\ t = 3$

23 **a** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$, $\overrightarrow{DC} = \begin{pmatrix} 2 \\ -5 \\ -1 \end{pmatrix}$

b ABCD is a parallelogram.

24 **a** S$(-2, 8, -3)$ **b** midpoints are at $(-\frac{1}{2}, 3, 1)$

25 **a** parallelogram **b** not a parallelogram

26 **a** R$(3, 1, 6)$ **b** X$(2, -1, 0)$

EXERCISE 9I

1 **a** parallel **b** not parallel **c** parallel
d parallel **e** parallel **f** not parallel

2 **a** $a = 3$ **b** $a = -6$

3 **a** $r = 3,\ s = -9$ **b** $r = -6,\ s = -4$

4 **a** $\overrightarrow{AB} \parallel \overrightarrow{CD}$, $|\overrightarrow{AB}| = 3|\overrightarrow{CD}|$

b $\overrightarrow{RS} \parallel \overrightarrow{KL}$, $|\overrightarrow{RS}| = \frac{1}{2}|\overrightarrow{KL}|$, $\overrightarrow{RS}$ and $\overrightarrow{KL}$ are in opposite directions.

c $\overrightarrow{AB} \parallel \overrightarrow{BC}$, $|\overrightarrow{AB}| = 2|\overrightarrow{BC}|$, A, B, and C are collinear.

5 $\overrightarrow{QS} = 2\overrightarrow{PR}$, PR : QS $= 1 : 2$

6 **a** $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$

7 **a** $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ **b** $\frac{2}{\sqrt{13}}\mathbf{i} - \frac{3}{\sqrt{13}}\mathbf{j}$ **c** $-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

8 **a** $\frac{1}{\sqrt{13}}(2\mathbf{i} + 3\mathbf{k})$ **b** $\frac{1}{\sqrt{6}}(-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

c $\frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

9 **a** $\begin{pmatrix} \frac{6}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{2}{\sqrt{17}} \\ \frac{8}{\sqrt{17}} \end{pmatrix}$ **c** $\sqrt{2}\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ **d** $\frac{5}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

10 **a** $\pm\frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ **b** $\pm\frac{2}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}$

11 **a** $\overrightarrow{AB} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}$ **b** $\overrightarrow{OB} = \begin{pmatrix} 3 + 2\sqrt{2} \\ 2 - 2\sqrt{2} \end{pmatrix}$

c B$(3 + 2\sqrt{2},\ 2 - 2\sqrt{2})$

12 **a** X$(4, -1)$ **b** X$(-\frac{13}{5}, 3, \frac{14}{5})$

c X$\left(2 + \frac{4}{\sqrt{6}},\ -1 - \frac{4}{\sqrt{6}},\ 4 + \frac{8}{\sqrt{6}}\right)$

13 **a** $\overrightarrow{AB} = -\frac{5}{3}\overrightarrow{BC}$ **b** $\overrightarrow{PQ} = -\frac{3}{5}\overrightarrow{QR}$ **c** $\overrightarrow{AB} = \frac{2}{5}\overrightarrow{BC}$

d $\overrightarrow{PQ} = -\frac{1}{2}\overrightarrow{QR}$

14 **a** $a = 7,\ b = -1$ **b** $a = -\frac{7}{2},\ b = -\frac{21}{2}$

15 **a** $\begin{pmatrix} x - X \\ y - Y \\ z - Z \end{pmatrix}$ **b** $(x - X)^2 + (y - Y)^2 + (z - Z)^2 = r^2$

EXERCISE 9J

1 **a** 11 **b** 6 **c** 2 **d** -9

2 **a** 7 **b** 22 **c** 29 **d** 66 **e** 52 **f** 3 **g** 5 **h** 1

3 **a** 2 **b** 2 **c** 14 **d** 14 **e** 4 **f** 4

4 **a** 1 **b** 1 **c** 3 **d** 0 **e** 0 **f** 5

5 **a** **i** 23 **ii** 12 **iii** -7 **iv** -2

6 $\mathbf{a} \bullet \mathbf{b}$ is a scalar and so $\mathbf{a} \bullet \mathbf{b} \bullet \mathbf{c}$ is the scalar product of a scalar and a vector, which is meaningless.

7 16

EXERCISE 9K

1 **a** 11 **b** $\approx 49.8°$

2 **a** $\approx 71.6°$ **b** $\approx 177°$ **c** $\approx 18.4°$ **d** $45°$

4 **Hint:** Show $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{c} = \mathbf{a} \bullet \mathbf{c} = 0$.

5 **a** $\approx 69.0°$ **b** $\approx 94.1°$ **c** $\approx 114°$

6 **a** $t = 6$ **b** $t = -8$ **c** $t = 0$ or 2

d $t = -\frac{3}{2}$ **e** $t = -\frac{5}{6}$ **f** $t = 0$ or -3

7 $r = -2,\ s = 5,\ t = -4$

8 **Note:** The negative of these vectors are also valid answers.

a $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

9 $\begin{pmatrix} -\frac{15}{\sqrt{13}} \\ \frac{10}{\sqrt{13}} \end{pmatrix}$ and $\begin{pmatrix} \frac{15}{\sqrt{13}} \\ -\frac{10}{\sqrt{13}} \end{pmatrix}$

10 ≈ 3.089 **11** **a** 5 **b** -9

12 **a** $\mathbf{a} \bullet \mathbf{b} \neq 0$ **b** $|\mathbf{a}| = 12$ **13** **a** $\mathbf{c} = \mathbf{d}$ **b** $\mathbf{c} = -\mathbf{d}$

14 $\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

15 **Hint:** Choose a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, where a and b are integers.

Solve for c such that $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$.

16 $\overrightarrow{AB} \bullet \overrightarrow{AC} = 0$, $\therefore$ $\widehat{BAC}$ is a right angle

17 **b** $|\overrightarrow{AB}| = \sqrt{14}$ units, $|\overrightarrow{BC}| = \sqrt{14}$ units, ABCD is a rhombus

c 0; the diagonals of a rhombus are perpendicular.

18 $\widehat{ABC} \approx 62.5°$, the exterior angle $\approx 117.5°$

19 **a** $\approx 54.7°$ **b** $60°$ **c** $\approx 35.3°$

20 **a** $\approx 30.3°$ **b** $\approx 54.2°$

EXERCISE 9L.1

1 **a** $\begin{pmatrix} 2 \\ 5 \\ 11 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ **c** $\begin{pmatrix} 3 \\ 14 \\ 1 \end{pmatrix}$

d $-\mathbf{i} - \mathbf{j} - \mathbf{k}$ **e** $\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$ **f** $\begin{pmatrix} -8 \\ -5 \\ 1 \end{pmatrix}$

3 **a** $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -11 \\ -2 \\ 5 \end{pmatrix}$ **b** $\mathbf{a} \bullet (\mathbf{a} \times \mathbf{b}) = 0 = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{b})$

c $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

4 **a** $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, $\mathbf{j} \times \mathbf{j} = \mathbf{0}$, $\mathbf{k} \times \mathbf{k} = \mathbf{0}$

$\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for all vectors $\mathbf{a}$.

b $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for all vectors $\mathbf{a}$ and $\mathbf{b}$.

c **i** $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$

ii $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

5 **a** $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ **b** 17

6 **a** $2\mathbf{i} - \mathbf{j} - \mathbf{k}$ **b** $5\mathbf{j}$ **c** $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ **d** $2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

7 **a** $k\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$

b $k\begin{pmatrix} 6 \\ 22 \\ -15 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$

c $(-\mathbf{i} + \mathbf{j} - 2\mathbf{k})n$, $n \in \mathbb{R}$, $n \neq 0$

d $(5\mathbf{i} + \mathbf{j} + 4\mathbf{k})n$, $n \in \mathbb{R}$, $n \neq 0$

8 $k\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$, $k \in \mathbb{R}$, $k \neq 0$, $\frac{\sqrt{10}}{6}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$ and $-\frac{\sqrt{10}}{6}\begin{pmatrix} 4 \\ -5 \\ -7 \end{pmatrix}$

EXERCISE 9L.2

1 **a** $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$

2 **a** $\mathbf{a} \bullet \mathbf{b} = -1$, $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$ **b** $\cos\theta = -\frac{1}{\sqrt{28}}$

c $\sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$ **d** $\sin\theta = \frac{\sqrt{27}}{\sqrt{28}}$

4 **a** $\mathbf{p} \bullet \mathbf{q} = 0 \quad \therefore \quad \mathbf{p} \perp \mathbf{q}$ **b** $\begin{pmatrix} -4 \\ -17 \\ -5 \end{pmatrix}$

c $|\mathbf{p} \times \mathbf{q}| = \sqrt{330}$, $|\mathbf{p}| = \sqrt{30}$, $|\mathbf{q}| = \sqrt{11}$

$\therefore \quad |\mathbf{p}||\mathbf{q}| = \sqrt{30 \times 11} = \sqrt{330} = |\mathbf{p} \times \mathbf{q}|$

5 **a** $\overrightarrow{OC}$ is parallel to $\overrightarrow{AB}$ **b** $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$

EXERCISE 9L.3

1 **a** $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -3 \\ 5 \end{pmatrix}$

c $\frac{\sqrt{83}}{2}$ units^2

2 **a** $\frac{\sqrt{101}}{2}$ units^2 **b** $\frac{\sqrt{133}}{2}$ units^2 **c** $\frac{\sqrt{69}}{2}$ units^2

3 **a** $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$ units^2

b **i** $\overrightarrow{BO} = -\mathbf{b}$, $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$

ii $\frac{1}{2}|-\mathbf{b} \times (\mathbf{a} - \mathbf{b})|$ units^2

4 $x = 2 \pm 2\sqrt{33}$

5 **a** $\overrightarrow{CB} = \overrightarrow{OA}$ and $\overrightarrow{AB} = \overrightarrow{OC}$

$\therefore$ opposite pairs of sides are parallel and equal in length.

b $\sqrt{222}$ units^2

6 $8\sqrt{2}$ units^2 **7** **a** D$(-4, 1, 3)$ **b** $\sqrt{307}$ units^2

8 **a** **i** $|\mathbf{b} \times \mathbf{c}|$ units^2 **ii** $|\mathbf{a}|\sin\theta$ units

b **Hint:** Let ϕ be the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

c 9 units^3

EXERCISE 9M

1 **a** **i** 2 **ii** $\sqrt{13}$ **b** **i** $-\frac{8}{\sqrt{14}}$ **ii** $\frac{\sqrt{171}}{\sqrt{7}}$

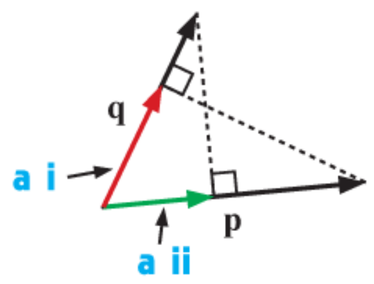
c **i** $\frac{5}{\sqrt{10}}$ **ii** $\frac{\sqrt{37}}{\sqrt{2}}$ **d** **i** $-\frac{6}{\sqrt{3}}$ **ii** $2\sqrt{2}$

2 **a** 0 **b** $\sqrt{14}$

$\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $|\mathbf{a}| = \sqrt{14}$

3 **a** **i** $-\frac{5}{\sqrt{26}}$ **b**

ii $-\frac{5}{\sqrt{14}}$



4 **a** Strongman A: 300 N, Strongman B: $60\sqrt{26} \approx 306$ N

b Strongman A: $\frac{2780}{\sqrt{86}}$, Strongman B: $\frac{2760}{\sqrt{86}}$

c No; the direction of Strongman A's exerted force is closer to the direction of the train's motion than that of Strongman B.

5 **a** The component of $\mathbf{a}$ perpendicular to $\mathbf{b}$ is in the plane containing $\mathbf{a}$ and $\mathbf{b}$. So, it is perpendicular to $\mathbf{a} \times \mathbf{b}$.
This component is perpendicular to both $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, so it has direction $\mathbf{b} \times (\mathbf{a} \times \mathbf{b})$.

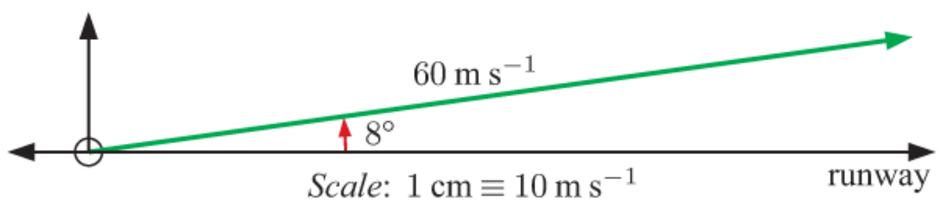
b **i** $\overrightarrow{OX} = \begin{pmatrix} \frac{12}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix}$ **ii** $\overrightarrow{XA} = \begin{pmatrix} -\frac{2}{5} \\ 1 \\ \frac{6}{5} \end{pmatrix}$

iii **(1)** $\overrightarrow{OX} \bullet \overrightarrow{XA} = 0 \quad \therefore \quad \overrightarrow{OX} \perp \overrightarrow{XA}$

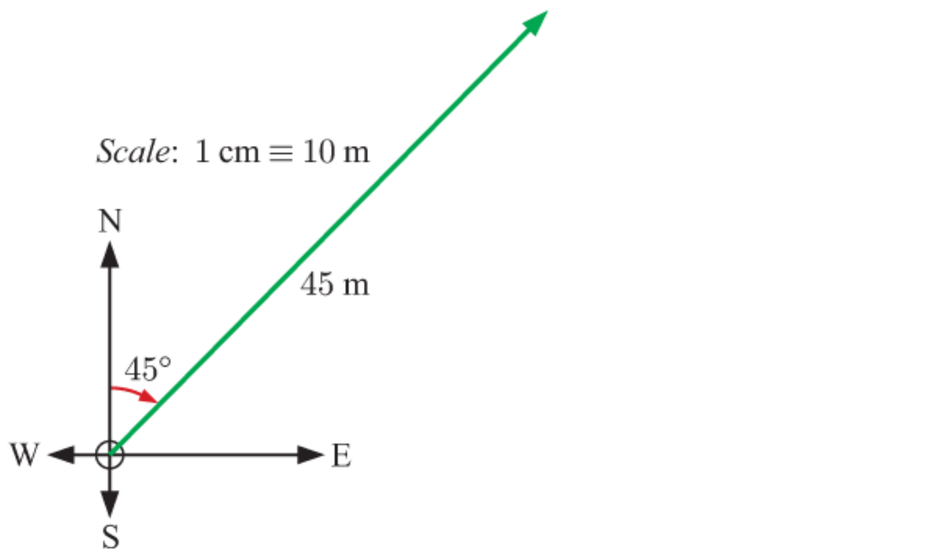
(2) $\overrightarrow{OX} + \overrightarrow{XA} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \mathbf{a}$

REVIEW SET 9A

1 **a**



b

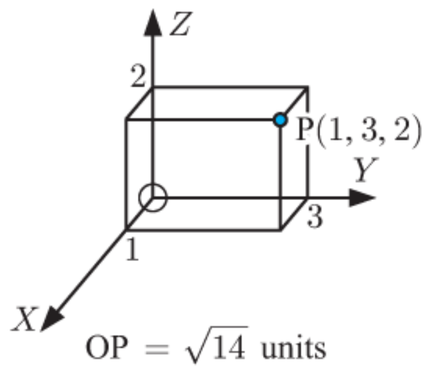


2 **a** $\overrightarrow{AC}$ **b** $\overrightarrow{AD}$

3 **a** $\begin{pmatrix}4\\3\end{pmatrix}$, $4\mathbf{i}+3\mathbf{j}$ **b** $\begin{pmatrix}-1\\-5\end{pmatrix}$, $-\mathbf{i}-5\mathbf{j}$

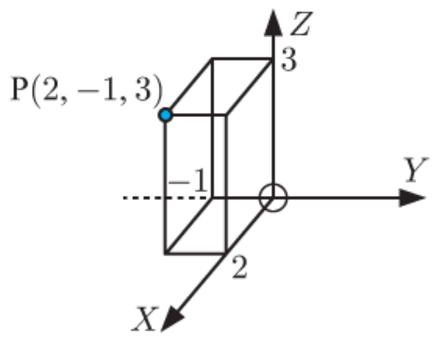
c $\begin{pmatrix}-4\\2\end{pmatrix}$, $-4\mathbf{i}+2\mathbf{j}$

4 **a**



OP $=\sqrt{14}$ units

b



OP $=\sqrt{14}$ units

5 $\begin{pmatrix}1\\4\\-9\end{pmatrix}$

6 **a** $\begin{pmatrix}-3\\0\\6\end{pmatrix}$ **b** $\sqrt{14}$ **c** $\begin{pmatrix}24\\-5\\-8\end{pmatrix}$ **d** $\begin{pmatrix}-1\\\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$

e $\begin{pmatrix}3\sqrt{53}\\-\sqrt{53}\\2\sqrt{53}\end{pmatrix}$ **f** $\sqrt{65}$

7 $a=1,\ b=1,\ c=-5$ **8** **a** 5 units **b** $2\mathbf{i}-5\mathbf{j}+4\mathbf{k}$

9 $m=5,\ n=-\frac{1}{2}$

11 $-\frac{5}{\sqrt{14}}\begin{pmatrix}3\\2\\-1\end{pmatrix}$ **12** **a** -13 **b** -36

13 **a** $\mathbf{a}\bullet\mathbf{b}=-4$ **b** $\mathbf{b}\bullet\mathbf{c}=10$ **c** $\mathbf{a}\bullet\mathbf{c}=-10$

14 **a** 8 **b** $\approx 62.2°$ **15** $\approx 64.0°$ **16** $t=-4$

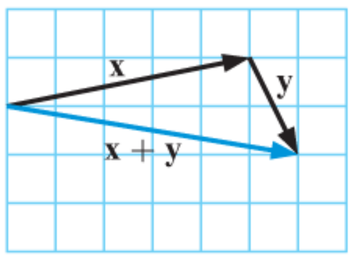
17 $\frac{1}{\sqrt{5}}\mathbf{i}+\frac{2}{\sqrt{5}}\mathbf{k}$ **18** $\approx 40.7°$

19 **a** $\begin{pmatrix}-3\\-7\\1\end{pmatrix}$ **b** 0 **c** $\begin{pmatrix}3\\7\\-1\end{pmatrix}$ **20** $\pm\frac{3}{\sqrt{59}}\begin{pmatrix}3\\7\\1\end{pmatrix}$

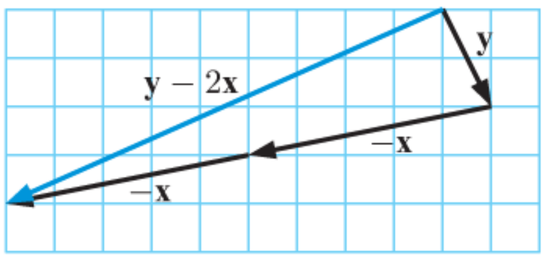
21 $\frac{\sqrt{66}}{2}$ units^2 **22** **a** $-\frac{\sqrt{7}}{\sqrt{2}}$ **b** $\frac{3\sqrt{3}}{\sqrt{2}}$

REVIEW SET 9B

1 **a**



b



2 **a** $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$

3 **a** $\begin{pmatrix}5\\2\end{pmatrix}$, $5\mathbf{i}+2\mathbf{j}$ **b** $\begin{pmatrix}-3\\3\end{pmatrix}$, $-3\mathbf{i}+3\mathbf{j}$

4 **a** $\mathbf{r}+\mathbf{q}$ **b** $-\mathbf{p}+\mathbf{r}+\mathbf{q}$ **c** $\mathbf{r}+\frac{1}{2}\mathbf{q}$ **d** $-\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{r}$

5 **a** $\begin{pmatrix}3\\-3\\11\end{pmatrix}$ **b** $\begin{pmatrix}7\\-3\\-26\end{pmatrix}$ **c** $\sqrt{74}$ **6** $\begin{pmatrix}8\\-8\\7\end{pmatrix}$

7 **a** $\overrightarrow{PQ}=\begin{pmatrix}-3\\12\\3\end{pmatrix}$ **b** $\sqrt{162}$ units **c** $\sqrt{61}$ units

8 $k=\pm\frac{1}{2}$ **9** $a=-2,\ b=0$ **10** $\pm\frac{4}{\sqrt{14}}(3\mathbf{i}-2\mathbf{j}+\mathbf{k})$

11 X$(6, 5, -13)$ **12** $\mathbf{v}\bullet\mathbf{w}=\pm 6$

14 **a** $\overrightarrow{PQ}=\begin{pmatrix}5\\-2\\-4\end{pmatrix}$ **b** $\approx 41.8°$ **15** $\approx 61.6°$

16 $t=2\pm\sqrt{2}$ **17** $\approx 80.3°$ **18** $\approx 16.1°$

19 $k\begin{pmatrix}0\\-2\\1\end{pmatrix},\ k\neq 0$ **20** $\mathbf{u}\bullet\mathbf{v}=\pm\sqrt{199}$

21 **a** $\begin{pmatrix}4\\5\\3\end{pmatrix}$ **b** C$\left(-\frac{24}{5\sqrt{2}}, -\frac{6}{\sqrt{2}}, -\frac{18}{5\sqrt{2}}\right)$

c $\frac{5}{\sqrt{2}}$ units^2 **d** $\frac{10}{\sqrt{2}}$ units^3

EXERCISE 10A

1 **a** **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=1-\lambda,\ y=4+2\lambda,\ \lambda\in\mathbb{R}$

b **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}5\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\5\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=5-2\lambda,\ y=2+5\lambda,\ \lambda\in\mathbb{R}$

c **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-6\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\7\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=-6+3\lambda,\ y=7\lambda,\ \lambda\in\mathbb{R}$

d **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\3\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=\lambda,\ y=2+3\lambda,\ \lambda\in\mathbb{R}$

e **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3\\0\end{pmatrix}+\lambda\begin{pmatrix}4\\2\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=3+4\lambda,\ y=2\lambda,\ \lambda\in\mathbb{R}$

f **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-2\\5\end{pmatrix}+\lambda\begin{pmatrix}6\\-11\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=-2+6\lambda,\ y=5-11\lambda,\ \lambda\in\mathbb{R}$

2 **a** $x=4-\lambda,\ y=-3+2\lambda,\ \lambda\in\mathbb{R}$

b Points are: $(4, -3), (3, -1), (2, 1), (5, -5), (7, -9)$

3 **a** $(0, 8)$

b It is parallel to $\begin{pmatrix}-1\\3\end{pmatrix}$ and in the opposite direction.

c $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\8\end{pmatrix}+\mu\begin{pmatrix}1\\-3\end{pmatrix},\ \mu\in\mathbb{R}$

4 **a** **i** $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\-7\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=1+2\lambda,\ y=3+\lambda,\ z=-7+3\lambda,\ \lambda\in\mathbb{R}$

b **i** $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-2\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=\lambda,\ y=1+\lambda,\ z=2-2\lambda,\ \lambda\in\mathbb{R}$

c **i** $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=-2+\lambda,\ y=2,\ z=1,\ \lambda\in\mathbb{R}$

d **i** $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\2\\-1\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix},\ \lambda\in\mathbb{R}$

ii $x=2\lambda,\ y=2-\lambda,\ z=-1+3\lambda,\ \lambda\in\mathbb{R}$

e **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

ii $x = 3, \ y = 2, \ z = -1 + \lambda, \ \lambda \in \mathbb{R}$

5 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$

b $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}, \ \lambda \in \mathbb{R}$

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$

d $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}, \ \lambda \in \mathbb{R}$

6 **a** $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$ **b** $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$

7 **a** $(0, 4, 1)$ **b** $(4, 0, 9)$ **c** $(-\frac{1}{2}, \frac{9}{2}, 0)$

8 **a** (x_0, y_0, z_0) **b** $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

9 **a** $\begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$ ✓

b $(9, 0, 10)$ and $(-3, 6, -2)$

10 $(0, 7, 3)$ and $(\frac{20}{3}, -\frac{19}{3}, -\frac{11}{3})$

11 **a** $\sqrt{3\lambda^2 + 2}$ units **b** $\lambda = 0$

12 **a** L_2: $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + s \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \ s \in \mathbb{R}$

b L_3: $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ -16 \\ 3 \end{pmatrix}, \ \lambda \in \mathbb{R}$

EXERCISE 10B

1 **a** $45°$ **b** $0°$ **c** $\approx 70.7°$ **d** $\approx 65.0°$ **2** $\approx 75.7°$

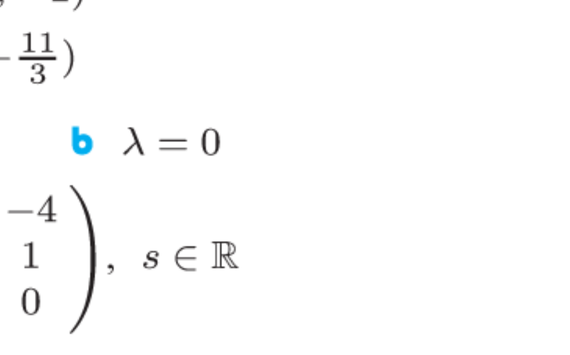
4 $\approx 75.5°$ **5** **a** $\approx 28.6°$ **b** **i** $a = -\frac{48}{7}$ **ii** $\approx 82.0°$

6 $\approx 76.7°$ **7** $\approx 49.9°$

EXERCISE 10C

1 **a** $(1, 2)$

b



c $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

d $\sqrt{29}$ cm s^{-1}

e $-\frac{8}{\sqrt{29}} \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

2 **a** **i** $(-4, 3, 0)$ **ii** $\begin{pmatrix} 12 \\ 5 \\ 6 \end{pmatrix}$ **iii** ≈ 14.3 m s^{-1}

b **i** $(3, 0, 4)$ **ii** $\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ **iii** 3 m s^{-1}

3 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \end{pmatrix}, \quad t \geqslant 0$ **b** $(8, -4.5)$

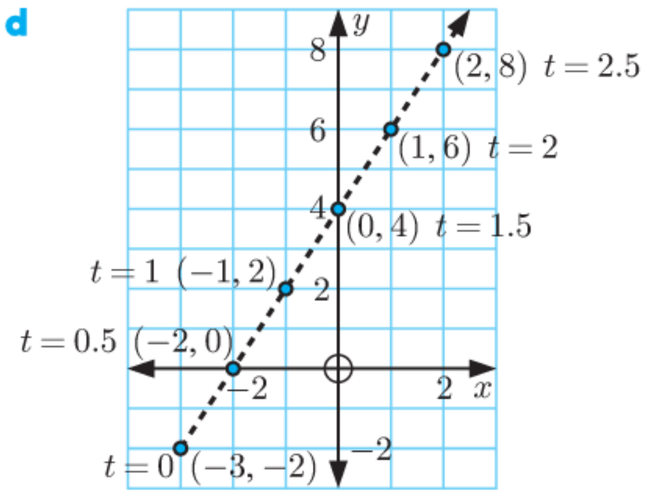
c 45 minutes

4 **a** $\begin{pmatrix} -3 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad t \geqslant 0$ **b** $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

c **i** $t = 1.5$ s

ii $t = 0.5$ s

d



5 **a** $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 7.2 \end{pmatrix}$ **c** $\begin{pmatrix} \frac{30\sqrt{85}}{17} \\ \frac{35\sqrt{85}}{17} \end{pmatrix}$

d $\begin{pmatrix} 2\sqrt{66} \\ \frac{7\sqrt{66}}{2} \\ \frac{\sqrt{66}}{2} \end{pmatrix}$ **e** $\begin{pmatrix} -12 \\ 30 \\ -84 \end{pmatrix}$

6 **a** A is at $(4, 5)$, B is at $(1, -8)$

b Yacht A: $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$; Yacht B: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

c Yacht A: $\sqrt{5}$ km h^{-1}; Yacht B: $\sqrt{5}$ km h^{-1}

d $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$

7 **a** $\begin{pmatrix} 3 \\ -3 \\ 1.5 \end{pmatrix}$ **b** $(100, -87, 45)$ **c** 810 m **d** $\approx 19.5°$

8 **a** $\begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}$ **b** ≈ 19.2 km h^{-1}

c $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix}, \ t \in \mathbb{R}$

d 1 hour **e** $\approx 8.98°$

EXERCISE 10D

1 **a** $\frac{3\sqrt{5}}{5}$ units **b** $\frac{\sqrt{2}}{2}$ units **c** $\frac{9\sqrt{2}}{2}$ units **d** 0 units

2 **a** $6\mathbf{i} - 6\mathbf{j}$ **b** $\begin{pmatrix} 6 - 6t \\ -6 + 8t \end{pmatrix}$ **c** when $t = \frac{3}{4}$ hours

d $t = 0.84$ and position is $(0.96, 0.72)$

3 **a** $\begin{pmatrix} -120 \\ -40 \end{pmatrix}$ **b** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 200 \\ 100 \end{pmatrix} + t \begin{pmatrix} -120 \\ -40 \end{pmatrix}$

c $(80, 60)$

d The aircraft is exactly 100 km from O at 1:00 pm.

e at 1:45 pm and $d_{\min} = 10\sqrt{10}$ km **f** 2:30 pm

4 **a** A$(1, 2, 0)$ **b** $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ **c** $\frac{5}{\sqrt{10}} = \frac{\sqrt{10}}{2}$ units

5 **a** $(2, -1, 4)$ **b** $\sqrt{27}$ units

6 **a** $(2, \frac{1}{2}, \frac{5}{2})$ **b** $\sqrt{\frac{3}{2}}$ units

c $\dfrac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|} = \dfrac{\sqrt{21}}{\sqrt{14}} = \sqrt{\frac{3}{2}}$ ✓

7 **a** $\begin{pmatrix} -0.1 \\ -0.8 \\ 0.4 \end{pmatrix}$ **b** **i** after 40 seconds **ii** $\sqrt{18}$ m

EXERCISE 10E

1 **a** **i** $\sqrt{58} \approx 7.62$ m **ii** 4 m

b $D = \sqrt{2t^2 - 32t + 136}$ m

c minimum distance $= \sqrt{8} \approx 2.83$ m when $t = 8$ s

2 **a** $\sqrt{2425} \approx 49.2$ m

b **i** 11 seconds after entering the enclosure

ii $\sqrt{5} \approx 2.24$ m **iii** $(-8, -4)$

3 **a** truck A: $\begin{pmatrix} -72 \\ 54 \end{pmatrix}$, truck B: $\begin{pmatrix} -75 \\ 40 \end{pmatrix}$

b truck A: 1:30 pm, truck B: 2:00 pm

c **i** $\approx$ 12:41 pm **ii** ≈ 64.4 km

4 **a** $\sqrt{89} \approx 9.43$ m **b** $\sqrt{35} \approx 5.92$ m when $t = 3$

5 **a** $a = 6$, $b = 8$; Yvonne throws the water balloon toward Alex, so $b > 0$.

b 2 seconds after Yvonne throws the water balloon

c 0 m; the water balloon hit Alex

EXERCISE 10F.1

1 $(2, 4)$

2 **a** A$(2, 3)$, B$(8, 6)$, C$(5, 0)$

b AB $=$ BC $= \sqrt{45}$ units, AC $= \sqrt{18}$ units

3 **a** A$(2, 5)$, B$(18, 9)$, C$(14, 25)$, D$(-2, 21)$

b **i** $\overrightarrow{AC} = \begin{pmatrix} 12 \\ 20 \end{pmatrix}$ and $\overrightarrow{DB} = \begin{pmatrix} 20 \\ -12 \end{pmatrix}$

ii $|\overrightarrow{AC}| = \sqrt{544}$ units, $|\overrightarrow{DB}| = \sqrt{544}$ units **iii** 0

c Diagonals are perpendicular and equal in length, so ABCD is a square.

4 no points of intersection; L_1 and L_2 are parallel.

5 **a** $(-1, 3)$

b Yes; both particles pass through this point when $t = 4$.

6 **a** Jove **b** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix} + t \begin{pmatrix} \frac{3}{2} \\ -2 \end{pmatrix}$ **c** $(-1, -5)$

d No; Jove passes through the intersection point after 1 second, and Fiona passes through it after 6 seconds.

7 **b** $\sqrt{10}$ km min^{-1}

c k minutes later, $(t - k)$ minutes have elapsed.

$$\therefore \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t-k) \begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

d $\approx 55.3°$

e **i** 1:35:28 pm **ii** 1:37:42 pm **iii** $\approx (6.08, 0.308)$

EXERCISE 10F.2

1 **a** $(1, 2, 3)$ **b, c** L_1 and L_2 do not intersect

d $(-4, 7, -7)$

2 **a** **i** $\sqrt{26} \approx 5.10$ m s^{-1} **ii** $\sqrt{35} \approx 5.92$ m s^{-1}

b The paths intersect at $(10, 15, 12)$.

c No; the bird passes through the intersection point after 3 seconds, and the bat passes through it after 7 seconds.

3 **a** $\begin{pmatrix} 12 \\ -6 \\ -4 \end{pmatrix}$ **b** $(25, -36, -30)$ **c** 42 m

4 **a** $\begin{pmatrix} 160 \\ 280 \\ 160 \end{pmatrix}$ **b** $\sqrt{341} \approx 18.5$ m s^{-1}

d **i** 1.9 seconds **ii** 36 m

REVIEW SET 10A

1 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad t \in \mathbb{R}$

b $x = -6 + 4t, \ y = 3 - 3t, \ t \in \mathbb{R}$

2 $m = 10$

3 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ -4 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

b $C\left(2 - \frac{2}{\sqrt{6}}, \ -1 + \frac{2}{\sqrt{6}}, \ 3 - \frac{4}{\sqrt{6}}\right)$, or

$C\left(2 + \frac{2}{\sqrt{6}}, \ -1 - \frac{2}{\sqrt{6}}, \ 3 + \frac{4}{\sqrt{6}}\right)$

4 $\approx 67.3°$

5 **a** $(-4, 3, 1)$ **b** $(28, 27, -7)$ **c** $\begin{pmatrix} 8 \\ 6 \\ -2 \end{pmatrix}$

d ≈ 10.2 m s^{-1}

6 $(7, -7)$

7 **a** $2\sqrt{5} \approx 4.47$ units **b** $\frac{8\sqrt{10}}{5} \approx 5.06$ units

8 **a** **i** $-6\mathbf{i} + 10\mathbf{j}$ **ii** $-5\mathbf{i} - 15\mathbf{j}$

iii $(-6 - 5t)\mathbf{i} + (10 - 15t)\mathbf{j}, \quad t \geqslant 0$

b $t = 0.48$ hours

c Shortest distance ≈ 8.85 km, so it will miss the reef.

9 **a** $x_1(t) = 2 + t, \ y_1(t) = 4 - 3t, \ t \geqslant 0$

b $x_2(t) = 13 - t, \ y_2(t) = (3 - 2a) + at, \ t \geqslant 2$

c 2:22:30 pm

d bearing $\approx 192.7°$, ≈ 4.54 km min^{-1} **e** $\approx 31.2°$

10 **a** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ **b** $(\frac{1}{6}, \frac{5}{6}, \frac{2}{3})$

11 **a** object A: $(2, 4, 2)$; object B: $(7, -10, 6)$

b $(4, -1, 3)$

c No; object A passes through the point of intersection after 3 seconds, and object B passes through it after 5 seconds.

12 **a** $\begin{pmatrix} -6 \\ 8 \end{pmatrix}$ **b** $\sqrt{170} \approx 13.0$ m

c shortest distance $= \sqrt{85} \approx 9.22$ m when Chelsea's puck is at $(7, 17)$.

REVIEW SET 10B

1 **a** $(5, 2)$ **b** $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

c $\mathbf{r} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s \begin{pmatrix} 4 \\ 10 \end{pmatrix}, \quad s \in \mathbb{R}$

2 **a** **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = 2 + 4\lambda, \ y = -3 + 2\lambda, \ z = 1 - \lambda, \ \lambda \in \mathbb{R}$

b **i** $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -8 \\ -3 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

ii $x = -1 + 6\lambda, \ y = 6 - 8\lambda, \ z = 3 - 3\lambda, \ \lambda \in \mathbb{R}$

3 $\approx 8.13°$ **4** $\approx 30.5°$

5 **a** $|\overrightarrow{AB}| = 3\sqrt{3}$ units

b A lies on the line **r** where $\lambda = -3$ and B lies on **r** where $\lambda = 0$ $\therefore$ the line between A and B is the same as line **r**, so it can be described by **r**.

c $\approx 70.5°$

6 $2\sqrt{10}(3\mathbf{i} - \mathbf{j})$

7 **a** $(-10, 5, 12)$ **b** $\begin{pmatrix} 6 \\ 14 \\ -0.4 \end{pmatrix}$ **c** $\approx 15.2 \text{ m s}^{-1}$

d 30 seconds **e** $(170, 425, 0)$ **f** $(50, 145, 8)$

8 **a** A$(5, 2)$, B$(6, 5)$, C$(8, 3)$

b $|\overrightarrow{AB}| = \sqrt{10}$ units, $|\overrightarrow{BC}| = \sqrt{8}$ units, $|\overrightarrow{AC}| = \sqrt{10}$ units

c isosceles

9 **a** No; $(-24\mathbf{i} + 7\mathbf{j}) \bullet (-3\mathbf{i} - 9\mathbf{j}) = 9$, so the velocity vectors are *not* perpendicular.

b $\begin{pmatrix} -9.6 \\ 2.8 \end{pmatrix}$ **c** $(54, -2)$ **d** 50 m **e** after 14.6 seconds

10 $2\sqrt{10} \approx 6.32$ cm when $t = 5$

11 **a** $(4, 3, 1)$ **b** no points of intersection

12 **a** Haley's drone: $\mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1.5 \end{pmatrix}, \quad t \geqslant 0$

Liam's drone: $\mathbf{r} = \begin{pmatrix} -12 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0.5 \end{pmatrix}, \quad t \geqslant 0$

b $\sqrt{129} \approx 11.4$ m **c** no

d closest after 5 seconds, the drones are $\sqrt{30} \approx 5.48$ m apart

EXERCISE 11A

1 **a** $3i$ **b** $8i$ **c** $11i$ **d** $\frac{1}{2}i$ **e** $i\sqrt{5}$

f $i\sqrt{17}$ **g** $2\sqrt{2}i$ **h** $-6i$ **i** $-i\sqrt{11}$ **j** $-\frac{1}{\sqrt{2}}i$

2 **a** $x = \pm 5$ **b** $x = \pm 5i$ **c** $x = \pm\sqrt{5}$

d $x = \pm i\sqrt{5}$ **e** $x = \pm\frac{3}{2}$ **f** $x = \pm\frac{3}{2}i$

3 **a** $x = 5 \pm 2i$ **b** $x = -3 \pm 4i$ **c** $x = -7 \pm i$

d $x = \frac{3}{2} \pm \frac{\sqrt{11}}{2}i$ **e** $x = \frac{1}{2} \pm \frac{\sqrt{15}}{2}i$ **f** $x = -1 \pm \sqrt{\frac{2}{3}}i$

g $x = \frac{3}{2} \pm \frac{1}{2}i$ **h** $x = \sqrt{3} \pm i$ **i** $x = \frac{1}{4} \pm \frac{\sqrt{7}}{4}i$

EXERCISE 11B

1

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$3 + 2i$	3	2	$3 - 2i$
$5 - i$	5	-1	$5 + i$
3	3	0	3
$4i$	0	4	$-4i$
0	0	0	0
$-3 + 4i$	-3	4	$-3 - 4i$
$-7 - 2i$	-7	-2	$-7 + 2i$
$-11i$	0	-11	$11i$
$i\sqrt{3}$	0	$\sqrt{3}$	$-i\sqrt{3}$
$1 - i\sqrt{2}$	1	$-\sqrt{2}$	$1 + i\sqrt{2}$

2 **a** 3, 0 **b** $4i$, $-11i$, $i\sqrt{3}$

3 **a** -1 **b** -5 **c** $-1 - 4i$ **d** $6 + 5i$ **e** -4 **f** 6

EXERCISE 11C

1 **a** $7 - i$ **b** $10 - 4i$ **c** $-1 + 2i$ **d** $3 - 3i$

e $4 - 7i$ **f** $12 + i$ **g** $3 + 4i$ **h** $21 - 20i$

2 **a** $-3 + 7i$ **b** $2i$ **c** $-2 + 2i$ **d** $-1 + i$

e $-5 - 12i$ **f** $-5 + i$ **g** $-6 - 4i$ **h** $-1 - 5i$

3 $(a + bi) + (a - bi) = 2a$ which is real

4 **a** $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$, $i^9 = i$, $i^{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$, $i^{-5} = -i$

b $i^{4n+3} = -i$

5 **a** -6 **b** No, $\sqrt{-4} \times \sqrt{-9} = -6 \neq \sqrt{36}$.

6 **a** $-\frac{1}{10} - \frac{7}{10}i$ **b** $-\frac{1}{5} + \frac{2}{5}i$ **c** $\frac{7}{5} + \frac{1}{5}i$ **d** $-1 - 2i$

7 **a** $-\frac{2}{5} + \frac{1}{5}i$ **b** $-\frac{1}{13} + \frac{8}{13}i$ **c** $-\frac{2}{5} + \frac{3}{5}i$

d $\dfrac{2}{x^2 + 1}$

8 **a** -2 **b** -4 **c** 3 **d** 0

9 **a** $-10 + 198i$ **b** $-\frac{7}{625} + \frac{24}{625}i$ **c** $404 - 1121i$

d $\frac{11}{125} + \frac{2}{125}i$ **e** $\frac{1}{50} - \frac{3}{200}i$ **f** $\frac{641}{10} - \frac{5277}{10}i$

10 $z = i\sqrt{2}$

11 **a** $w = \dfrac{a^2 - b^2 - 1}{(a + 1)^2 + b^2} + \dfrac{2ab}{(a + 1)^2 + b^2}\,i$

b purely imaginary if $a^2 - b^2 = 1$ and $a \neq 0$, $b \neq 0$

EXERCISE 11D

1 **a** $a = 3$, $b = -1$ **b** $a = -\sqrt{2}$, $b = 1$

c $a = 0$, $b = 0$

2 **a** $x = 0$, $y = -2$ **b** $x = -2$ **c** $x = 3$, $y = 2$

d $x = -\frac{2}{13}$, $y = -\frac{3}{13}$ **e** $x = 0$, $y = 0$

f $x = 3$, $y = -2$ or $x = 4$, $y = -\frac{3}{2}$

g $x = 2$, $y = -5$ or $x = -\frac{5}{3}$, $y = 6$

h $x = -1$, $y = 0$

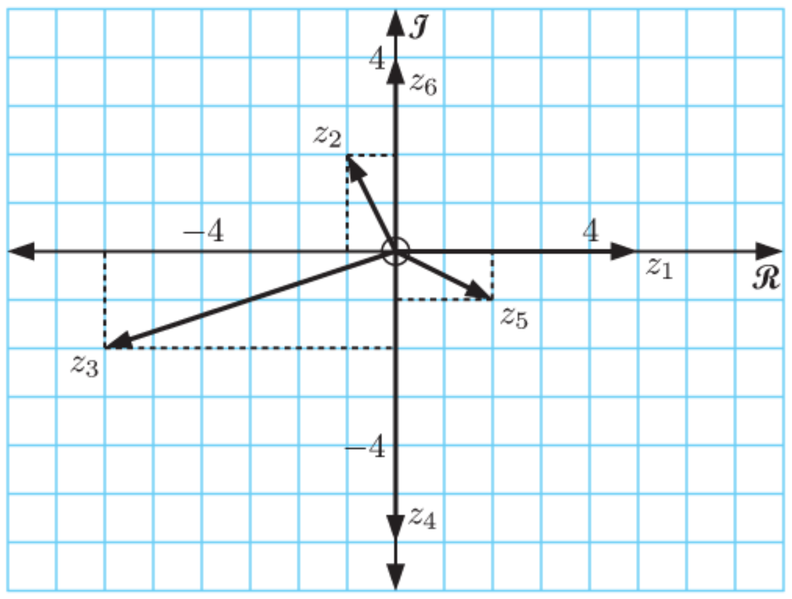
3 $z = 5 - 4i$ **4** $m = -\frac{1}{11}$, $n = \frac{8}{11}$

5 $a = 3$, $b = -5$ or $a = -3$, $b = 5$

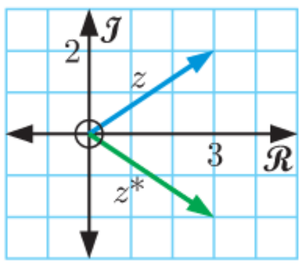
6 $z = \frac{4}{3} + \frac{2}{3}i$ or $z = -\frac{4}{3} - \frac{2}{3}i$

EXERCISE 11E

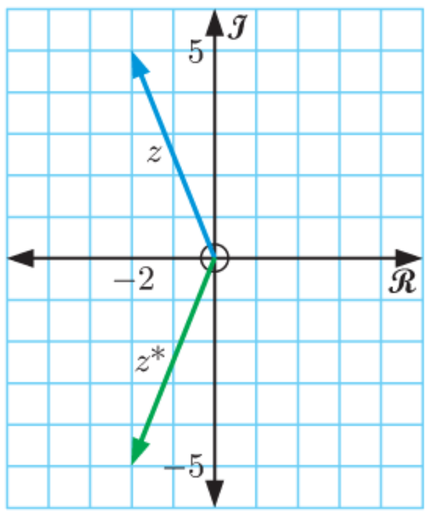
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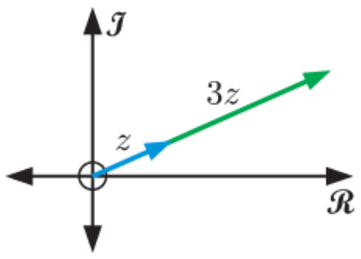
2 **a**



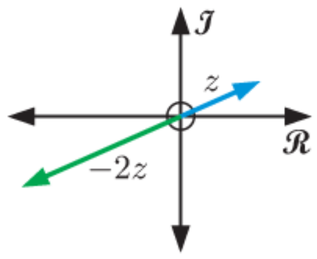
b



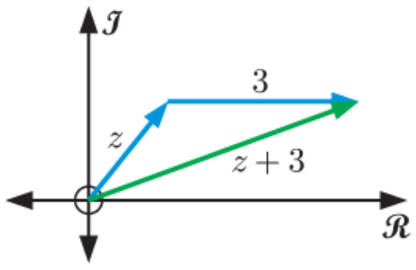
3 **a**



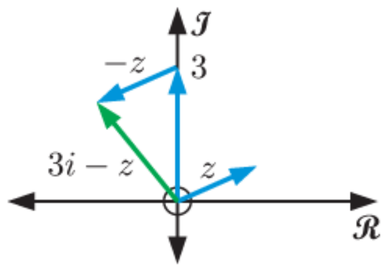
b



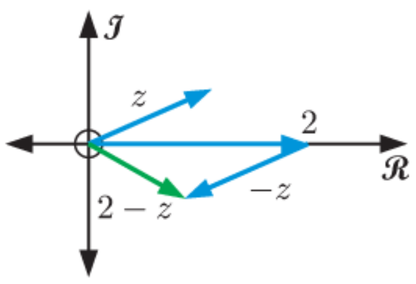
c



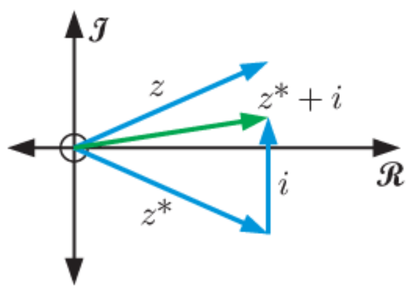
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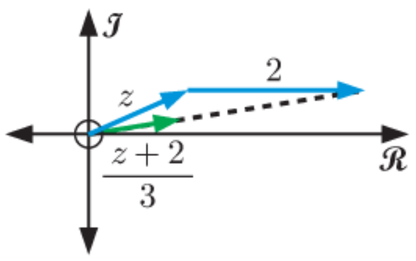
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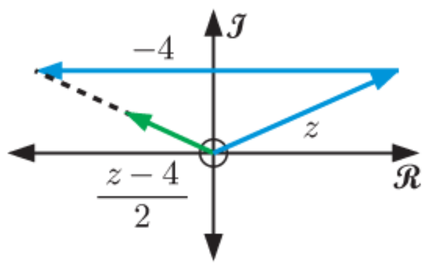
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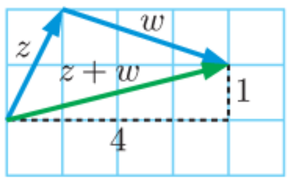
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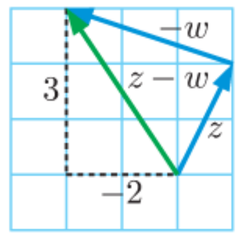
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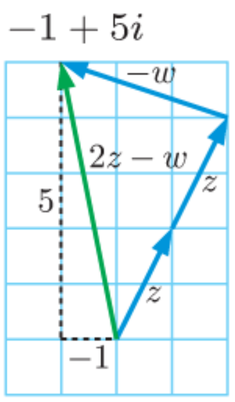
4 **a** $4+i$



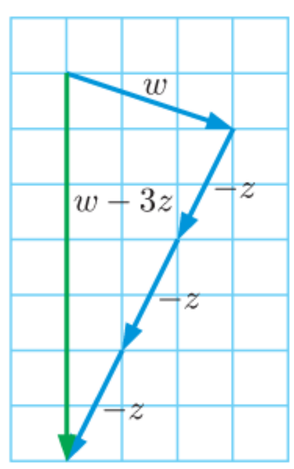
b $-2+3i$



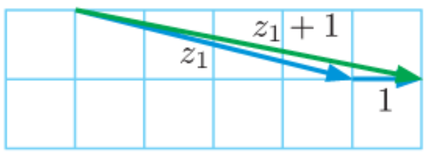
c $-1+5i$



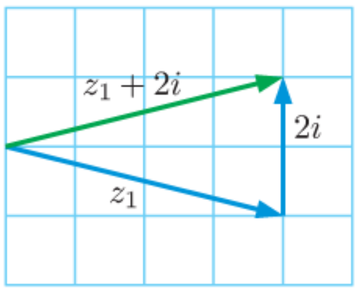
d $-7i$



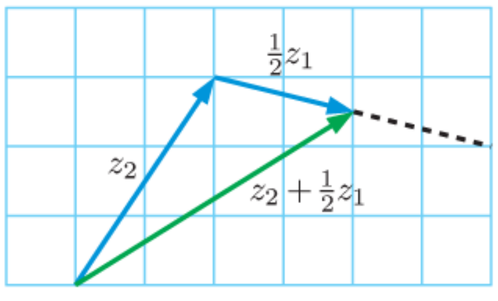
5 **a** $5-i$



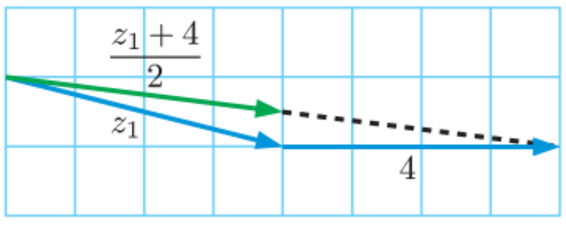
b $4+i$



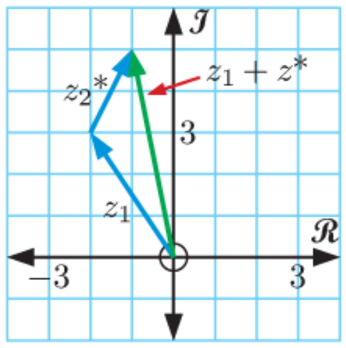
c $4+\frac{5}{2}i$



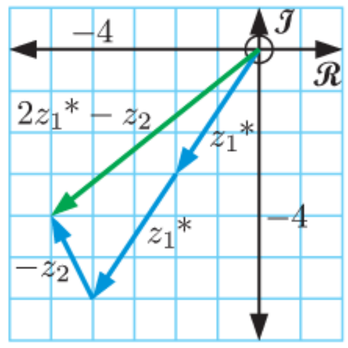
d $4-\frac{1}{2}i$



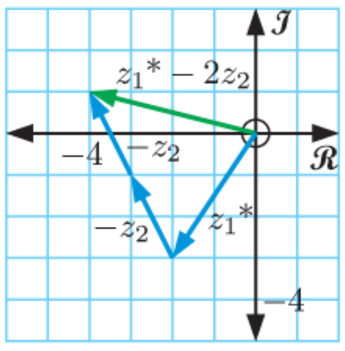
6 **a** $-1+5i$



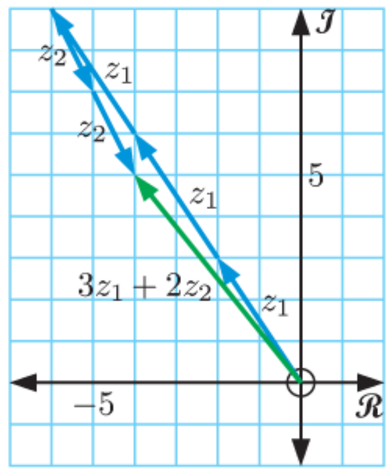
b $-5-4i$



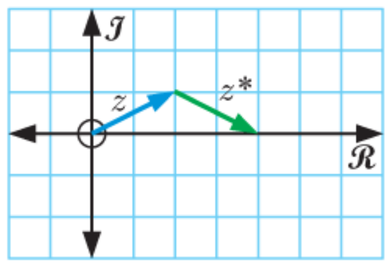
c $-4+i$



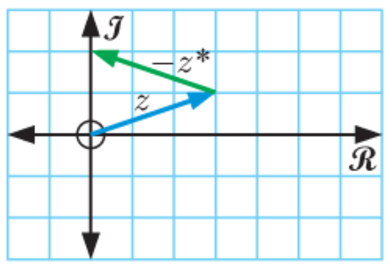
d $-4+5i$



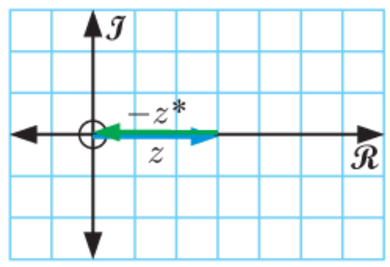
7 **a**



b

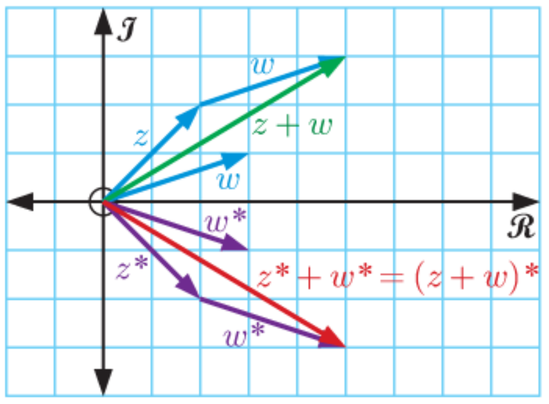


$z-z^*$ is purely imaginary when $\mathcal{Im}(z) \neq 0$

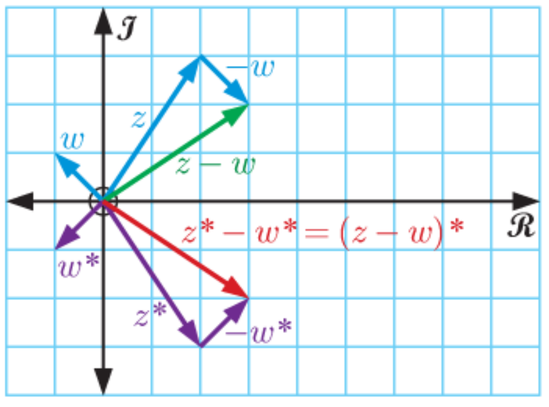


$z-z^*=0$ when $\mathcal{Im}(z)=0$

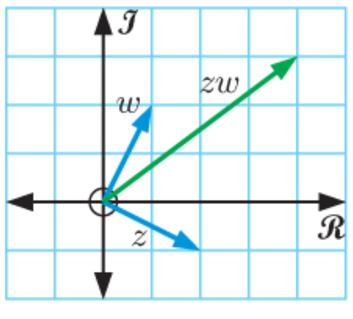
8 **a**



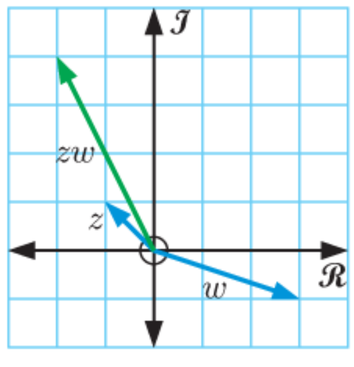
b



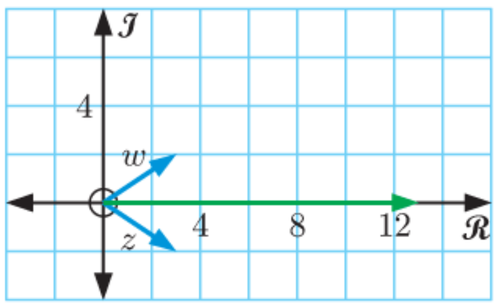
9 **a** $zw=4+3i$



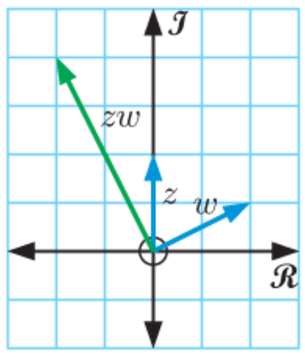
b $zw=-2+4i$



c $zw = 13$



d $zw = -2 + 4i$



EXERCISE 11F

1 **a** 5 **b** 13 **c** $2\sqrt{17}$ **d** 3 **e** 4

2 **a** $|z| = \sqrt{5}$, $\arg z \approx 0.464$

b $|z| = \sqrt{5}$, $\arg z \approx -0.464$

c $|z| = \sqrt{5}$, $\arg z \approx 2.68$

d $|z| = \sqrt{5}$, $\arg z \approx -2.68$

3 **a** $|w| = \sqrt{13}$, $\arg w \approx 0.588$

b $|w| = \sqrt{13}$, $\arg w \approx 2.55$

c $|w| = \sqrt{13}$, $\arg w \approx -0.983$

d $|w| = \sqrt{13}$, $\arg w \approx 2.16$

4 $a = -\sqrt{3}$

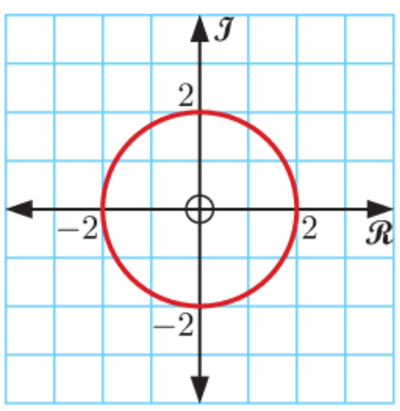
5 **a** $\sqrt{5}$ **b** $\sqrt{5}$ **c** 5 **d** 5 **e** $5\sqrt{2}$ **f** $5\sqrt{2}$

g $\frac{1}{\sqrt{2}}$ **h** $\frac{1}{\sqrt{2}}$ **i** 5 **j** 5 **k** $5\sqrt{5}$ **l** $5\sqrt{5}$

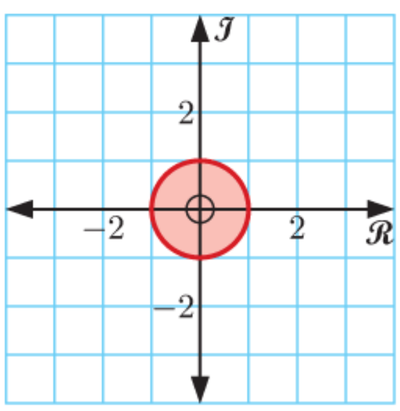
6 **a** 1 **b** $|r|$

7 **a** 6 **b** 4 **c** $2\sqrt{5}$ **d** 2 **e** $\frac{1}{2}$ **f** $\frac{4}{3}$

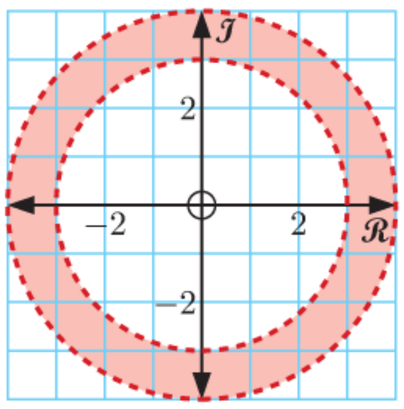
8 **a**



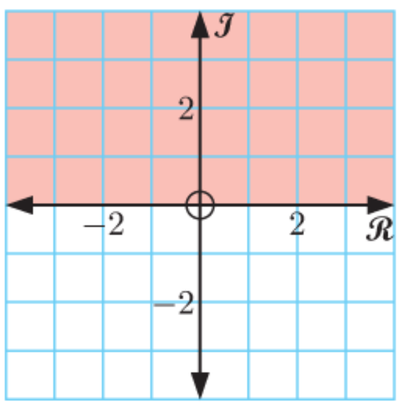
b



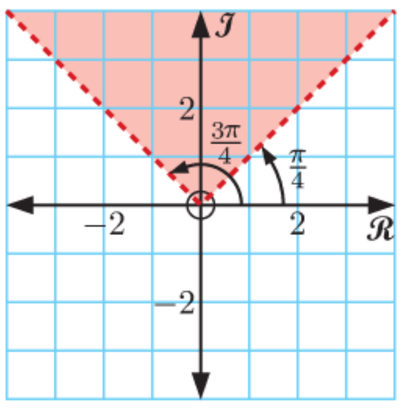
c



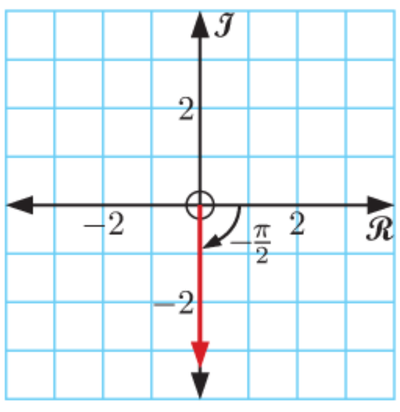
d



e



f



9 **a** $4 + 5i$ **b** $|z + w| = \sqrt{41}$, $\arg(z + w) \approx 0.896$

c $\theta \approx 0.353$

10 **a** $|w| = 3$ **b** $\arg w = \frac{9\pi}{10}$ **c** $|z + w| = 3\sqrt{2}$

d $\arg(z + w) = \frac{13\pi}{20}$

11 **a** $w = \left(\frac{a^2 + b^2 - 1}{a^2 + (b-1)^2}\right) + \left(\frac{2a}{a^2 + (b-1)^2}\right) i$ **b** 0

12 $z = 2 + 6i$

EXERCISE 11G

1 **a** **i** $4\sqrt{2}$ units **ii** $(1, 4)$

b **i** $5\sqrt{5}$ units **ii** $(-\frac{3}{2}, 2)$

2 **a** $\sqrt{32 - 32\cos\frac{\pi}{5}}$

b perimeter $= \sqrt{32 - 32\cos\frac{\pi}{5}} + 8$ units

area $= 8\sin\frac{\pi}{5}$ units^2

3 **a** reflection in the $\mathcal{R}$-axis **b** rotation of π about O

c reflection in the $\mathcal{I}$-axis **d** clockwise rotation of $\frac{\pi}{2}$ about O

4 **a** **i** 2 **ii** 4 **iii** $2\sqrt{3}$ **iv** $\frac{2\pi}{3}$ **v** $\frac{5\pi}{6}$

b $\sqrt{5}$, the point representing z_4 is $\sqrt{5}$ units from the point representing i.

EXERCISE 11H.1

1 **a** $4\,\text{cis}\,0$ **b** $4\,\text{cis}\,\frac{\pi}{2}$ **c** $6\,\text{cis}\,\pi$ **d** $3\,\text{cis}\left(-\frac{\pi}{2}\right)$

e $\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ **f** $2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ **g** $2\,\text{cis}\,\frac{5\pi}{6}$ **h** $4\,\text{cis}\,\frac{\pi}{6}$

2 0, no angle exists with the positive real axis.

3 $k\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ if $k > 0$, $-k\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$ if $k < 0$,
not possible if $k = 0$

4 **a** $2i$ **b** $4\sqrt{2} + 4\sqrt{2}i$ **c** $2\sqrt{3} + 2i$ **d** 3

e $1 - i$ **f** $-\frac{\sqrt{3}}{2} + \frac{3}{2}i$ **g** -5 **h** $-5 - 5\sqrt{3}i$

5 **a** $z_1 = 4\,\text{cis}\,\frac{2\pi}{3}$

b **i** $z_2 = 8\,\text{cis}\,\frac{7\pi}{6}$ **ii** $z_2 = -4\sqrt{3} - 4i$

6 **a** $-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ **b** 1 **c** 1

7 **a** $\approx 3.06 + 2.58i$ **b** $\approx -1.41 + 1.01i$

c $\approx 1.27 - 3.06i$ **d** $\approx -1.86 - 1.35i$

e $\approx -3.48 + 0.426i$ **f** $\approx -2.55 - 1.25i$

8 **a** $\approx 5\,\text{cis}\,(-0.927)$ **b** $\approx 13\,\text{cis}\,(-1.97)$

c $\approx 3\,\text{cis}\,(0.491)$ **d** $\approx 2.69\,\text{cis}\,(2.12)$

e $\approx 2.78\,\text{cis}\,(-0.207)$ **f** $\approx 17.7\,\text{cis}\,(2.29)$

9 **a** $\sqrt{2} + \sqrt{2}i$, $2\,\text{cis}\,\frac{\pi}{4}$

b $-\frac{7}{2} - \frac{3\sqrt{3}}{2}i$, $\approx 4.36\,\text{cis}\,(-2.50)$

10 **a** $|z - 1| = 2\sin\frac{\phi}{2}$, $\arg(z - 1) = \frac{\phi}{2} + \frac{\pi}{2}$

b $z - 1 = \left(2\sin\frac{\phi}{2}\right)\text{cis}\left(\frac{\phi}{2} + \frac{\pi}{2}\right)$,

$(z - 1)^* = \left(2\sin\frac{\phi}{2}\right)\text{cis}\left(-\frac{\phi}{2} - \frac{\pi}{2}\right)$

11 **a** $|z + 2i| = 2$ **b** **ii** $-\frac{\pi}{3}$ **iii** $w = \sqrt{3} - 3i$

EXERCISE 11H.2

1 **a** $\text{cis}\,3\theta$ **b** $\text{cis}\,2\theta$ **c** $\text{cis}\,3\theta$ **d** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

e $\sqrt{2} + i\sqrt{2}$ **f** 8 **g** $-\sqrt{3} + 3i$ **h** $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

i $-2i$ **j** $-1 - i$ **k** -4 **l** $4i$

2 **a** $\text{cis}\,\pi = -1$ **b** $\text{cis}\,\pi = -1$ **c** $\text{cis}\,\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

3 **a** $|z| = 2$, $\arg z = \theta$

b **i** $2\,\text{cis}\,(-\theta)$ **ii** $2\,\text{cis}\,(\theta + \pi)$ **iii** $2\,\text{cis}\,(\pi - \theta)$

iv $2\,\text{cis}\,(\pi - \theta)$

4 **a** $\text{cis}\left(-\frac{\pi}{2}\right)$ **b** $r\,\text{cis}\left(\theta - \frac{\pi}{2}\right)$

c A clockwise rotation through $\frac{\pi}{2}$ about the origin.

5 **a** $\text{cis}\,\pi$ **b** $r\,\text{cis}(\theta+\pi)$
c An anticlockwise rotation through π about the origin.

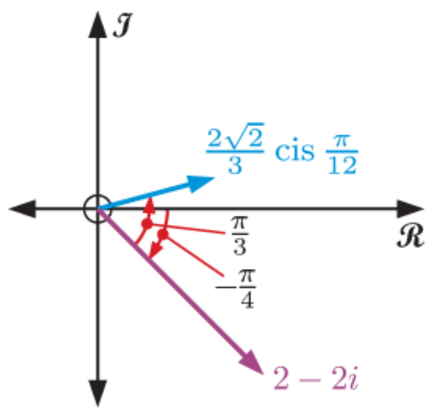
6 z^* is the reflection of z in the $\mathcal{R}$-axis.
$\therefore\ \ z^* = r\,\text{cis}(-\theta)$

EXERCISE 11H.3

1 **a** $z = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

b $\frac{2\sqrt{2}}{3}\,\text{cis}\,\frac{\pi}{12}$

c

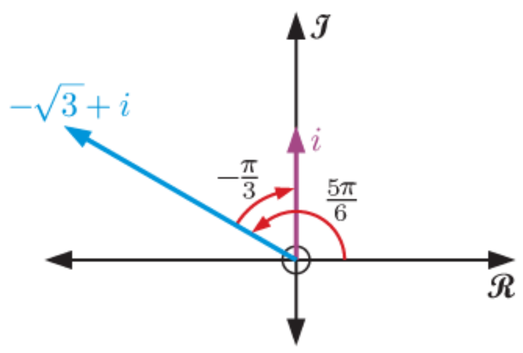


d z has been dilated with scale factor $\frac{1}{3}$, then rotated anticlockwise through $\frac{\pi}{3}$ about the origin.

2 **a** $z = 2\,\text{cis}\,\frac{5\pi}{6}$

b $\text{cis}\,\frac{\pi}{2} = i$

c



d z has been dilated with scale factor $\frac{1}{2}$, then rotated clockwise through $\frac{\pi}{3}$ about the origin.

3 **a** $z = \sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$, $w = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$ **b** $2\sqrt{2}\,\text{cis}\,\frac{5\pi}{12}$

c z has been dilated with scale factor 2, then rotated clockwise through $\frac{\pi}{3}$ about the origin.

4 **a** $\cos\frac{\pi}{12} = \frac{\sqrt{2}+\sqrt{6}}{4}$, $\sin\frac{\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

b $\cos\frac{11\pi}{12} = \frac{-\sqrt{2}-\sqrt{6}}{4}$, $\sin\frac{11\pi}{12} = \frac{\sqrt{6}-\sqrt{2}}{4}$

6 **a** $|z| = 2$, $\arg z = \frac{\pi}{5}$ **b** $|w| = 5$, $\arg w = -\frac{2\pi}{7}$

c $|zw| = 10$, $\arg zw = -\frac{3\pi}{35}$

d $\left|\frac{z}{w}\right| = \frac{2}{5}$, $\arg\left(\frac{z}{w}\right) = \frac{17\pi}{35}$

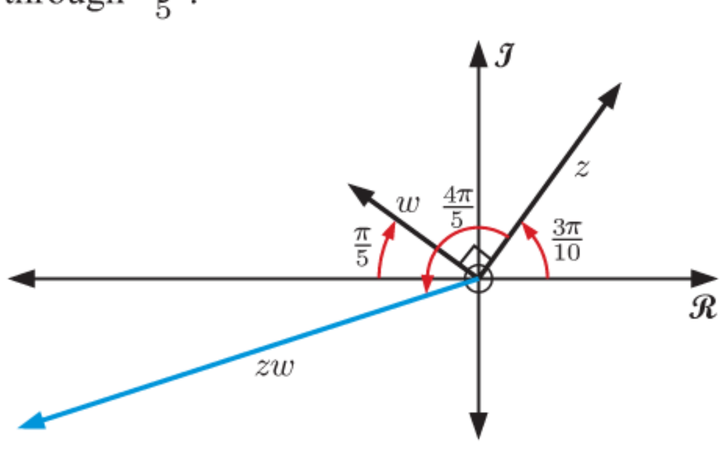
e $\left|z^4\right| = 16$, $\arg(z^4) = \frac{4\pi}{5}$

f $\left|w^3\right| = 125$, $\arg(w^3) = -\frac{6\pi}{7}$

7 **a** $z = 3\,\text{cis}\,\frac{3\pi}{10}$, $w = 2\,\text{cis}\,\frac{4\pi}{5}$

b z is dilated with scale factor 2, then rotated anticlockwise through $\frac{4\pi}{5}$.

c



8 **a** $|-z| = 3$, $\arg(-z) = \theta - \pi$

b $\left|z^2\right| = 9$, $\arg(z^2) = 2\theta$

c $|iz| = 3$, $\arg(iz) = \theta + \frac{\pi}{2}$

d $|(1+i)z| = 3\sqrt{2}$, $\arg[(1+i)z] = \theta + \frac{\pi}{4}$

e $\left|\frac{z}{i}\right| = 3$, $\arg\left(\frac{z}{i}\right) = \theta - \frac{\pi}{2}$

f $\left|\frac{z}{1-i}\right| = \frac{3}{\sqrt{2}}$, $\arg\left(\frac{z}{1-i}\right) = \theta + \frac{\pi}{4}$

9 **a** $z_2 - z_1 = \overrightarrow{AB}$, $z_3 - z_2 = \overrightarrow{BC}$

b $\left|\frac{z_2-z_1}{z_3-z_2}\right| = 1$, $\arg\left(\frac{z_2-z_1}{z_3-z_2}\right) = \frac{2\pi}{3}$ **c** 1

EXERCISE 11I

1 **a** -1 **b** $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **c** $-i$

2 **a** $\sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$, $\sqrt{2}e^{i\frac{3\pi}{4}}$ **b** $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$, $3e^{-i\frac{\pi}{6}}$

c $-1+\sqrt{3}i$, $2\,\text{cis}\,\frac{2\pi}{3}$

3 **a** $\approx 0.362 + 0.932i$ **b** $\approx 3.06 - 2.58i$

c $\approx -7.32 + 4.11i$ **d** $\approx 0.324 - 0.528i$

4 **a** $\approx 5.39e^{0.381i}$ **b** $\approx 9.85e^{2.72i}$ **c** $\approx 3.16e^{-2.54i}$

d $\approx 4.90e^{-1.10i}$

6 **a** 3θ **b** $\frac{\theta}{2}$ **c** $4\theta + \frac{\pi}{2}$ **d** $\frac{2}{5}\theta - \frac{\pi}{2}$

7 **a** $\left|\sqrt{w}\right| = \sqrt{3}$, $\arg(\sqrt{w}) = \frac{\phi}{2}$

b $\left|z^4 w\right| = 48$, $\arg(z^4 w) = 4\theta + \phi$

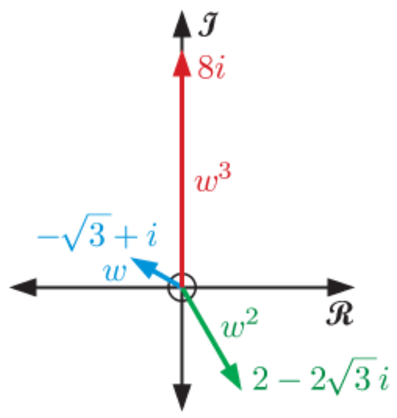
c $\left|\frac{z}{w^2}\right| = \frac{2}{9}$, $\arg\left(\frac{z}{w^2}\right) = \theta - 2\phi$

d $\left|w^3 \times \sqrt{z}\right| = 27\sqrt{2}$, $\arg(w^3 \times \sqrt{z}) = 3\phi + \frac{1}{2}\theta$

8 **a** $\approx 0.540 + 0.841i$ **b** $\approx 0.540 - 0.841i$

c $\approx 0.455 + 0.891i$ **d** ≈ 0.208

10 **a**



b $k = 9$

11 $A = \dfrac{\sin\frac{\pi}{5} + \cos\frac{\pi}{5}}{\sin\frac{\pi}{7} + \cos\frac{\pi}{7}}$, $B = \dfrac{-\sin\frac{2\pi}{35}}{\sin\frac{\pi}{7} + \cos\frac{\pi}{7}}$

EXERCISE 11J

1 **a** $\approx 1.11\cos(3t + 1.39)$ **b** $\approx 2.17\cos(2t + 0.331)$

2 **b** $\approx 4.67\sin(t - 0.338)$

3 **a** $\approx 7.01\sin(4t + 2.98)$ **b** $\approx 7.59\sin(2t - 2.64)$

4 **a** $\approx 4.64\cos(x - 1.37)$

b P(1.37, 4.64), Q(4.51, -4.64), R(6.08, 0)

5 $\approx 10\cos(x - 0.927)$

6 **a** $V(t) \approx 20.3\cos(30t - 0.790)$ **b** ≈ 20.3 volts

7 **a** $H(t) \approx 0.231\sin(2t + 0.183)$

b at ≈ 1.48, 3.05, 4.62, 6.19, 7.76, and 9.33 s

c **i** at ≈ 0.694 s **ii** $\approx 85.2\%$

REVIEW SET 11A

1 **a** $7i$ **b** $i\sqrt{23}$ **c** $-\frac{1}{3}i$

2 **a** $x = 3 \pm i\sqrt{2}$ **b** $x = \sqrt{2} \pm i$ **c** $x = -\frac{3}{8} \pm \frac{\sqrt{7}}{8}i$

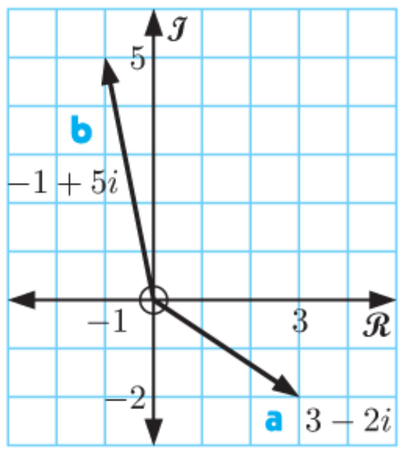
3 **a** $6 + 2i$ **b** -7 **c** -3

4 **a** $12 + 5i$ **b** $-1 + i$ **c** $\frac{3}{5} + \frac{4}{5}i$

5 real part $\frac{7\sqrt{3}}{4}$, imaginary part $-\frac{3}{4}$ **6** $z = \frac{3}{5} - \frac{1}{5}i$

7 $a = 0,\ b = -1$

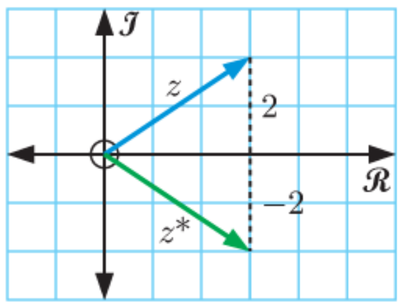
8



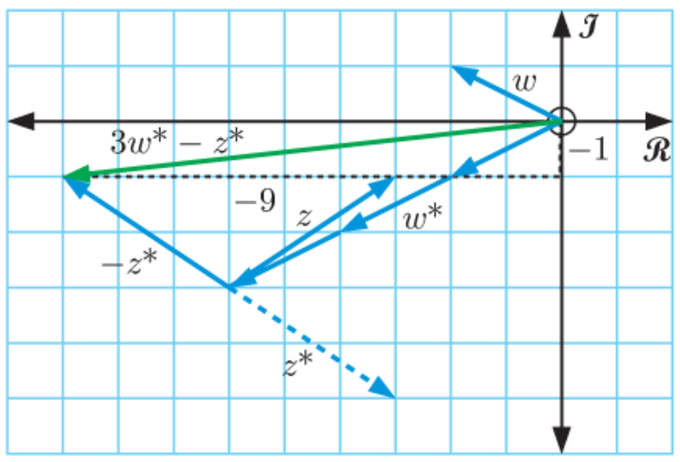
9 **a** $1+3i$

b $8+3i$

c $3-2i$



d $-9-i$



10 **a** $|z| = \sqrt{29},\quad \arg z \approx 0.381$

b $|z| = \sqrt{29},\quad \arg z \approx 2.76$

c $|z| = \sqrt{29},\quad \arg z \approx -0.381$

d $|z| = \sqrt{29},\quad \arg z \approx 1.95$

11 **a** 25 **b** 20 **c** $5\sqrt{10}$ **d** $\frac{1}{5}$ **e** $\frac{3}{25}$ **f** 1

12 **a** $3\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ **b** $7\,\text{cis}\left(-\frac{\pi}{2}\right)$ **c** $12\,\text{cis}\,\frac{5\pi}{6}$

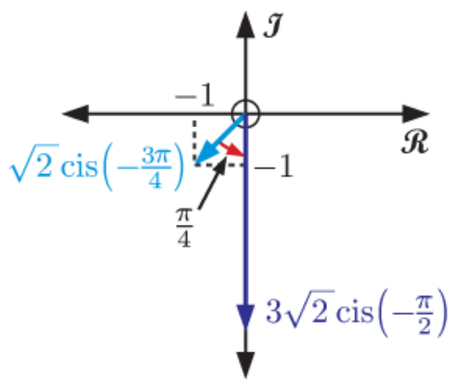
13 **a** $\approx 8.06\,\text{cis}\,(-2.09)$ **b** $\approx 2.35 + 0.855i$

14 **a** $\phi = \frac{\pi}{10}$ **b** $-\frac{2\pi}{5}$ **c** $2\,\text{cis}\,\frac{4\pi}{15}$

16 **a** $z = \sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

b $z = 3\sqrt{2}\,\text{cis}\left(-\frac{\pi}{2}\right)$

c



d z has been dilated with scale factor 3, then rotated anticlockwise through $\frac{\pi}{4}$ about the origin.

17 $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

18 **a** $5+2i$ **b** $2\sqrt{2}$ **c** 17^5 **d** ≈ -2.03

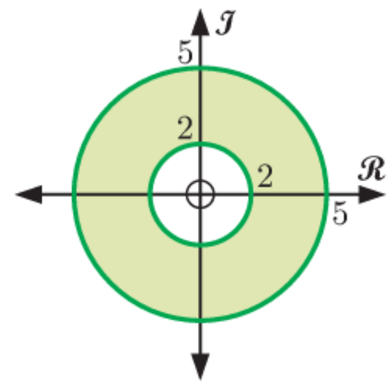
19 $4\,\text{cis}\left(-\frac{\pi}{3}\right),\quad n = 3(k+\frac{1}{2}),\quad k \in \mathbb{Z}$

20 **a** $\left|z^3\right| = 64,\quad \arg(z^3) = 3\theta$

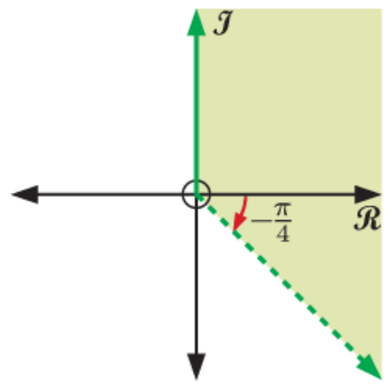
b $\left|\frac{1}{z}\right| = \frac{1}{4},\quad \arg\left(\frac{1}{z}\right) = -\theta$

c $|iz^*| = 4,\quad \arg(iz^*) = \frac{\pi}{2} - \theta$

21 **a**



b



22 **b** $\left|\frac{z_2 - z_1}{z_3 - z_2}\right| = 1,\quad \arg\left(\frac{z_2 - z_1}{z_3 - z_2}\right) = \frac{\pi}{3}$

23 **a** $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ **b** -5 **24** $12\,\text{cis}\left(-\frac{\pi}{12}\right)$

25 $\approx 3.37\sin(3t - 2.14)$

REVIEW SET 11B

1 **a** $x = \pm 3i$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{3}}{2}i$ **c** $x = \frac{3}{2} \pm \frac{3}{2}i$

2 **a** $-3+8i$ **b** $\frac{1}{2} - \frac{5}{2}i$ **c** $13+6i$

3 **a** $-16+6i$ **b** $-15-16i$ **c** $-\frac{9}{13} + \frac{20}{13}i$ **d** $-3-2i$

4 **a** $-39-80i$ **b** $\frac{1}{5} + \frac{2}{5}i$ **c** $-119+120i$ **d** $-\frac{1}{4}$

5 $-1+2i$

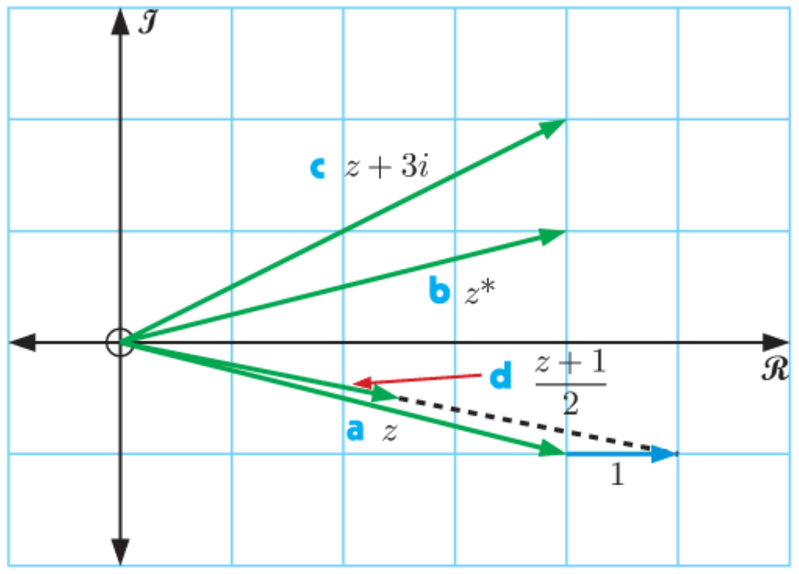
6 **a** $x = 0,\ y = 0$ **b** $x = 5,\ y = -7$

c $x = 0,\ y = 0$ or $x = 1,\ y = 0$ or $x = -\frac{1}{2},\ y = \frac{\sqrt{3}}{2}$ or $x = -\frac{1}{2},\ y = -\frac{\sqrt{3}}{2}$

d $x = -1,\ y = 2$

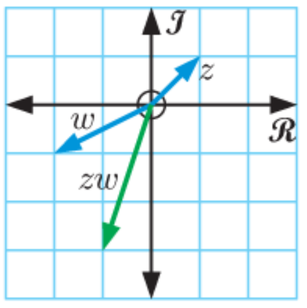
7 $z = 3-2i$ or $-3+2i$

10



11 **a** 12 **b** 8 **c** $\frac{\sqrt{2}}{4}$ **d** $4\sqrt{10}$

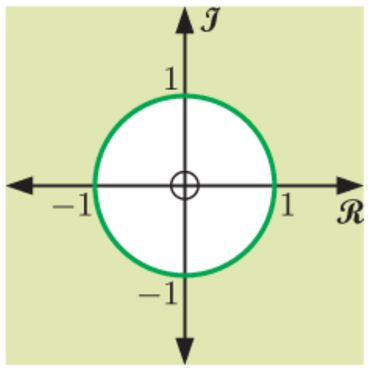
12 $zw = -1-3i$



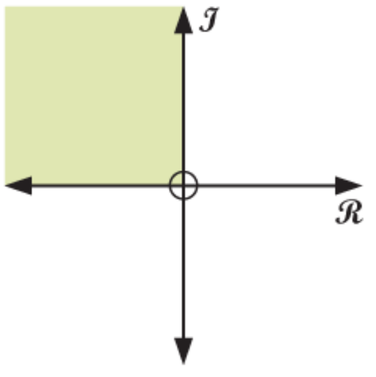
13 **a** $\sqrt{10}$ **b** $\sqrt{10}$ **c** 10 **d** 10 **e** 10

f 1 **g** 10 **h** $10\sqrt{10}$

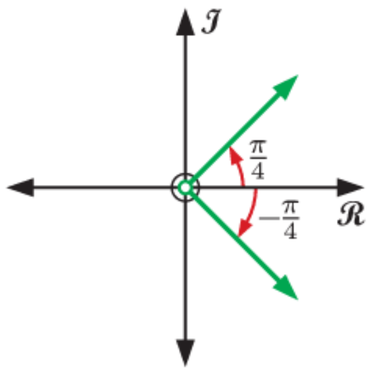
14 **a**



b



c



15 **a** $5\,\text{cis}\left(-\frac{\pi}{2}\right)$ **b** $4\,\text{cis}\left(-\frac{\pi}{3}\right)$ **c** $-k\sqrt{2}\,\text{cis}\,\frac{3\pi}{4},\ k<0$

16 **a** $4i$ **b** $-1+i$ **c** $4-4\sqrt{3}i$

17 **a** $\text{cis}\,\frac{13\pi}{15}$ **b** $3\,\text{cis}\left(-\frac{7\pi}{20}\right)$ **c** $\text{cis}\,\frac{\pi}{3}$

18 **a** $3r\,\text{cis}\,\frac{4\pi}{5}$ **b** $r\,\text{cis}\left(-\frac{\pi}{5}\right)$ **c** $r\,\text{cis}\left(-\frac{4\pi}{5}\right)$

d $r^2\,\text{cis}\left(-\frac{2\pi}{5}\right)$ **e** $r\,\text{cis}\left(-\frac{7\pi}{10}\right)$

19 $b=\frac{1}{\sqrt{3}}$

20 $(1-i)z=4\,\text{cis}\left(\alpha-\frac{\pi}{4}\right),\quad \arg[(1-i)z]=\alpha-\frac{\pi}{4}$

21 **a** $\left|\frac{z_1^{\,2}}{z_2^{\,2}}\right|=1,\quad \arg\left(\frac{z_1^{\,2}}{z_2^{\,2}}\right)=\pi$

22 **a** $\frac{5\pi}{12}$ **b** $-\frac{11\pi}{12}$ **c** $k=\sqrt{2}$

23 **a** $3\sqrt{2}\,\text{cis}\left(-\frac{3\pi}{4}\right),\quad 3\sqrt{2}e^{-i\,\frac{3\pi}{4}}$

b $-\frac{5\sqrt{3}}{2}-\frac{5}{2}i,\quad 5e^{-i\,\frac{5\pi}{6}}$ **c** $7i,\quad 7\,\text{cis}\,\frac{\pi}{2}$

24 **a** $I(t)\approx 2.65\cos(10t+1.24)$

b ≈ 2.65 mA at ≈ 0.505 s

EXERCISE 12A

1 **a** 1×4 **b** 2×1 **c** 2×2 **d** 3×3

2 **a** -1 **b** 0 **c** -3 **d** 1

3 **a** $\mathbf{Q}=\begin{pmatrix}2 & 1 & 6 & 1\end{pmatrix}$ **b** $\mathbf{P}=\begin{pmatrix}1.95\\2.35\\0.45\\2.95\end{pmatrix}$

c It is the total cost of groceries.

4 **a** $\mathbf{P}=\begin{pmatrix}1000 & 1500 & 1250\\1500 & 1000 & 1000\\800 & 2300 & 1300\\1200 & 1200 & 1200\end{pmatrix}$ **b** 4×3

5 $\begin{pmatrix}40 & 50 & 55 & 40\\25 & 65 & 44 & 30\\35 & 40 & 40 & 35\\35 & 40 & 35 & 50\end{pmatrix}$

EXERCISE 12B

1 **a** $a_{12}\neq b_{12}$ **b** $\mathbf{A}$ and $\mathbf{C}$ do not have the same order.

2 **a** $x=0,\ y=1$ **b** $x=3,\ y=2$

c $x=-2,\ y=-2$ **d** $x=0,\ y=0$

EXERCISE 12C

1 **a** $\begin{pmatrix}9 & 1\\3 & 3\end{pmatrix}$ **b** $\begin{pmatrix}6 & 8\\-1 & 1\end{pmatrix}$ **c** $\begin{pmatrix}3 & 4\\-6 & -1\end{pmatrix}$

d $\begin{pmatrix}0 & 0\\-11 & -3\end{pmatrix}$

2 **a** $\begin{pmatrix}20 & 1 & -8\\8 & 10 & -2\\1 & -5 & 18\end{pmatrix}$ **b** $\begin{pmatrix}-14 & 9 & -14\\12 & -6 & 14\\-5 & 3 & -4\end{pmatrix}$

c $\begin{pmatrix}14 & -9 & 14\\-12 & 6 & -14\\5 & -3 & 4\end{pmatrix}$

3 **a** $\begin{matrix}\text{Friday}\\\text{Saturday}\end{matrix}\begin{pmatrix}85 & 102\\92 & 137\\52 & 49\end{pmatrix}$ **b** 187 men, 229 women, 101 children

4 **a** **i** $\begin{pmatrix}1.72\\27.85\\0.92\\2.53\\3.56\end{pmatrix}$ **ii** $\begin{pmatrix}1.79\\28.75\\1.33\\2.25\\3.51\end{pmatrix}$

b subtract cost price from selling price **c** $\begin{pmatrix}0.07\\0.90\\0.41\\-0.28\\-0.05\end{pmatrix}$

5 **a**
$\begin{matrix}&\text{L}&\text{R}\\\\\\\end{matrix}\begin{matrix}\\\\\\\end{matrix}$
$\begin{pmatrix}23 & 19\\17 & 29\\31 & 24\end{pmatrix}\begin{matrix}\text{fridge}\\\text{stove}\\\text{microwave}\end{matrix}$

b
$\begin{matrix}&\text{L}&\text{R}\\\\\\\end{matrix}\begin{matrix}\\\\\\\end{matrix}$
$\begin{pmatrix}18 & 25\\7 & 13\\36 & 19\end{pmatrix}\begin{matrix}\text{fridge}\\\text{stove}\\\text{microwave}\end{matrix}$

c
$\begin{matrix}&\text{L}&\text{R}\\\\\\\end{matrix}\begin{matrix}\\\\\\\end{matrix}$
$\begin{pmatrix}41 & 44\\24 & 42\\67 & 43\end{pmatrix}\begin{matrix}\text{fridge}\\\text{stove}\\\text{microwave}\end{matrix}$

6 **a** $\mathbf{A}+\mathbf{B}=\begin{pmatrix}1 & 3\\5 & 2\end{pmatrix},\quad \mathbf{B}+\mathbf{A}=\begin{pmatrix}1 & 3\\5 & 2\end{pmatrix}$

7 **a** $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\begin{pmatrix}6 & 3\\-1 & 6\end{pmatrix},$

$\mathbf{A}+(\mathbf{B}+\mathbf{C})=\begin{pmatrix}6 & 3\\-1 & 6\end{pmatrix}$

EXERCISE 12D

1 **a** $\begin{pmatrix}12 & 24\\48 & 12\end{pmatrix}$ **b** $\begin{pmatrix}2 & 4\\8 & 2\end{pmatrix}$ **c** $\begin{pmatrix}\frac{1}{2} & 1\\2 & \frac{1}{2}\end{pmatrix}$

d $\begin{pmatrix}-3 & -6\\-12 & -3\end{pmatrix}$

2 **a** $\begin{pmatrix}3 & 5 & 6\\2 & 8 & 7\end{pmatrix}$ **b** $\begin{pmatrix}1 & 1 & 4\\0 & 4 & 1\end{pmatrix}$

c $\begin{pmatrix}5 & 8 & 11\\3 & 14 & 11\end{pmatrix}$ **d** $\begin{pmatrix}5 & 7 & 14\\2 & 16 & 9\end{pmatrix}$

3 $12\mathbf{A}=\begin{pmatrix}12\\48\\24\\12\end{pmatrix}$

4 **a** $\mathbf{A}=\begin{pmatrix}75\\27\\102\end{pmatrix}\begin{matrix}\leftarrow\text{DVD}\\\leftarrow\text{Blu-ray}\\\leftarrow\text{games}\end{matrix},\quad \mathbf{B}=\begin{pmatrix}136\\43\\129\end{pmatrix}\begin{matrix}\leftarrow\text{DVD}\\\leftarrow\text{Blu-ray}\\\leftarrow\text{games}\end{matrix}$

b $5\mathbf{A}+2\mathbf{B}=\begin{pmatrix}647\\221\\768\end{pmatrix}\begin{matrix}\leftarrow\text{DVD}\\\leftarrow\text{Blu-ray}\\\leftarrow\text{games}\end{matrix}$

It represents the total weekly average hirings.

5 **a**
$\begin{matrix}&\text{A}&\text{B}&\text{C}&\text{D}\\\\\\\end{matrix}$
$\begin{pmatrix}360 & 480 & 480 & 720\\600 & 480 & 360 & 900\\480 & 480 & 600 & 600\\120 & 240 & 240 & 180\end{pmatrix}$

b **i**
$\begin{matrix}&\text{A}&\text{B}&\text{C}&\text{D}\\\\\\\end{matrix}$
$\begin{pmatrix}35 & 46 & 46 & 69\\58 & 46 & 35 & 86\\46 & 46 & 58 & 58\\12 & 23 & 23 & 17\end{pmatrix}$

ii
$\begin{matrix}&\text{A}&\text{B}&\text{C}&\text{D}\\\\\\\end{matrix}$
$\begin{pmatrix}26 & 34 & 34 & 51\\43 & 34 & 26 & 64\\34 & 34 & 43 & 43\\9 & 17 & 17 & 13\end{pmatrix}$

EXERCISE 12E

1 **a** $3A$ **b** O **c** $-C$ **d** O **e** $2A + 2B$
f $-A - B$ **g** $-2A + C$ **h** $4A - B$ **i** $3B$

2 **a** $X = A - B$ **b** $X = C - B$ **c** $X = 2C - 4B$
d $X = \frac{1}{2}A$ **e** $X = \frac{1}{3}B$ **f** $X = A - B$
g $X = 2C$ **h** $X = \frac{1}{2}B - A$ **i** $X = \frac{1}{4}(A - C)$

3 **a** $X = \begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix}$ **b** $X = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}$

c $X = \begin{pmatrix} -1 & -6 \\ 1 & -\frac{1}{2} \end{pmatrix}$

EXERCISE 12F.1

1 **a** 11 **b** 22 **c** 16

2 **b** $\begin{pmatrix} w & x & y & z \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$

3 **a** $\mathbf{P} = \begin{pmatrix} 27 & 35 & 39 \end{pmatrix}$, $\mathbf{Q} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

b total cost $= \begin{pmatrix} 27 & 35 & 39 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} = \$291$

4 **a** $\mathbf{P} = \begin{pmatrix} 10 & 6 & 3 & 1 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 3 \\ 2 \\ 4 \\ 2 \end{pmatrix}$

b total points $= \begin{pmatrix} 10 & 6 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 4 \\ 2 \end{pmatrix} = 56$ points

EXERCISE 12F.2

1 Number of columns in **A** does not equal number of rows in **B**.

2 **a** when $m = n$ **b** 2×3
c **B** has 3 columns, **A** has 2 rows, and $3 \neq 2$

3 **a** does not exist **b** $\begin{pmatrix} 28 & 29 \end{pmatrix}$

4 **a** 8 **b** $\begin{pmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{pmatrix}$

5 **a** $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -7 \\ -2 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & 5 & 3 \end{pmatrix}$
d $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

6 **a** $\mathbf{Q} = \begin{pmatrix} 32 & 24 \\ 25 & 16 \\ 13 & 9 \end{pmatrix}$ **b** $\mathbf{P} = \begin{pmatrix} 1.19 \\ 1.55 \end{pmatrix}$

c $\mathbf{QP} = \begin{pmatrix} 32 & 24 \\ 25 & 16 \\ 13 & 9 \end{pmatrix} \begin{pmatrix} 1.19 \\ 1.55 \end{pmatrix} = \begin{pmatrix} 75.28 \\ 54.55 \\ 29.42 \end{pmatrix}$

It represents the total value of sales for each pen colour for the week.

d $\$75.28 + \$54.55 + \$29.42 = \$159.25$

7 **a** $\mathbf{P} = \begin{pmatrix} 12.5 \\ 9.5 \end{pmatrix}$, $\mathbf{N} = \begin{pmatrix} 2375 & 5156 \\ 2502 & 3612 \end{pmatrix}$

b $\begin{pmatrix} 78\,669.5 \\ 65\,589 \end{pmatrix}$ income from day 1 / income from day 2 **c** £144 258.50

8 **a** $\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{pmatrix}$ **b** $\mathbf{P} = \begin{pmatrix} 14 & 6 & 38 \\ 12 & 4 & 44 \end{pmatrix}$

c $\begin{pmatrix} 96 & 140 \\ 104 & 152 \end{pmatrix}$ **d** **i** \$96 **ii** \$152

e The elements of **PR** tell us that, if all the items are to be bought at one store, it is cheapest to do so at store A for both you and your friend. However, the cheapest way is to buy paint from store A, and hammers and screwdrivers from store B.

EXERCISE 12F.3

1 **a** $\mathbf{A}^2 + \mathbf{A}$ **b** $\mathbf{B}^2 + 2\mathbf{B}$ **c** $\mathbf{A}^3 - 2\mathbf{A}^2 + \mathbf{A}$
d $\mathbf{A}^3 + \mathbf{A}^2 - 2\mathbf{A}$ **e** $\mathbf{AC} + \mathbf{BC} + \mathbf{AD} + \mathbf{BD}$
f $\mathbf{A}^2 + \mathbf{BA} + \mathbf{AB} + \mathbf{B}^2$ **g** $\mathbf{A}^2 + \mathbf{BA} - \mathbf{AB} - \mathbf{B}^2$
h $\mathbf{A}^2 + 2\mathbf{A} + \mathbf{I}$ **i** $9\mathbf{I} - 6\mathbf{B} + \mathbf{B}^2$

2 **a** $\mathbf{A}^3 = 3\mathbf{A} - 2\mathbf{I}$, $\mathbf{A}^4 = 4\mathbf{A} - 3\mathbf{I}$
b $\mathbf{B}^3 = 3\mathbf{B} - 2\mathbf{I}$, $\mathbf{B}^4 = 6\mathbf{I} - 5\mathbf{B}$, $\mathbf{B}^5 = 11\mathbf{B} - 10\mathbf{I}$
c $\mathbf{C}^3 = 13\mathbf{C} - 12\mathbf{I}$, $\mathbf{C}^5 = 121\mathbf{C} - 120\mathbf{I}$

3 **a** **i** $\mathbf{I} + 2\mathbf{A}$ **ii** $2\mathbf{I} - 2\mathbf{A}$ **iii** $10\mathbf{A} + 6\mathbf{I}$
b $\mathbf{A}^2 + \mathbf{A} + 2\mathbf{I}$ **c** **i** $-3\mathbf{A}$ **ii** $-2\mathbf{A}$ **iii** $\mathbf{A}$

4 **a** $\mathbf{A}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

b false as $\mathbf{A}(\mathbf{A} - \mathbf{I}) = \mathbf{O}$ does not imply that $\mathbf{A} = \mathbf{O}$ or $\mathbf{A} - \mathbf{I} = \mathbf{O}$

c $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & b \\ \frac{a - a^2}{b} & 1 - a \end{pmatrix}$, $b \neq 0$

5 For example, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

6 **a** $a = 3,\ b = -4$ **b** $a = 1,\ b = 8$

7 **a** $\mathbf{A}^2 = -2\mathbf{A} + \mathbf{I}$ **b** $\mathbf{A}^3 = 5\mathbf{A} - 2\mathbf{I}$
c $\mathbf{A}^4 = -12\mathbf{A} + 5\mathbf{I}$

EXERCISE 12G.1

1 $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = 3\mathbf{I}$, $\begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{pmatrix}$

2 $\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = 10\mathbf{I}$, $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{pmatrix}$

3 **a** -2 **b** -1 **c** 0 **d** 1

4 **a** 26 **b** 6 **c** -1 **d** $a^2 + a$

5 **a** 2 **b** $\mathbf{A}^{-1} = \begin{pmatrix} \frac{7}{2} & -\frac{3}{2} \\ -2 & 1 \end{pmatrix}$

6 **a** $\frac{1}{14}\begin{pmatrix} 5 & -4 \\ 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ **c** does not exist
d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **e** $\frac{1}{10}\begin{pmatrix} 2 & 0 \\ 1 & 5 \end{pmatrix}$ **f** does not exist
g $-\frac{1}{15}\begin{pmatrix} 7 & -2 \\ -4 & -1 \end{pmatrix}$ **h** $\frac{1}{10}\begin{pmatrix} 2 & -4 \\ 1 & 3 \end{pmatrix}$

i $\begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix}$ **j** does not exist **k** $\begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix}$

l $\begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$

7 **a** $\frac{1}{2k+6}\begin{pmatrix} 2 & -1 \\ 6 & k \end{pmatrix}$, $k \neq -3$ **b** $\frac{1}{3k}\begin{pmatrix} k & 1 \\ 0 & 3 \end{pmatrix}$, $k \neq 0$

c $\frac{1}{(k+2)(k-1)}\begin{pmatrix} k & -2 \\ -1 & k+1 \end{pmatrix}$, $k \neq -2$ or 1

d $\frac{1}{k(k+1)}\begin{pmatrix} k & -k \\ 3 & k-2 \end{pmatrix}$, $k \neq 0$ or -1

e $\frac{1}{k(2-k)}\begin{pmatrix} 1 & 1-k \\ -2k & k^2 \end{pmatrix}$, $k \neq 0$ or 2

f $\frac{1}{(k+4)(k-1)}\begin{pmatrix} 3k & -2 \\ -k^2-2 & k+1 \end{pmatrix}$, $k \neq -4$ or 1

EXERCISE 12G.2

1 $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2\mathbf{I}$, $\begin{pmatrix} -\frac{11}{2} & \frac{9}{2} & \frac{15}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & -3 & -5 \end{pmatrix}$

2 **a** $\begin{pmatrix} \frac{5}{4} & \frac{3}{4} & -\frac{7}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{3}{4} \\ -\frac{3}{4} & -\frac{1}{4} & \frac{5}{4} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{7}{19} & -\frac{12}{19} & \frac{13}{19} \\ \frac{2}{19} & \frac{2}{19} & \frac{1}{19} \\ \frac{3}{19} & \frac{3}{19} & -\frac{8}{19} \end{pmatrix}$

c $\begin{pmatrix} \frac{477}{9497} & -\frac{109}{9497} & -\frac{630}{9497} \\ \frac{4}{9497} & \frac{257}{18\,994} & \frac{527}{18\,994} \\ -\frac{284}{9497} & \frac{747}{18\,994} & \frac{571}{18\,994} \end{pmatrix}$

d $\approx \begin{pmatrix} 1.60 & -0.996 & -0.169 \\ -3.22 & 1.93 & 0.629 \\ 2.00 & -1.09 & -0.396 \end{pmatrix}$

e $\begin{pmatrix} -\frac{21}{16} & -\frac{17}{16} & \frac{5}{4} & \frac{11}{16} \\ -\frac{17}{16} & -\frac{29}{16} & \frac{5}{4} & \frac{15}{16} \\ \frac{5}{4} & \frac{5}{4} & -1 & -\frac{3}{4} \\ \frac{11}{16} & \frac{15}{16} & -\frac{3}{4} & -\frac{5}{16} \end{pmatrix}$

f $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -1 & \frac{3}{2} & -\frac{1}{2} \\ -\frac{15}{34} & \frac{1}{2} & -\frac{4}{17} & \frac{29}{34} & -\frac{23}{34} \\ -\frac{29}{34} & \frac{3}{2} & -\frac{61}{17} & \frac{149}{34} & -\frac{83}{34} \\ \frac{39}{34} & -\frac{3}{2} & \frac{58}{17} & -\frac{157}{34} & \frac{87}{34} \\ \frac{1}{17} & 0 & -\frac{4}{17} & \frac{6}{17} & -\frac{3}{17} \end{pmatrix}$

EXERCISE 12H

1 **a** $\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

b $\begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}$

c $\begin{pmatrix} 3 & -1 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

2 **a** $x = \frac{32}{7}$, $y = \frac{22}{7}$ **b** $x = -\frac{37}{23}$, $y = -\frac{75}{23}$

c $x = \frac{17}{13}$, $y = -\frac{37}{13}$ **d** $x = \frac{59}{13}$, $y = -\frac{25}{13}$

e $x = -40$, $y = -24$ **f** $x = \frac{1}{34}$, $y = \frac{55}{34}$

3 **a** **i** $\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, $\det \mathbf{A} = 10$

ii yes, $x = 2.5$, $y = -1$

b **i** $\begin{pmatrix} 2 & k \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, $\det \mathbf{A} = -2 - 4k$

ii $k \neq -\frac{1}{2}$, $x = \frac{8+11k}{2+4k}$, $y = \frac{5}{1+2k}$

iii $k = -\frac{1}{2}$, no solutions

4 **a** $\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 3 \\ 9 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix}$

b $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 13 \end{pmatrix}$

c $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \\ -2 \end{pmatrix}$

5 **a** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$ **b** $a = 2$, $b = -1$, $c = 3$

6 **a** $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ **b** $u = -1$, $v = 3$, $w = 5$

7 **a** $x = \frac{23}{10}$, $y = \frac{13}{10}$, $z = -\frac{9}{2}$

b $x = -\frac{1}{3}$, $y = -\frac{95}{21}$, $z = \frac{2}{21}$

c $x = 2$, $y = 4$, $z = -1$ **d** $x = 2$, $y = -1$, $z = 5$

e $x = 4$, $y = -2$, $z = 1$ **f** $x = 4$, $y = -3$, $z = 2$

g $x = 4$, $y = 6$, $z = -7$ **h** $x = 3$, $y = 11$, $z = -7$

i $x \approx 0.326$, $y \approx 7.65$, $z \approx 4.16$

8 **a** x is the cost of one football, y is the cost of one baseball, and z is the cost of one basketball.

b 11 basketballs

9 **a** $\begin{cases} 2x + 3y + 8z = 903 \\ x + 5y + 4z = 749 \\ x + 2y + 11z = 865 \end{cases}$

b $x = 112$, $y = 85$, $z = 53$ **c** €2 977 000

10 **a** cashews: \$12 per kg, macadamias: \$15 per kg, Brazil nuts: \$10 per kg

b \$11.80

11 class P: 24 students, class Q: 27 students, class R: 25 students

12 $x^2 - 0.799xy + 0.538y^2 - 0.616x - 1.07y - 3.69 = 0$

REVIEW SET 12A

1 **a** $\begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$ **c** $\begin{pmatrix} -2 & 0 \\ 4 & -8 \end{pmatrix}$

d $\begin{pmatrix} 2 & 2 \\ 2 & -5 \end{pmatrix}$ **e** $\begin{pmatrix} -5 & -4 \\ -2 & 6 \end{pmatrix}$ **f** $\begin{pmatrix} 7 & 6 \\ 4 & -11 \end{pmatrix}$

g $\begin{pmatrix} -1 & 8 \\ 2 & -4 \end{pmatrix}$ **h** $\begin{pmatrix} 3 & 2 \\ -6 & -8 \end{pmatrix}$ **i** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{pmatrix}$

j $\begin{pmatrix} 9 & 4 \\ 0 & 1 \end{pmatrix}$ **k** $\begin{pmatrix} -3 & -10 \\ 6 & 8 \end{pmatrix}$ **l** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{pmatrix}$

2 **a** $a = 0$, $b = 5$, $c = 1$, $d = -4$

b $a = 2$, $b = -1$, $c = 3$, $d = 8$

3 **a** $\mathbf{Y} = \mathbf{B} - \mathbf{A}$ **b** $\mathbf{Y} = \frac{1}{2}(\mathbf{A} - \mathbf{C})$ **c** $\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$
d $\mathbf{Y} = \mathbf{C}\mathbf{B}^{-1}$ **e** $\mathbf{Y} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$ **f** $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$

4 **a** $4\mathbf{L}$ **b** $-2\mathbf{L}$

5 **a** $\begin{pmatrix} 10 & -12 \\ -10 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{pmatrix}$ **c** not possible

6 **a** $\mathbf{A} - \mathbf{A}^2$ **b** $\mathbf{AB} - \mathbf{B}^2 + \mathbf{A}^2 - \mathbf{BA}$ **c** $4\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}$

7 $\mathbf{A}^3 = 27\mathbf{A} + 10\mathbf{I}$, $\mathbf{A}^4 = 145\mathbf{A} + 54\mathbf{I}$ **8** $a = 4$, $b = -7$

9 **a** $\begin{pmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{pmatrix}$ **b** does not exist **c** $\begin{pmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{pmatrix}$

10 $k \neq \frac{3}{4}$

11 **a** $x = 0$, $y = -\frac{1}{2}$ **b** $x = \frac{12}{7}$, $y = \frac{13}{7}$
c $x = 1$, $y = -2$, $z = -1$

12 **a** $a = -3$, $b = 18$, $c = 48$ **b** 48 m **c** 8 seconds

13 **a** $\begin{cases} 3x + 2y + 5z = 267 \\ 2x + 3y + z = 145 \\ x + 5y + 4z = 230 \end{cases}$
b opera: €32, play: €18, concert: €27 **c** €200

REVIEW SET 12B

1 **a** $\begin{pmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{pmatrix}$

2 **a** $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 & 2 \\ 4 & 1 \\ 3 & 1 \end{pmatrix}$
b **i** book 2 (hard cover) **ii** \$101

4 **a** $\mathbf{X} = \frac{1}{2}(\mathbf{B} - \mathbf{A})$ **b** $\mathbf{X} = \frac{2}{3}\mathbf{B} - \mathbf{A}$ **c** $\mathbf{X} = \frac{1}{4}(\mathbf{B} - \mathbf{A})$

5 $\mathbf{X} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{3}{2} \end{pmatrix}$

6 **a** $\begin{pmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 11 & 12 \end{pmatrix}$
d $\mathbf{BA}$ does not exist

7 **a** $\det \mathbf{B} \neq 0$ **b** $\mathbf{AB} = \mathbf{BA}$

8 **a** $\det \mathbf{A} = 5$ **b** $\det(-2\mathbf{A}) = 20$ **c** $\det(\mathbf{A}^2) = 25$

9 **a** $x = \frac{14}{3}$, $y = \frac{1}{3}$ **b** $x = -1$, $y = 3$
c $x = 2$, $y = 1$, $z = 3$

10 $k \in \mathbb{R}$, $k \neq 3$, -2, or 2

11 **a** $k \neq -3$ or 1 **b** $x = \dfrac{-6}{k-1}$, $y = \dfrac{2}{k-1}$

12 **a** $\begin{cases} 2a + 2b + c = -8 \\ 2a - 4b - c = 20 \\ a + 3b + c = -10 \end{cases}$ **b** $x^2 + y^2 + 4x + 2y - 20 = 0$

13 **a** adult: \$68, student: \$52, senior: \$56, child: \$46
b \$320

EXERCISE 13A.1

1 **a** $\lambda^2 - 7\lambda + 6$ **b** 1, 6

2 **a** -4, 6 **b** 3 **c** $-2 \pm \sqrt{11}$ **d** $-3 \pm i\sqrt{2}$
e 4 **f** $\frac{9}{2} \pm \frac{\sqrt{7}}{2}i$

3 **b** $\det(\lambda\mathbf{I} - \mathbf{I}) = (\lambda - 1)^2$
$= 0$ when $\lambda = 1$

4 **a** 12, 1 **b** $\lambda^2 - 13\lambda + 12$

5 **a** $\mathbf{A}^2 = \begin{pmatrix} 16 & -12 \\ -20 & 24 \end{pmatrix}$, $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{12} & -\frac{1}{12} \end{pmatrix}$
b $\mathbf{A}$ has eigenvalues -2 and 6, $\mathbf{A}^2$ has eigenvalues 4 and 36, $\mathbf{A}^{-1}$ has eigenvalues $-\frac{1}{2}$ and $\frac{1}{6}$.

6 **a** The eigenvalues of $-\mathbf{A}$ are the negatives of the eigenvalues of $\mathbf{A}$.
b The eigenvalues of $\mathbf{A}^2$ are the squares of the eigenvalues of $\mathbf{A}$.
c The eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$.

EXERCISE 13A.2

1 **a** Eigenvalues are 4 and 3.
For $\lambda = 4$, $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$.
For $\lambda = 3$, $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} t$, $t \neq 0$.
b Eigenvalues are 4 and 0.
For $\lambda = 4$, $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t$, $t \neq 0$.
For $\lambda = 0$, $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} t$, $t \neq 0$.
c Eigenvalues are 3 and -2.
For $\lambda = 3$, $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$, $t \neq 0$.
For $\lambda = -2$, $\mathbf{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} t$, $t \neq 0$.

2 **a** $\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - 2)^2$
$= 0$ when $\lambda = 2$
b $\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

3 $\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)(\lambda - \lambda_2)$
$= 0$ when $\lambda = \lambda_1$ or λ_2
The corresponding eigenvectors are $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$, $t \neq 0$ and $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} t$, $t \neq 0$.

4 **a** $\det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - a)^2$
$= 0$ when $\lambda = a$
c For any non-zero vector $\mathbf{x}$, $\mathbf{Ax} = a\mathbf{Ix} = a\mathbf{x}$.

5 **a** Eigenvalues are 3 and -2.
For $\lambda = 3$, $\mathbf{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} t$, $t \neq 0$.
For $\lambda = -2$, $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $t \neq 0$.
b Eigenvalues are 9 and 4.
For $\lambda = 9$, $\mathbf{x} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} t$, $t \neq 0$.
For $\lambda = 4$, $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$, $t \neq 0$.

6 **a** Both $\mathbf{A}$ and $\mathbf{B}$ have eigenvalues 3 and 1.
b For $\lambda = 3$, $\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} t$, $t \neq 0$. **c** no
For $\lambda = 1$, $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} t$, $t \neq 0$.

7 $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 0$

8 **a** i **b** $-i$ **c** $\begin{pmatrix} -i \\ 1 \end{pmatrix} t, \quad t \neq 0$

9 **a** Eigenvalues are $1+\sqrt{2}$ and $1-\sqrt{2}$.

For $\lambda = 1+\sqrt{2}$, $\mathbf{x} = \begin{pmatrix} 1 \\ \sqrt{2}-2 \end{pmatrix} t, \quad t \neq 0.$

For $\lambda = 1-\sqrt{2}$, $\mathbf{x} = \begin{pmatrix} 1 \\ -\sqrt{2}-2 \end{pmatrix} t, \quad t \neq 0.$

b Eigenvalues are $3+4i$ and $3-4i$.

For $\lambda = 3+4i$, $\mathbf{x} = \begin{pmatrix} \frac{1}{2}+i \\ 1 \end{pmatrix} t, \quad t \neq 0.$

For $\lambda = 3-4i$, $\mathbf{x} = \begin{pmatrix} \frac{1}{2}-i \\ 1 \end{pmatrix} t, \quad t \neq 0.$

c Eigenvalues are $i\sqrt{5}$ and $-i\sqrt{5}$.

For $\lambda = i\sqrt{5}$, $\mathbf{x} = \begin{pmatrix} -\frac{1}{2}+\frac{\sqrt{5}}{2}i \\ 1 \end{pmatrix} t, \quad t \neq 0.$

For $\lambda = -i\sqrt{5}$, $\mathbf{x} = \begin{pmatrix} -\frac{1}{2}-\frac{\sqrt{5}}{2}i \\ 1 \end{pmatrix} t, \quad t \neq 0.$

EXERCISE 13B

3 **a** **i** $\lambda_1 = 2, \quad \mathbf{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \lambda_2 = -3, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b **i** $\lambda_1 = 2+\sqrt{3}, \quad \mathbf{x}_1 = \begin{pmatrix} -3-\sqrt{3} \\ 1 \end{pmatrix},$

$\lambda_2 = 2-\sqrt{3}, \quad \mathbf{x}_2 = \begin{pmatrix} -3+\sqrt{3} \\ 1 \end{pmatrix}$

4 **a** $\lambda_1 = 10, \quad \mathbf{x}_1 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}, \quad \lambda_2 = 5, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

EXERCISE 13C

1 $\mathbf{A}^5 = \frac{1}{3}\begin{pmatrix} 2+4^5 & 2-2\times 4^5 \\ 1-4^5 & 1+2\times 4^5 \end{pmatrix} = \begin{pmatrix} 342 & -682 \\ -341 & 683 \end{pmatrix}$

2 **a** $\lambda_1 = 7, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = -2, \quad \mathbf{x}_2 = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$

b $\mathbf{P} = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}$

c $\mathbf{A}^n = \begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 7^n & 0 \\ 0 & (-2)^n \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}^{-1}$

d $\mathbf{P}^{-1}\mathbf{A}^{13}\mathbf{P} = \begin{pmatrix} 7^{13} & 0 \\ 0 & (-2)^{13} \end{pmatrix}$

3 $\mathbf{A}^{60} = \begin{pmatrix} 3^{30} & 0 \\ 0 & 3^{30} \end{pmatrix}$

4 $\mathbf{C}^{2015} = \frac{1}{3}\begin{pmatrix} -2+2^{2015} & 2+2^{2016} \\ 1+2^{2015} & -1+2^{2016} \end{pmatrix}$

EXERCISE 13D.1

1 **a** $\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$

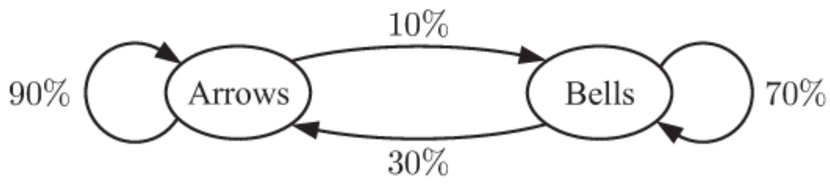
b **i** 20% of customers who buy from Bioblatt one week will buy from its competitors the next.

ii 90% of customers who buy from one of the competitors one week will buy from one of the competitors the next week.

c $\mathbf{Ts}_0 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}$ After one week Bioblatt has 10% of the market share.

d **i** 17% **ii** $\approx 25.3\%$

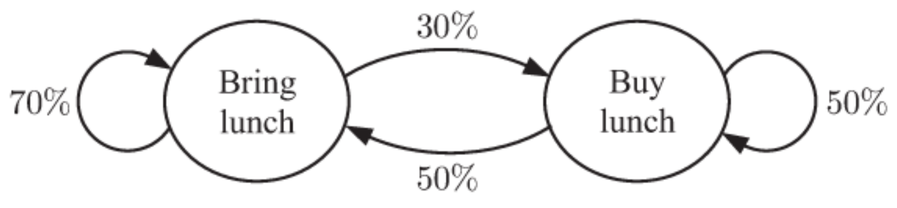
2 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}$

b $\mathbf{s}_0 = \begin{pmatrix} 600 \\ 400 \end{pmatrix}$ **c** **i** 660 people **ii** ≈ 718 people

3 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{pmatrix}$

b $\mathbf{s}_0 = \begin{pmatrix} 243 \\ 157 \end{pmatrix}$

c **i** ≈ 151 students **ii** ≈ 150 students

4 **a** 40% of customers who bought Sheez this week will buy Baaah next week.

b $\mathbf{T} = \begin{pmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{pmatrix}$

c **i** $\mathbf{Ts}_1 = \begin{pmatrix} 0.64 \\ 0.36 \end{pmatrix}$

In the second week, Baaah's market share is 64% and Sheez' market share is 36%.

ii 59.2% **iii** $\approx 42.8\%$

5 **a** $\mathbf{s}_0 = \begin{pmatrix} 62 \\ 38 \end{pmatrix}$

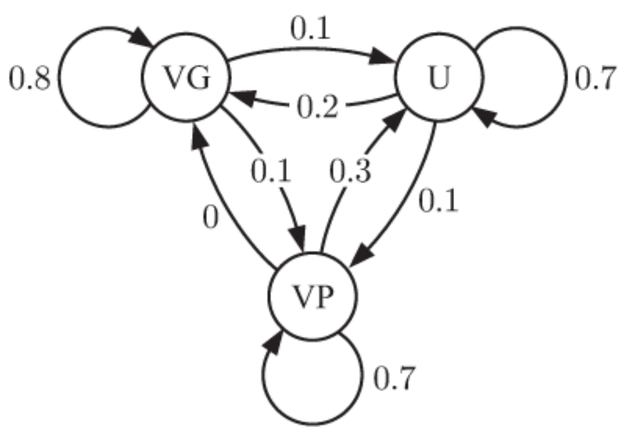
b **i** $\mathbf{Ts}_0 = \begin{pmatrix} 53.4 \\ 46.6 \end{pmatrix}$ **ii** ≈ 47 members

iii $\mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 47.38 \\ 52.62 \end{pmatrix}$

We expect ≈ 47 smokers and ≈ 53 non-smokers at the March meeting.

c ≈ 34 members

6 **a**



b **i** 10% of the land which is very good this year will be very poor next year.

ii No land which is very poor this year will be very good next year.

c $\mathbf{T}^2 = \begin{pmatrix} 0.66 & 0.3 & 0.06 \\ 0.18 & 0.54 & 0.42 \\ 0.16 & 0.16 & 0.52 \end{pmatrix}$

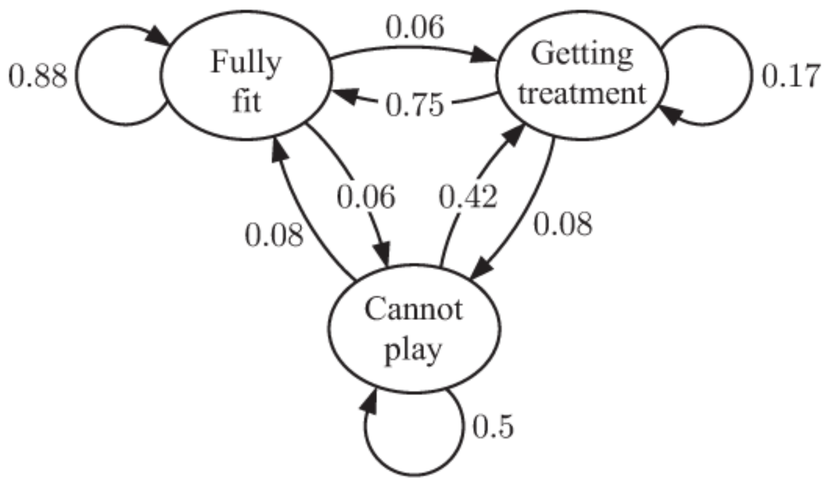
i 16% of land which is usable now will be very poor in two years' time.

ii 42% of land which is very poor now will be usable in two years' time.

d $\mathbf{s}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

e After 3 years, 13.2% is very good, 45.6% is usable, and 41.2% is very poor.

7 **a**



b $\mathbf{T} = \begin{pmatrix} 0.88 & 0.75 & 0.08 \\ 0.06 & 0.17 & 0.42 \\ 0.06 & 0.08 & 0.50 \end{pmatrix}$

c **i** 88% of players who are fully fit this week will be fully fit next week.

ii 42% of players who cannot play this week will be getting treatment next week.

d $\mathbf{T}^2 = \begin{pmatrix} 0.8242 & 0.7939 & 0.4254 \\ 0.0882 & 0.1075 & 0.2862 \\ 0.0876 & 0.0986 & 0.2884 \end{pmatrix}$

79.39% of players getting treatment this week will be fully fit in two weeks' time.

e **i** 31 fully fit, 4 getting treatment, 3 cannot play

ii 30 fully fit, 4 getting treatment, 4 cannot play

EXERCISE 13D.2

1 **a** $\mathbf{s}_0$ describes the current state of the system. The door is currently open 10% of the time and closed 90% of the time. $\mathbf{T}$ is the transition matrix which predicts how the system will change each time someone visits.

b $\mathbf{s} = \begin{pmatrix} \frac{1}{15} \\ \frac{14}{15} \end{pmatrix}$

c In the long term the door is left open $\frac{1}{15} \times 100\% \approx 6.67\%$ of the time. Jada's concern is not justified.

2 **a** $\mathbf{s}_0 = \begin{pmatrix} 425 \\ 716 \end{pmatrix}$

b Clydes: ≈ 648 passengers, Roos: ≈ 493 passengers

c We assume that the preferences of the passengers remain the same from month to month, and that the total number of passengers travelling between the cities does not change. The first assumption may be reasonable, but the second assumption may not be if the number of people travelling between the cities varies throughout the year.

d Clydes: $\approx 56.8\%$, Roos: $\approx 43.2\%$ **e** $\mathbf{s} = \begin{pmatrix} \frac{21}{37} \\ \frac{16}{37} \end{pmatrix}$

3 **a** $1, -\frac{3}{10}$ **b** $\begin{pmatrix} \frac{4}{9} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ **c** $\mathbf{s} = \begin{pmatrix} \frac{4}{13} \\ \frac{9}{13} \end{pmatrix}$

4 **a** **i** 29 fully fit, 6 getting treatment, 4 injured

ii 29 fully fit, 6 getting treatment, 4 injured

iii 30 fully fit, 6 getting treatment, 3 injured

Note: Due to rounding, the number of players in each category does not add up to 40.

b The squad should have enough fully fit players to field three teams of 9.

c In the long term, $\approx 75.8\%$ of players are available.
$\therefore$ the squad should have at least 36 players.

5 **a** $\mathbf{T} = \begin{pmatrix} 0.75 & 0.2 & 0.15 \\ 0.15 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.65 \end{pmatrix}$

b $\mathbf{T}^2 = \begin{pmatrix} 0.6075 & 0.3 & 0.25 \\ 0.2225 & 0.43 & 0.2725 \\ 0.17 & 0.27 & 0.4775 \end{pmatrix}$

Of the birds currently living on Chalk, in two years' time 25% of them will be living on Paua, 27.25% will be living on Manu, and 47.75% will be living on Chalk.

c **i** Paua: 32.55%, Manu: 34.3%, Chalk: 33.15%

ii Paua: $\approx 38.4\%$, Manu: $\approx 31.0\%$, Chalk: $\approx 30.6\%$

d Paua: $\approx 41.2\%$, Manu: $\approx 29.9\%$, Chalk: $\approx 28.9\%$

e No, $\approx 32.4\%$ of birds will migrate each year.

6 **a** **i** 10% **ii** 7% **iii** 34%

b **i** C **ii** D **c** B

7 **c** In the long term, the two populations will have the proportions $\frac{q}{p+q}$ and $\frac{p}{p+q}$.

REVIEW SET 13A

1 **a** $\lambda_2 + \lambda - 4$ **b** $-\frac{1}{2} \pm \frac{\sqrt{17}}{2}$

2 **a** $\mathbf{Ax} = \begin{pmatrix} -42 \\ 18 \end{pmatrix}$, eigenvalue $= 6$ **b** -4

3 **a** $\lambda_1 = 12, \quad \mathbf{x}_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \lambda_2 = 2, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

4 **a** $\lambda = 5, \quad \mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

b $\lambda_1 = -4 + \sqrt{3}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 + \sqrt{3} \\ 1 \end{pmatrix},$

$\lambda_2 = -4 - \sqrt{3}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 - \sqrt{3} \\ 1 \end{pmatrix}$

5 **a** $\lambda_1 = 7, \quad \mathbf{x}_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \quad \lambda_2 = -2, \quad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

6 $\mathbf{A}^6 = \frac{1}{3} \begin{pmatrix} -2 \times 7^6 + 5 \times 4^6 & -10 \times 7^6 + 10 \times 4^6 \\ 7^6 - 4^6 & 5 \times 7^6 - 2 \times 4^6 \end{pmatrix}$

7 $\mathbf{C}^{10} = \frac{1}{10} \begin{pmatrix} 4 \times 8^{10} + 6 \times 2^{10} & -3 \times 8^{10} + 3 \times 2^{10} \\ -8 \times 8^{10} + 8 \times 2^{10} & 6 \times 8^{10} + 4 \times 2^{10} \end{pmatrix}$

8 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}$

b **i** 225 passengers **ii** ≈ 288 passengers

9 **a** $\mathbf{T} = \begin{pmatrix} 0.55 & 0.05 \\ 0.45 & 0.95 \end{pmatrix}$

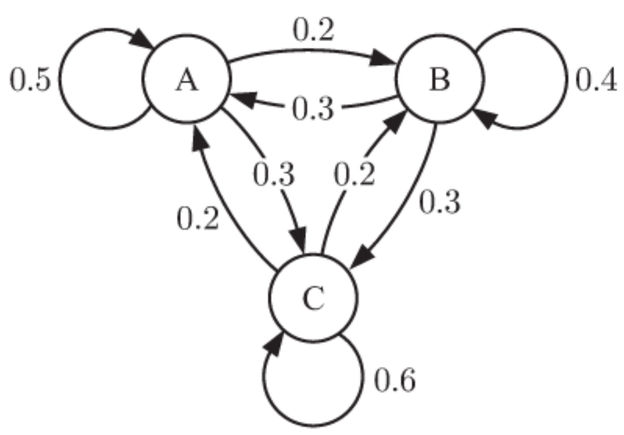
b 45% of the children of smokers will be non-smokers.

c $\mathbf{T}^2\mathbf{s}_0 = \begin{pmatrix} 0.225 \\ 0.775 \end{pmatrix}$

22.5% of the grandchildren of this generation will be smokers, and 77.5% will be non-smokers.

d $\mathbf{s} = \begin{pmatrix} \frac{1}{10} \\ \frac{9}{10} \end{pmatrix}$, in the long term 10% are adult smokers.

10 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.3 & 0.3 & 0.6 \end{pmatrix}$

b After leaving taxi rank C, the probability that Bill will return to taxi rank A is 0.2.

c C **d** C

REVIEW SET 13B

1 **a** $\pm\sqrt{22}$ **b** -3 **c** $\frac{7}{2} \pm \frac{\sqrt{3}}{2}i$ **2** $k = -7$

3 **a** $\lambda_1 = 7$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -2$, $\mathbf{x}_2 = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

b $\lambda_1 = 21$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -6$, $\mathbf{x}_2 = \begin{pmatrix} -5 \\ 4 \end{pmatrix}$

4 **a** i **b** $-i$ **c** $\begin{pmatrix} 3-i \\ 2 \end{pmatrix} t$, $t \neq 0$

5 **a** $\lambda = -4$, $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

b $\lambda_1 = 2 + \sqrt{7}$, $\mathbf{x}_1 = \begin{pmatrix} 3+\sqrt{7} \\ 1 \end{pmatrix}$,
$\lambda_2 = 2 - \sqrt{7}$, $\mathbf{x}_2 = \begin{pmatrix} 3-\sqrt{7} \\ 1 \end{pmatrix}$

c $\lambda_1 = 3 + i$, $\mathbf{x}_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$,
$\lambda_2 = 3 - i$, $\mathbf{x}_2 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$

6 **a** $\lambda_1 = 5$, $\mathbf{x}_1 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$, $\lambda_2 = 4$, $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

b $\mathbf{P} = \begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}$, $\mathbf{PBP}^{-1} = \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix}$

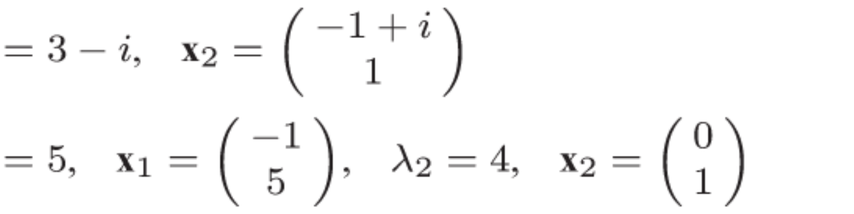
c **i** $\mathbf{B}^3 = \begin{pmatrix} 5^3 & 0 \\ -5^4 + 5 \times 4^3 & 4^3 \end{pmatrix} = \begin{pmatrix} 125 & 0 \\ -305 & 64 \end{pmatrix}$

ii $\mathbf{B}^6 = \begin{pmatrix} 5^6 & 0 \\ -5^7 + 5 \times 4^6 & 4^6 \end{pmatrix} = \begin{pmatrix} 15\,625 & 0 \\ -57\,645 & 4096 \end{pmatrix}$

7 **a** $\mathbf{P} = \begin{pmatrix} -1 & 3 \\ 1 & 4 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 15 \end{pmatrix}$

b $\mathbf{A}^{10} = \frac{1}{7}\begin{pmatrix} 4 + 3 \times 15^{10} & -3 + 3 \times 15^{10} \\ -4 + 4 \times 15^{10} & 3 + 4 \times 15^{10} \end{pmatrix}$

8 **a** **i**



ii $\mathbf{T} = \begin{pmatrix} 0.8 & 0.15 \\ 0.2 & 0.85 \end{pmatrix}$

b $\mathbf{T}^2 = \begin{pmatrix} 0.67 & 0.2475 \\ 0.33 & 0.7525 \end{pmatrix}$, $\mathbf{T}^3 = \begin{pmatrix} 0.5855 & 0.310\,875 \\ 0.4145 & 0.689\,125 \end{pmatrix}$

c The 2nd column of $\mathbf{T}^2$ tells us how the people who supported Suzie this week will respond in 2 weeks' time. 24.75% will support Chris, and 75.25% will support Suzie.

d **i** Chris **ii** Suzie

9 **a** $\mathbf{T} = \begin{pmatrix} 0.43 & 0.18 & 0.09 \\ 0.47 & 0.56 & 0.55 \\ 0.10 & 0.26 & 0.36 \end{pmatrix}$,

$\mathbf{T}^2 = \begin{pmatrix} 0.2785 & 0.2016 & 0.1701 \\ 0.5203 & 0.5412 & 0.5483 \\ 0.2012 & 0.2572 & 0.2816 \end{pmatrix}$

b **i** 47% **ii** 20.12%

c **i** $\approx$ 19.1% will be underweight, $\approx$ 54.4% will be of healthy weight, $\approx$ 26.5% will be overweight.

ii 20.4% will be underweight, $\approx$ 54.0% will be of healthy weight, $\approx$ 25.6% will be overweight.

d $\mathbf{s} \approx \begin{pmatrix} 0.210 \\ 0.539 \\ 0.252 \end{pmatrix}$

In the long term, $\approx$ 21.0% of people will be underweight, $\approx$ 53.9% will be of healthy weight, and $\approx$ 25.2% will be overweight.

e No, it is unlikely that the transition matrix will remain constant over a number of generations, as attitudes and lifestyles will change.

10 **a** $\mathbf{s}_0 = \begin{pmatrix} 60 \\ 60 \end{pmatrix}$, $\mathbf{T} = \begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix}$

b $\mathbf{Ts}_0 = \begin{pmatrix} 48 \\ 72 \end{pmatrix}$

After 1 week 48 students are painting and 72 students are doing pottery.

c $\lambda_1 = 1$, $\mathbf{x}_1 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$, $\lambda_2 = \frac{2}{5}$, $\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

d $20\left(2 + \left(\frac{2}{5}\right)^n\right)$ students **e** 40 students

EXERCISE 14A

1 **a** (4, 6) **b** (−1, 2) **c** (5, −1) **d** (0, −3)
e $(3 + h,\ 2 + k)$

2 **a** (−2, 3) **b** (2, 4) **c** (0, 1) **d** (−5, −1)
e $(x - 2,\ y + 3)$

3 **a** (−2, 1) **b** (−1, 4) **c** (0, −2) **d** (1, 6)
e $(-1 - h,\ 3 - k)$

4 **a** (−4, 1) **b** (−1, −1) **c** (3, 5) **d** (−5, −4)
e $(x - 4,\ y + 1)$

5 **a** $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$ **c** $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$
e $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ **f** $\begin{pmatrix} 2-x \\ -3-y \end{pmatrix}$

6 **a** $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} + \begin{pmatrix} e_2 \\ f_2 \end{pmatrix}$ **b** $\begin{pmatrix} e_1 + e_2 \\ f_1 + f_2 \end{pmatrix}$

7 $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$, $\det \mathbf{A} = 1$

EXERCISE 14B

1 **a** $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ **d** $\mathbf{A} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$

e $\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

2 **a** The transformation is a clockwise rotation about O through $\frac{\pi}{4}$.

b The transformation is a clockwise rotation about O through $\approx 1.97^c$.

3 **a** $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ **b** $\mathbf{A}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

c The inverse of the transformation matrix for an anticlockwise rotation about O through θ is the matrix for an anticlockwise rotation about O through $-\theta$.

4 **a** $\left(\frac{-5-\sqrt{3}}{2}, \frac{-5\sqrt{3}+1}{2} \right)$

b $\left(\frac{3\sqrt{3}-1}{2}, \frac{-3-\sqrt{3}}{2} \right)$ **c** $\left(-\frac{1}{\sqrt{2}}, \frac{5}{\sqrt{2}} \right)$

d $\left(-\frac{3}{\sqrt{2}}, \frac{5}{\sqrt{2}} \right)$

EXERCISE 14C

1 **a** $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} -\frac{12}{13} & \frac{5}{13} \\ \frac{5}{13} & \frac{12}{13} \end{pmatrix}$ **d** $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

2 **a** The transformation is a reflection in the line $y = (\tan\frac{\pi}{8})x$.

b The transformation is a reflection in the line $y = 4x$.

3 **a** Yes, it can be an anticlockwise rotation about O through $\frac{3\pi}{4}$.

b Yes, it can be a reflection in the line $y = (\tan\frac{\pi}{8})x$.

4 $\mathbf{A}$ is its own inverse. If an object is reflected in $y = (\tan\alpha)x$, then the resulting image is reflected in $y = (\tan\alpha)x$, then we obtain the original object.

5 **a** $(\frac{4}{5}, \frac{22}{5})$ **b** $(-1, -3)$ **c** $(-2, 4)$

EXERCISE 14D

1 **a** a horizontal stretch with scale factor $\frac{1}{2}$

b a vertical stretch with scale factor 2

c a horizontal stretch with scale factor 3

d a vertical stretch with scale factor $\frac{1}{4}$

2 **a** $\mathbf{A} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & 1 \end{pmatrix}$ **b** $(\frac{15}{2}, 4)$

c $\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{5} & 0 \\ 0 & 1 \end{pmatrix}$; a horizontal stretch with scale factor $\frac{2}{5}$.

3 **a** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$ **b** $(-1, \frac{3}{2})$

c $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$; a vertical stretch with scale factor $\frac{4}{3}$.

EXERCISE 14E

1 **a** an enlargement with scale factor 3

b a reduction with scale factor $\frac{1}{4}$

2 **a** $\mathbf{A} = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$ **b** $(4, 8)$

c $\mathbf{A}^{-1} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix}$; a reduction with scale factor $\frac{3}{4}$.

3 **a** $\mathbf{x}' = \begin{pmatrix} kx \\ y \end{pmatrix}$ **b** $\mathbf{x}'' = \begin{pmatrix} kx \\ ky \end{pmatrix}$

c $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, which is the transformation matrix of an enlargement with scale factor k.

EXERCISE 14F

1 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

b **i** a reflection in $y = (\tan\frac{\pi}{12})x$

ii a reflection in $y = (\tan\frac{5\pi}{12})x$

c In general for 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{AB} \neq \mathbf{BA}$, so the corresponding linear transformations are different.

2 **a** a reflection in $y = x$

b a reflection in $y = (\tan\frac{5\pi}{12})x$

c an anticlockwise rotation about O through $\frac{\pi}{3}$

d an anticlockwise rotation about O through $\approx 0.927^c$

3 **a** an enlargement with scale factor k; order does not matter

b a rotation about O through $(\theta + \phi)$; order does not matter

c a reflection in the line $y = (\tan\frac{\theta}{2})x$; if the transformations are reversed, then the composite transformation would be a reflection in the line $y = (\tan(-\frac{\theta}{2}))x$, which is different for $\theta \neq k\pi$, $k \in \mathbb{Z}$.

4 **a** $\mathbf{x}' = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

b $\mathbf{x}' = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

c $\mathbf{x}' = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 5 \\ 0 \end{pmatrix}$

5 **Hint:** Let $\mathbf{A}$ be the transformation matrix for a reflection in the line $y = (\tan\alpha)x$, and $\mathbf{B}$ be the transformation matrix for a rotation about O through θ. Find $\mathbf{BA}$.

6 **a** $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\left(\mathbf{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right)$

b $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ -4 \end{pmatrix}$, which is of the form $\mathbf{x}' = \mathbf{Ax} + \mathbf{b}$.

c $\mathbf{x}' = \mathbf{A}(\mathbf{x} + \mathbf{c}) = \mathbf{Ax} + \mathbf{Ac}$, that is $\mathbf{b} = \mathbf{Ac}$.

7 a reflection in $y = 3x$

EXERCISE 14G

1 **a** 3 **b** 4 **c** 7 **d** 10

2 **a** **i** $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ **ii** 1

b **i** $\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\mathbf{x}$ **ii** 1

c **i** $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbf{x}$ **ii** 1

d **i** $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$ **ii** 2

e **i** $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}\mathbf{x}$ **ii** $\frac{1}{3}$

f **i** $\mathbf{x}' = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}\mathbf{x}$ **ii** $\frac{16}{9}$

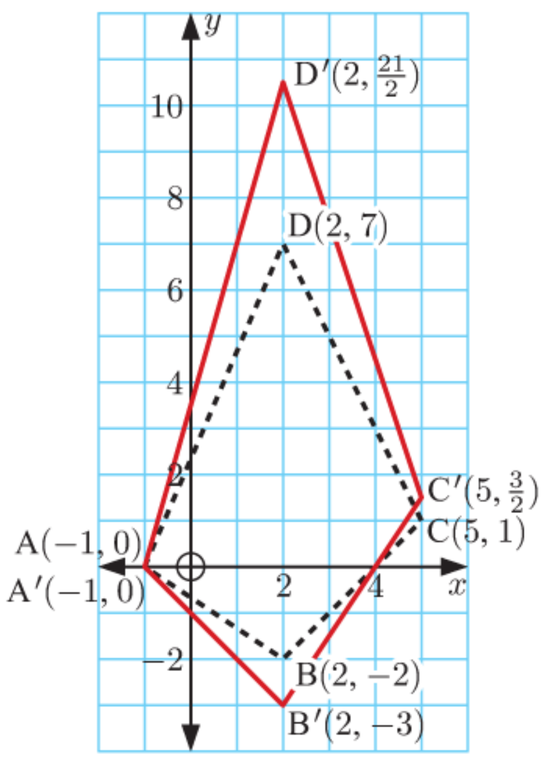
3 no; $|\det \mathbf{A}| = 1$

4 **a** a horizontal stretch with scale factor 3

b $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x}$

c $\dfrac{\text{area A'B'C'D'}}{\text{area ABCD}} = \dfrac{9}{3} = 3 = |\det \mathbf{A}|$

5 **a**

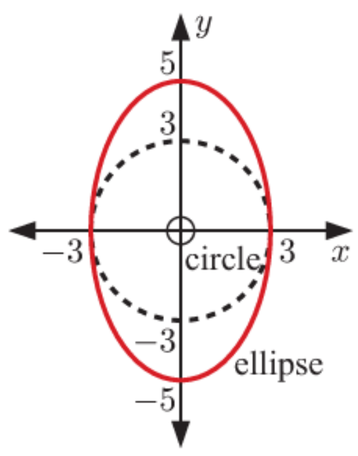


b 27 units^2

c $\mathbf{x}' = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\mathbf{x}$

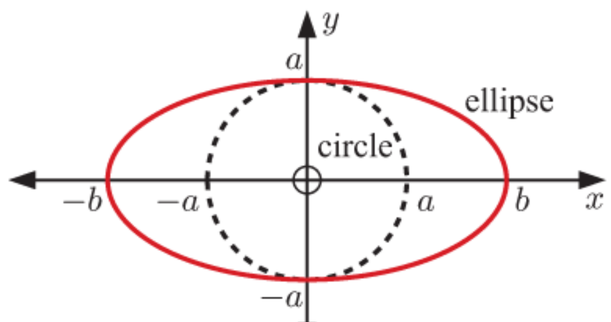
d 40.5 units^2

6 **a**



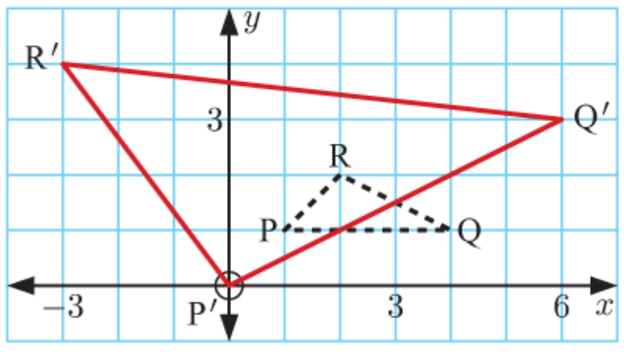
b 15π units^2

7 **a**



8 **a** P'(0, 0), Q'(6, 3), and R'(−3, 4)

b



c $16\frac{1}{2}$ units^2

9 **a** $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{3}{5} & -\frac{8}{5} \\ -\frac{4}{5} & \frac{6}{5} \end{pmatrix}$

c The combination of transformations doubles area.

10 **a** $\begin{pmatrix} 0 & -\frac{3}{2} \\ -3 & 0 \end{pmatrix}$

b O'(0, 0), A'$(-\frac{3}{2}, -3)$, B'$(-\frac{3}{2}, -9)$, C'$(0, -6)$

c 9 units^2

11 No, $\det(\mathbf{AB}) = \det(\mathbf{BA})$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

REVIEW SET 14A

1 **a** $(-2, 6)$ **b** $(-5, 4)$ **c** $(-4, -1)$ **d** $(-8, -4)$

2 **a** $(3, 2)$ **b** $(2, -4)$ **c** $(-4, -6)$

3 **a** $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$

4 **a** a reflection in the line $y = 2x$

b an anticlockwise rotation about O through $\approx 0.395^c$

5 $\left(-\frac{3\sqrt{3}}{2} - 2,\ -\frac{3}{2} + 2\sqrt{3}\right)$

6 **a** a vertical stretch with scale factor 5

b a horizontal stretch with scale factor $\frac{2}{3}$

c a reduction with scale factor $\frac{1}{4}$

7 **a** $\mathbf{A} = \begin{pmatrix} \frac{7}{2} & 0 \\ 0 & 1 \end{pmatrix}$ **b** $(21, -4)$

c $\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & 1 \end{pmatrix}$; a horizontal stretch with scale factor $\frac{2}{7}$.

8 **a** $(-1 - 2\sqrt{3},\ \sqrt{3} - 2)$ **b** $(-\frac{9}{2}, 9)$

9 **a** a reflection in the line $y = \frac{1}{\sqrt{3}}x$

b an anticlockwise rotation about O through $\frac{2\pi}{3}$

10 **a** $\mathbf{x}' = \begin{pmatrix} 0 & -4 \\ -1 & 0 \end{pmatrix}\mathbf{x}$ **b** $(-20, 3)$

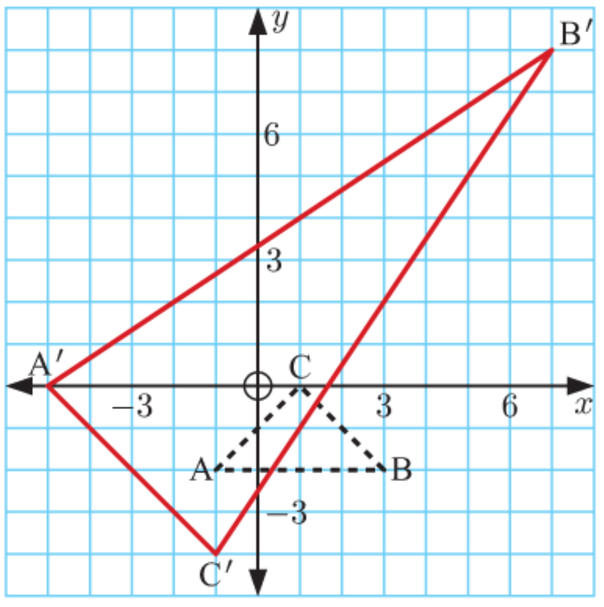
11 **a** **i** $\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}$ **ii** 5

b **i** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **ii** 1

c **i** $\begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ **ii** $\frac{1}{16}$

12 **a** A'$(-5, 0)$, B'$(7, 8)$, C'$(-1, -4)$

b



c 40 units^2

REVIEW SET 14B

1 **a** $(7, 3)$ **b** $(4, -2)$ **c** $(-2, -5)$

2 **a** $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 6 \\ -1 \end{pmatrix}$ **c** $\begin{pmatrix} -9 \\ -3 \end{pmatrix}$

3 **a** $\mathbf{A} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} \frac{5}{3} & 0 \\ 0 & 1 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

4 **a** an enlargement with scale factor $\frac{4}{3}$

b an anticlockwise rotation about O through $\approx 0.464^c$

c a reflection in the line $y \approx -1.87x$

5 $(-\frac{11}{5}, \frac{23}{5})$

6 **a** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{5} \end{pmatrix}$ **b** $(-4, -\frac{6}{5})$

c $\mathbf{A}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}$; a vertical stretch with scale factor $\frac{5}{3}$.

7 **a** **i** B$(-1, 3)$ **ii** OA $=$ OB $= \sqrt{10}$ units

8 **a** a reflection in the line $y = (\tan \frac{\pi}{12})x$

b a rotation about O through π

9 **a** $\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\mathbf{x} + \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

b $\mathbf{x}' = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 12 \\ -7 \end{pmatrix}$

10 **a** $\mathbf{A} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix}$

b $\left(-\frac{3}{2\sqrt{2}}, -\frac{27}{2\sqrt{2}}\right)$

c Yes, the order in which the transformations are applied does not matter.

11 **a** ellipse **b** 6π cm^2

12 **a** $\mathrm{P}'\left(-2, \frac{\sqrt{3}}{2}\right)$, $\mathrm{Q}'(-8 + 2\sqrt{3}, \frac{1}{2} + 2\sqrt{3})$,

$\mathrm{R}'(-8 + 6\sqrt{3}, \frac{3}{2} + 2\sqrt{3})$, $\mathrm{S}'\left(-2 + 4\sqrt{3}, 1 + \frac{\sqrt{3}}{2}\right)$

b 24 units^2

EXERCISE 15A

1 **a** 6 vertices, 8 edges

b **i** B, E

ii B, E, F

c **i** AE, DE, and EF

ii BD, DE, and EF

d

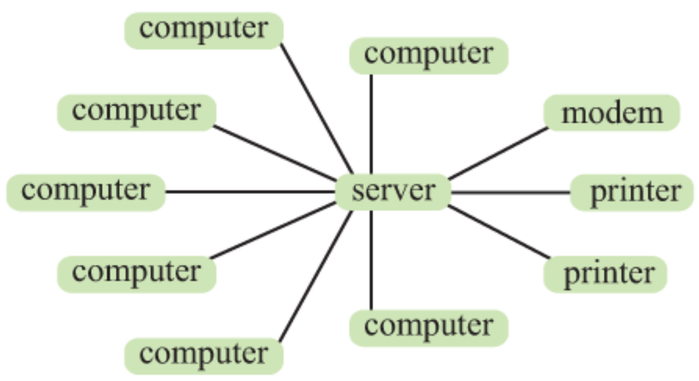
Vertex	*Degree*
A	2
B	4
C	1
D	3
E	4
F	2

2 **a**

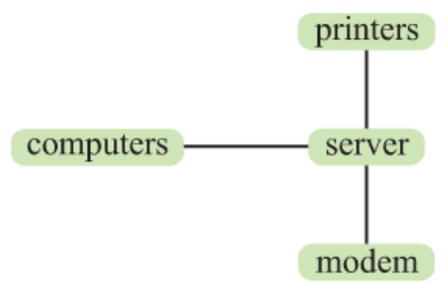
Vertex	*in degree*	*out degree*
P	1	2
Q	2	3
R	1	1
S	1	1
T	3	2
U	2	1

b S $\rightarrow$ T $\rightarrow$ P $\rightarrow$ Q $\rightarrow$ R

3 **a**

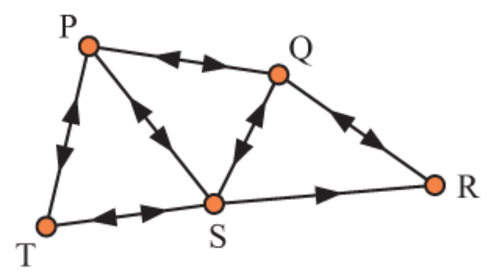


b

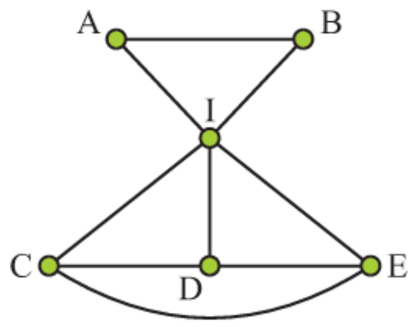


4 **a** S is connected to R, but R is not connected to S.

b



5 **a**

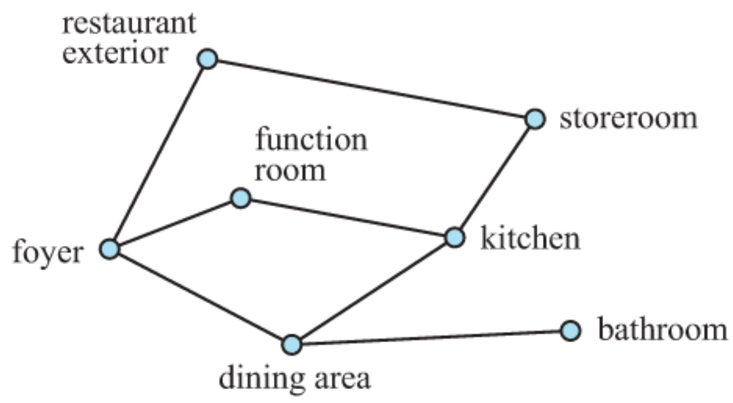


b **i** 2 people

ii 3 people

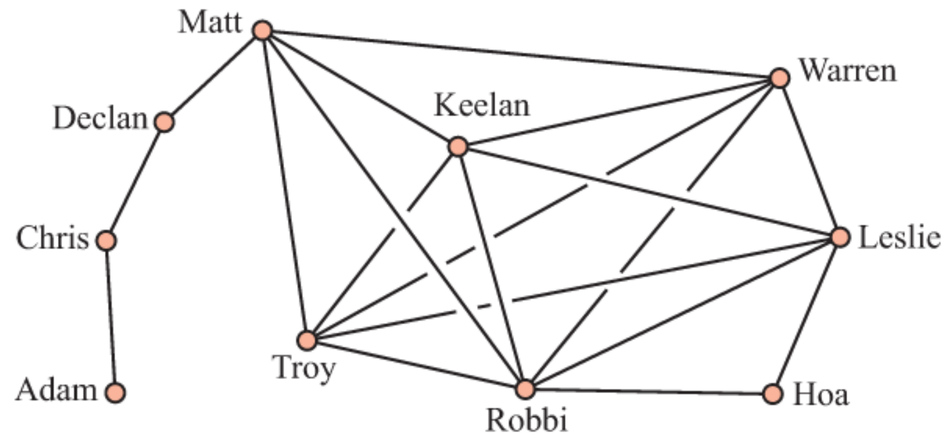
iii 5 people

6 **a**



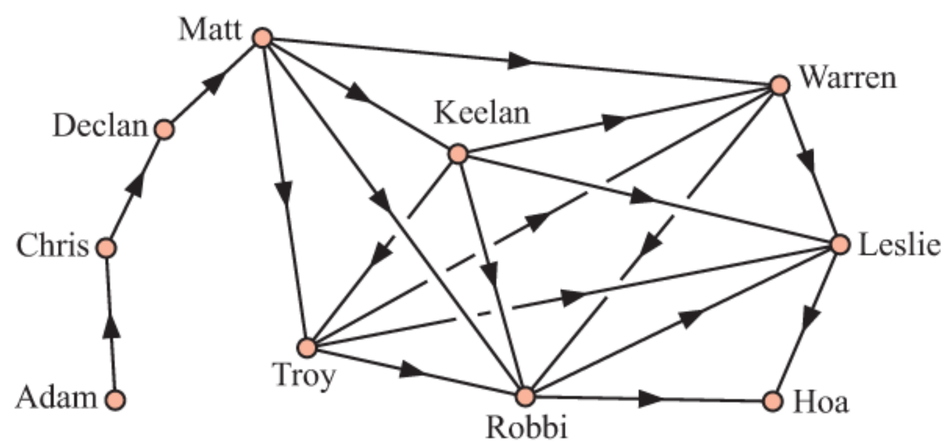
b **i** 3 rooms **ii** 3 rooms

7 **a**



b 5, Warren can compete against 5 other members.

c



d in degree 1, out degree 4

Of his 5 potential opponents, Keelan weighs less than 1 of them, and weighs more than 4 of them.

8 **a** **i** 9 **ii** 18

b $d = 2e$, every edge contributes 2 to the total sum of the degrees.

c 11 edges

d If the undirected graph has an odd number of vertices with odd degree, then d is odd.
However, $d = 2e$ which must be even.
$\therefore$ an undirected graph must have an even number of vertices with odd degree.

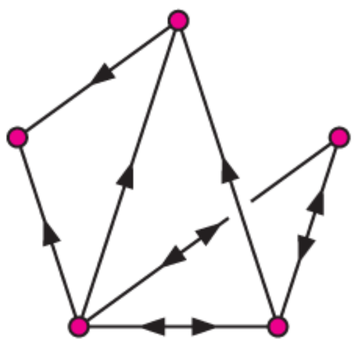
9 No, there may be loops, or more than one edge between the same pair of vertices.

10 **a**

Vertex	*in degree*	*out degree*
P	2	2
Q	2	0
R	1	1
S	1	1
T	3	2
U	2	4
V	2	3

b **i** 13 **ii** 13
In a directed graph, each movement *to* a vertex has a corresponding movement *from* another vertex. The sums of the in and out degrees will be the same.

11 **a** $k = 2$ **b**



EXERCISE 15B

1 **a** **i** simple **ii** connected **iii** not complete

b **i** not simple **ii** connected **iii** not complete

c **i** not simple **ii** connected **iii** not complete

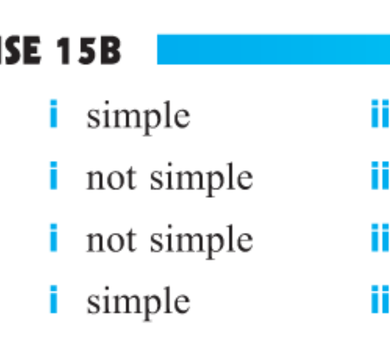
d **i** simple **ii** connected **iii** complete

2 **a** **i** 3 min **ii** 17 min

b

	P	Q	R	S	T
P	–	8	–	7	6
Q	8	–	9	–	3
R	–	9	–	5	–
S	7	–	5	–	3
T	6	3	–	3	–

3 **a**

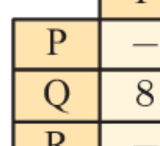


b 3, there are 3 railway stations connected to station B.

c **i** connected
ii not complete

d A → B → D which costs £60

4 **a** False, for example,



is a subgraph of

but the latter is not connected.

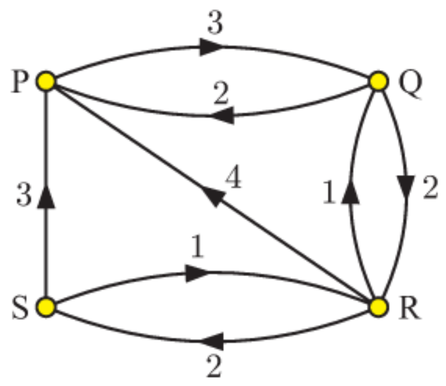
b True, a complete graph is by definition simple. A subset of a complete graph will contain no loops and a maximum of one edge joining any pair of distinct vertices.

5 **a** not strongly connected **b** strongly connected
c not strongly connected

6 **a** yes **b**

	A	B	C	D	E
A	–	4	–	–	6
B	5	–	–	8	–
C	–	6	–	–	–
D	–	–	5	–	3
E	–	9	–	4	–

7

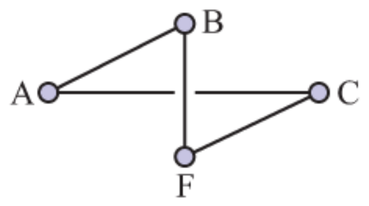


8 **a** It is not possible to travel from every vertex to every other vertex since we cannot travel from right to left on the graph.

b 2 chairlifts, for example from I to A, and A to D.

9 **a** It is not possible to travel from every vertex to every other vertex.

b



10 **a** $n - 1$ **b** $\dfrac{n(n-1)}{2}$ **c** $n - 1 \leqslant e \leqslant \dfrac{n(n-1)}{2}$

$\therefore \quad 2n - 2 \leqslant 2e \leqslant n^2 - n$

EXERCISE 15C

1 **a** Yes, all edges between adjacent vertices exist.

b Yes, no edges are repeated. **c** No, vertex E is repeated.

d Yes, no edges are repeated, and it starts and finishes at the same vertex.

e No, it does not finish at the same vertex at which it started.

2 **a** A → C → D **b** A → B → C → D

c A → B → C → E → D

d B → A → C → E → A → D

e A → B → C → D → E → A **f** not possible

g A → B → C → D → E → F → D → A

h A → D → F → E → A → B → C → D → E → C → A

3 **a** True, a trail only requires that no edge is repeated.

b False, a trail can have repeated vertices, but a path cannot.

c True, by definition.

d True, every circuit has a repeated vertex.

4 **a, b**

Path	*Weight*
P → Q → R → T	17
P → R → T	14
P → S → T	11
P → S → V → T	14
P → S → V → U → T	16

c Q → P → S → V → U, weight 16

5 **a** E → C → A → A → B → C → B → D → C

b Not possible, for example we cannot include edge EC without traversing it twice.

EXERCISE 15D.1

1 **a** $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

d $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

2 Terms opposite the leading diagonal correspond to the same pair of vertices.
In an undirected graph we can move between a pair of vertices in either direction, and in the same number of ways.

3 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

c $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$

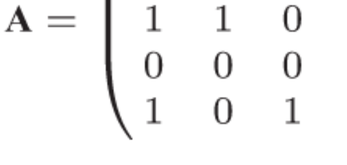
4 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$

b It is symmetric about the leading diagonal.

5 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

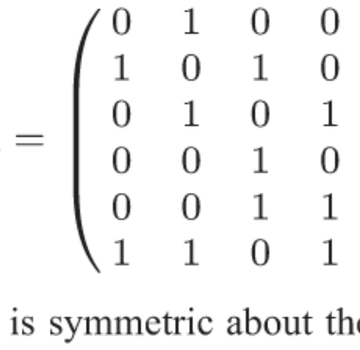
b 4, this is the number of 1-step routes starting at Q.
c 4, this is the number of 1-step routes ending at R.

6 **a**



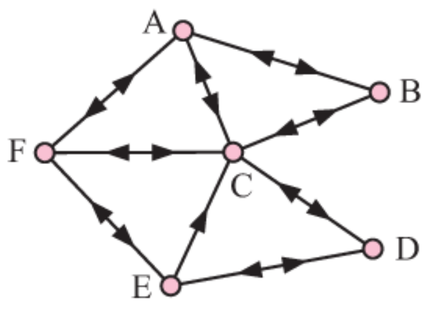
b

c



7 **a** the street from E to C

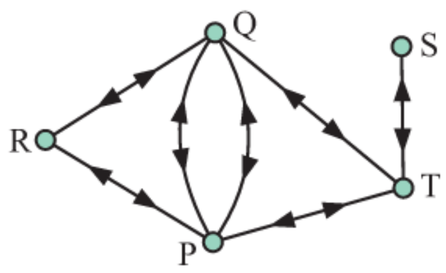
b



8 **a** No, the element in the 2nd row, 4th column of the connectivity matrix is 0, so there are no 1-step routes from Q to S.

b Q → T → S

c



EXERCISE 15D.2

1 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$ **b** $\mathbf{A}^2 = \begin{pmatrix} 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 2 \\ 2 & 0 & 3 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 & 3 \end{pmatrix}$

c two 2-step routes **d** E → C → B and E → A → B

2 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$ **b** $\mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}$

c **i** There are two 2-step routes from D to D.
ii There is one 2-step route from A to B (A → D → B), but no 2-step routes from B to A.

d $\mathbf{A}^3 = \begin{pmatrix} 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$ This shows the number of 3-step routes between each of the vertices.

3 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$

b $\mathbf{A}^2 = \begin{pmatrix} 3 & 0 & 1 & 1 & 3 \\ 0 & 2 & 2 & 2 & 0 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 2 & 3 & 1 \\ 3 & 0 & 1 & 1 & 3 \end{pmatrix}$ $\mathbf{A}^2$ shows the number of "mutual friends" between each pair of class members.

$\mathbf{A}^3 = \begin{pmatrix} 2 & 6 & 7 & 7 & 2 \\ 6 & 0 & 2 & 2 & 6 \\ 7 & 2 & 4 & 5 & 7 \\ 7 & 2 & 5 & 4 & 7 \\ 2 & 6 & 7 & 7 & 2 \end{pmatrix}$ $\mathbf{A}^3$ shows the number of ways class members are connected via 3 friendships.

c In **A**, there is a 1 in columns 2 and 3 of row 1, but a 0 in row 2, column 3 (and in row 3, column 2).

4 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$

b $\mathbf{A}^3 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 0 & 3 \\ 3 & 1 & 3 & 2 & 1 & 2 \\ 3 & 2 & 3 & 2 & 1 & 3 \\ 0 & 6 & 1 & 2 & 2 & 3 \end{pmatrix}$

c three 3-step routes;
E → A → B → C, E → F → B → C, E → F → D → C

d six 3-step routes; F → D → C → B, F → E → A → B

5 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

b **i** $\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{pmatrix}$

ii $\mathbf{A}^3 = \begin{pmatrix} 3 & 3 & 1 & 1 & 2 \\ 1 & 3 & 3 & 0 & 2 \\ 2 & 2 & 3 & 1 & 3 \\ 0 & 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 2 & 3 \end{pmatrix}$

iii $\mathbf{A}^4 = \begin{pmatrix} 3 & 5 & 6 & 1 & 5 \\ 5 & 3 & 4 & 3 & 6 \\ 6 & 5 & 4 & 3 & 6 \\ 3 & 1 & 1 & 2 & 3 \\ 4 & 6 & 4 & 1 & 4 \end{pmatrix}$

c The 4th row and 4th column is positive only in $\mathbf{A}^4$. It will take 4 steps for Sam's rumour to get back to him, passing through 3 people.

6 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

b $\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

These matrices show the number of 2-step and 3-step routes between each of the locations.

c **i** 3 train rides **ii** 3 train rides **iii** 6 train rides

7 **a**

$$\mathbf{A} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \\ \text{F} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

b **i** $\mathbf{A}^2 = \begin{pmatrix} 2 & 0 & 1 & 0 & 2 & 2 \\ 0 & 3 & 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \\ 0 & 3 & 1 & 4 & 0 & 1 \\ 2 & 0 & 1 & 0 & 2 & 2 \\ 2 & 0 & 1 & 1 & 2 & 3 \end{pmatrix}$

ii $\mathbf{A}^4 = \begin{pmatrix} 13 & 1 & 8 & 3 & 13 & 15 \\ 1 & 19 & 8 & 22 & 1 & 4 \\ 8 & 8 & 9 & 10 & 8 & 10 \\ 3 & 22 & 10 & 27 & 3 & 8 \\ 13 & 1 & 8 & 3 & 13 & 15 \\ 15 & 4 & 10 & 8 & 15 & 19 \end{pmatrix}$

c 2 combinations: Dublin, Edinburgh, Barcelona
Dublin, Frankfurt, Barcelona

d The value in the first row and first column of $\mathbf{A}^4$ is 13, so there are 13 4-step routes starting and finishing at Amsterdam. However, this includes routes which revisit cities, such as A → D → A → D → A, and routes which visit the same combinations of cities, such as A → D → F → B → A and A → B → F → D → A.

8 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

b $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 & 1 \end{pmatrix}$

c You cannot travel from K to K, K to O, or M to M in 2 steps or less.

d There would be no way to visit O. It would also take longer for people at N to get to K.

9 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

b $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 & 0 \\ 1 & 3 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 1 & 1 & 2 \end{pmatrix}$

There are 0s in the matrix $\mathbf{A} + \mathbf{A}^2$ which indicates that there are some pairs of squares on the board for which it is not possible to move between in at most two moves.

c $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 2 & 6 & 1 & 5 & 2 & 3 \\ 6 & 3 & 6 & 2 & 8 & 2 \\ 1 & 6 & 2 & 3 & 2 & 5 \\ 5 & 2 & 3 & 2 & 6 & 1 \\ 2 & 8 & 2 & 6 & 3 & 6 \\ 3 & 2 & 5 & 1 & 6 & 2 \end{pmatrix}$

There are no 0s in the matrix $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3$ which indicates that it is possible to move between any two squares in at most three moves.

10 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$

c **i** If the service from C to D is cancelled, it will not be possible to travel from B to D in at most 2 trips.

ii C to B, or C to E

11 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

b $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 1 & 2 & 2 \\ 2 & 3 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 3 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

This shows the number of ways a signal can be sent between stations in at most 3 steps.

c $\mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{4}\mathbf{A}^3 = \begin{pmatrix} 0.25 & 1.25 & 0.5 & 0.75 & 0.75 \\ 0.75 & 1 & 1.25 & 1.75 & 2 \\ 0.25 & 0.5 & 0.25 & 1.25 & 0.75 \\ 0.75 & 1.5 & 0.5 & 0.75 & 1.75 \\ 1 & 0.5 & 0.25 & 0.25 & 0.25 \end{pmatrix}$

This could represent the strength of a signal between two stations.

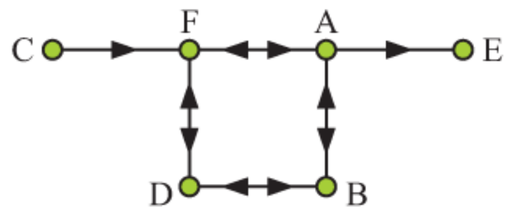
d **i** Cannot get from A to A, or E to E in at most 3 steps.

ii Cannot get from C to B, C to C, D to C, or D to D in at most 3 steps.

e The effect in **d ii** creates more serious problems.

12 **a** No, there are some one-way doorways.

b



c **i** C **ii** A **iii** E

d **i** $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 1 & 2 \end{pmatrix}$

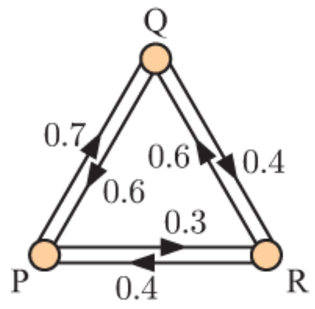
$\mathbf{A}^3 = \begin{pmatrix} 0 & 4 & 0 & 0 & 2 & 4 \\ 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 2 \\ 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 \end{pmatrix}$

ii From $\mathbf{A} + \mathbf{A}^2$, you cannot get from the entrance (C) to rooms B, C, or E passing through at most 2 rooms, and from $\mathbf{A}^3$ you can get to rooms B, E, and F from C by passing through 3 rooms. Hence the dinosaur room must be in either B or E.

iii B, as E is the exit.

EXERCISE 15E

1 **a**



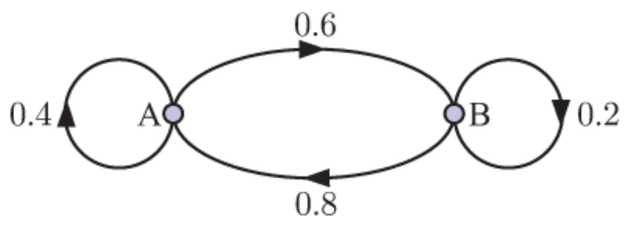
b $\mathbf{T} = \begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.6 & 0 & 0.4 \\ 0.4 & 0.6 & 0 \end{pmatrix}$

c A train at station P will go to station R with probability 0.3.

d $\mathbf{s}_0\mathbf{T}^3 = \begin{pmatrix} 0.22 & 0.546 & 0.234 \end{pmatrix}$

Three stops from now, a bus currently at P has probability 0.22 of being at P, probability 0.546 of being at Q, and probability 0.234 of being at R.

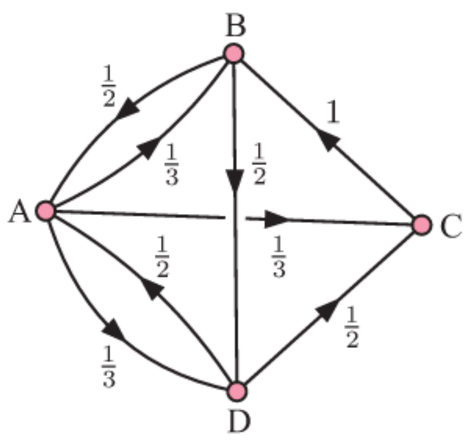
2 **a**



b $\mathbf{T} = \begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix}$

c 0.5824

3 **a**



b $\mathbf{T} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}$

c $\frac{5}{18}$ (A → B → A → D, A → C → B → D, or A → D → A → D)

4 **a** $\mathbf{T} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \end{pmatrix}$

b **i** $\mathbf{s}_0\mathbf{T}^2 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

The mouse is most likely to be at junction X. No matter which tunnel the mouse takes first, it is always possible to return to X after 2 tunnels.

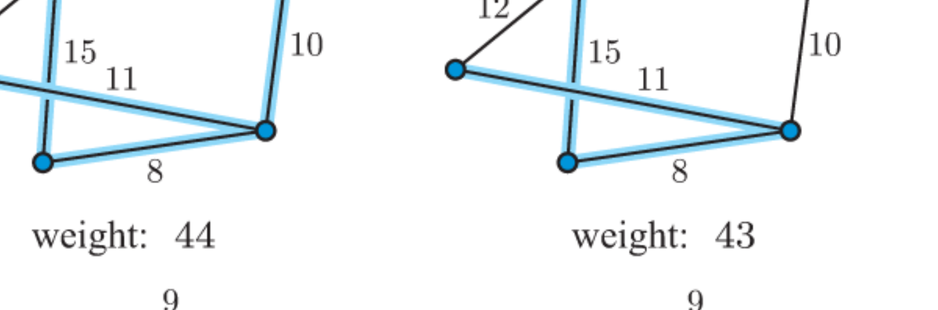
EXERCISE 15F

1 **a** tree **b** not a tree **c** tree **d** not a tree

2 **a** tree, the subgraph has no cycles

b not a spanning tree, the subgraph does not contain all vertices of the graph

3 **a** **i** **ii** **iii**

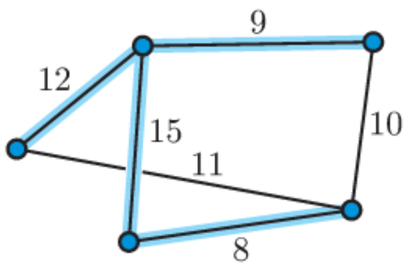


b

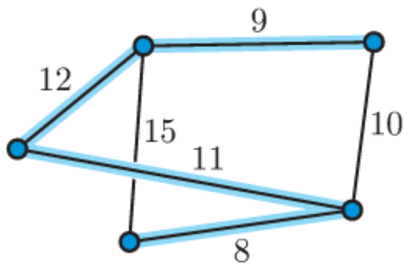
	Number of vertices	*Number of edges in spanning tree*
i	4	3
ii	5	4
iii	7	6

c *If a graph has n vertices, its spanning tree has $n-1$ edges.*
The first edge connects two vertices, and each additional edge adds exactly one vertex to the tree. We therefore need $n-1$ edges to connect all n vertices.

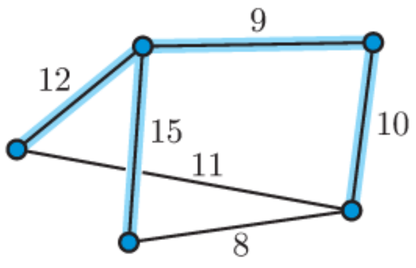
4



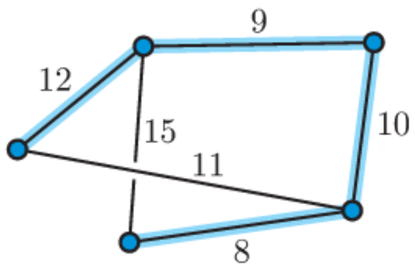
weight: 44



weight: 40



weight: 46

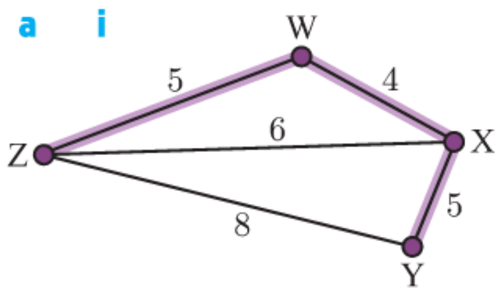


weight: 39

5 No, disconnected graphs do not have a spanning tree.

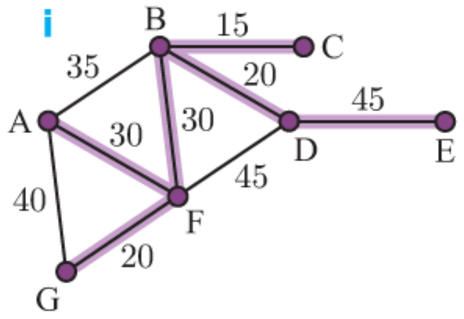
EXERCISE 15G.1

1 a i



ii 14 units

b i



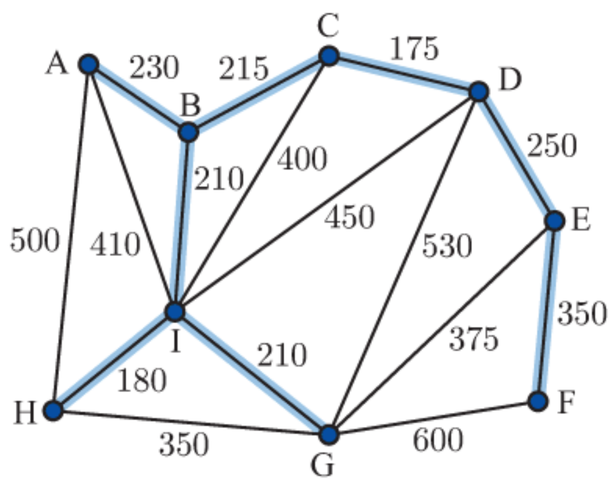
ii 160 units

2 AF, BF, CF, CE, DE

3 a 29.1 m **b** no

c

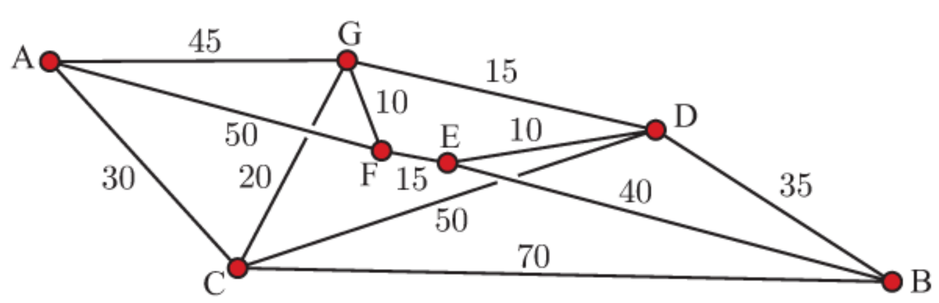
4 a



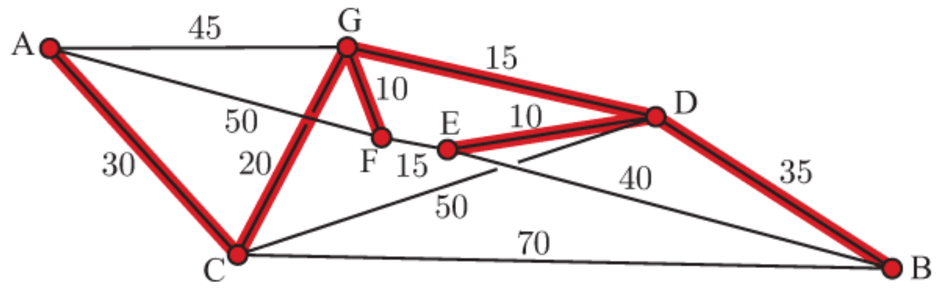
b 1820 m

5 a EF, FG, CD, DF, FK, HJ, AJ, CJ, AB **b** 19 km

6 a

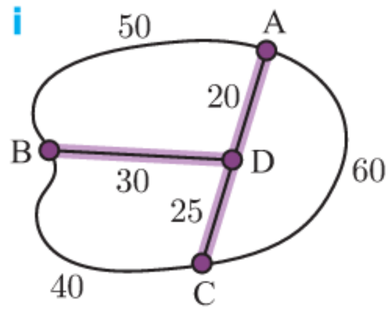


b



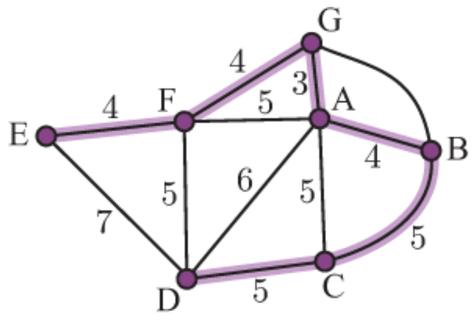
EXERCISE 15G.2

1 a i



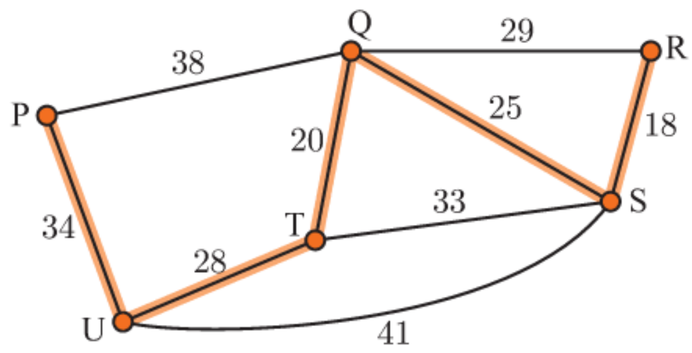
ii 75 units

b i **Note:** Other solutions are possible.



ii 25 units

2 a

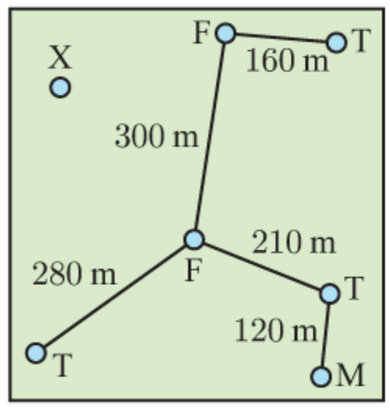


b 125 m

3 a AB, BD, CD, DE, and EF should be cleared, 43 km

b the same except BE instead of DE

4 a 1070 m



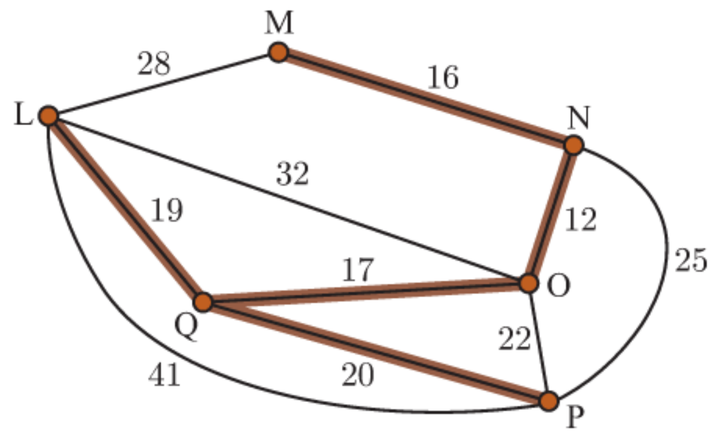
Scale 1 cm ≡ 200 m

b 1310 m

c We assume it is possible to dig in straight lines between the fountains and taps, and that it is equally difficult to dig everywhere.

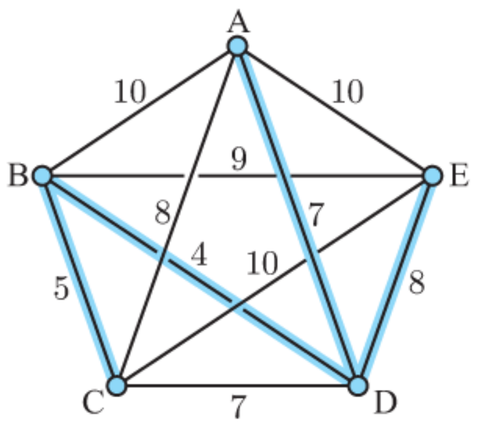
5 a $70 million

b $84 million



6 a There is a weight for every edge from every vertex to every other vertex.

b



7 a AE, CE, BC, BD **b** $73 000

EXERCISE 15H

1 a Eulerian, A → B → F → E → A → C → F → D → A

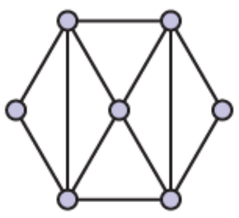
b neither **c** neither

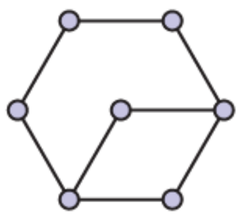
d semi-Eulerian,
A → B → D → C → B → A → C → E → A → E → D

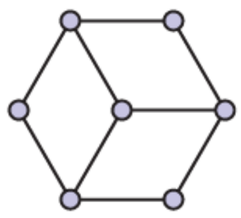
e Eulerian,
A → B → F → A → C → B → D → E → C → D → A

f semi-Eulerian, G → F → H → D → E → F → A → B → D → C → B → G → C → H

2 **Note:** These are examples only.

a 

b 

c 

3 **a** More than two vertices have odd degree, so the graph is neither Eulerian nor semi-Eulerian.

b Exactly two vertices (A and B) have odd degree, so the graph is semi-Eulerian.
A → H → G → A → B → C → D → F → G → E → B

c There are no vertices of odd degree, so the graph is Eulerian.
A → H → G → F → D → C → B → E → G → A

4 **a** No, the graph contains vertices of odd degree, so it is not Eulerian and thus such a tour does not exist.

b BF, E → D → C → B → D → F → B → A → I → J → H → G → J → A → F → G → E

5 **a** The graph does not contain vertices of odd degree, so it is Eulerian and thus such a route is possible.

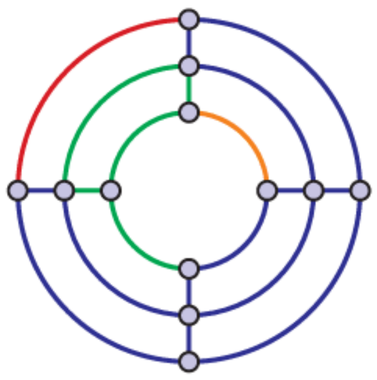
b In any such route, every street is traversed exactly once, so the total distance travelled is the same regardless of the order of the vertices.

c 2570 m or 2.57 km

6 **a** no

b Yes, adding a bridge between the largest island and the southern bank, *or* removing the existing bridge between the largest island and southern bank, will create a Eulerian circuit.

7 **a** 4 pen strokes are needed.
An example is shown alongside.



b If a graph has 2 vertices of odd degree, it is semi-Eulerian, and the graph can be drawn with a single pen stroke.
This graph has 8 vertices of odd degree, so we could make the graph semi-Eulerian by adding 3 new edges to the graph.
Equivalently, we can think of adding an edge between two vertices as lifting the pen at one vertex and moving it to the other.
So, an additional 3 pen strokes are required to complete the diagram, making 4 in total.

8 **a** neither **b** m, n even, length is mn

c $m = n = 1$ *or* one of m, n is odd and the other is 2

EXERCISE 15I

1 **a** There are no vertices of odd degree.
$\therefore$ the graph is Eulerian.

b 69 km, L → M → N → L → O → N → O → L

2 **a** semi-Eulerian

3 5.4 km, O → D → C → O → A → F → E → A → O → B → E → C → O

4 **a** AB and CD, AC and BD, AD and BC **b** 41 km

5 71 units, A → B → C → D → B → D → F → E → C → G → E → F → G → I → H → F → I → F → B → A

EXERCISE 15J

1 **a** A → B → C → E → F → G → D → H → A

b **i** E → F → G → D → C → B → A → H

ii E → F → G → A → H → D → C → B

c The graph is Hamiltonian since it contains a Hamiltonian cycle (in **a**).

2 **a** semi-Hamiltonian **b** Hamiltonian
c semi-Hamiltonian **d** neither **e** Hamiltonian
f neither

3 **a** A → B → C → F → D → E,
A → B → E → D → C → F,
A → B → E → D → F → C,
A → C → F → D → B → E,
A → C → F → D → E → B,
A → C → B → E → D → F

b A → C → F → D → B → E (total weight = 23)

c Hamiltonian, A → B → E → D → F → C → A is a Hamiltonian cycle.

4 **a** 67 km

b **i** A → B → C → D → E → G → F, 69 km
ii A → B → C → F → G → E → D, 70 km

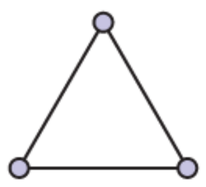
5 **a** P → Q → R → S → T → U → V → W → P, 405 km

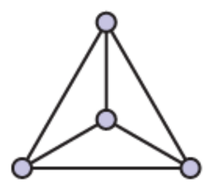
b **i** no **ii** P → W → V → U → T → S → Q → R

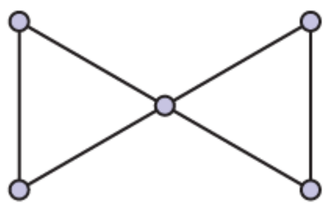
6 **a**

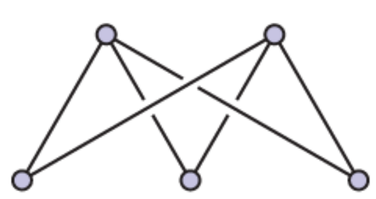
c

b not possible

7 **a** 

b 

c 

d 

8 **a** Each vertex is connected to every other vertex.
It is always possible to travel to a vertex that has not been visited, and return to the starting vertex immediately, without travelling along the same edge (if $n \geqslant 3$).

b k_2 is  which only has one edge and no Hamiltonian cycle.

EXERCISE 15K.1

1 **a** P → S → Q → R → P, upper bound is 224

b lower bound is 214

c **i** $214 \leqslant 216 \leqslant 224$ ✓

ii P → Q → R → S → P, weight 216

2 **a** C → B → A → D → E → C, upper bound is 48

b lower bound is 41

c C → A → B → E → D → C, weight 43

3 **a** **i** 62 **ii** 57 **iii** 57 **iv** 57 **v** 57
57 is the best upper bound.

b **i** 54 **ii** 57 **iii** 57 **iv** 55 **v** 54
57 is the best lower bound.

c C → B → D → A → E → C, weight 57

4 **a**

Vertex deleted	*Lower bound*
A	31
B	31
C	31
D	31
E	34

34 is the best lower bound.

b A → E → D → C → B → A, upper bound is 36

c **i** Town E, as the roads between E and the other towns are the shortest on the graph.

ii E → D → C → B → A → E, upper bound is 36

5 A → E → D → C → F → B → A, with total distance 52 km

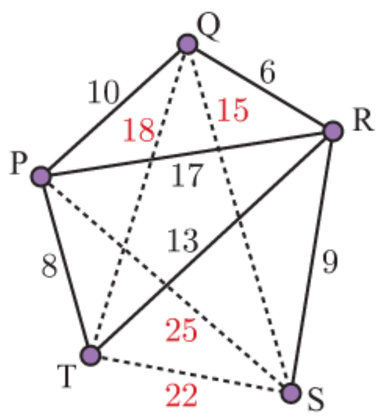
6 **a** **i** T → B → P → O → D → L → S → M → C → T

ii C → O → P → B → T → M → S → L → D → C

b It makes no difference. The cycles given in **a** are not necessarily optimal, and any cycle starting and ending in Calais could also be travelled starting and ending at Toulouse.

EXERCISE 15K.2

1 **a**



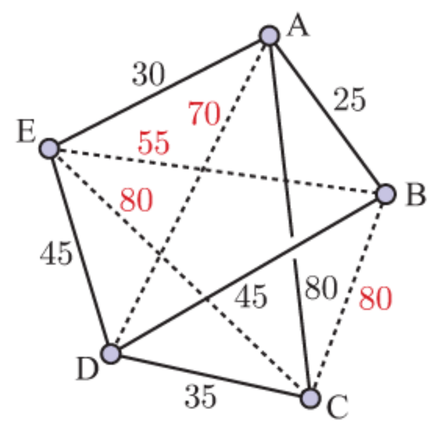
b P → T → R → Q → S → P, upper bound is 67

c lower bound is 46

d P → Q → R → S → R → T → P, weight 55

$46 \leqslant 55 \leqslant 67$ ✓

2 **a**



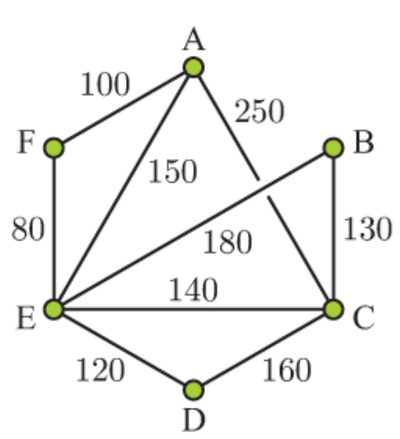
b **i** B → A → E → D → C → B, upper bound is 215

ii lower bound is 215

iii £215, B → A → E → D → C → D → B, town D is visited twice

3 **a** 3 cities

b



c

	A	B	C	D	E	F
A	0	330	250	270	150	100
B	330	0	130	290	180	260
C	250	130	0	160	140	220
D	270	290	160	0	120	200
E	150	180	140	120	0	80
F	100	260	220	200	80	0

d **i** D → E → F → A → C → B → D, upper bound is 970

ii lower bound is 730

iii For example, D → E → F → A → E → B → C → D has total distance 920 km, E is visited twice.

REVIEW SET 15A

1 **a** not strongly connected **b** strongly connected

2 **a** **i** yes

ii yes

iii no

c P → T → S

b

	P	Q	R	S	T	U
P	–	70	–	–	110	90
Q	70	–	60	–	–	30
R	–	60	–	90	–	50
S	–	–	90	–	50	–
T	110	–	–	50	–	80
U	90	30	50	–	80	–

3 **a** Yes, no edges are repeated.

b Yes, no edges or vertices are repeated.

c No, edge EF is repeated.

d Yes, it starts and ends at the same vertex, and no vertex is repeated (except the starting vertex B).

4 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$ **b** $\mathbf{A}^2 = \begin{pmatrix} 2 & 1 & 2 & 2 & 1 \\ 1 & 4 & 2 & 2 & 3 \\ 2 & 2 & 3 & 2 & 2 \\ 2 & 2 & 2 & 3 & 2 \\ 1 & 3 & 2 & 2 & 4 \end{pmatrix}$

c three 2-step routes

d T → P → Q, T → R → Q, T → S → Q

5 **a** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

b **i** $\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 1 & 3 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 2 \\ 1 & 3 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 \end{pmatrix}$

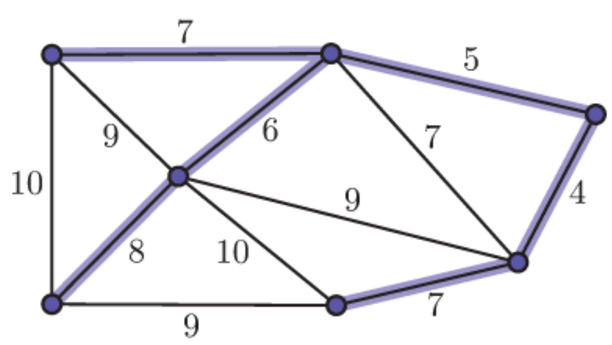
ii No, items cannot be moved from D to A in at most 3 steps.

c **i** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

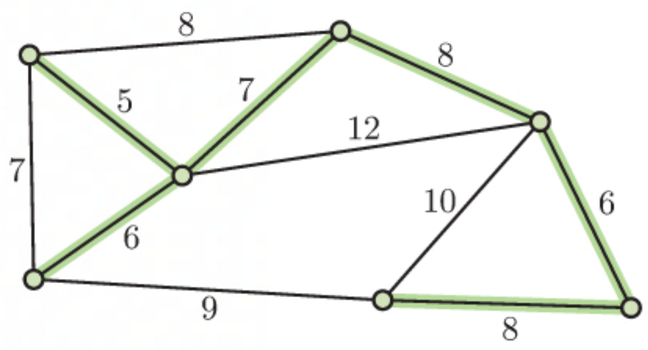
$\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 = \begin{pmatrix} 1 & 2 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 1 & 2 \end{pmatrix}$

ii Items cannot be moved from C, D, or E to A or B at all.

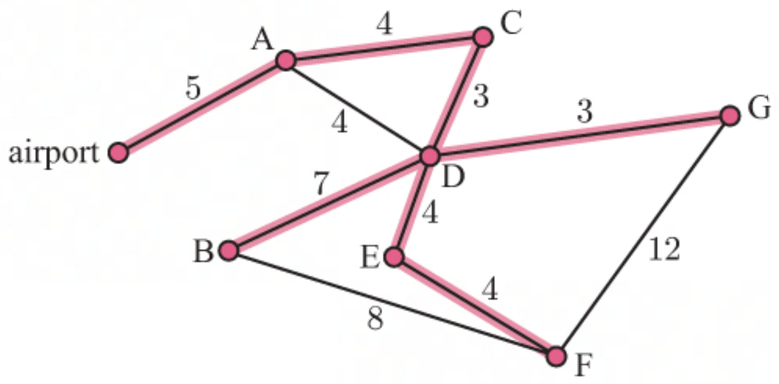
6 **a**



b



7



8 **a**

$$\mathbf{T} = \begin{array}{c} \\ \text{A} \\ \text{B} \\ \text{C} \\ \text{D} \\ \text{E} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{A} & \text{B} & \text{C} & \text{D} & \text{E} \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 & \frac{5}{6} & 0 \\ \frac{1}{6} & 0 & \frac{2}{3} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

b **i** ≈ 0.204

Possible routes: A → A → A → A,
A → A → B → A, A → B → A → A,
A → B → D → A

ii ≈ 0.0185

Possible route: A → B → D → E

9 **a** The graph has exactly two vertices (A and D) of odd degree.

b A → B → E → D → C → B → D **c** AD

10 Example solution:

O → A → E → B → A → B → C → E → D → C → D → O

minimum distance = 136 units

11 yes, A → B → F → G → H → G → F → B → C → D → H → K → J → G → C → F → J → I → F → E → I → E → A (119 minutes)

12 **a** O → D → C → A → B → O, upper bound is 48

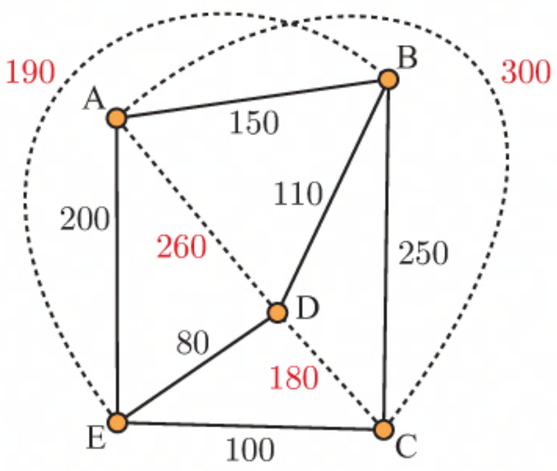
b lower bound is 37

13 **a** Hamiltonian cycle

b A → F → E → D → C → B → G → A

c 163 minutes or 2 hours 43 minutes

14 **a**



b upper bound = 840 km

c lower bound = 640 km > 600 km

REVIEW SET 15B

1 **a** **i** A and AB **ii** A and O

b 2, people with type B blood can donate kidneys to 2 different blood types.

c **i** O **ii** AB

2 **a** E → B → C → E → F → G

b A → G → D → C → B → E → A

3 33 vertices

4 **a** 9 **b**

$$\mathbf{A} = \begin{array}{c} \\ \text{B} \\ \text{F} \\ \text{H} \\ \text{K} \\ \text{M} \end{array} \begin{array}{c} \begin{array}{ccccc} \text{B} & \text{F} & \text{H} & \text{K} & \text{M} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

c **i** 5 **ii** 3

d Frankfurt ↔ Köln reduces the answer to **c i** to 3.
Köln → Berlin reduces the answer to **c i** to 4
∴ making Frankfurt to Köln "two-way" would be the better decision.

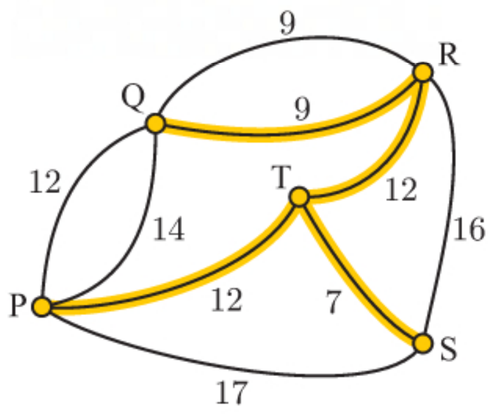
5 **a** No. This is shown by the element 0 in row 3, column 4 of **A**.

b $\mathbf{A} + \mathbf{A}^2 = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{pmatrix}$

c To get from B to C, C to D, or D to D requires going through 2 other towns.
Going from C to C requires going through 3 other towns. By contrast there are many ways to get from A to B.

d **i** 5 **ii** yes, B → A → D → C → B

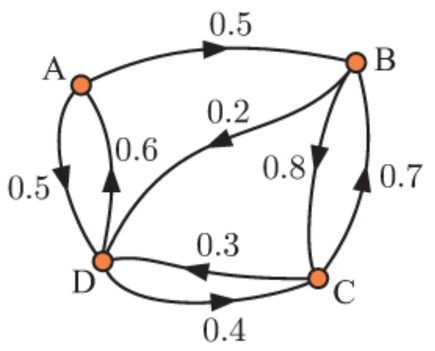
6



minimum weight = 40

7 **a** yes

b

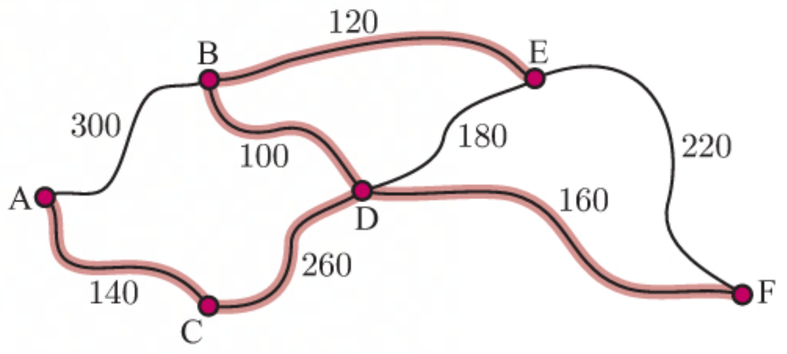


c

$$\mathbf{T} = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0.8 & 0.2 \\ 0 & 0.7 & 0 & 0.3 \\ 0.6 & 0 & 0.4 & 0 \end{pmatrix}$$

d 0.116

8



9 **a** CD, BC, CE, AD; total length = 1070 km

b The most popular flight routes should be considered. For example, there should be more flights to and from the most populated town.

10 **a** None of the vertices have odd degree.

b B → C → D → F → A → E → F → D → E → C → A → B

11 Example solution:

A → B → C → D → C → B → E → D → A → C → E → A

minimum distance = 1361 units

12 Example route:

A → B → C → D → J → C → B → J → A → I → H → G → K → H → J → K → E → F → G → H → I → A

minimum length = 2770 m = 2.77 km

13 **a** Evan and Heidi's hotel → Sacré Cœur → Opéra Garnier → Louvre Museum → Notre-Dame de Paris → Orsay Museum → Hôtel des Invalides → Eiffel Tower → Arc de Triomphe → Evan and Heidi's hotel

b **i** No; the graph contains vertices of odd degree, so it is not Eulerian.

ii Yes; Evan and Heidi's hotel → Arc de Triomphe → Opéra Garnier → Sacré Cœur → Evan and Heidi's hotel → Eiffel Tower → Arc de Triomphe → Louvre Museum → Opéra Garnier → Orsay Museum → Notre-Dame de Paris → Louvre Museum → Orsay Museum → Hôtel des Invalides → Eiffel Tower

14 **a** **i** H → D → G → E → F → H
{total time = 37 minutes}

ii D → H → G → E → F → D
{total time = 35 minutes}
upper bound = 35 minutes

b lower bound = 32 minutes

c D → H → G → E → F → D, 35 minutes

15 **a**

	A	B	C	D	E	F
A	0	190	150	680	450	430
B	190	0	100	630	400	380
C	150	100	0	530	300	280
D	680	630	530	0	350	250
E	450	400	300	350	0	100
F	430	380	280	250	100	0

b **i** C **ii** upper bound = \$1620

iii lower bound = \$1070

iv D → F → C → A → B → C → E → F → D, \$1620, F and C visited twice

v D → F → E → C → B → A → C → D, \$1490

EXERCISE 16A

1 **a** **i** 3 **ii** 3 **iii** 1

b **i** Site B as P lies in cell B.

ii PB $= \sqrt{10} \approx 3.16$ units, PA $= 3\sqrt{2} \approx 4.24$ units, PC $= \sqrt{26} \approx 5.10$ units

c **i** Q lies on the edge between cells A and C. Q is equally closest to sites A and C.

ii QA $= \sqrt{10} \approx 3.16$ units, QC $= \sqrt{10} \approx 3.16$ units, QB $= 3\sqrt{2} \approx 4.24$ units

2 **a** All points in the green cell are closest to site A than any other site.

b All points on the blue edge are equally closest to sites D and C.

c The red vertex is equally closest to sites A, B, and C.

3 **a** **i** site B **ii** site D **iii** site C **iv** site A

b **i** distance to site A $= \sqrt{5}$ units
distance to site D $= \sqrt{5}$ units

ii distance to site A $= \sqrt{13}$ units
distance to site D $= \sqrt{13}$ units

c $(-3, 0)$ is equally closest to sites A and D, hence it lies on an edge. $(-3, 2)$ is closest to site B and does not lie on an edge.

d 32 units^2

4 **a** true **b** true **c** not necessarily true

d not necessarily true

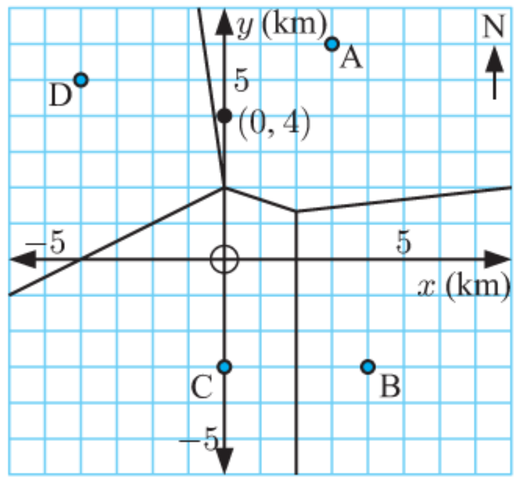
5 If the circle passed through another site then P would lie on an edge. If another site was contained in the circle, P would be closer to that site than X. P would not lie in cell X.

6 **a** **i** post office E **ii** post office B **iii** post office A **iv** post office C

b $(-1, \frac{1}{2})$ and $(-1, -2)$

7 **a** **i** school C **ii** school A **iii** school D **iv** school B

b **i** school D **ii**

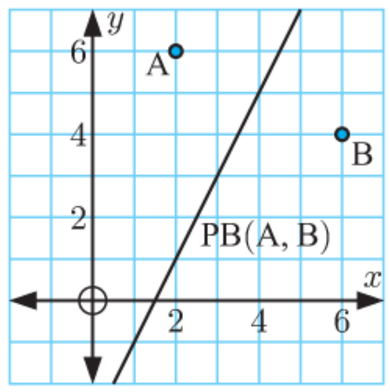


c **i** (0, 2) **ii** 5 km

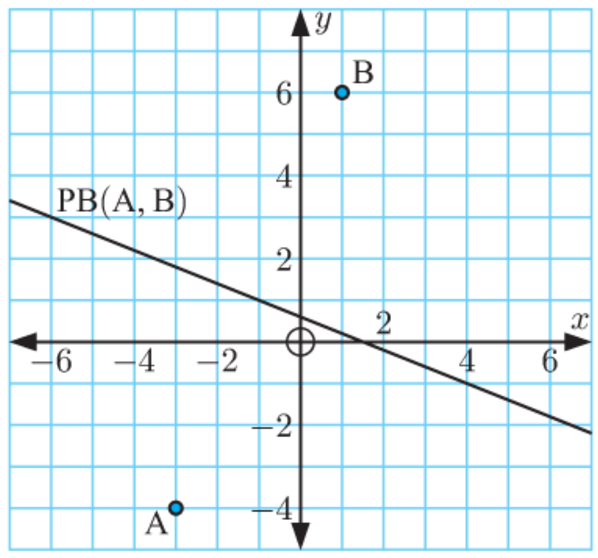
8 A vertex of a Voronoi diagram is equally closest to at least 3 sites (whose cells meet at that vertex). The circle's edge passes through site X, which is one of the sites V is closest to, and since every point on the edge is equidistant from the centre, the other closest sites must also lie on the edge of the circle.

EXERCISE 16B

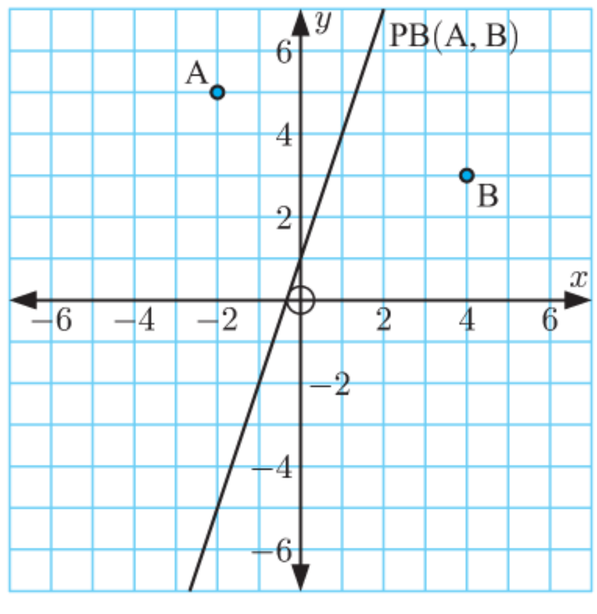
1 **a**



b



2 **a**



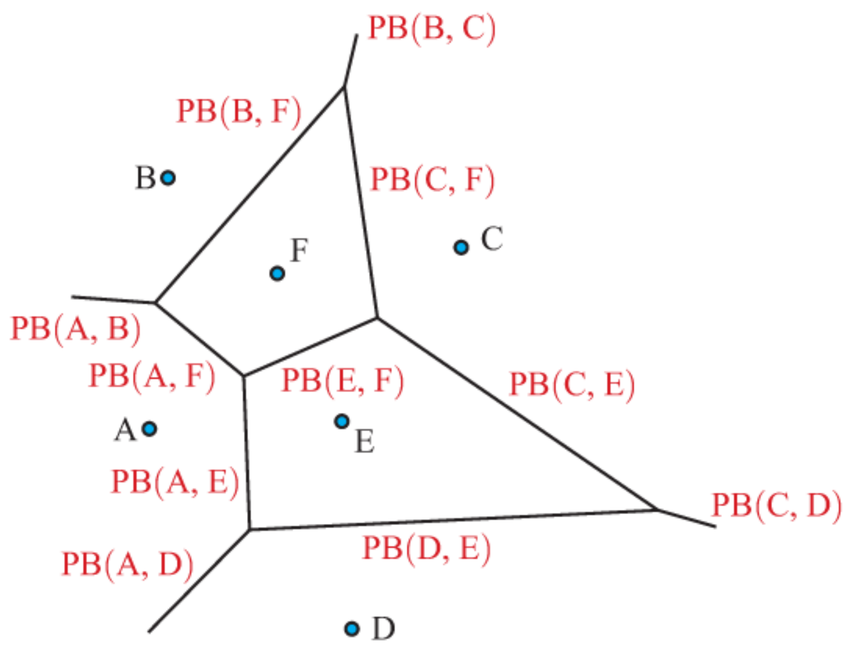
b $y = 3x + 1$

c **i** $-5 = 3(-2) + 1$ ✓

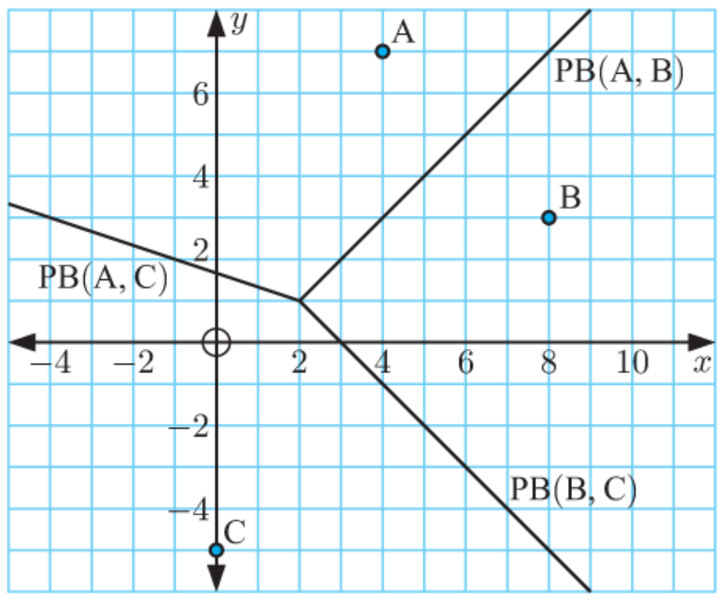
ii distance to site A = 10 units
distance to site B = 10 units

d **i** site B **ii** site B **iii** site A

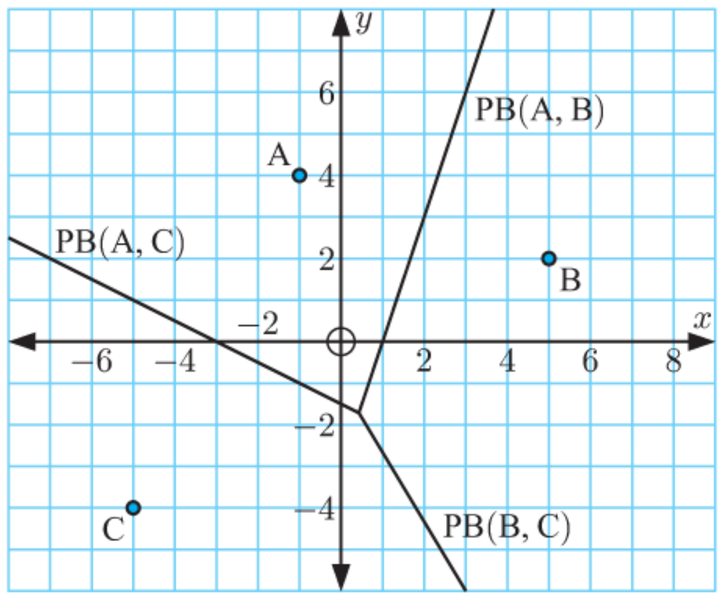
3



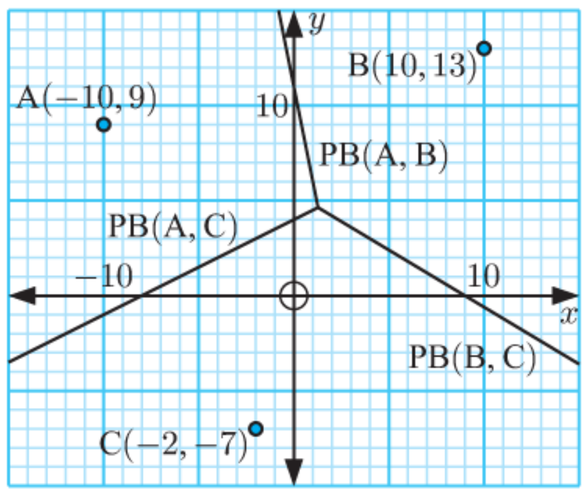
4 **a**



b



5 **a**

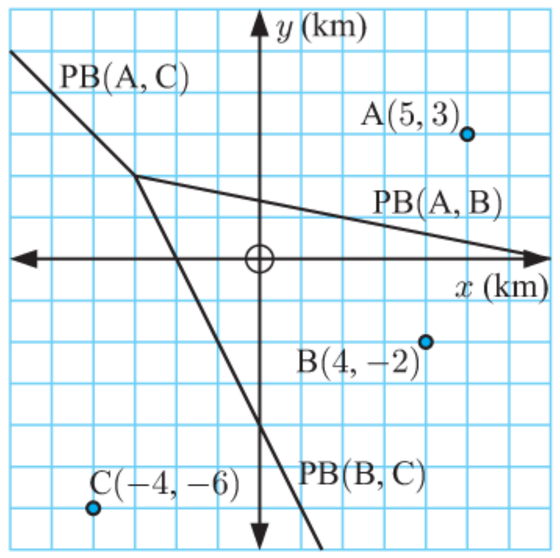


b PB(A, B): $y = -5x + 11$
PB(A, C): $y = \frac{1}{2}x + 4$
PB(B, C): $y = -\frac{3}{5}x + \frac{27}{5}$

c $\left(\frac{14}{11}, \frac{51}{11}\right)$
distance to A = distance to B = distance to C $\approx$ 12.1 units

d **i** site A **ii** site B **iii** site C

6 **a**

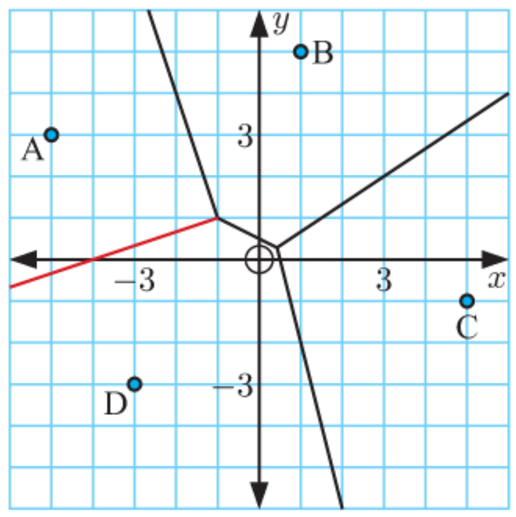


b **i** store A
ii store A
iii store C

c **i** $(-3, 2)$ **ii** $\sqrt{65} \approx 8.06$ km

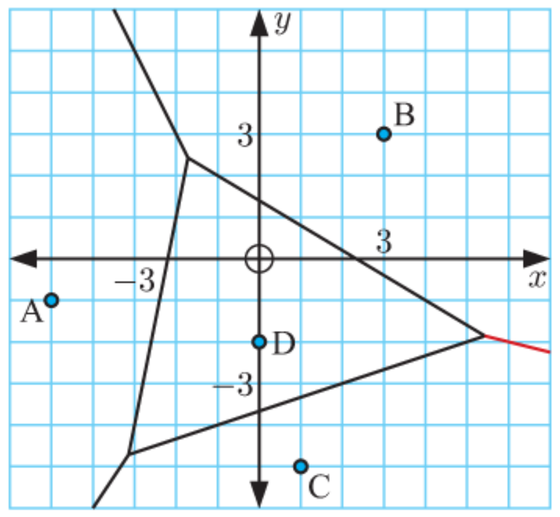
7 **a** Sites A and D are currently in the same cell.

b



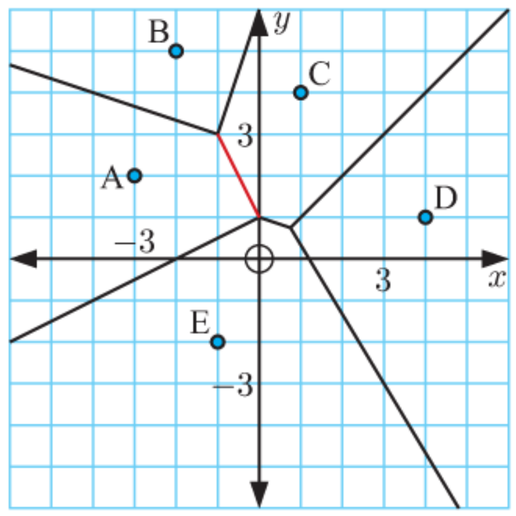
PB(A, D):
$y = \frac{1}{3}x + \frac{4}{3}$

8 **a**



PB(B, C):
$x + 4y + 2 = 0$

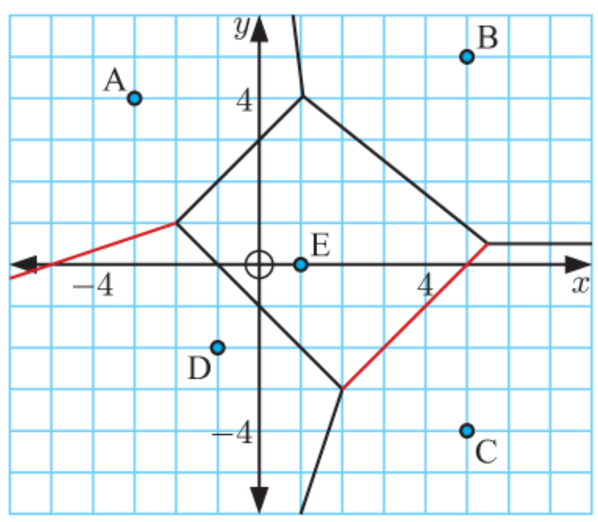
b



PB(A, C):
$2x + y - 1 = 0$

9 **a** 2 edges

b



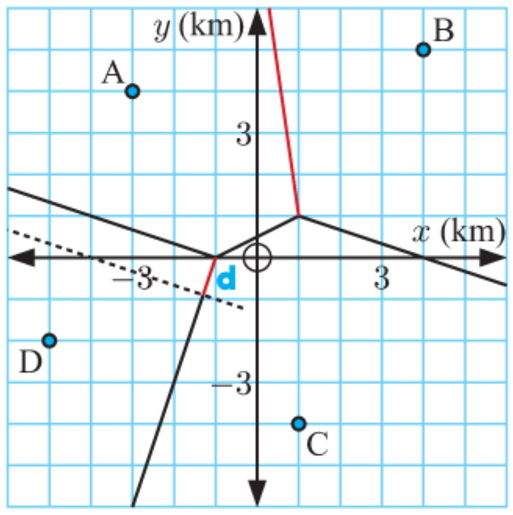
PB(A, D): $y = \frac{1}{3}x + \frac{5}{3}$, PB(C, E): $y = x - 5$

c **i** site A **ii** site E

d **i** sites C and E **ii** sites A, D, and E

10 **a** $y = 3x + 3$

b, d



PB(A, B): $y = -7x + 8$

c **i** camp B **ii** $\sqrt{13}$ km ≈ 3.61 km

11 **a** There is a cell which does not contain a site.

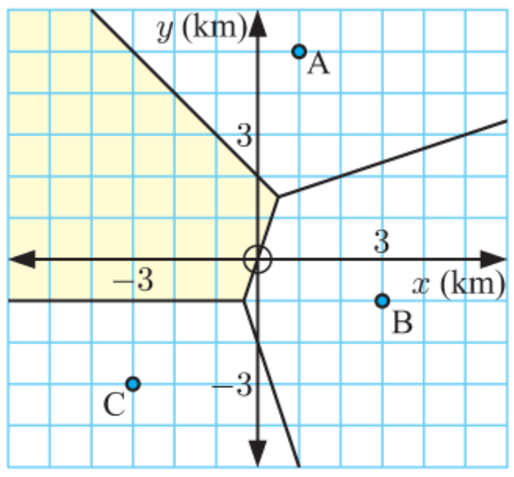
b The edge of the missing site and site B is perpendicular to the x-axis.

$\therefore$ the missing site lies on the x-axis.

c $(-3, 0)$

12 **a** $(-1, -4)$ **b** $(0, -1)$

13 **a**



b Distance from $(-2, 1)$ to $(0, -1) = 2\sqrt{2} \approx 2.83$ km

Distance from $(0, -1)$ to B $= 3$ km

If the vet clinic was at $(-2, 1)$ then the cell would include the point $(0, -1)$.

c $(-3, 1)$ **d** **i** vet A **ii** vets B and C

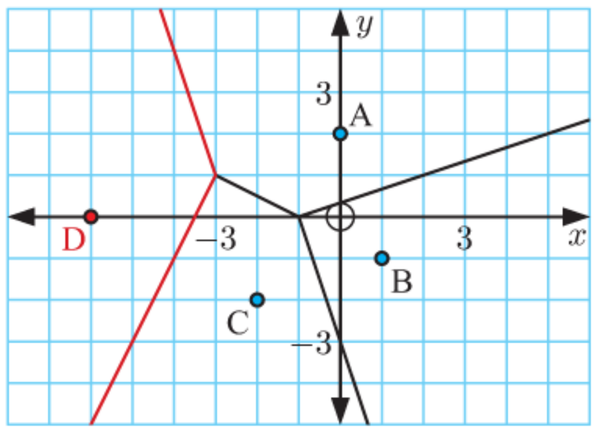
e $2\sqrt{2} \approx 2.83$ km

EXERCISE 16C

1 **a** cell C

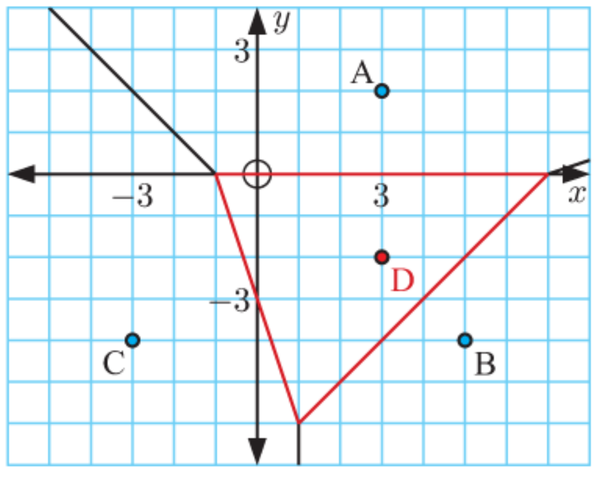
b cells A and C as they share an edge

c



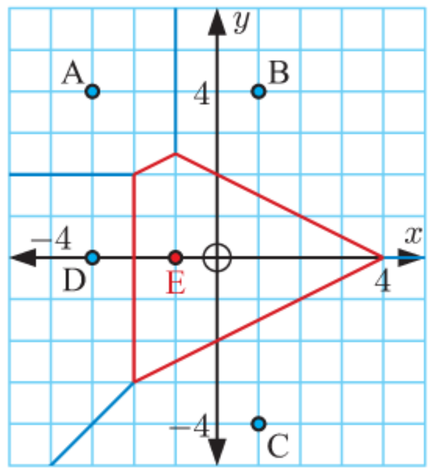
2 **a** Site D is relatively central to the other sites.

b

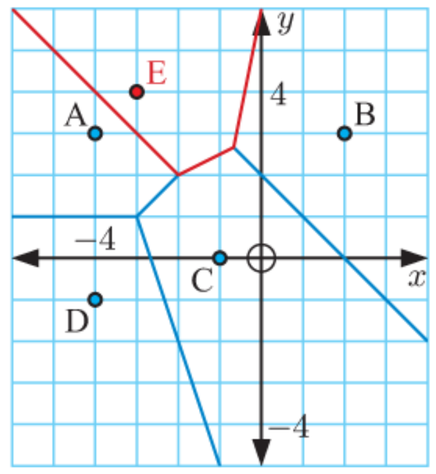


c 24 units^2

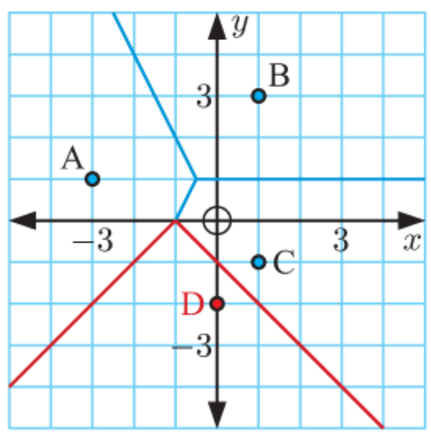
3 **a**



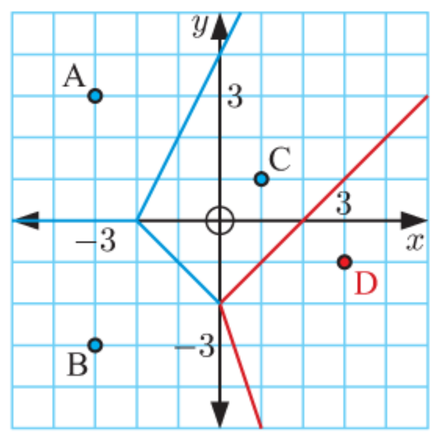
b



4 **a**

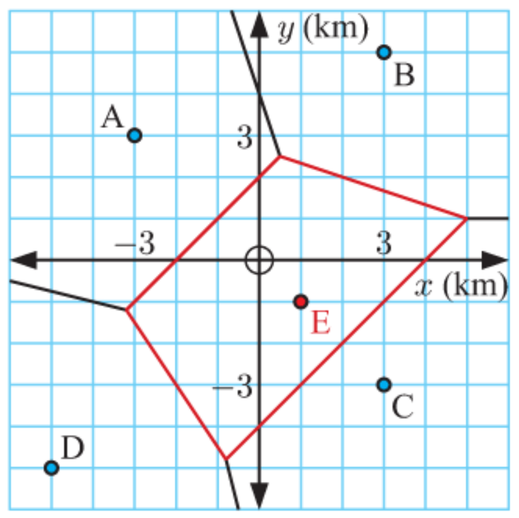


b



5 **a** **i** A **ii** B

b **i**

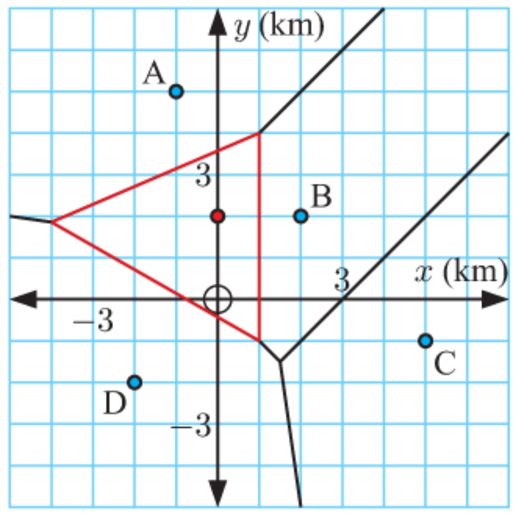


ii Yes, the edge is now closer to ATM D, implying that some people are now closer to ATM E.

iii (5, 1)

6 **a** **i** polling booth A **ii** polling booth D

b **i**



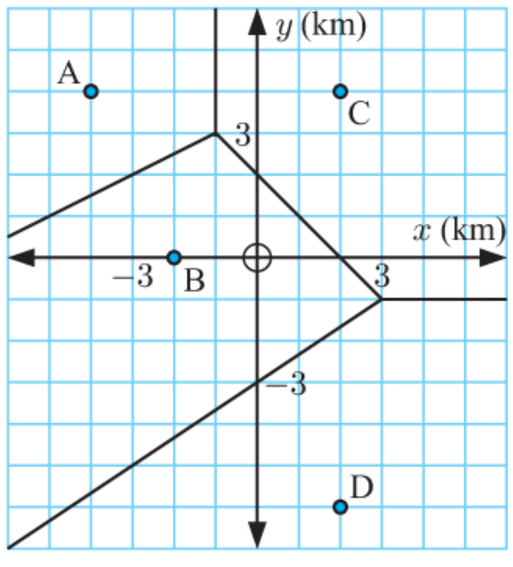
ii cell C

iii 12.5 km^2

EXERCISE 16D

1 **a** 27.3°C **b** 25.6°C **c** 28.4°C

2 **a**



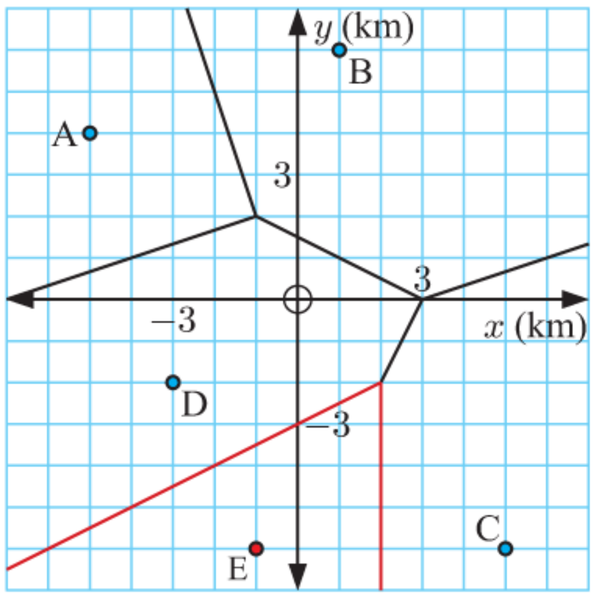
b **i** 48 m

ii 57 m

iii 36 m

3 **a** **i** 9.3 inches **ii** 5.5 inches **iii** ≈ 10.8 inches

b **i**



ii Yes, in **i** and **iii** we would now estimate 10.6 inches of snow at $(-1, -4)$ and $(1, -4)$.

EXERCISE 16E

1 **a** $(1, 1)$, radius $\sqrt{13}$ units **b** $(0, -1)$, radius $2\sqrt{5}$ units

2 **a** $(20, 10)$ **b** $4\sqrt{65}$ km ≈ 32.2 km

c towns B, C, and E

d Distance from proposed location to B $= 5\sqrt{26} \approx 25.5$ km

$\therefore$ **a** is preferable.

3 Let P lie in cell X. The largest empty circle centred at P would touch X. As P lies within cell X, the largest empty circle does not touch any other site, meaning that a larger empty circle could be created with centre closer to the edge of the cell.

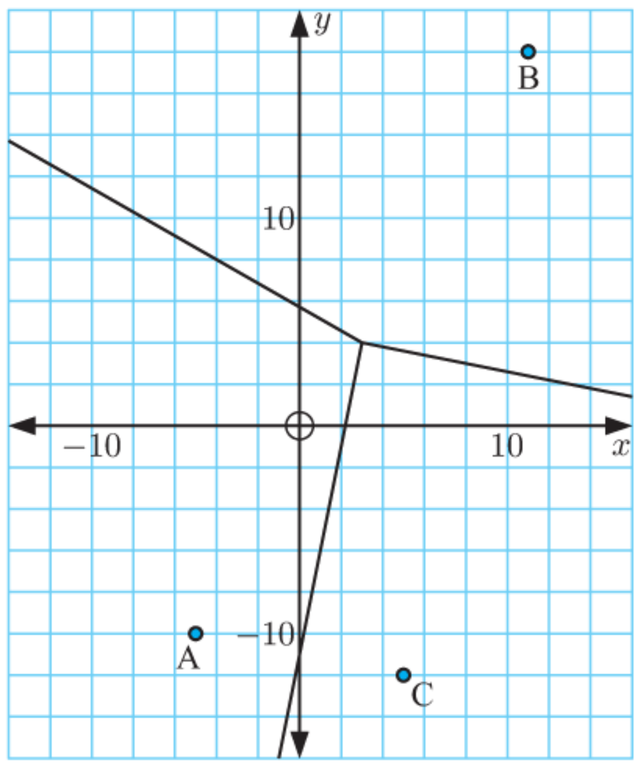
$\therefore$ the circle centred at P is not the largest empty circle.

$\therefore$ the largest empty circle cannot lie within a cell.

4 **a** **i** PB(A, D): $y = \frac{3}{7}x - \frac{1}{7}$ **ii** PB(C, D): $y = -x - 3$

b $V_2(-2, -1)$ **c** $V_1(3, 14)$

5 **a**



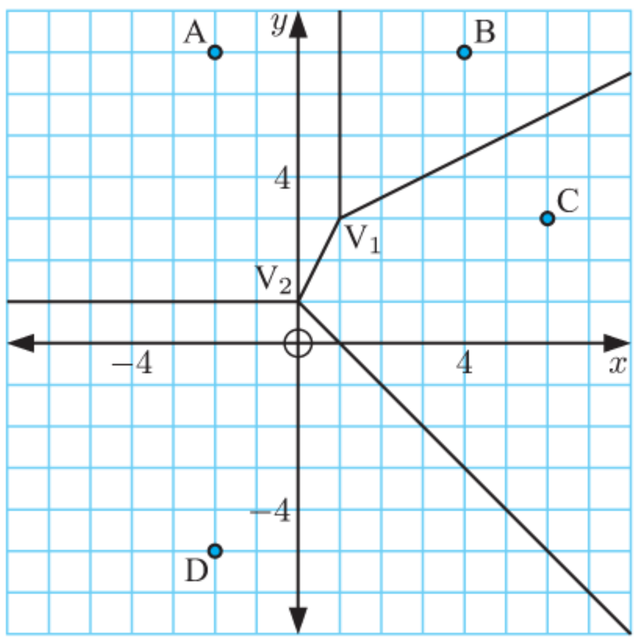
PB(A, B): $y = -\frac{4}{7}x + \frac{40}{7}$,

PB(A, C): $y = 5x - 11$,

PB(B, C): $y = -\frac{1}{5}x + \frac{23}{5}$

b $(3, 4)$ **c** centre $(3, 4)$, radius $= 2\sqrt{65}$ units

6 **a**



b $V_1(1, 3)$, $V_2(0, 1)$

c centre $(0, 1)$, radius $= 2\sqrt{10}$ units

7 **a** **i** station B **ii** station E **b** $V_3(-1, -\frac{17}{4})$

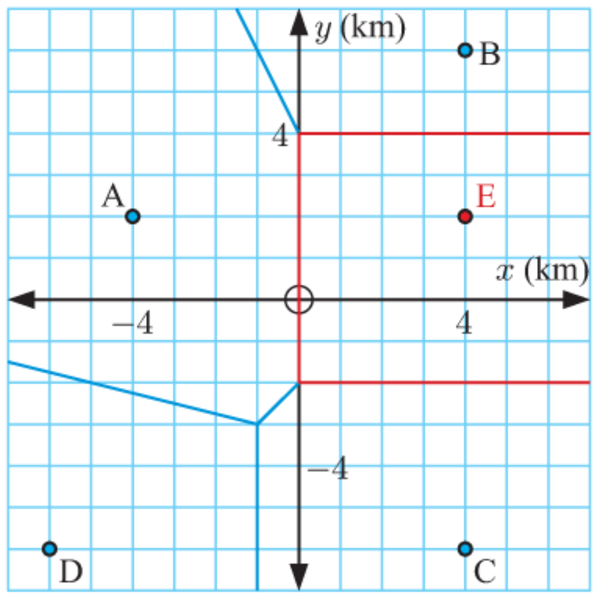
c **i** $V_3(-1, -\frac{17}{4})$

ii Distance from $(-3, -3)$ to E ≈ 2.83 km

Distance from $(-3, -3)$ to $V_3 \approx 2.36$ km

8 **a** $(2, 0)$, $2\sqrt{10} \approx 6.32$ km from A, B, and C

b **i**



ii $(-1, -3)$

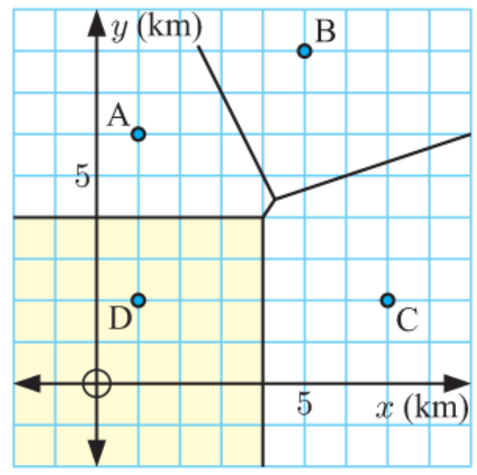
9 **a** **i** fire station C

ii fire station B

c We could create a Voronoi diagram like in **b**.

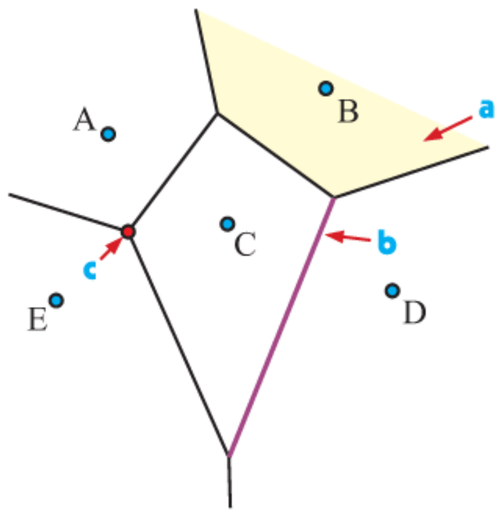
d $(\frac{30}{7}, \frac{31}{7})$

b



REVIEW SET 16A

1

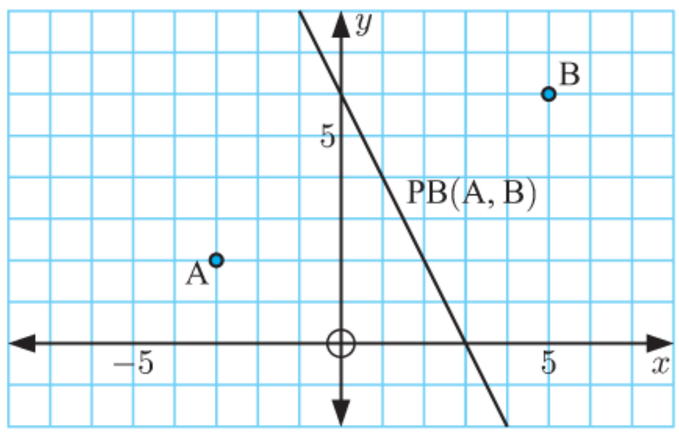


2 **a** **i** airport C **ii** airport A

b **i** The aeroplane is equally closest to airports B and D.

ii ≈ 361 km **iii** 300 km

3 **a**



b PB(A, B): $y = -2x + 6$

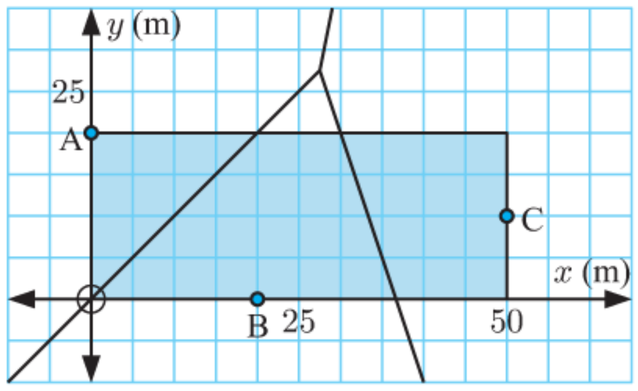
c **i** $0 = -2(3) + 6$ ✓

ii Distance from $(3, 0)$ to A $= 2\sqrt{10}$ units

Distance from $(3, 0)$ to B $= 2\sqrt{10}$ units

d **i** site A **ii** site A

4 **a**

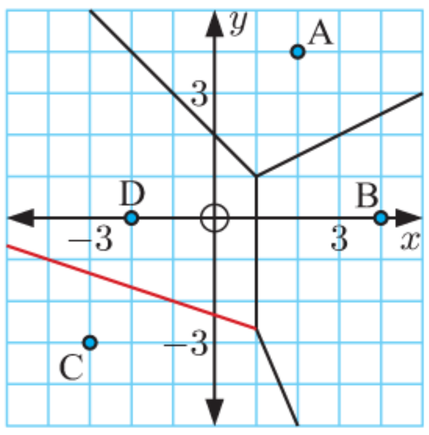


b No, the vertex lies outside the pool.

c **i** exit C **ii** 15 m **d** **i** $\frac{1}{5}$ **ii** $\frac{7}{15}$ **iii** $\frac{1}{3}$

5 **a** Sites C and D are currently in the same cell.

b



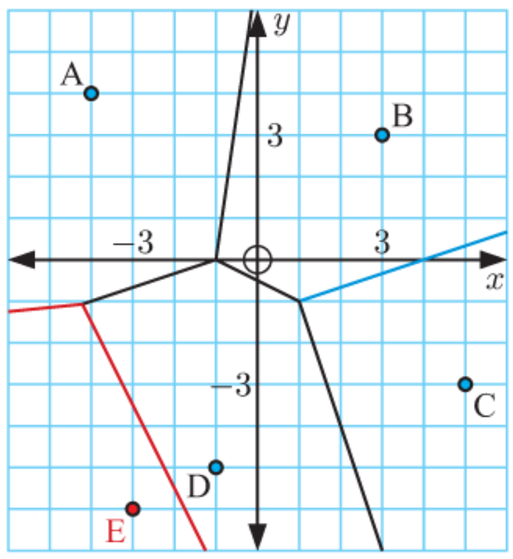
PB(C, D): $x + 3y + 7 = 0$

6 **a** $x - 3y = 4$

b **i** site C

ii site D

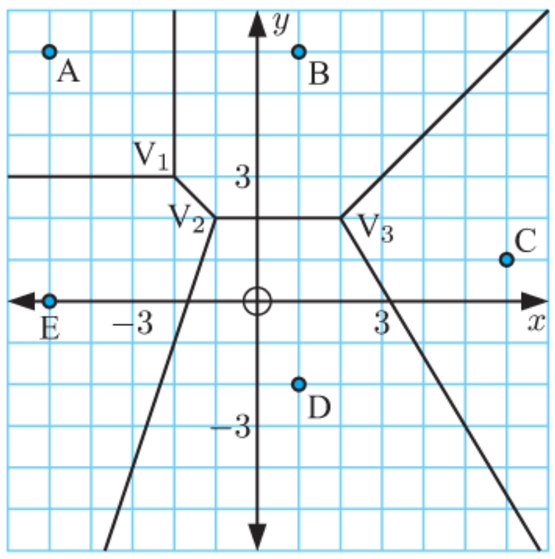
c



d Yes, in **ii**, $(-5, -2)$ would now be closest to site E.

7 **a** 11 km h^{-1} **b** 14 km h^{-1} **c** 19 km h^{-1}

8 **a**



b $V_1(-2, 3)$, $V_2(-1, 2)$, and $V_3(2, 2)$

c $V_2(-1, 2)$, radius $= 2\sqrt{5}$ units

9 **a** **i** $y = \frac{1}{2}x - 10$ **ii** $x = 10$ **b** $V_2(10, -5)$

c **i** $V_1(0, 5)$ **ii** $10\sqrt{2} \approx 14.1$ km

iii towns A, B, and D

REVIEW SET 16B

1 **a** **i** site C **ii** site E **iii** site C **iv** sites A, B, and E

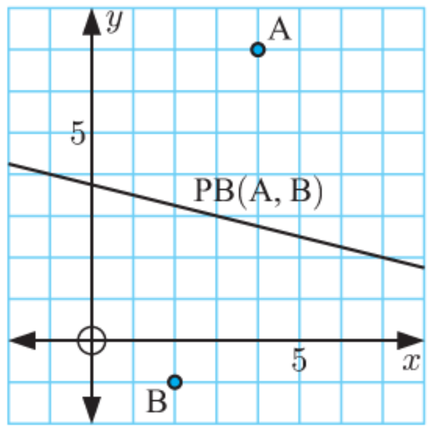
b No, they do not share a common edge. **c** (1, 2)

2 A point on an edge is equally closest to two sites. If the circle passes through a site, then the radius of the circle is the distance to that closest site.

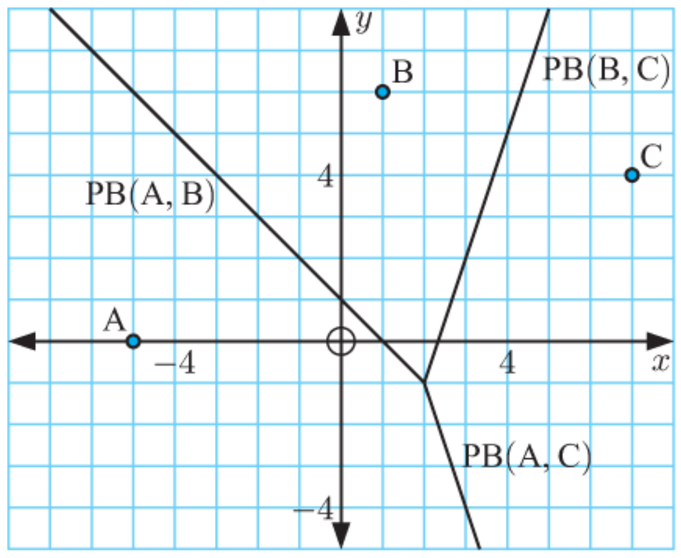
$\therefore$ the radius would also be the distance to the other site.

$\therefore$ the circle passes through two sites.

3 **a**



b



4 **a** There is an edge with no corresponding site. **b** (3, 1)

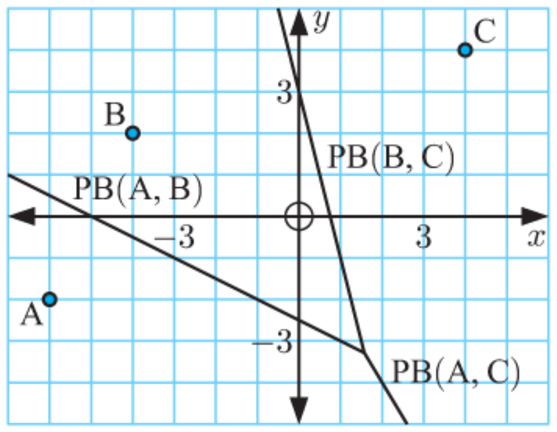
c **i** taxi rank A

ii $\approx$ 26 min 50 s; we are assuming that she can walk there in a straight line, and at a constant speed.

d **i** taxi ranks B and D

ii
- Whether he can walk there in a straight line.
- The price.
- His direction of travel once he gets into the taxi.

5 **a**



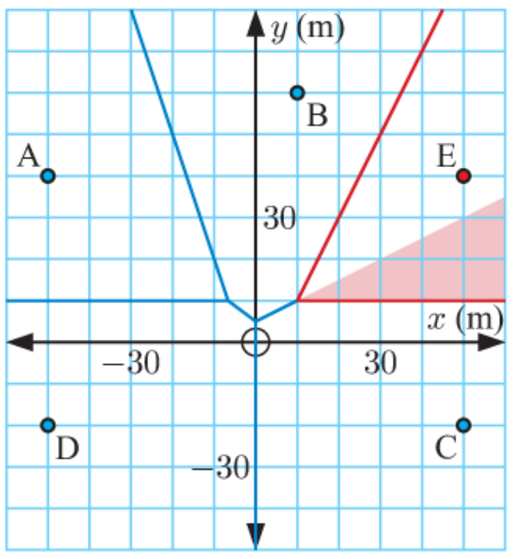
b **i** site C

ii site A

iii sites A and B

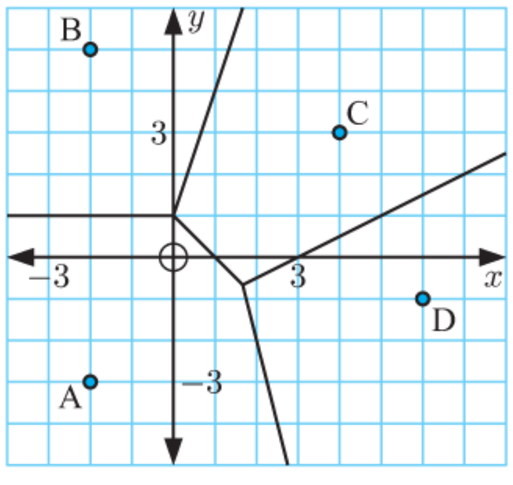
6 **a** ≈ 41.2 m

b **i**, **iii**



ii cells B and C

7 **a**



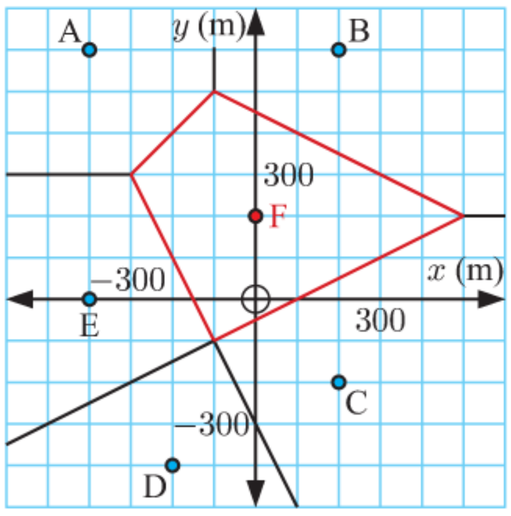
b **i** 9.2 m

ii 6.9 m

8 **a** **i** parking lot E **ii** parking lot D

b **i** (0, 200) **ii** ≈ 447 m

iii



iv 240 000 m^2

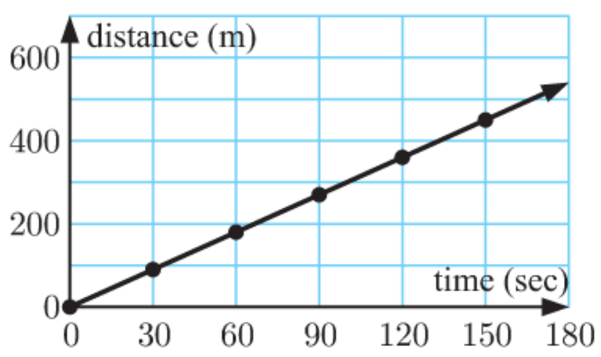
c **i** (200, 200) is in the cell containing the parking lot F.

ii 200 m closer

EXERCISE 17A.1

1 **a** Yes. The distance increases by the same amount each time interval.

b



c 3 m per s

2 **a** Yes. The height increases by the same amount each time interval.

b 5 cm per week

3 **a** 3 **b** -2 **c** $-\frac{1}{4}$ **4** $\frac{5}{2}$

EXERCISE 17A.2

1 **a** No. The graph is not a straight line.

b **i** 60 km per hour **ii** 100 km per hour

2 **a** 100 m per hour **b** 100 m per hour (downwards)

3 **a** $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** $-\frac{5}{4}$ **d** -2

4 **a** **i** 3 **ii** 2.5 **iii** 2.1 **iv** 2.01 **v** 2.001

b The average rate of change approaches 2.

EXERCISE 17B

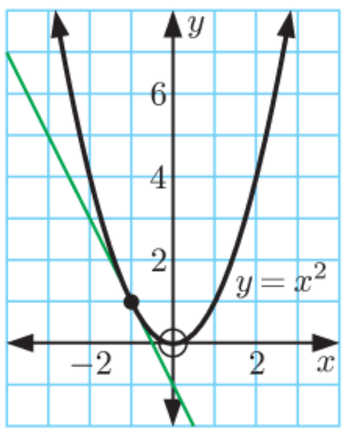
1 **a** 0.5 m s^{-1} **b** 2 m s^{-1}

2 **a** 8000 L **b** 3000 L **c** 10 667 L per hour

d 3000 L per hour

e The rate at which water is leaking is decreasing.

3 **a**, **b**



c -2

EXERCISE 17C.1

1 **a** as $x \to 1^-$, $f(x) \to 3^-$
as $x \to 1^+$, $f(x) \to 3^+$

b 3 **c** **i** 0 **ii** -1

2 **a** 7 **b** 7 **c** 11 **d** 16 **e** 0 **f** 5

3 **a** 5 **b** 7 **c** c

4 **a** -2 **b** 7 **c** -1

5 As $x \neq 0$, we can cancel the xs, to give

$$\lim_{x \to 0} \frac{x}{x} = \lim_{x \to 0} 1 = 1$$

6 **a**

x	$\frac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999
1.999 99	3.999 99

x	$\frac{x^2-4}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001
2.000 01	4.000 01

b $\lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

7 **a** -3 **b** 5 **c** -1 **d** 6 **e** -4 **f** -8

g 1 **h** 2 **i** 5

EXERCISE 17C.2

1 **a** **i** as $x \to 0^-$, $f(x) \to -\infty$
as $x \to 0^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0^-$
as $x \to \infty$, $f(x) \to 0^+$
vertical asymptote $x = 0$, horizontal asymptote $y = 0$

ii $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = 0$

b **i** as $x \to -3^-$, $f(x) \to \infty$
as $x \to -3^+$, $f(x) \to -\infty$
as $x \to -\infty$, $f(x) \to 3^+$
as $x \to \infty$, $f(x) \to 3^-$
vertical asymptote $x = -3$,
horizontal asymptote $y = 3$

ii $\lim_{x \to -\infty} f(x) = 3$, $\lim_{x \to \infty} f(x) = 3$

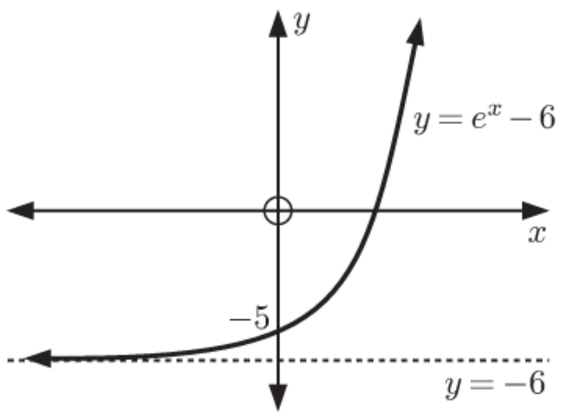
c **i** as $x \to -\frac{2}{3}^-$, $f(x) \to -\infty$
as $x \to -\frac{2}{3}^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to -\frac{2}{3}^-$
as $x \to \infty$, $f(x) \to -\frac{2}{3}^+$
vertical asymptote $x = -\frac{2}{3}$,
horizontal asymptote $y = -\frac{2}{3}$

ii $\lim_{x \to -\infty} f(x) = -\frac{2}{3}$, $\lim_{x \to \infty} f(x) = -\frac{2}{3}$

d **i** as $x \to 1^-$, $f(x) \to \infty$,
as $x \to 1^+$, $f(x) \to -\infty$,
as $x \to -\infty$, $f(x) \to -1^+$
as $x \to \infty$, $f(x) \to -1^-$
vertical asymptote $x = 1$,
horizontal asymptote $y = -1$

ii $\lim_{x \to -\infty} f(x) = -1$, $\lim_{x \to \infty} f(x) = -1$

2 **a**



b **i** $\lim_{x \to -\infty} (e^x - 6) = -6$

ii $\lim_{x \to \infty} (e^x - 6)$ does not exist

$y = -6$ is a horizontal asymptote of $y = e^x - 6$.

3 **a** does not exist **b** -3 **c** does not exist **d** 30
e does not exist **f** 10

4 **a**

x	xe^{-x}
10	$\approx 0.000\,454$
50	$\approx 9.64 \times 10^{-21}$
100	$\approx 3.72 \times 10^{-42}$
200	$\approx 2.77 \times 10^{-85}$

b $\lim_{x \to \infty} xe^{-x} = 0$

c yes

EXERCISE 17D

1 **a** $(3+h)^2$ **b** $\dfrac{(3+h)^2 - 9}{(3+h) - 3} = 6 + h$ for $h \neq 0$

c **i** 7 **ii** 6.5 **iii** 6.1 **iv** 6.01 **d** 6

2 **a** **i** 2 **ii** 8

b

x-coordinate	Gradient of tangent to $f(x) = x^2$
1	2
2	4
3	6
4	8

c $2a$

3 **a** 4 **b** 5 **c** 0 **d** -3

4 **a** $(1+h)^3 = 1 + 3h + 3h^2 + h^3$ **b** **ii** 3

5 **b** **i** -1 **ii** $-\frac{1}{9}$

EXERCISE 17E

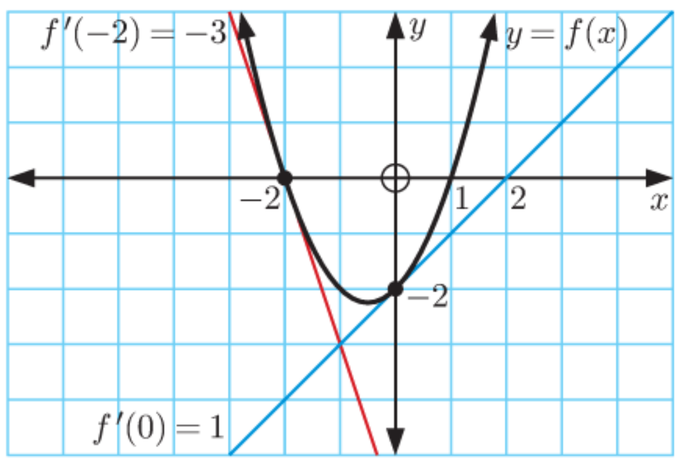
1 **a** $f(0) = 4$ **b** $f'(0) = -1$ **2** $f'(2) = 1$

3 **a** positive **b** negative **c** negative **d** positive

4 **a** **i** $f'(-2) = -3$
At $x = -2$, the derivative function is -3, *or* the gradient of the tangent to $y = f(x)$ at the point where $x = -2$ is -3.

ii $f'(0) = 1$
At $x = 0$, the derivative function is 1, *or* the gradient of the tangent to $y = f(x)$ at the point where $x = 0$ is 1.

b



EXERCISE 17F

1 **a** $f'(x) = 0$ **b** $f'(x) = 1$ **c** $f'(x) = 2$
d $f'(x) = -1$

2 **a** $\dfrac{dy}{dx} = 2x$ **b** $\dfrac{dy}{dx} = -2x$ **c** $\dfrac{dy}{dx} = 4x + 1$
d $\dfrac{dy}{dx} = -2x + 5$

3 **a** $f'(x) = 6x$

b $f'(2) = 12$. The gradient of the tangent to $y = f(x)$ at the point where $x = 2$ is 12.

4 **a** **i** 3 **ii** -1 **b** $f'(x) = -2x + 3$

c $f'(0) = 3$, $f'(2) = -1$

5 **a** $f'(x) = x - 1$ **b** **i** -3 **ii** $(5, \frac{11}{2})$

6 **a** $f'(x) = -\frac{1}{2}x + 1$

b $f'(-2) = 2$, $f'(3) = -\frac{1}{2}$
Gradients of tangents are 2 and $-\frac{1}{2}$, and $2 \times -\frac{1}{2} = -1$
$\therefore$ tangents are perpendicular.

7 **a**

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	27	12	3	0	3	12	27

c $\dfrac{dy}{dx} = 3x^2$

8 **a**

x	-3	-2	-1	0	1	2	3
$\frac{dy}{dx}$	$-\frac{1}{9}$	$-\frac{1}{4}$	-1	undefined	-1	$-\frac{1}{4}$	$-\frac{1}{9}$

b $\dfrac{dy}{dx} = -\dfrac{1}{x^2}$

9 **a**

$f(x)$	$f'(x)$
x^1	1
x^2	$2x$
x^3	$3x^2$
x^{-1}	$-\dfrac{1}{x^2}$
x^0	0

b If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

REVIEW SET 17A

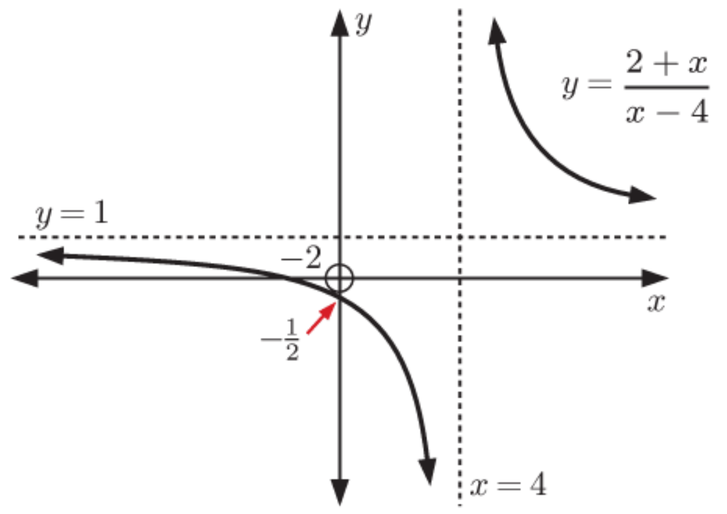
1 $\frac{4}{3}$

2 **a** Yes. The height increases by the same amount each time interval.

b 1.6 m s^{-1}

3 **a** -1 **b** -1 **c** 8

4 **a**



b as $x \to 4^-$, $y \to -\infty$, as $x \to -\infty$, $y \to 1^-$,
as $x \to 4^+$, $y \to \infty$, as $x \to \infty$, $y \to 1^+$
vertical asymptote $x = 4$, horizontal asymptote $y = 1$

c $\lim_{x \to -\infty} \dfrac{2+x}{x-4} = 1$, $\lim_{x \to \infty} \dfrac{2+x}{x-4} = 1$

5 **b**

h	$\dfrac{f(3+h) - f(3)}{h}$
0.1	12.2
0.01	12.02
0.001	12.002
0.0001	12.0002

c $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h} = 12$

The gradient of the tangent to $y = 2x^2$ at $(3, 18)$ is 12.

6 $f'(3) = -\frac{1}{2}$ **7** **a** $f'(x) = 2x + 2$ **b** $\dfrac{dy}{dx} = -6x$

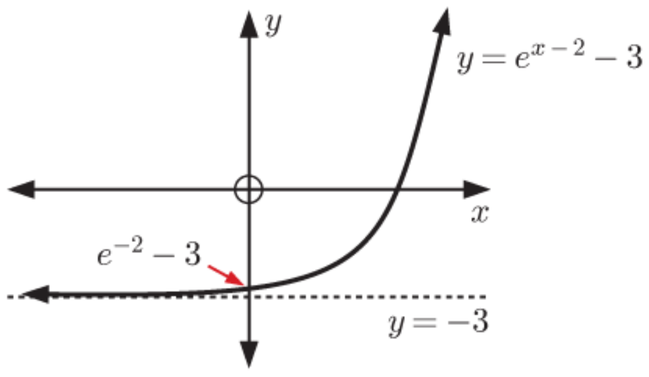
8 **a** $\dfrac{dy}{dx} = 4x$ **b** gradient $= 16$ **c** $x = -3$

REVIEW SET 17B

1 $\frac{1}{2}$ **2** **a** 2°C per hour **b** $-2\frac{1}{2}$°C per hour

3 **a** -3 **b** 3 **c** -1

4 **a**

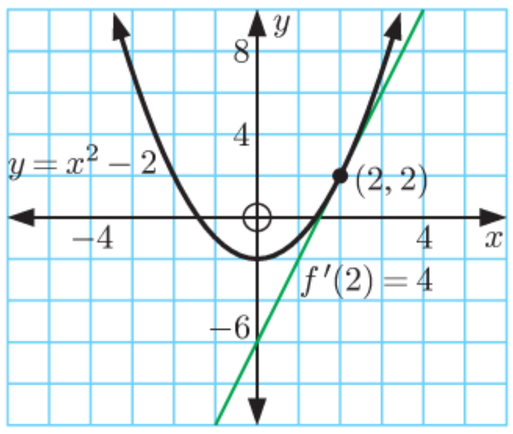


b **i** $\lim_{x\to -\infty} (e^{x-2} - 3) = -3$

ii $\lim_{x\to \infty} (e^{x-2} - 3)$ does not exist

5 **a** negative **b** positive **c** positive **d** negative

6 **a**, **b**



c 4

7 **a** $\dfrac{dy}{dx} = 2x + 5$ **b** $(-4, -6)$

8 **a** **i** 447.2 m **ii** 432.8 m **b** $f'(t) = -9.6t$

c **i** 9.6 m s^{-1} **ii** 19.2 m s^{-1}

EXERCISE 18A

1 **a** $3x^2$ **b** $8x^7$ **c** $11x^{10}$ **d** 6

e $6x^2$ **f** $14x$ **g** $15x^4$ **h** $30x^5$

i 5 **j** $2x$ **k** $2x + 1$ **l** $2x + 3$

m $4x + 1$ **n** $6x - 7$ **o** $-4x$ **p** $2x^3 - 12x$

q $3x^2 - 8x + 6$ **r** $6x^2 + 1$ **s** $-1 - 12x^2$ **t** $\frac{3}{5}x^2 - 7x$

2 **a** $-\dfrac{2}{x^3}$ **b** $-\dfrac{5}{x^6}$ **c** $-\dfrac{1}{3x^2}$ **d** $-\dfrac{12}{x^4}$ **e** $\dfrac{28}{x^5}$

f $2 - \dfrac{6}{x^3}$ **g** $2x + \dfrac{6}{x^2}$ **h** $-\dfrac{4}{x^3} - \dfrac{36}{x^5}$ **i** $3 + \dfrac{1}{x^2} - \dfrac{4}{x^3}$

j $\dfrac{16}{x^3} - \dfrac{3}{2x^4}$ **k** $2x - \dfrac{4}{x^2}$ **l** $-\dfrac{2}{x^2} + \dfrac{10}{x^3}$

3 **a** $\dfrac{1}{2\sqrt{x}}$ **b** $\dfrac{1}{3\sqrt[3]{x^2}}$ **c** $2 - \dfrac{1}{2\sqrt{x}}$

d $\frac{3}{2}\sqrt{x}$ **e** $\dfrac{-1}{2x\sqrt{x}}$ **f** $\dfrac{3}{2x^2\sqrt{x}}$

g $4x + \dfrac{3}{2x\sqrt{x}}$ **h** $-\dfrac{1}{2x\sqrt{x}} + \dfrac{4}{x^2}$

i $\dfrac{1}{2\sqrt{x}} - \dfrac{5}{2x\sqrt{x}}$ **j** $-\dfrac{7}{2x\sqrt{x}} - \dfrac{3\sqrt{x}}{2}$

k $2 + \dfrac{15}{2x^3\sqrt{x}}$ **l** $\dfrac{5\sqrt[3]{x^2}}{3} - \dfrac{2}{3\sqrt[3]{x}} - \dfrac{2}{3x\sqrt[3]{x}}$

4 **a** $\dfrac{dy}{dx} = 2\pi x$ **b** $\dfrac{dy}{dx} = 7.5x^2 - 2.8x$ **c** $\dfrac{dy}{dx} = 6x + \dfrac{16}{x^3}$

d $\dfrac{dy}{dx} = 2x - 1$ **e** $\dfrac{dy}{dx} = 12x - 1$ **f** $\dfrac{dy}{dx} = 2x - 10$

g $\dfrac{dy}{dx} = 8x - 4$ **h** $\dfrac{dy}{dx} = 6x^2 - 6x - 5$

i $\dfrac{dy}{dx} = \frac{3}{2}\sqrt{x} - \dfrac{3}{\sqrt{x}} - \dfrac{9}{2x\sqrt{x}}$

5 **a** $\dfrac{dy}{dt} = 2t^3 - \frac{1}{3}$ **b** $\dfrac{dy}{dt} = 3t^{-\frac{3}{2}} = \dfrac{3}{t\sqrt{t}}$

c $\dfrac{dT}{dt} = \frac{1}{3}t^{-\frac{2}{3}} + 4t^{-3} = \dfrac{1}{3\sqrt[3]{t^2}} + \dfrac{4}{t^3}$

d $\dfrac{dP}{du} = -5u^{-2} - 15u^{\frac{1}{2}} = -\dfrac{5}{u^2} - 15\sqrt{u}$

6 **a** 4 **b** 22 **c** $-\frac{16}{729}$ **d** -7 **e** $\frac{3}{2}$

f $\frac{13}{4}$ **g** $\frac{1}{8}$ **h** -11

7 $a = 7$ **8** $a = -9,\ b = 8$ **9** $a = 3,\ b = 6$

10 $a = 2,\ b = -3$

11 $\dfrac{dy}{dx} = 4 + \dfrac{3}{x^2}$, $\dfrac{dy}{dx}$ is the gradient function of $y = 4x - \dfrac{3}{x}$ from which the gradient of the tangent at any point can be found.

12 **a** $\{x \mid x > 0\}$ **b** $f'(x) = \dfrac{1}{2\sqrt{x}} + \dfrac{2}{x\sqrt{x}}$

c $\{x \mid x > 0\}$

d $f'(1) = 2.5$ The gradient of the tangent to the curve $f(x) = \sqrt{x} - \dfrac{4}{\sqrt{x}}$ at $x = 1$ is 2.5.

13 **a** $\dfrac{dS}{dt} = 4t + 4$ m s^{-1}, $\dfrac{dS}{dt}$ is the instantaneous rate of change in the car's position at the time t.

b When $t = 3$, $\dfrac{dS}{dt} = 16$ m s^{-1}. This is the instantaneous rate of change in position at the time $t = 3$ seconds.

14 When $x = 1000$, $\dfrac{dC}{dx} = 7$. When 1000 toasters per week are being produced, the cost of production is increasing by £7 per toaster.

EXERCISE 18B.1

1 **a** $g(f(x)) = (2x + 7)^2$ **b** $g(f(x)) = 2x^2 + 7$

c $g(f(x)) = \sqrt{3 - 4x}$ **d** $g(f(x)) = 3 - 4\sqrt{x}$

e $g(f(x)) = \dfrac{2}{x^2 + 3}$ **f** $g(f(x)) = \dfrac{4}{x^2} + 3$

2 **Note:** There may be other answers.

a $g(x) = x^3,\ f(x) = 3x + 10$

b $g(x) = x^5,\ f(x) = 7 - 2x$

c $g(x) = \dfrac{1}{x},\ f(x) = 2x + 4$

d $g(x) = \sqrt{x},\ f(x) = x^2 - 3x$

e $g(x) = \dfrac{1}{x^4},\ f(x) = 5x - 1$

f $g(x) = \dfrac{10}{x^3},\ f(x) = 3x - x^2$

EXERCISE 18B.2

1 **a** u^{-2}, $u = 2x - 1$ **b** $u^{\frac{1}{2}}$, $u = x^2 - 3x$

c $2u^{-\frac{1}{2}}$, $u = 2 - x^2$ **d** $u^{\frac{1}{3}}$, $u = x^3 - x^2$

e $4u^{-3}$, $u = 3 - x$ **f** $10u^{-1}$, $u = x^2 - 3$

2 **a** $\dfrac{dy}{dx} = 4(2x + 3) = 8x + 12$ **b** $\dfrac{dy}{dx} = 8x + 12$

3 **a** $\dfrac{dy}{dx} = 8(4x - 5)$ **b** $\dfrac{dy}{dx} = 2(5 - 2x)^{-2}$

c $\dfrac{dy}{dx} = \frac{1}{2}(3x - x^2)^{-\frac{1}{2}}(3 - 2x)$

d $\dfrac{dy}{dx} = -12(1 - 3x)^3$ **e** $\dfrac{dy}{dx} = -18(5 - x)^2$

f $\dfrac{dy}{dx} = \frac{1}{3}(2x^3 - x^2)^{-\frac{2}{3}}(6x^2 - 2x)$

g $\dfrac{dy}{dx} = -60(5x - 4)^{-3}$

h $\dfrac{dy}{dx} = 6\left(x^2 - \dfrac{2}{x}\right)^2\left(2x + \dfrac{2}{x^2}\right)$

4 **a** $-\frac{1}{\sqrt{3}}$ **b** -18 **c** -8 **d** -4 **e** $-\frac{3}{32}$ **f** 0

5 $a = 3,\ b = 1$ **6** $a = 2,\ b = 1$ **7** $a = 2,\ b = 3$

EXERCISE 18C

1 **a** $f'(x) = 2x - 1$ **b** $f'(x) = 4x + 2$

c $f'(x) = 2x(x + 1)^{\frac{1}{2}} + \frac{1}{2}x^2(x + 1)^{-\frac{1}{2}}$

d $f'(x) = 2x + 2$

e $f'(x) = (x^2 - 1)^{\frac{1}{2}} + x^2(x^2 - 1)^{-\frac{1}{2}}$

f $f'(x) = (x + 1)^2 + 2x(x + 1)$

2 **a** $\dfrac{dy}{dx} = 2x(2x - 1) + 2x^2$

b $\dfrac{dy}{dx} = 4(2x + 1)^3 + 24x(2x + 1)^2$

c $\dfrac{dy}{dx} = 2x(3 - x)^{\frac{1}{2}} - \frac{1}{2}x^2(3 - x)^{-\frac{1}{2}}$

d $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x - 3)^2 + 2\sqrt{x}(x - 3)$

e $\dfrac{dy}{dx} = 10x(3x^2 - 1)^2 + 60x^3(3x^2 - 1)$

f $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x - x^2)^3 + 3\sqrt{x}(x - x^2)^2(1 - 2x)$

3 **a** -48 **b** $406\frac{1}{4}$ **c** $\frac{13}{3}$ **d** $\frac{11}{2}$

4 **b** $x = 3$ or $\frac{3}{5}$

c Domain of $\dfrac{dy}{dx}$ is $\{x \mid x > 0\}$. Domain of original function is $\{x \mid x \geqslant 0\}$. $\dfrac{dy}{dx}$ is undefined when $x = 0$.

5 $x = -1$ and $x = -\frac{5}{3}$ **6** $x = 1$ and $x = -\frac{1}{3}$

7 **a** $x = \frac{2}{3}$ **b** $x = 0$ **8** $a = -\frac{7}{4}$ and $a = -\frac{20}{9}$

EXERCISE 18D

1 **a** $\dfrac{dy}{dx} = \dfrac{7}{(2 - x)^2}$ **b** $\dfrac{dy}{dx} = \dfrac{2x^2 + 2x}{(2x + 1)^2}$

c $\dfrac{dy}{dx} = \dfrac{-x^2 - 3}{(x^2 - 3)^2}$ **d** $\dfrac{dy}{dx} = \dfrac{2x + 1}{2\sqrt{x}(1 - 2x)^2}$

e $\dfrac{dy}{dx} = \dfrac{3x^2 - 6x + 9}{(3x - x^2)^2}$ **f** $\dfrac{dy}{dx} = \dfrac{2 - 3x}{2(1 - 3x)^{\frac{3}{2}}}$

2 **a** $\dfrac{4}{(3 - x)^2}$ **b** $\dfrac{-3x^2 - 3}{(x^2 - 1)^2}$ **c** $\dfrac{4x^3 - 3x^2}{(2x - 1)^2}$

d $\dfrac{2x - 20}{(x - 5)^{\frac{3}{2}}}$ **e** $\dfrac{3x^2 + 3}{2\sqrt{x}(3 - x^2)^2}$ **f** $\dfrac{-x^3 - 6x}{(x^2 + 3)^{\frac{3}{2}}}$

3 **a** 1 **b** 1 **c** $-\frac{7}{324}$ **d** $-\frac{28}{27}$

4 $f'(x) = \dfrac{x - 2}{2(x - 1)^{\frac{3}{2}}}$

5 **b** **i** never $\left\{\dfrac{dy}{dx}\right.$ is undefined at $\left.x = -1\right\}$

ii $x \leqslant 0$ and $x = 1$

6 **b** **i** $x = -3$ and $x = 2$ **ii** $x = -\frac{1}{2}$

7 **b** **i** $x = -2 \pm \sqrt{11}$ **ii** $x = -2$

EXERCISE 18E

1 **a** $f'(x) = 4e^{4x}$ **b** $f'(x) = e^x$ **c** $f'(x) = -2e^{-2x}$

d $f'(x) = \frac{1}{2}e^{\frac{x}{2}}$ **e** $f'(x) = -e^{-\frac{x}{2}}$ **f** $f'(x) = 2e^{-x}$

g $f'(x) = 2e^{\frac{x}{2}} + 3e^{-x}$ **h** $f'(x) = \dfrac{e^x - e^{-x}}{2}$

i $f'(x) = -2xe^{-x^2}$ **j** $f'(x) = -\dfrac{e^{\frac{1}{x}}}{x^2}$

k $f'(x) = 20e^{2x}$ **l** $f'(x) = 40e^{-2x}$

m $f'(x) = 2e^{2x+1}$ **n** $f'(x) = \frac{1}{4}e^{\frac{x}{4}}$

o $f'(x) = -4xe^{1-2x^2}$ **p** $f'(x) = -0.02e^{-0.02x}$

2 **a** $e^x + xe^x$ **b** $3x^2e^{-x} - x^3e^{-x}$ **c** $\dfrac{xe^x - e^x}{x^2}$

d $\dfrac{1 - x}{e^x}$ **e** $2xe^{3x} + 3x^2e^{3x}$ **f** $\dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $20e^{-0.5x} - 10xe^{-0.5x}$ **h** $\dfrac{e^x + 2 + 2e^{-x}}{(e^{-x} + 1)^2}$

3 **a** $\dfrac{dy}{dx} = 4e^x(2 + e^x)^3$ **b** $\dfrac{dy}{dx} = \dfrac{e^x}{2\sqrt{e^x - 1}}$

c $\dfrac{dy}{dx} = \frac{3}{2}(e^x + e^{-x})^{\frac{1}{2}}(e^x - e^{-x})$

d $\dfrac{dy}{dx} = -e^{2x}(e^{2x} + 2)^{-\frac{3}{2}}$

4 **a** 108 **b** -1 **c** $\frac{9}{\sqrt{19}}$ **d** $-\dfrac{4}{e^3}$

5 **a** $\{y \mid 0 \leqslant y < \sqrt{6}\}$ **b** **i** P$(\ln 2,\ 2)$ **ii** $-\frac{1}{2}$

6 $k = -9$

7 **a** $\dfrac{dy}{dx} = 2^x \ln 2$

c **i** $\dfrac{dy}{dx} = 5^x \ln 5$ **ii** $\dfrac{dy}{dx} = 8 \times 10^x \ln 10$

8 P$(0,\ 0)$ or P$\left(2,\ \dfrac{4}{e^2}\right)$ **9** **c** $S(x)$ **d** $\dfrac{1}{[C(x)]^2}$

EXERCISE 18F

1 **a** $\dfrac{dy}{dx} = \dfrac{1}{x}$ **b** $\dfrac{dy}{dx} = \dfrac{2}{2x + 1}$ **c** $\dfrac{dy}{dx} = \dfrac{1 - 2x}{x - x^2}$

d $\dfrac{dy}{dx} = -\dfrac{2}{x}$ **e** $\dfrac{dy}{dx} = 2x \ln x + x$ **f** $\dfrac{dy}{dx} = \dfrac{1 - \ln x}{2x^2}$

g $\dfrac{dy}{dx} = e^x \ln x + \dfrac{e^x}{x}$ **h** $\dfrac{dy}{dx} = \dfrac{2\ln x}{x}$

i $\dfrac{dy}{dx} = \dfrac{1}{2x\sqrt{\ln x}}$ **j** $\dfrac{dy}{dx} = \dfrac{e^{-x}}{x} - e^{-x}\ln x$

k $\dfrac{dy}{dx} = \dfrac{\ln(2x)}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}$ **l** $\dfrac{dy}{dx} = \dfrac{4}{1-x}$

m $\dfrac{dy}{dx} = \dfrac{\ln x - 2}{\sqrt{x}(\ln x)^2}$ **n** $\dfrac{dy}{dx} = \ln(x^2+1) + \dfrac{2x^2}{x^2+1}$

o $\dfrac{dy}{dx} = \dfrac{1-2\ln x}{x^3}$

2 $f'(x) = \dfrac{1}{x}$, since $\ln(kx) = \ln k + \ln x$ and $\ln k$ is a constant.

3 **a** $\dfrac{dy}{dx} = \ln 5$ **b** $\dfrac{dy}{dx} = \dfrac{3}{x}$ **c** $\dfrac{dy}{dx} = \dfrac{4x^3+1}{x^4+x}$

d $\dfrac{dy}{dx} = \dfrac{1}{x-2}$ **e** $\dfrac{dy}{dx} = \dfrac{6}{2x+1}[\ln(2x+1)]^2$

f $\dfrac{dy}{dx} = \dfrac{1-\ln(4x)}{x^2}$ **g** $\dfrac{dy}{dx} = -\dfrac{1}{x}$ **h** $\dfrac{dy}{dx} = \dfrac{-1}{x(\ln x)^2}$

4 **a** $\dfrac{dy}{dx} = \dfrac{1}{2x-1}$ **b** $\dfrac{dy}{dx} = \dfrac{-2}{2x+3}$

c $\dfrac{dy}{dx} = 1 + \dfrac{1}{2x}$ **d** $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{2(2-x)}$

e $\dfrac{dy}{dx} = \dfrac{1}{x+3} - \dfrac{1}{x-1}$ **f** $\dfrac{dy}{dx} = \dfrac{2}{x} + \dfrac{1}{3-x}$

5 **a** $f'(x) = \dfrac{9}{3x-4}$ **b** $f'(x) = \dfrac{1}{x} + \dfrac{2x}{x^2+1}$

c $f'(x) = \dfrac{2x+2}{x^2+2x} - \dfrac{1}{x-5}$

6 2 **7** $a = 3,\ b = \dfrac{1}{e}$ **8** $a = 4,\ b = e^3$ **9** $(-3, \ln 6)$

EXERCISE 18G

1 **a** $\dfrac{dy}{dx} = 2\cos 2x$ **b** $\dfrac{dy}{dx} = \cos x - \sin x$

c $\dfrac{dy}{dx} = -3\sin 3x - \cos x$ **d** $\dfrac{dy}{dx} = \cos(x+1)$

e $\dfrac{dy}{dx} = 2\sin(3-2x)$ **f** $\dfrac{dy}{dx} = \dfrac{5}{\cos^2 5x}$

g $\dfrac{dy}{dx} = \frac{1}{2}\cos\frac{x}{2} + 3\sin x$ **h** $\dfrac{dy}{dx} = 4\cos x + 2\sin 2x$

i $\dfrac{dy}{dx} = -3\sin 6x - 20\cos 4x$

2 **a** $2x - \sin x$ **b** $\dfrac{1}{\cos^2 x} - 3\cos x$ **c** $e^x\cos x - e^x\sin x$

d $-e^{-x}\sin x + e^{-x}\cos x$ **e** $\dfrac{\cos x}{\sin x}$

f $2e^{2x}\tan x + \dfrac{e^{2x}}{\cos^2 x}$ **g** $\dfrac{3\pi}{\cos^2 \pi x}$ **h** $-\frac{1}{2}\sin\frac{x}{2}$

i $\dfrac{6}{\cos^2 2x}$ **j** $\cos x - x\sin x$ **k** $\dfrac{x\cos x - \sin x}{x^2}$

l $\tan x + \dfrac{x}{\cos^2 x}$ **m** $-\dfrac{\sin\sqrt{x}e^{\cos\sqrt{x}}}{2\sqrt{x}}$

n $\dfrac{\sin x\cos x - 2x}{\sqrt{x}\sin^2 x}$

o $\dfrac{2x\cos 2x - x\sin x - 3\cos x - 3\sin 2x}{x^4}$

3 **a** $2x\cos(x^2)$ **b** $-\dfrac{1}{2\sqrt{x}}\sin(\sqrt{x})$ **c** $-\dfrac{\sin x}{2\sqrt{\cos x}}$

d $e^x\cos(e^x)$ **e** $2\sin x\cos x$

f $-3\sin x\cos^2 x$ **g** $-20\sin 2x\cos 2x$

h $-\sin x\sin 2x + 2\cos x\cos 2x$ **i** $\sin x\sin(\cos x)$

j $-12\sin 4x\cos^2 4x$ **k** $-\dfrac{16\cos 4x}{\sin^3 4x}$ **l** $\dfrac{4\tan 2x}{3\cos^2 2x}$

4 **a** $-\frac{9}{8}$ **b** $\frac{8}{3}$ **c** 0

5 **a** $f'(x) = 0$ **b** $f(x) = 2(\cos^2 x + \sin^2 x) + 1$
$= 2(1) + 1$
$= 3$, a constant

6 **b** 1 **c** $\dfrac{\cos\theta + i\sin\theta}{e^{i\theta}} = 1$
$\therefore\ e^{i\theta} = \cos\theta + i\sin\theta$

EXERCISE 18H

1 **a** $f''(x) = 6$ **b** $f''(x) = \dfrac{3}{2x^2\sqrt{x}}$

c $f''(x) = 12x - 6$ **d** $f''(x) = \dfrac{12-6x}{x^4}$

e $f''(x) = 8$ **f** $f''(x) = \dfrac{20}{(2x-1)^3}$

2 **a** $\dfrac{d^2y}{dx^2} = -6x$ **b** $\dfrac{d^2y}{dx^2} = 2 - \dfrac{30}{x^4}$

c $\dfrac{d^2y}{dx^2} = -\dfrac{9}{4x^2\sqrt{x}}$ **d** $\dfrac{d^2y}{dx^2} = \dfrac{8}{x^3}$

e $\dfrac{d^2y}{dx^2} = 12x^2 - 36x + 18$

f $\dfrac{d^2y}{dx^2} = 2 + \dfrac{2}{(1-x)^3}$ **g** $\dfrac{d^2y}{dx^2} = 9e^{3x}$

h $\dfrac{d^2y}{dx^2} = \dfrac{-x^2e^{-x} - 2xe^{-x} + 2 - 2e^{-x}}{x^3}$

i $\dfrac{d^2y}{dx^2} = -\dfrac{x^3 - 3x^2 - 6x - 6}{x^3e^x}$

3 **a** $f(2) = 9$ **b** $f'(2) = 10$ **c** $f''(2) = 12$

4 **a** $x = 1$ **b** $x = 2$ or $x = 3$

5

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

6 **a** $f(1) = 0$ **b** $f'(1) = 3$ **c** $f''(1) = 0$

7 **a** $f(1) = 3e - 2$ **b** $f'(1) = 3e - 2$ **c** $f''(1) = 3e$

8 **a** $\dfrac{d^2y}{dx^2} = \dfrac{6\sin x}{\cos^3 x}$ **b** $\dfrac{d^2y}{dx^2} = 2\cos x - x\sin x$

c $\dfrac{d^2y}{dx^2} = 18(\cos^2 3x - \sin^2 3x)$

d $\dfrac{d^2y}{dx^2} = \dfrac{4\sin x - 2}{x^3} - \dfrac{\cos x}{x^2} + \dfrac{6\cos x}{x^4}$

e $\dfrac{d^2y}{dx^2} = -2e^{-x}\cos x$ **f** $\dfrac{d^2y}{dx^2} = \dfrac{2(4x^2\tan(x^2) + 1)}{\cos^2(x^2)}$

10 **a** $k = 3$ **b** $f''(x) = 12\cos^2 x\sin x - 6\sin^3 x + 3\sin x$

11 $-3\sqrt{3}$

12 **a** $\dfrac{d^2y}{dx^2} = \dfrac{1}{x^2}$ **b** $\dfrac{d^2y}{dx^2} = \dfrac{1}{x}$ **c** $\dfrac{d^2y}{dx^2} = \dfrac{2}{x^2}(1 - \ln x)$

REVIEW SET 18A

1 **a** $f'(x) = 15x^2$ **b** $f'(x) = 6x^5 - 5$

c $f'(x) = 14x + \dfrac{3}{x^2}$ **d** $f'(x) = 3 + \dfrac{8}{x^3}$

e $f'(x) = 3\sqrt{x}$ **f** $f'(x) = \dfrac{2}{\sqrt{x}} + \dfrac{1}{2x\sqrt{x}}$

2 **a** $\dfrac{dy}{dx} = 6x - 4x^3$ **b** $\dfrac{dy}{dx} = 1 + \dfrac{1}{x^2}$

c $\dfrac{dy}{dx} = 2x\sqrt{x-2} + \dfrac{x^2}{2\sqrt{x-2}}$

3 **a** $f'(x) = (x^2+1)^{-\frac{3}{2}}$ **b** $(0, 0)$

4 **a** $\dfrac{dy}{dx} = 3x^2 e^{x^3+2}$ **b** $\dfrac{dy}{dx} = \dfrac{1}{x+3} - \dfrac{2}{x}$

c $\dfrac{dy}{dx} = 3x^2e^{2x} + 2x^3e^{2x}$

5 **a** 10 **b** -15

6 **a** $5 + 3x^{-2}$ **b** $4(3x^2 + x^{\frac{1}{2}})^3(6x + \frac{1}{2}x^{-\frac{1}{2}})$

c $2x(1-x^2)^3 - 6x(x^2+1)(1-x^2)^2$

7 $(-2, 19)$ and $(1, -2)$

8 **a** $(5\cos 5x)\ln x + \dfrac{\sin 5x}{x}$ **b** $\cos x \cos 2x - 2\sin x \sin 2x$

c $e^{-2x}\left(\dfrac{1}{\cos^2 x} - 2\tan x\right)$

9 $\frac{\sqrt{3}}{2}$

10 **a** $f'(x) = \dfrac{-x^2 + 6x - 3}{e^x}$ **b** $\dfrac{2}{e}$ **c** $x = 3 \pm \sqrt{6}$

12 **a** $f'(x) = 8x(x^2+3)^3$

b $g'(x) = \dfrac{\frac{1}{2}x(x+5)^{-\frac{1}{2}} - 2(x+5)^{\frac{1}{2}}}{x^3}$

c $h'(x) = \dfrac{6e^{4x} - 8xe^{4x}}{(1-2x)^2}$

13 **a** $\dfrac{dy}{dx} = \cos^2 x - \sin^2 x$ **b** $-\frac{1}{2}$

14 **a** $f(\frac{\pi}{2}) = 1$ **b** $f'(\frac{\pi}{2}) = 0$ **c** $f''(\frac{\pi}{2}) = 2$

15 2

16 **a** $\dfrac{d^2y}{dx^2} = \frac{3}{2}x^2 + x - \frac{1}{2}$ **b** $\dfrac{d^2y}{dx^2} = -2e^{-x} + xe^{-x}$

17 **a** $f'(x) = \dfrac{1}{2\sqrt{x}}\cos 4x - 4\sqrt{x}\sin 4x,$

$f''(x) = -\dfrac{1}{4x\sqrt{x}}\cos 4x - \dfrac{4}{\sqrt{x}}\sin 4x - 16\sqrt{x}\cos 4x$

b **i** $\frac{1}{\sqrt{2}}(\frac{2}{\sqrt{\pi}} - \sqrt{\pi})$ **ii** $-\frac{8\sqrt{2}}{\sqrt{\pi}}$

REVIEW SET 18B

1 **a** $f'(x) = 6x - 7$ **b** $f'(x) = 2x + 10$

c $f'(x) = \dfrac{1}{\sqrt{x}} + \dfrac{3}{x^2}$ **d** $f'(x) = 15x\sqrt{x}$

2 **a** $\dfrac{dy}{dx} = 6x^2 - 12x + 7$ **b** $\dfrac{dy}{dx} = -\dfrac{3}{x^2} + \dfrac{15}{x^4}$

c $\dfrac{dy}{dx} = -\dfrac{5}{x\sqrt[3]{x}}$

3 **a** $f(3) = -17$ **b** $f'(3) = -17$ **c** $f''(3) = -6$

4 **a** $\dfrac{dy}{dx} = 3x^2(1-x^2)^{\frac{1}{2}} - x^4(1-x^2)^{-\frac{1}{2}}$

b $\dfrac{dy}{dx} = \dfrac{(2x-3)(x+1)^{\frac{1}{2}} - \frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$

c $\dfrac{dy}{dx} = 3x^2 - 1 - \dfrac{1}{2x\sqrt{x}}$

5 **a** $\dfrac{dy}{dx} = e^x + xe^x$ **b** $(1, e)$

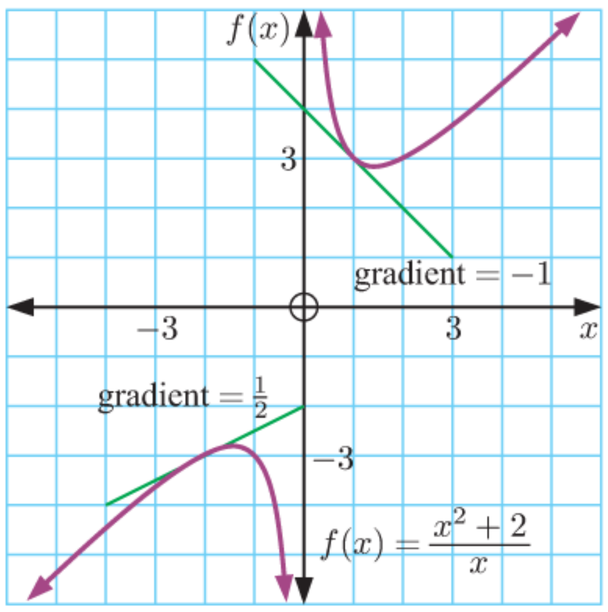
6 **a** $f'(x) = \dfrac{e^x}{e^x + 3}$ **b** $f'(x) = \dfrac{3}{x+2} - \dfrac{1}{x}$

c $f'(x) = \dfrac{1}{\cos x \sin x}$

7 when $x = 1$, $\dfrac{dy}{dx} = 0$

8 **a** $f'(x) = 1 - \dfrac{2}{x^2}$ **b** **i** -1 **ii** $\frac{1}{2}$

c



9 **a** $\dfrac{dy}{dx} = \dfrac{3x^2 - 3}{x^3 - 3x}$ **b** $\dfrac{dy}{dx} = \dfrac{e^x(x-2)}{x^3}$

c $\dfrac{dy}{dx} = 2e^{2x}\sin x + e^{2x}\cos x$

10 **a** $f''(x) = 24x^2 - 24x - 18$ **b** $x = -\frac{1}{2}$ or $\frac{3}{2}$

11 **a** $10 - 10\cos 10x$ **b** $\tan x$

c $(5\cos 5x)\ln(2x) + \dfrac{\sin 5x}{x}$

12 **a** $\frac{28}{9}$ **b** 16 **13** $a = 4,\ b = e^2$

14 **a** $-\frac{1}{4}$ **b** **Hint:** Show that $\dfrac{dy}{dx} = -\dfrac{2\sin x + 1}{(\sin x + 2)^2}$.

15 **b** **i** $x = \frac{1}{2}$ **ii** $x \leqslant 0$

17 **a** $x = -6 \pm \sqrt{33}$ **b** $x = \pm\sqrt{3}$ **c** $x = -3,\ 0,$ or 3

EXERCISE 19A

1 **a** $f'(x) = 2x - 4$ **b** $y = -2x - 1$

2 **a** $y = -7x + 11$ **b** $y = \frac{1}{4}x + 2$ **c** $y = -2x - 2$

d $y = -2x + 6$ **e** $y = -5x - 9$ **f** $y = -5x - 1$

3 **a** $y = 21$ and $y = -6$ **b** $y = 23$ and $y = -9$

c $y = 2$

4 **a** $k = -5$ **b** $y = 4x - 15$ **5** $y = -3x + 1$

6 $a = -4,\ b = 7$ **7** $a = 2,\ b = \frac{1}{2}$

9 **a** $f'(x) = 2x - \dfrac{8}{x^3}$ **b** $x = \pm\sqrt{2}$

c When $x = \sqrt{2}$, $y = 4$ and when $x = -\sqrt{2}$, $y = 4$.
$\therefore$ tangents are $y = 4$.

10 $a = 4,\ b = 3$

11 **a** $y = -e^{-2}x + 3e^{-2}$ **b** $y = -\frac{1}{3}x - \frac{1}{3} + \ln 3$

c $y = 4ex - e$ **d** $y = \dfrac{e^2}{2}x - \frac{3}{2}$ **e** $y = 3ex - 5e$

12 **a** Domain is $\{x \mid x < 0 \text{ or } x > 2\}$

b $f'(x) = \dfrac{1}{x} + \dfrac{1}{x-2}$ **c** $y = \frac{4}{3}x - 4 + \ln 3$

13 x-intercept $\frac{2}{3}$, y-intercept $-2e$ **14** $\dfrac{3}{4\sqrt{e}}$ units^2

15 **a** $y = x$ **b** $y = x$ **c** $y = -\frac{1}{2}x + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$

d $y = 1$ **e** $y = 2$

16 **Hint:** Show that there are no tangents which have gradient $= 0$.

17 $\frac{1}{2}\left(\frac{\pi}{2} + 1\right)^2$ units^2

18 $(-4, -64)$ **19** $(4, -31)$ **20** $(-1, -2)$

21 **a** $y = (2a - 1)x - a^2 + 9$

b $y = 5x$, point of contact $(3, 15)$, and
$y = -7x$, point of contact $(-3, 21)$

22 **a** $y = (2a + 4)x - a^2$

b $y = 12x - 16$, point of contact $(4, 32)$, and
$y = -4$, point of contact $(-2, -4)$

23 $y = 5x - 15$ and $y = -7x - 3$

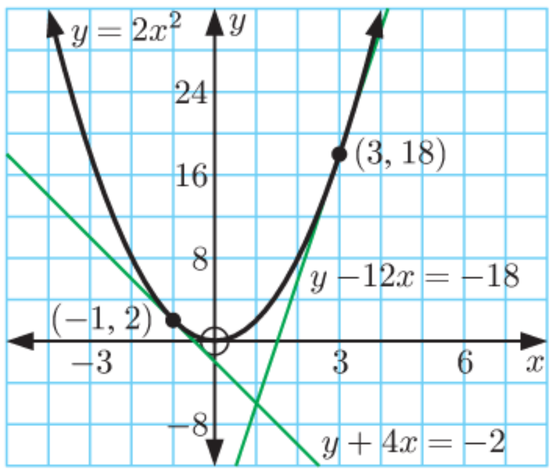
24 **a** $y = e^a x + e^a(1 - a)$ **b** $y = ex$

25 **a** $y + 4x = -2$ and $y - 12x = -18$

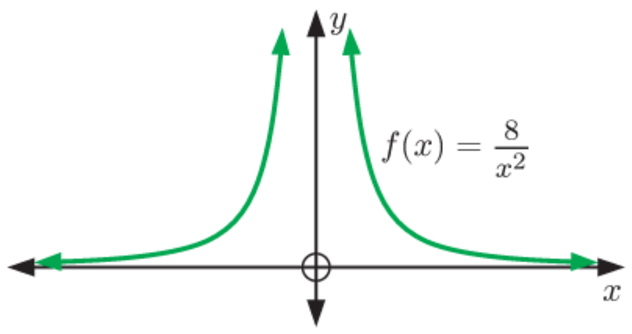
b $(-1, 2)$ for $y + 4x = -2$ and
$(3, 18)$ for $y - 12x = -18$

c For a tangent to pass through $(1, 4)$, $4 = 4a - 2a^2$ must have real solutions. But $\Delta < 0$, so no real solutions.

d



26 **a**



b $16x + a^3y = 24a$ **c** A is $(\frac{3}{2}a, 0)$, B is $\left(0, \dfrac{24}{a^2}\right)$

d area $= \dfrac{18}{|a|}$ units^2; as $a \to \infty$, area $\to 0$

27 $\approx 63.43°$

28 **a** **Hint:** They must have the same y-coordinate at $x = b$ and the same gradient.

c $a = \frac{1}{2e}$ **d** $y = e^{-\frac{1}{2}}x - \frac{1}{2}$

EXERCISE 19B

1 **a** $x + 8y = 132$ **b** $x + 7y = 26$ **c** $x - 3y = -11$

d $x + 6y = 43$ **e** $64x + 4y = -65$ **f** $x = 2$

2 **a** $y = 4 - 2x$ **b** $y = -\frac{9}{62}x + \frac{1259}{186}$

3 **a** $y = x + 1$ **b** $ex + y = e^2 + 1$

c $x + 2ey = 1 + 2e^2$ **d** $2x + y = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

e $y = -\frac{1}{6}x + \frac{\pi}{24} - 1$ **f** $2x - 2\sqrt{3}y = \pi + \sqrt{3}$

4 $a = 2,\ b = 4$ **5** $(-1, -2)$ and $(2, 1)$

6 $x = 0$ **7** $y = -\sqrt{14}x + 4\sqrt{14}$

EXERCISE 19C

1 **a** **i** $x \geqslant 0$ **ii** never **b** **i** never **ii** $-2 < x \leqslant 3$

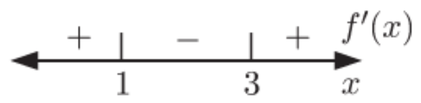
c **i** $x \leqslant 2$ **ii** $x \geqslant 2$ **d** **i** $x \in \mathbb{R}$ **ii** never

e **i** $1 \leqslant x \leqslant 5$ **ii** $x \leqslant 1,\ x \geqslant 5$

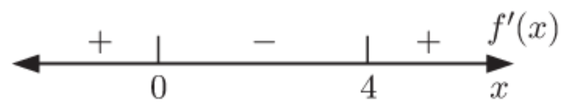
f **i** $2 \leqslant x < 4,\ x > 4$ **ii** $x < 0,\ 0 < x \leqslant 2$

2 **a** **i** $x \leqslant 1,\ x \geqslant 3$ **ii** $1 \leqslant x \leqslant 3$

b $f'(x) = 3x^2 - 12x + 9$
$= 3(x - 3)(x - 1)$



3 **a** $f'(x) = 3x^2 - 12x$
$= 3x(x - 4)$



b increasing for $x \leqslant 0$ and $x \geqslant 4$
decreasing for $0 \leqslant x \leqslant 4$

4 **a** increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

b decreasing for all $x \in \mathbb{R}$

c increasing for $x \geqslant -\frac{3}{4}$, decreasing for $x \leqslant -\frac{3}{4}$

d decreasing for all $x \neq 0$

e increasing for $x > 0$, decreasing for $x < 0$

f decreasing for all $x > 0$

g increasing for $x \leqslant 0$ and $x \geqslant 4$,
decreasing for $0 \leqslant x \leqslant 4$

h increasing for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$,
decreasing for $x \leqslant -\sqrt{\frac{2}{3}},\ x \geqslant \sqrt{\frac{2}{3}}$

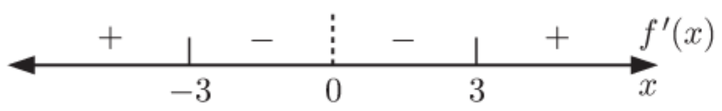
i increasing for $-\frac{1}{2} \leqslant x \leqslant 3$,
decreasing for $x \leqslant -\frac{1}{2},\ x \geqslant 3$

j increasing for all $x \in \mathbb{R}$

k increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

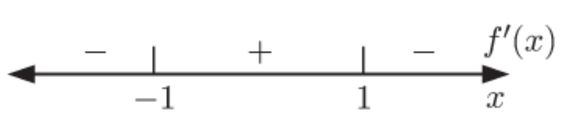
l increasing for $x \leqslant 2 - \sqrt{3},\ x \geqslant 2 + \sqrt{3}$,
decreasing for $2 - \sqrt{3} \leqslant x \leqslant 2 + \sqrt{3}$

5 **a**



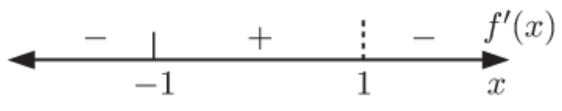
b increasing for $x \leqslant -3$ and $x \geqslant 3$,
decreasing for $-3 \leqslant x < 0$ and $0 < x \leqslant 3$

6 **a**



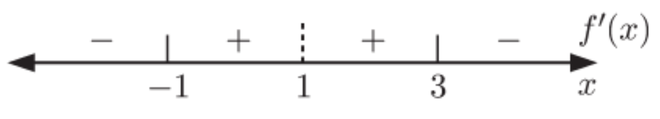
b increasing for $-1 \leqslant x \leqslant 1$,
decreasing for $x \leqslant -1$ and $x \geqslant 1$

7 **a**



b increasing for $-1 \leqslant x < 1$,
decreasing for $x \leqslant -1$ and $x > 1$

8 **a**



b increasing for $-1 \leqslant x < 1$ and $1 < x \leqslant 3$,
decreasing for $x \leqslant -1$ and $x \geqslant 3$

9 In this case, $f(x)$ is only defined when $x > 0$.
$\therefore$ $f'(x)$ is only defined when $x > 0$.

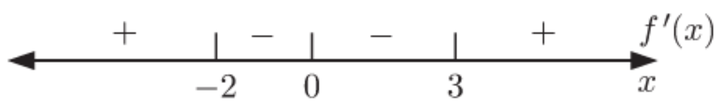
10 **a** increasing for all $x \in \mathbb{R}$, never decreasing
b increasing for $x > -2$, never decreasing
c never increasing, decreasing for all $x \in \mathbb{R}$
d increasing for $x \geqslant -1$, decreasing for $x \leqslant -1$
e increasing for $x \geqslant 1$, decreasing for $0 < x \leqslant 1$
f increasing for $x \geqslant e^{-\frac{1}{3}}$, decreasing for $0 < x \leqslant e^{-\frac{1}{3}}$
g increasing for $x \geqslant \sqrt{3}$ and $x \leqslant -\sqrt{3}$,
decreasing for $-\sqrt{3} \leqslant x < -1$ and $-1 < x < 1$ and $1 < x \leqslant \sqrt{3}$
h increasing for $x \leqslant 0$, decreasing for $x \geqslant 0$
i increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$
j increasing for $x \geqslant 2$,
decreasing for $x < 1$ and $1 < x \leqslant 2$
k increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$
l increasing for $x \leqslant -1$,
decreasing for $-1 \leqslant x < 0$ and $x > 0$

11 **a** increasing for $(2k+1)\pi \leqslant x \leqslant (2k+2)\pi$, $k \in \mathbb{Z}$,
decreasing for $2k\pi \leqslant x \leqslant (2k+1)\pi$, $k \in \mathbb{Z}$
b increasing for $\dfrac{(4k-1)\pi}{2} \leqslant x \leqslant \dfrac{(4k+1)\pi}{2}$, $k \in \mathbb{Z}$,
decreasing for $\dfrac{(4k+1)\pi}{2} \leqslant x \leqslant \dfrac{(4k+3)\pi}{2}$, $k \in \mathbb{Z}$
c increasing for $x \neq \dfrac{(2k+1)\pi}{2}$, $k \in \mathbb{Z}$, never decreasing
d increasing for $\dfrac{(6k+1)\pi}{3} \leqslant x \leqslant \dfrac{(6k+5)\pi}{3}$, $k \in \mathbb{Z}$,
decreasing for $\dfrac{(6k-1)\pi}{3} \leqslant x \leqslant \dfrac{(6k+1)\pi}{3}$, $k \in \mathbb{Z}$

EXERCISE 19D

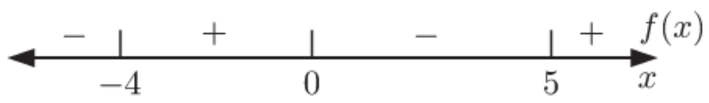
1 **a** A - local max, O - stationary inflection, B - local min.

b



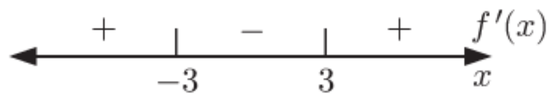
c **i** $x \leqslant -2$ and $x \geqslant 3$ **ii** $-2 \leqslant x \leqslant 3$

d

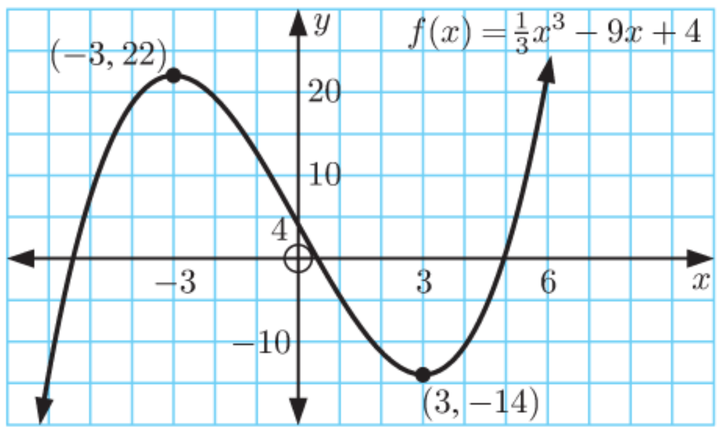


2 **a** P is a local maximum, Q is a local minimum.
b $f'(x) = 3x^2 + 12x - 15 = 3(x+5)(x-1)$
c P$(-5, 60)$, Q$(1, -48)$

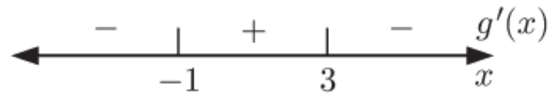
3 **a** $f'(x) = x^2 - 9$
$= (x+3)(x-3)$



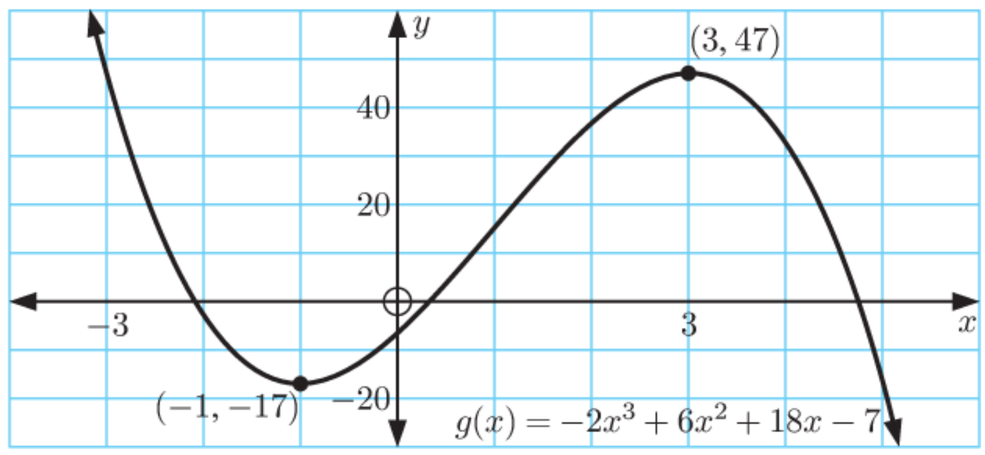
b increasing for $x \leqslant -3$ and $x \geqslant 3$,
decreasing for $-3 \leqslant x \leqslant 3$
c $(-3, 22)$ is a local maximum, $(3, -14)$ is a local minimum
d As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$.
e



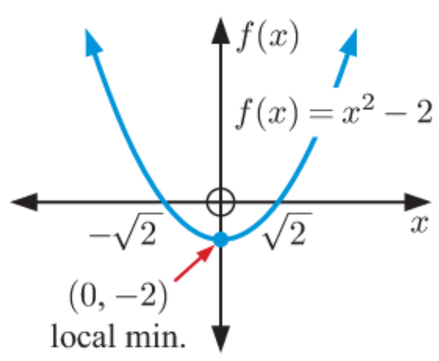
4 **a** $g'(x) = -6x^2 + 12x + 18 = -6(x-3)(x+1)$



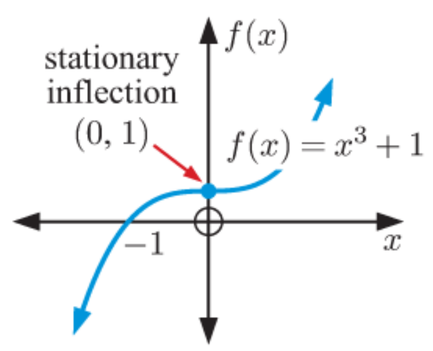
b increasing for $-1 \leqslant x \leqslant 3$,
decreasing for $x \leqslant -1$ and $x \geqslant 3$
c $(3, 47)$ is a local maximum, $(-1, -17)$ is a local minimum
d As $x \to \infty$, $g(x) \to -\infty$, as $x \to -\infty$, $g(x) \to \infty$.
e



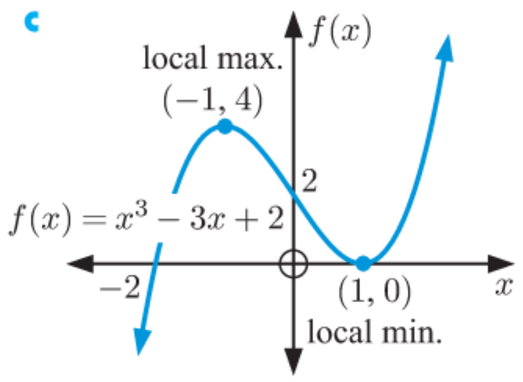
5 **a**



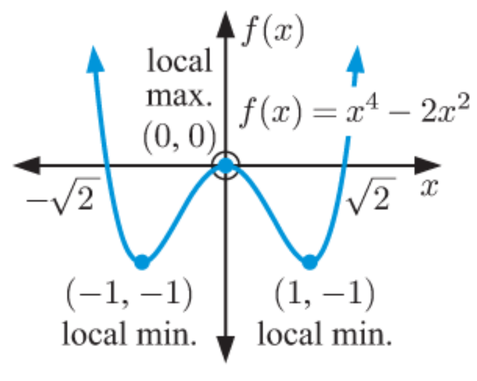
b



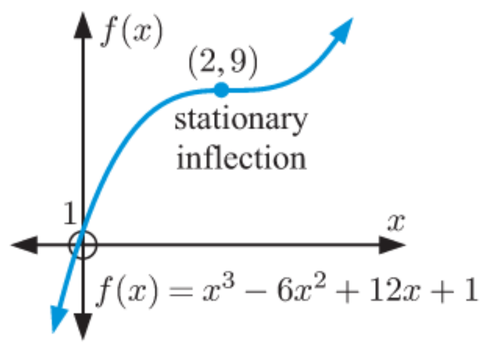
c



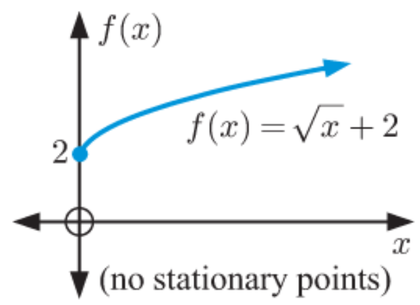
d



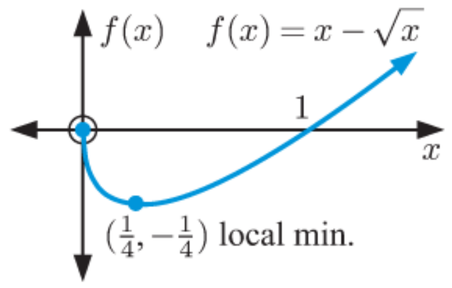
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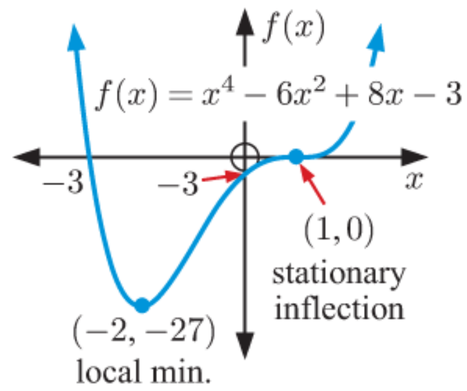
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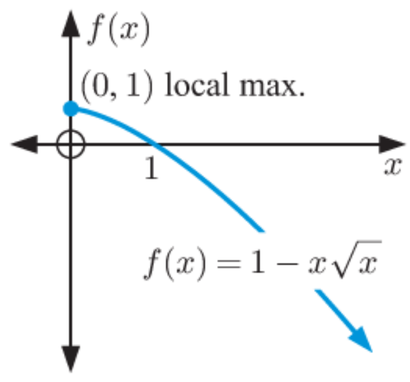
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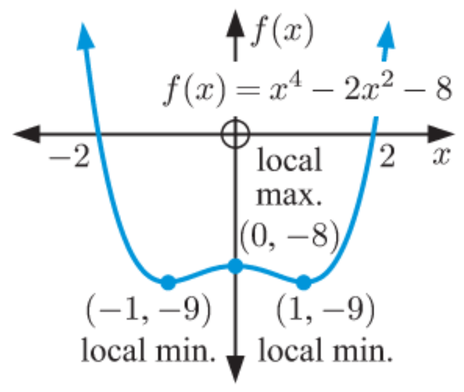
h



i



j



6 **a** $x = -\dfrac{b}{2a}$ **b** local min. if $a > 0$, local max. if $a < 0$

7 **a** $\left(1, \frac{1}{e}\right)$ is a local maximum

b $\left(-2, \frac{4}{e^2}\right)$ is a local maximum, $(0, 0)$ is a local minimum

c $(1, e)$ is a local minimum **d** $(-1, e)$ is a local maximum

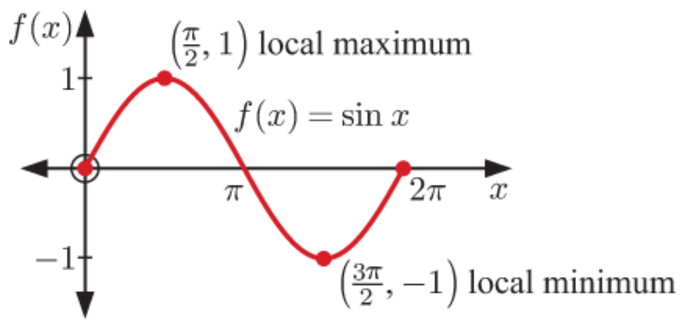
8 **a** $a = 9$ **b** $(-4, 113)$

9 **a** $a = -12,\ b = -13$

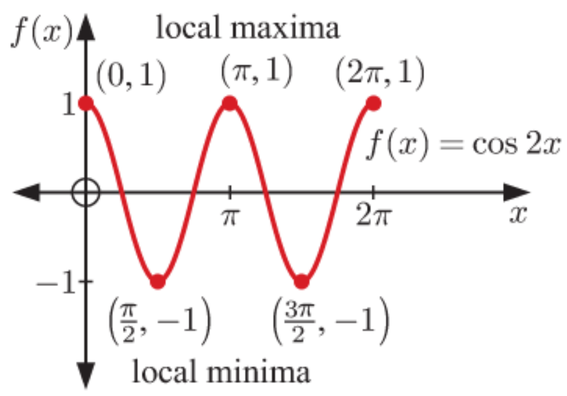
b $(-2, 3)$ is a local maximum, $(2, -29)$ is a local minimum

10 **a** $a = 3,\ b = 6$ **b** local minimum **11** **a** $x > 0$

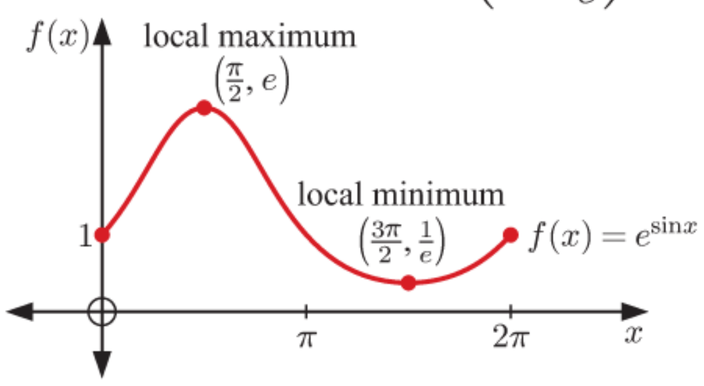
12 **a** $\left(\frac{\pi}{2}, 1\right)$ is a local maximum, $\left(\frac{3\pi}{2}, -1\right)$ is a local minimum



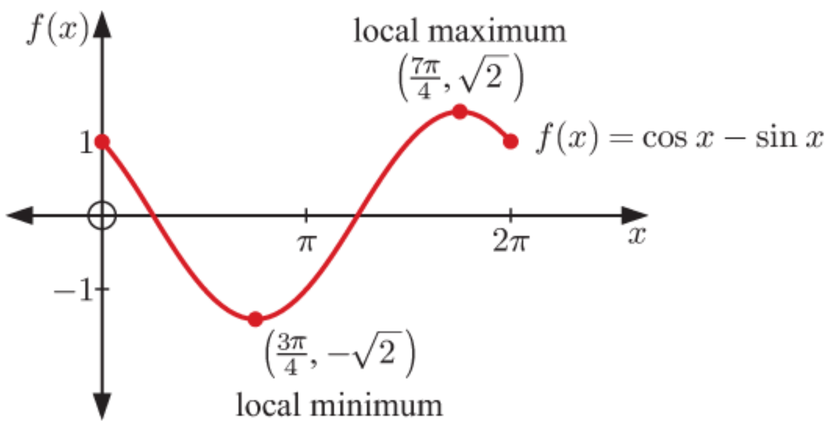
b $(0, 1)$, $(\pi, 1)$, and $(2\pi, 1)$ are local maxima, $\left(\frac{\pi}{2}, -1\right)$ and $\left(\frac{3\pi}{2}, -1\right)$ are local minima



c $\left(\frac{\pi}{2}, e\right)$ is a local maximum, $\left(\frac{3\pi}{2}, \frac{1}{e}\right)$ is a local minimum



d $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$ is a local minimum, $\left(\frac{7\pi}{4}, \sqrt{2}\right)$ is a local maximum



13 $P(x) = -9x^3 - 9x^2 + 9x + 2$

14 **a** greatest value is 63 when $x = 5$, least value is -18 when $x = 2$

b greatest value is 4 when $x = 3$ and $x = 0$, least value is -16 when $x = -2$

c greatest value is 20 when $x = 4$, least value is 12 when $x = 2$

d greatest value is 0 when $x = 0$, least value is -4 when $x = 4$

16 $a = \frac{\sqrt{e}}{2},\quad b = -\frac{1}{8}$

17 **Hint:** Show that $\frac{\ln x}{x}$ has only one stationary point, which is a local maximum.

18 **Hint:** Show that $f(x) \geqslant 1$ for all $x > 0$.

EXERCISE 19E

1 **a**

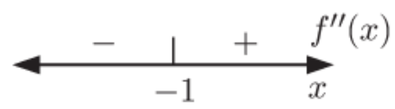
Point	$f(x)$	$f'(x)$	$f''(x)$
A	+	−	+
B	−	0	+
C	+	+	0
D	+	0	−
E	0	−	−

b B is a local minimum, D is a local maximum

c C

2 **a** $f'(x) = 3x^2 + 6x - 5$
$f''(x) = 6x + 6$

b



c **i** $x \geqslant -1$ **ii** $x \leqslant -1$

3 **a** concave up **b** concave down **c** concave down

d concave up

4 **a** **i** $x \geqslant 0$ **ii** $x \leqslant 0$ **iii** $x \in \mathbb{R}$ **iv** never

b **i** never **ii** $x \in \mathbb{R}$ **iii** $x \leqslant 0$ **iv** $x \geqslant 0$

c **i** $x \in \mathbb{R}$ **ii** never **iii** $x \in \mathbb{R}$ **iv** never

d **i** $x > 0$ **ii** never **iii** never **iv** $x > 0$

e **i** $x > 0$ **ii** never **iii** never **iv** $x > 0$

f **i** $-\sqrt{6} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{6}$

ii $x \leqslant -\sqrt{6}$ and $0 \leqslant x \leqslant \sqrt{6}$

iii $x \leqslant -\sqrt{2}$ and $x \geqslant \sqrt{2}$ **iv** $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

5 **a** x-intercept $\ln\sqrt{3}$, y-intercept -2

b $f'(x) = 2e^{2x} > 0$ for all $x \in \mathbb{R}$

c $f''(x) = 4e^{2x} > 0$ for all $x \in \mathbb{R}$

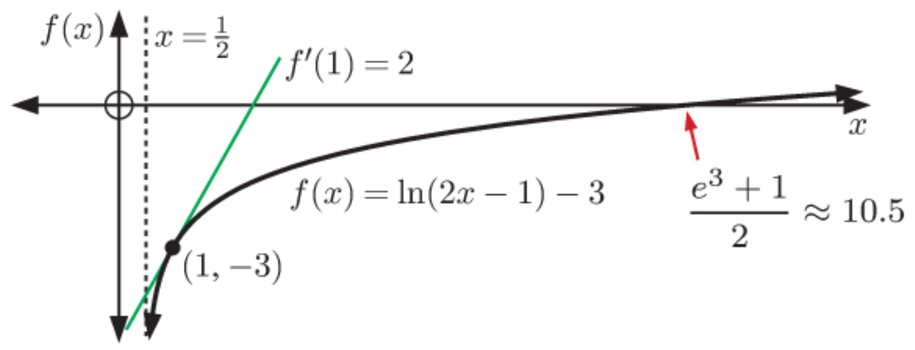
6 **a** x-intercept is $\frac{e^3 + 1}{2} \approx 10.5$

b no, $\therefore$ there is no y-intercept

c Domain is $\{x \mid x > \frac{1}{2}\}$ **d** gradient $= 2$

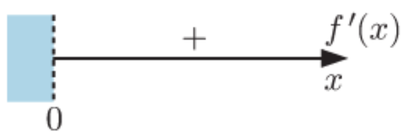
e $f''(x) = \frac{-4}{(2x-1)^2} < 0$ for all $x > \frac{1}{2}$, so $f(x)$ is concave down.

f



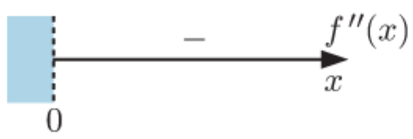
7 **a** $x > 0$

b $f'(x) = \frac{1}{x}$



$f(x)$ is increasing for $x > 0$.

$f''(x) = -\frac{1}{x^2}$



$f(x)$ is concave down for $x > 0$.

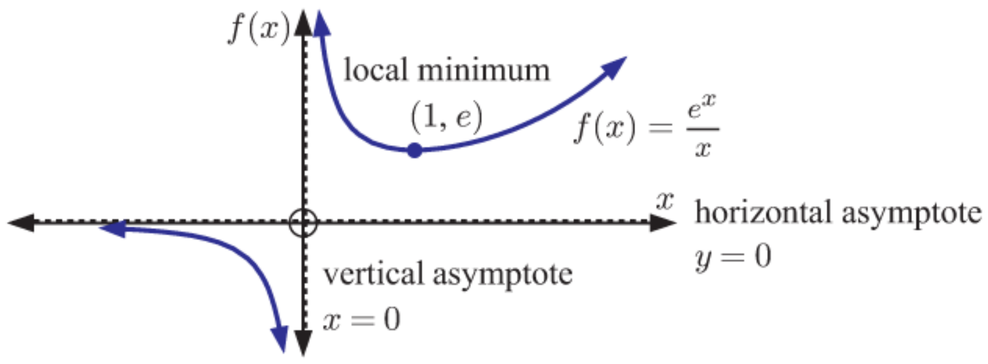
c $ex + y = e^2 + 1$

8 **a** $f(x)$ does not have any x or y-intercepts.

b as $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to 0^-$

c local minimum at $(1, e)$ **d** **i** $x > 0$ **ii** $x < 0$

e



f $2x + ey = -3$

EXERCISE 19F

1 **a**

Point	$f(x)$	$f'(x)$	$f''(x)$
A	0	−	+
B	−	0	+
C	0	+	0
D	+	0	0

b B is a local minimum.

c C is a non-stationary inflection point,
D is a stationary inflection point.

2 **a** no points of inflection **b** stationary inflection at $(0, 2)$

c non-stationary inflection at $(2, 3)$

d stationary inflection at $(0, 2)$
non-stationary inflection at $(-1\frac{1}{3}, 11\frac{13}{27})$

e no points of inflection

f stationary inflection at $(-2, -3)$

g non-stationary inflection at $(1, 9)$

h non-stationary inflections at $(-1, 5)$ and $(1, 5)$

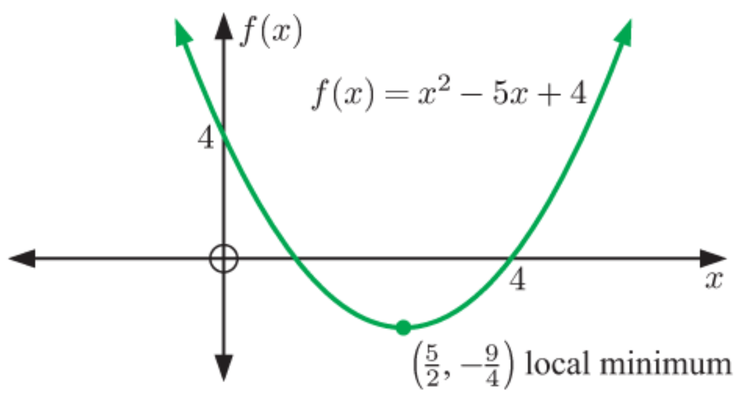
3 **a** **i** local minimum at $(\frac{5}{2}, -\frac{9}{4})$

ii no points of inflection

iii increasing for $x \geqslant \frac{5}{2}$, decreasing for $x \leqslant \frac{5}{2}$

iv concave up for all $x \in \mathbb{R}$

v



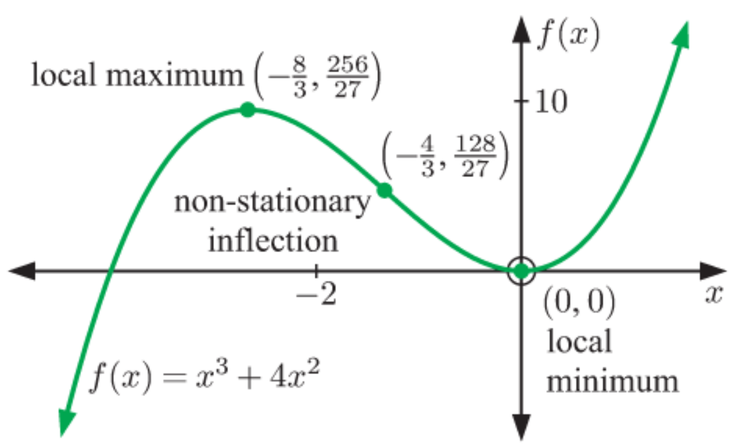
b **i** local maximum at $(-\frac{8}{3}, \frac{256}{27})$,
local minimum at $(0, 0)$

ii non-stationary inflection at $(-\frac{4}{3}, \frac{128}{27})$

iii increasing for $x \leqslant -\frac{8}{3}$ and $x \geqslant 0$,
decreasing for $-\frac{8}{3} \leqslant x \leqslant 0$

iv concave up for $x \geqslant -\frac{4}{3}$, concave down for $x \leqslant -\frac{4}{3}$

v



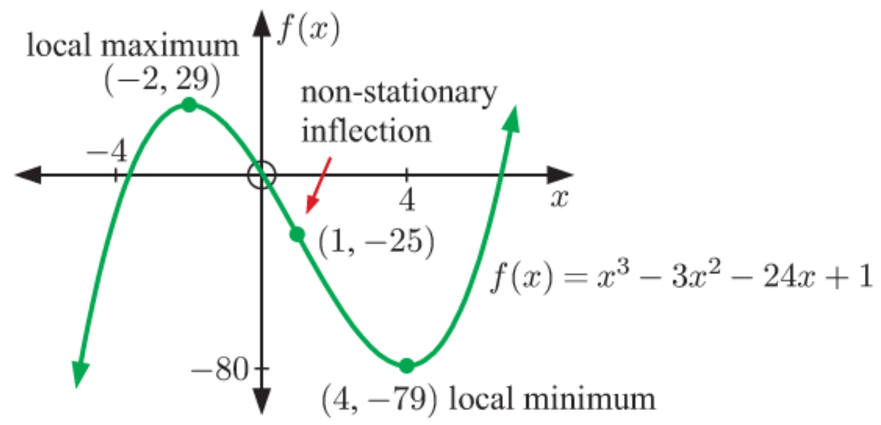
c **i** local maximum at $(-2, 29)$
local minimum at $(4, -79)$

ii non-stationary inflection at $(1, -25)$

iii increasing for $x \leqslant -2$ and $x \geqslant 4$
decreasing for $-2 \leqslant x \leqslant 4$

iv concave down for $x \leqslant 1$, concave up for $x \geqslant 1$

v

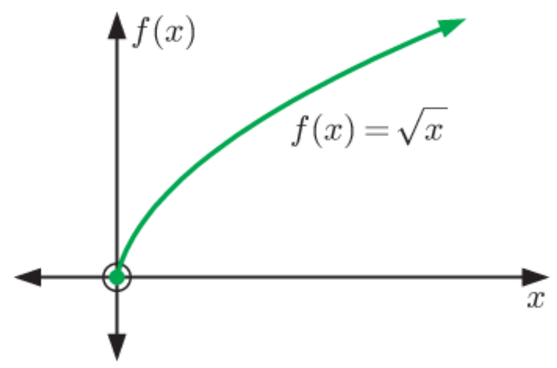


d **i** no turning points **ii** no points of inflection

iii increasing for $x > 0$, never decreasing

iv concave down for $x > 0$, never concave up

v

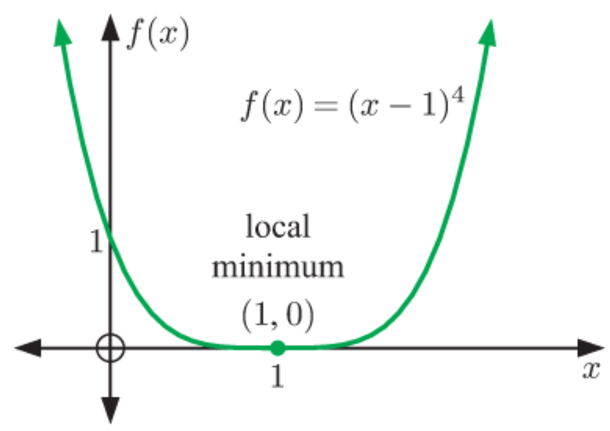


e **i** local minimum at $(1, 0)$ **ii** no points of inflection

iii increasing for $x \geqslant 1$, decreasing for $x \leqslant 1$

iv concave up for all $x \in \mathbb{R}$

v



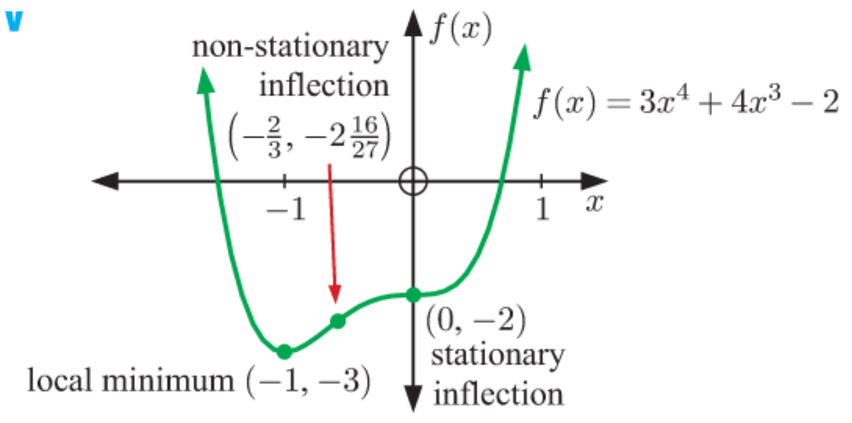
f **i** local minimum at $(-1, -3)$

ii non-stationary inflection at $\left(-\frac{2}{3}, -2\frac{16}{27}\right)$
stationary inflection at $(0, -2)$

iii increasing for $x \geqslant -1$, decreasing for $x \leqslant -1$

iv concave down for $-\frac{2}{3} \leqslant x \leqslant 0$
concave up for $x \leqslant -\frac{2}{3}$ and $x \geqslant 0$

v



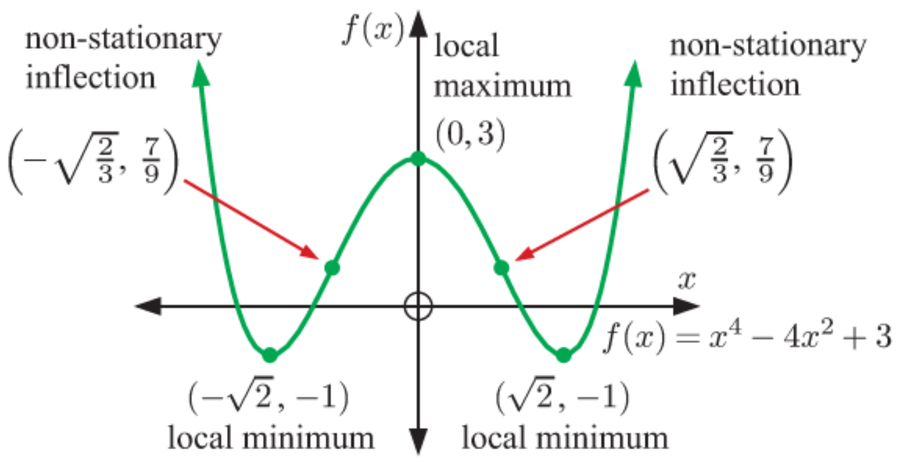
g **i** local minima at $(-\sqrt{2}, -1)$ and $(\sqrt{2}, -1)$,
local maximum at $(0, 3)$

ii non-stationary inflections at $(\sqrt{\frac{2}{3}}, \frac{7}{9})$ and
$(-\sqrt{\frac{2}{3}}, \frac{7}{9})$

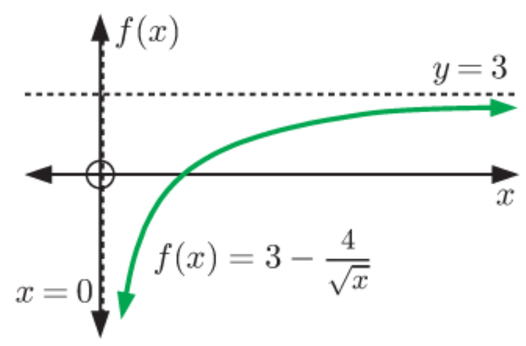
iii increasing for $-\sqrt{2} \leqslant x \leqslant 0$ and $x \geqslant \sqrt{2}$
decreasing for $x \leqslant -\sqrt{2}$ and $0 \leqslant x \leqslant \sqrt{2}$

iv concave down for $-\sqrt{\frac{2}{3}} \leqslant x \leqslant \sqrt{\frac{2}{3}}$
concave up for $x \leqslant -\sqrt{\frac{2}{3}}$ and $x \geqslant \sqrt{\frac{2}{3}}$

v



h **i** no turning points **ii** no points of inflection
iii increasing for $x > 0$, never decreasing
iv concave down for $x > 0$, never concave up
v



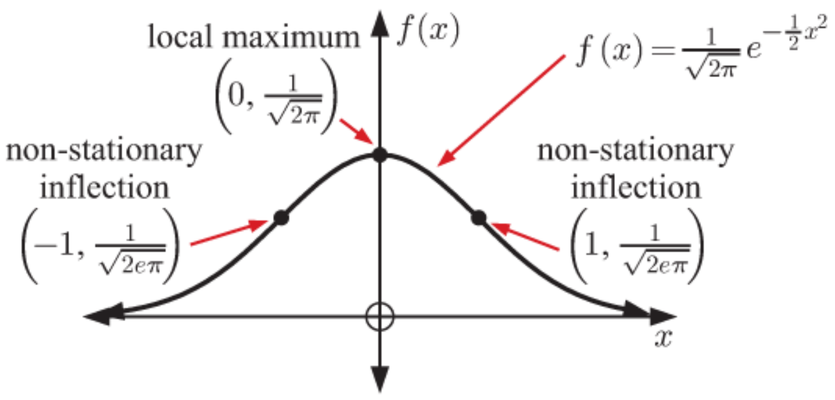
4 **a** local maximum at $\left(0, \frac{1}{\sqrt{2\pi}}\right)$

b increasing for $x \leqslant 0$, decreasing for $x \geqslant 0$

c non-stationary inflections at $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$

d as $x \to \infty$, $f(x) \to 0^+$, as $x \to -\infty$, $f(x) \to 0^+$

e

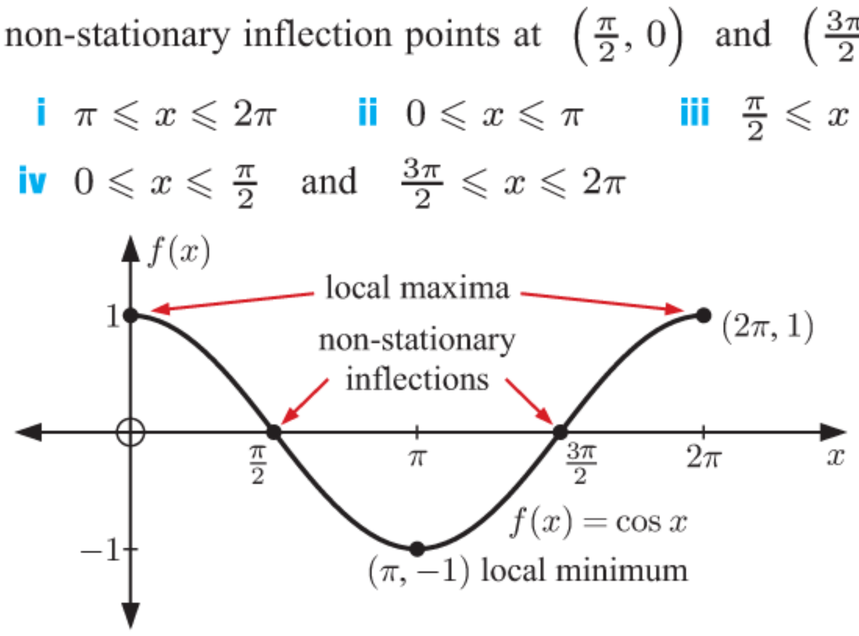


5 **a** The inflection points coincide with the x-intercepts.

b non-stationary inflection points at $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$

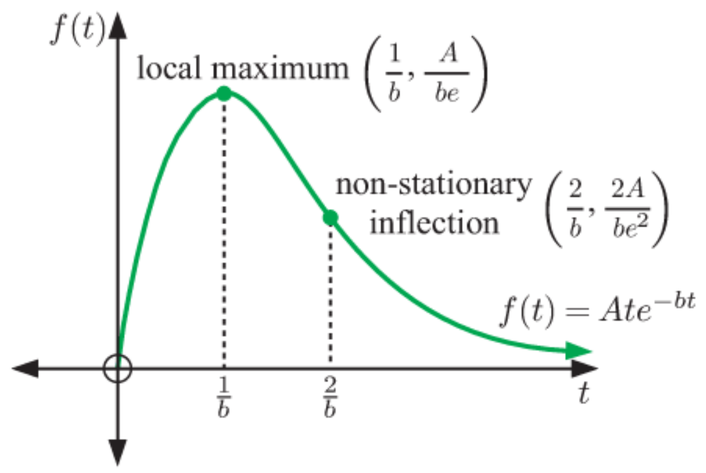
c **i** $\pi \leqslant x \leqslant 2\pi$ **ii** $0 \leqslant x \leqslant \pi$ **iii** $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$
iv $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

d



6 **a** **i** **Hint:** Show that $f'(t) = Ae^{-bt}(1 - bt)$.
ii **Hint:** Show that $f''(t) = Abe^{-bt}(bt - 2)$.

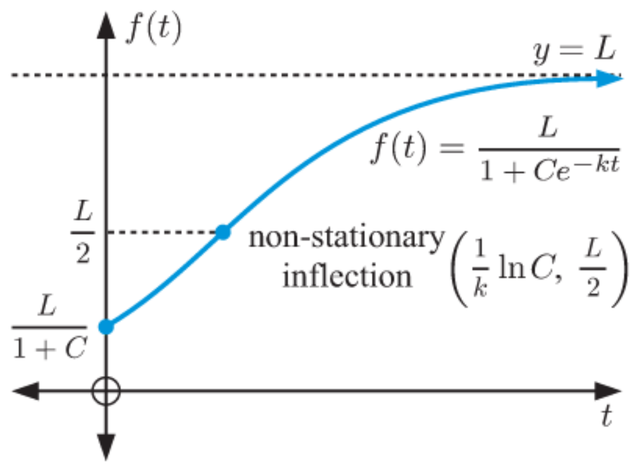
b



7 **a** y-intercept is $\frac{L}{1 + C}$

b As $t \to \infty$, $e^{-kt} \to 0$, $\frac{L}{1 + Ce^{-kt}} \to L$

d



REVIEW SET 19A

1 **a** $y = 4x + 2$ **b** $y = 7x - 14$ **c** $y = -1$
d $3x - ey = -1$ **e** $y = \frac{2}{e}x$

2 **a** $8x + 3y = 44$ **b** $y = -\frac{1}{6e^2}x + \frac{18e^4 + 1}{6e^2}$

3 $p = 1$, $q = -8$

4 $(-2, 19)$ and $(1, -2)$

5 $(-2, -25)$

6 **a** $y = -\frac{1}{2e^{2a}}x + \frac{a}{2e^{2a}} + e^{2a}$ **b** $y \approx -1.12x$

7 P(0, 7.5), Q(3, 0)

8 $\frac{3267}{152} \approx 21.5$ units^2

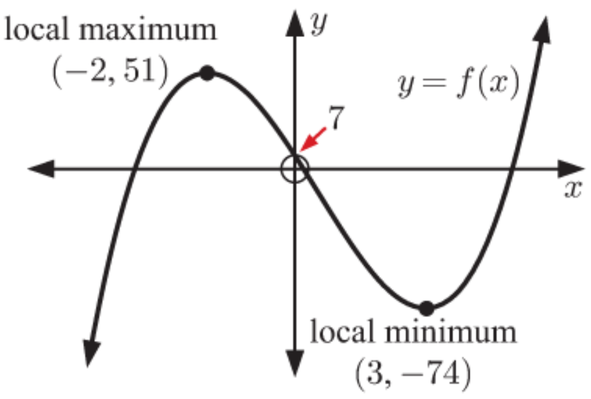
9 **a** $-6 \leqslant x \leqslant 2$ **b** $x \leqslant -6$ and $x \geqslant 2$

10 **a** local maximum at $(-2, 51)$, local minimum at $(3, -74)$

b increasing for $x \leqslant -2$, $x \geqslant 3$
decreasing for $-2 \leqslant x \leqslant 3$

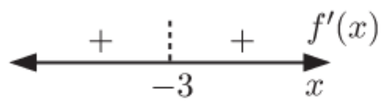
c as $x \to \infty$, $f(x) \to \infty$,
as $x \to -\infty$, $f(x) \to -\infty$

d



11 **a** $\{x \mid x \neq -3\}$ **b** x-intercept $\frac{2}{3}$, y-intercept $-\frac{2}{3}$

c $f'(x) = \frac{11}{(x + 3)^2}$



d no stationary points

12 greatest value is ≈ 10.3 when $x = 10$,
least value is 6 when $x = 4$

13 **b** **i** $x > 0$ **ii** $x < \frac{1}{2}$

c local maximum at $(\frac{1}{2}, \frac{1}{2})$

14 **a** increasing for $x \leqslant -\sqrt{2}$ and $x \geqslant \sqrt{2}$,
decreasing for $-\sqrt{2} \leqslant x \leqslant \sqrt{2}$

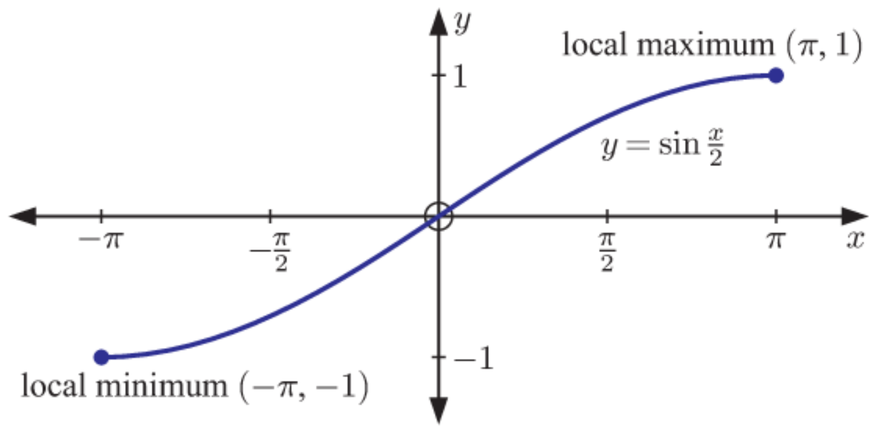
b increasing for $x \geqslant 1$, decreasing for $x \leqslant 1$

c increasing for all $x \in \mathbb{R}$

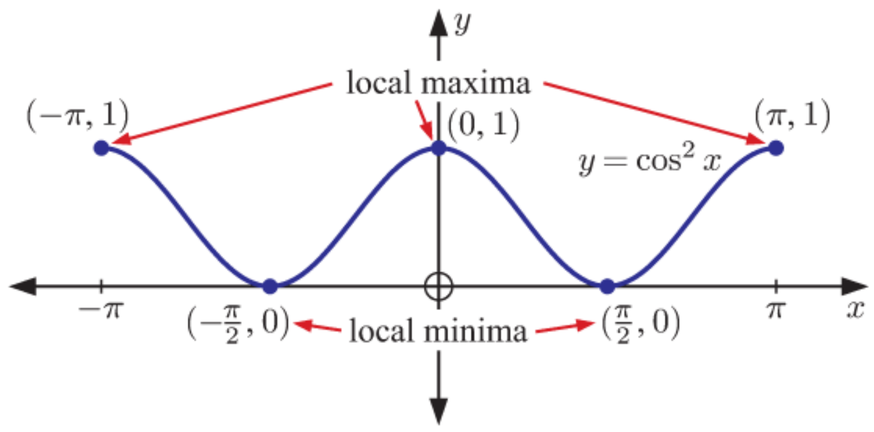
15 **a** local maximum at $(1, 3)$, local minimum at $(\frac{1}{3}, \frac{77}{27})$

b local maximum at $(-6, -12)$, local minimum at $(0, 0)$

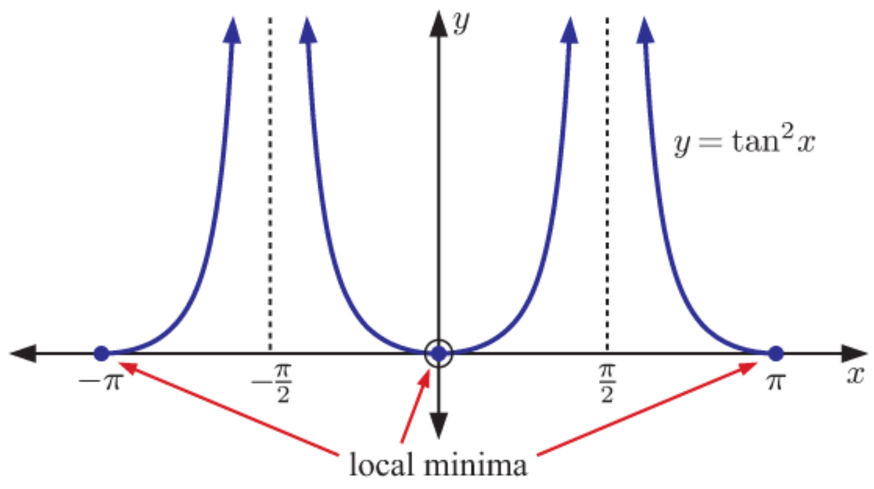
16 **a** $(-\pi, -1)$ is a local minimum, $(\pi, 1)$ is a local maximum



b $(-\pi, 1)$, $(0, 1)$, and $(\pi, 1)$ are local maxima, $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, 0)$ are local minima

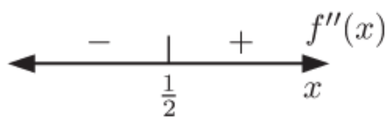


c $(-\pi, 0)$, $(0, 0)$, and $(\pi, 0)$ are local minima



17 **a** $f'(x) = 6x^2 - 6x + 1$,
$f''(x) = 12x - 6$

b



c $x \leqslant \frac{1}{2}$ **d** $(\frac{1}{2}, -12)$

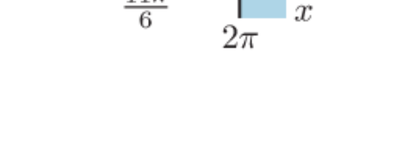
18 **a** concave up for $x \geqslant \frac{4}{3}$, concave down for $x \leqslant \frac{4}{3}$

b concave up for $x \leqslant -3$,
concave down for $-3 \leqslant x < 0$ and $x > 0$

19 **a** $x > 0$

b $f'(x) = 1 + \dfrac{1}{x}$ $\qquad f''(x) = -\dfrac{1}{x^2}$





$f(x)$ is increasing and concave downwards for all $x > 0$.

c

20 **a** $f'(x) = e^{x\sqrt{3}}(\cos x + \sqrt{3}\sin x)$ **b** $x = \frac{5\pi}{6}$ or $\frac{11\pi}{6}$

c

d **i** $0 \leqslant x \leqslant \frac{5\pi}{6}$ and $\frac{11\pi}{6} \leqslant x \leqslant 2\pi$

ii $\frac{5\pi}{6} \leqslant x \leqslant \frac{11\pi}{6}$

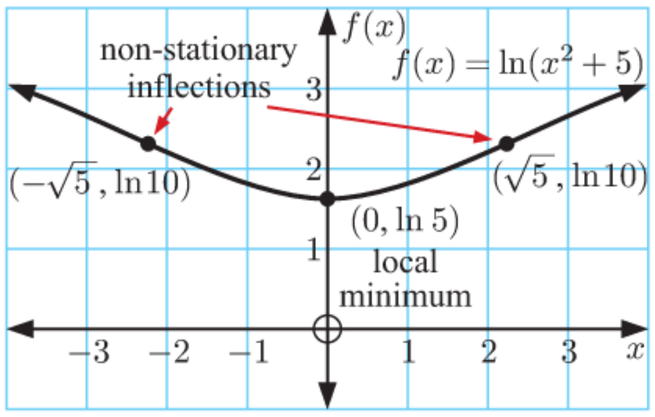
21 **a** $(0, \ln 5)$ is a local minimum

b $(-\sqrt{5}, \ln 10)$ and $(\sqrt{5}, \ln 10)$ are non-stationary inflections

c increasing for $x \geqslant 0$, decreasing for $x \leqslant 0$

d concave up for $-\sqrt{5} \leqslant x \leqslant \sqrt{5}$
concave down for $x \leqslant -\sqrt{5}$ and $x \geqslant \sqrt{5}$

e



REVIEW SET 19B

1 **a** $y = 31x - 43$ **b** $x + 128y = 41$

c $y = 3x + \frac{3\sqrt{3}}{2} - \frac{\pi}{2}$ **d** $y = \frac{3}{4}x + \frac{1}{2}$

2 **a** $x = 1$ **b** $x = 0$ **4** $a = -14,\ b = 21$

6 **a** $f(3) = 2,\ f'(3) = -1$ **b** $f(x) = x^2 - 7x + 14$

7 **a** $a = 2$ **b** $y = 3x - 1$ **c** $(-4, -13)$ **8** $(\frac{1}{2}, \frac{1}{4})$

9 **a** $2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$ **b** $x + 2\sqrt{2}y = \frac{\pi}{2} + 2$

10 $a = \frac{5}{2},\ b = -\frac{3}{2}$

11 **a** $-1 \leqslant x \leqslant 0$ and $x \geqslant 4$ **b** $x \leqslant -1$ and $0 \leqslant x \leqslant 4$

12 **a** $a = -9$

b local maximum at $(-1, 55)$, local minimum at $(3, 23)$

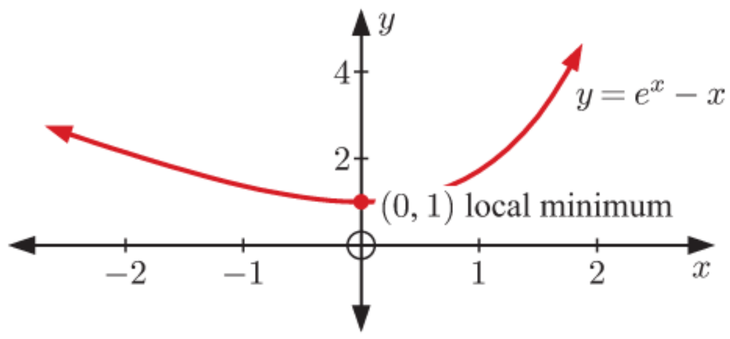
13 **a** local minimum at $(0, 1)$ **b** as $x \to \infty,\ f(x) \to \infty$

c $f''(x) = e^x$



$f(x)$ is concave up for all $x \in \mathbb{R}$.

d



14 **a**



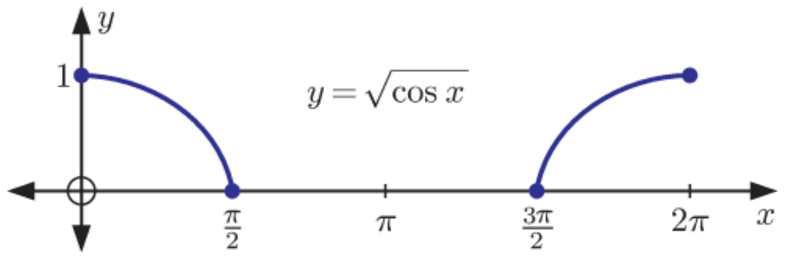
15 $(1 - a,\ e^{a-1})$ is a local maximum **16** $a = 2,\ b = 3$

17 stationary inflection at $(0, 9)$,
non-stationary inflection at $(\frac{3}{2}, \frac{63}{16})$

18 **a** $0 \leqslant x \leqslant \frac{\pi}{2}$ and $\frac{3\pi}{2} \leqslant x \leqslant 2\pi$

b $f'(x) = -\dfrac{\sin x}{2\sqrt{\cos x}}$, increasing for $\frac{3\pi}{2} < x \leqslant 2\pi$,
decreasing for $0 \leqslant x < \frac{\pi}{2}$

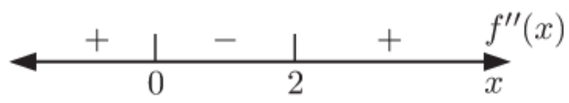
c



19 **a** $f'(x) = 4x^3 - 12x^2$



$f''(x) = 12x^2 - 24x$



b $(3, -20)$ is a local minimum

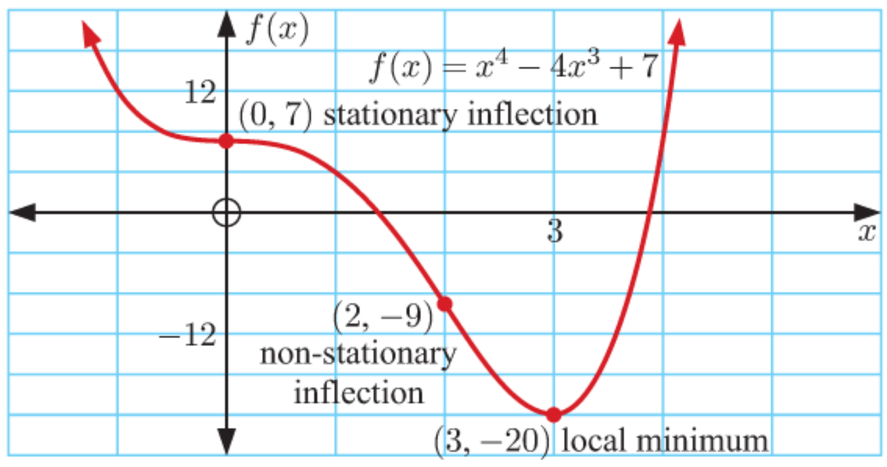
c $(0, 7)$ is a stationary inflection

$(2, -9)$ is a non-stationary inflection

d **i** $x \geqslant 3$ **ii** $x \leqslant 3$ **iii** $x \leqslant 0$ and $x \geqslant 2$

iv $0 \leqslant x \leqslant 2$

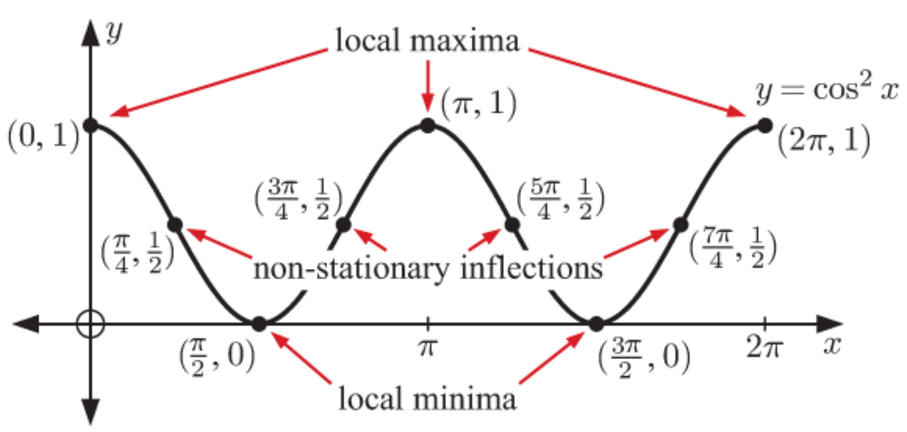
e



20 **a** local maxima at $(0, 1)$, $(\pi, 1)$, and $(2\pi, 1)$,
local minima at $(\frac{\pi}{2}, 0)$ and $(\frac{3\pi}{2}, 0)$

b non-stationary inflections at $(\frac{\pi}{4}, \frac{1}{2})$, $(\frac{3\pi}{4}, \frac{1}{2})$, $(\frac{5\pi}{4}, \frac{1}{2})$, and $(\frac{7\pi}{4}, \frac{1}{2})$

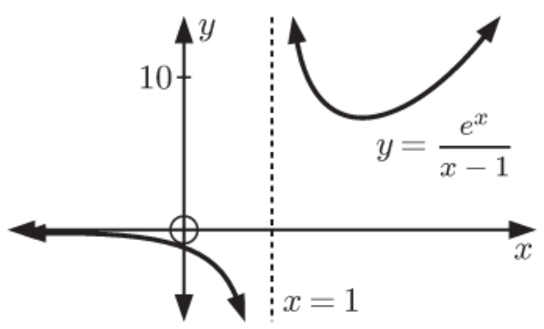
c



21 **a** -1 **b** $x \neq 1$

c $f'(x) \leqslant 0$ for $x < 1$ and $1 < x \leqslant 2$
and $f'(x) \geqslant 0$ for $x \geqslant 2$
$f''(x) > 0$ for $x > 1$, $f''(x) < 0$ for $x < 1$
The function is decreasing for all defined values of $x \leqslant 2$, and increasing for all $x \geqslant 2$. The curve is concave down for $x < 1$ and concave up for $x > 1$.

d



e tangent is $y = e^2$

EXERCISE 20A

1 **a** \$118 000 **b** $\dfrac{dP}{dt} = 4t - 12$ thousand dollars per year

c 20, which means that in 8 years from now, profits will be increasing at a rate of \$20 000 per year.

2 **a** 190 m^3 per day **b** 180 m^3 per day

3 **a** **i** $Q(0) = 100$ **ii** $Q(25) = 50$ **iii** $Q(100) = 0$

b **i** decreasing by 1 unit per year

ii decreasing by $\frac{1}{\sqrt{2}}$ units per year

c $Q'(t) = -\dfrac{5}{\sqrt{t}} < 0$ for all $t > 0$

4 **a** **i** $\dfrac{dh}{dt} = 0.2t + 0.15$ metres per year

ii 0.35 m per year **iii** after 3 years **iv** 2.2 m

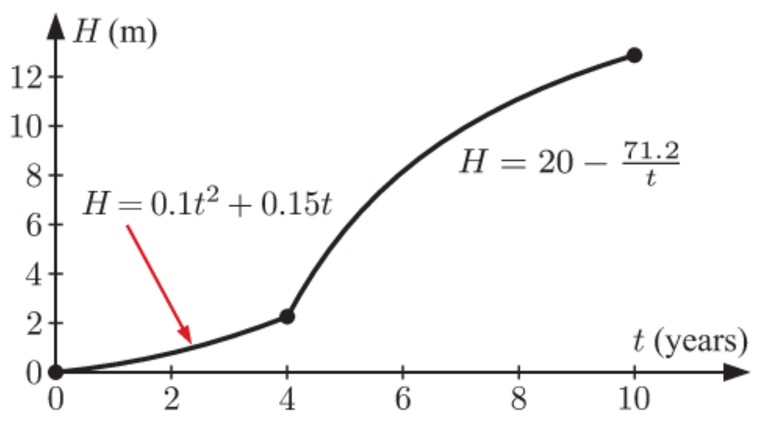
b **i** $k = 71.2$ **ii** $\dfrac{dH}{dt} = \dfrac{71.2}{t^2}$

iii Yes, since $\dfrac{dh}{dt} \neq \dfrac{dH}{dt}$.

iv It means that according to the model the tree will never stop growing.

v After 10 years, height = 12.88 m
After 20 years, height = 16.44 m
After 50 years, height = 18.576 m
The rate of growth is slowing as the years progress.

c



5 **a** $C'(x) = 6 + 8.4x^{-0.3}$ dollars per pair

b $C'(220) \approx \$7.67$, this estimates the cost of making the 221st pair of jeans if 220 pairs are currently being made.

c $C(221) - C(220) \approx \$7.66$, this is the actual cost of making the 221st pair of jeans. The answer in **b** is a very good estimate.

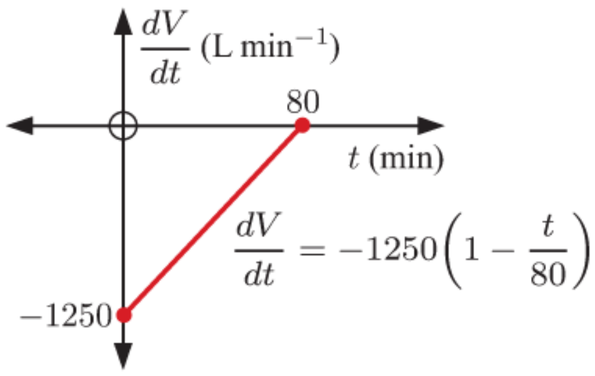
6 **a** **i** 4500 euros **ii** 4000 euros

b **i** decreasing at ≈ 210.22 euros per km h^{-1}

ii increasing at ≈ 11.31 euros per km h^{-1}

c ≈ 79.4 km h^{-1}

7 **a** $\dfrac{dV}{dt} = -1250\left(1 - \dfrac{t}{80}\right)$ L min^{-1}



b $t = 0$ when the tap was first opened

c $\dfrac{d^2V}{dt^2} = \dfrac{125}{8} > 0$

This shows that the rate of change of V is increasing at a constant rate.

8 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

b $\dfrac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$

When $x = \frac{1}{2}$, $\dfrac{dy}{dx} = 0.175$, the height of the hill is increasing as the gradient is positive.

When $x = 1\frac{1}{2}$, $\dfrac{dy}{dx} = -0.225$, the height of the hill is decreasing as the gradient is negative.

c ≈ 2.55 km from the sea, ≈ 63.1 m deep

9 **a** **i** 20 grams **ii** ≈ 15.7 grams **iii** ≈ 3.73 grams

b ≈ 300 hours or ≈ 12.5 days **c** $b = -0.01$

d **i** ≈ -0.0736 g h^{-1} **ii** -9.08×10^{-6} g h^{-1}

10 **a** $P = 8000\sqrt{t+1} - 8000$ kg m s^{-1}

b $\dfrac{dP}{dt} = \dfrac{4000}{\sqrt{t+1}}$; when $t = 3$, $\dfrac{dP}{dt} = 2000$

After 3 seconds, the momentum of the car is increasing at 2000 kg m s^{-1} per second.

c after 24 seconds

11 **a** $\approx 20.7°$C **b** $c = -0.12$

c **i** decreasing at 11.4°C min^{-1}

ii decreasing at ≈ 3.43°C min^{-1}

iii decreasing at ≈ 1.03°C min^{-1}

12 **a** ≈ 43.9 cm **b** ≈ 10.4 years

c **i** growing at ≈ 5.45 cm per year

ii growing at ≈ 1.88 cm per year

13 $\frac{21}{\sqrt{2}}$ cm^2 per radian ≈ 0.259 cm^2 per degree

14 **a** $l = \sqrt{800 - 800\cos\theta}$ cm

b 10 cm per radian ≈ 0.175 cm per degree

15 **a** **i** 0 volts **ii** 340 volts

b **i** $-34\,000\pi$ volts per second **ii** 0 volts per second

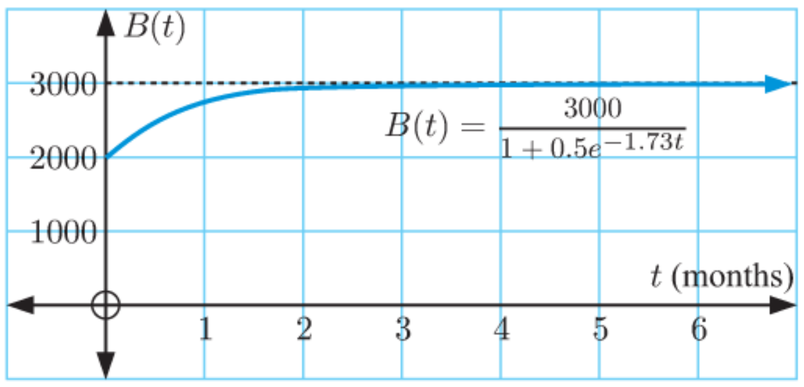
16 **a** 2000 bees **b** $\approx 37.8\%$ **c** yes, 3000 bees

d $B'(t) = \dfrac{2595}{e^{1.73t}(1 + 0.5e^{-1.73t})^2} > 0$ for all $t \geqslant 0$

$\therefore$ $B(t)$ is increasing over time.

e ≈ 0.0806 bees per month

f



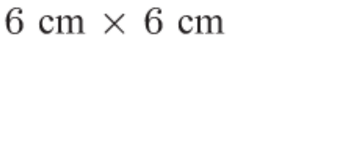
EXERCISE 20B

1 250 items per day **2** 10 blankets per day

3 **b** 15 m $\times$ 30 m

4 **b** $L_{\min} \approx 28.3$ m, when $x \approx 7.07$

c

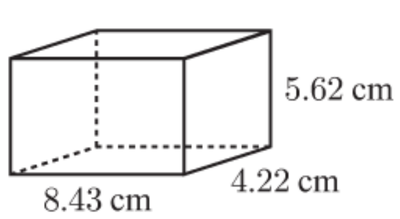


5 **a** **Hint:** $V = 200 = 2x \times x \times h$

b **Hint:** Show $h = \dfrac{100}{x^2}$ and substitute into the surface area equation.

c $A_{\min} \approx 213$ cm^2, when $x \approx 4.22$

d



6 **a** **Hint:** $V_{\text{cylinder}} = \pi r^2 h$ and 1 L $= 1000$ cm^3.

b **Hint:** $SA_{\text{cylinder}} = 2\pi r^2 + 2\pi rh$

c $r \approx 5.42$, $h \approx 10.84$

7 **b** 6 cm $\times$ 6 cm

8 **a** $0 \leqslant x \leqslant \frac{200}{\pi} \approx 63.7$

b $l = 100$, $x = \frac{100}{\pi} \approx 31.8$, $A = \frac{20\,000}{\pi} \approx 6370$ m^2

9 20 kettles

10 **a** area $= 4x\sqrt{25 - x^2}$ cm^2 **b** $5\sqrt{2}$ cm $\times$ $5\sqrt{2}$ cm

11 **b** $\theta \approx 1.91$, $A \approx 237$ cm^2

12 **c** $\theta = \frac{\pi}{6}$, area ≈ 130 cm^2

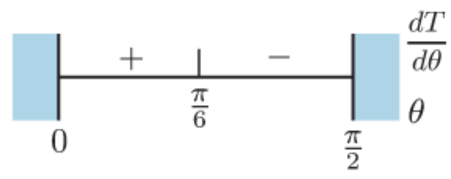
13 **a** $E'(t) = 750e^{-1.5t}(1 - 1.5t)$

b 40 minutes after the injection

14 $3\frac{1}{3}$ km

15 **a** **Hint:** Use the cosine rule. **b** ≈ 3550 **c** $\approx 5{:}36$ pm

16 **c** $\theta = \frac{\pi}{6}$ **d**



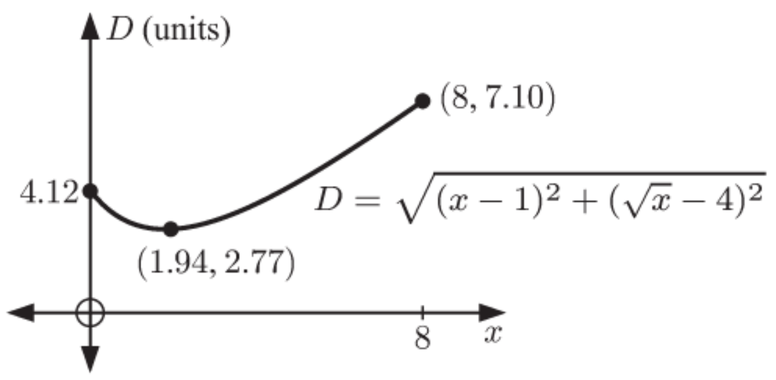
e **i** Row from P to Q at an angle of $\frac{\pi}{6}$ to the diameter of the lake, then walk from Q to R.

ii Walk from P to R.

17 **a** X is between A and C.

c $x \approx 2.67$ This is the distance in km from A to X which minimises the time taken to get from B to C.

18 **b**



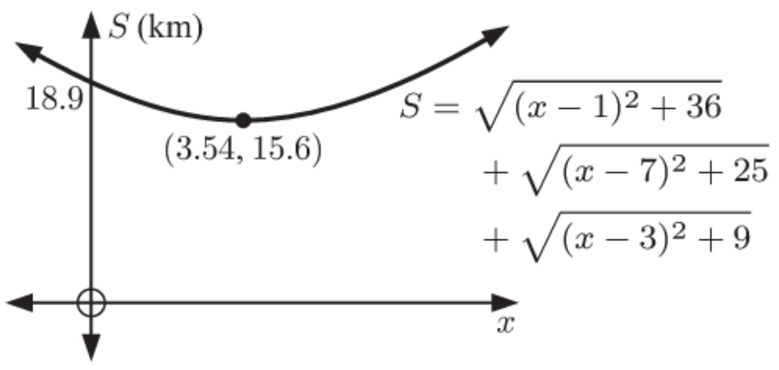
c $D \approx 2.77$ units, when $x \approx 1.94$

d $\dfrac{dD}{dx} = 0$ when $x \approx 1.94$

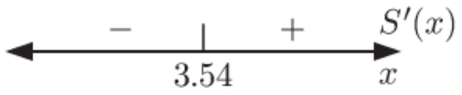
19 **b** PA $= \sqrt{(x-1)^2 + 36}$, PB $= \sqrt{(x-7)^2 + 25}$

c $S = \sqrt{(x-1)^2 + 36} + \sqrt{(x-7)^2 + 25} + \sqrt{(x-3)^2 + 9}$

d



e



f at (3.54, 8)

20 **a** $P'(t) = \dfrac{25\,000\,000e^{-0.5t}}{(1 + 1000e^{-0.5t})^2}$

b ≈ 2813 wasps per month

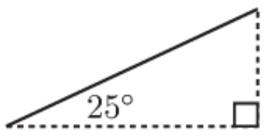
c after ≈ 9.13 weeks and 18.5 weeks

d after ≈ 13.8 weeks

EXERCISE 20C

1 **a** $h(0) = 3$ $\therefore$ $c = 3$ **b** $h'(x) = 2ax + b$

c



$h'(0) =$ gradient $= \tan 25°$

$\therefore$ $b = \tan 25°$

d $x \approx 14.5$

2 **a** $C(x) \approx -0.178x^3 + 15.3x^2 + 3280x + 24\,500$

b €331 700

3 **a** $P(0) = 21.5 \quad \therefore \quad d = 21.5$

b $P'(t) = 3at^2 + 2bt + c$ **c** $t = 15$

d $3375a + 225b + 15c = 5$

$125\,000a + 2500b + 50c = -13.9$

$675a + 30b + c = 0$

$\therefore \quad P(t) \approx 0.000\,136t^3 - 0.0263t^2 + 0.697t + 21.5$

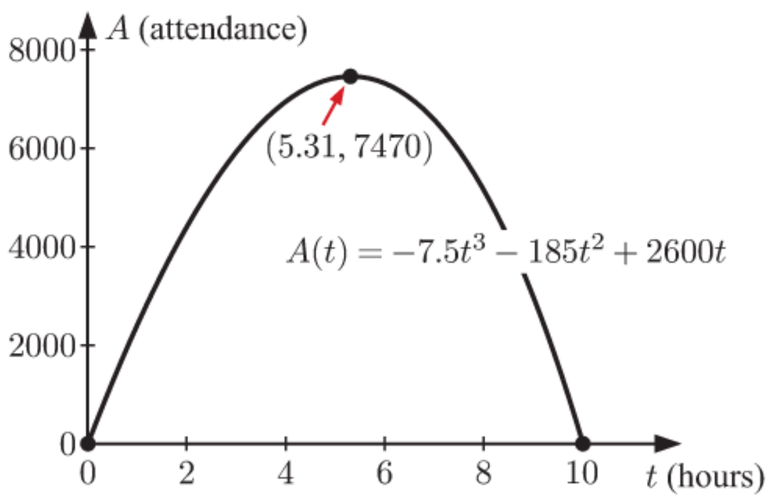
e **i** ≈ 22.4 million **ii** ≈ -1.99 million

f The model provided a reasonable estimate when interpolating between data points. The extrapolation in **e ii** however predicted a negative population, which is not possible.

4 **a** $A(0) = 0, \ A(10) = 0, \ A'(2) = 1770, \ A'(8) = -1800$

b $A(t) = -7.5t^3 - 185t^2 + 2600t, \quad 0 \leqslant t \leqslant 10$

c



The model seems reasonable; the attendance was always increasing until shortly after 1 pm, then it began to decrease.

d ≈ 7470 people at 1:19 pm

5 **a** **B**, Ir^2 is approximately constant

b $I \approx \dfrac{180\,266}{r^2}$ **c** $\dfrac{dI}{dr} \approx -\dfrac{360\,532}{r^3}$

d decreasing by ≈ 0.361 lux per cm

e The surface area of a sphere is proportional to the square of its radius. The total illuminance from a point is a sphere. Since $SA \propto r^2$ and $I \propto \dfrac{1}{r^2}$ the total illuminance is a constant.

EXERCISE 20D

1 **a** $\dfrac{dy}{dx} = 6x^2 \dfrac{dx}{dt}$ **b** increasing at 54 units per second

2 **a** $A = x^2$ **b** $\dfrac{dA}{dt} = 2x \dfrac{dx}{dt}$ **c** 24 cm^2 per second

3 **a** $y = \dfrac{100}{x}$ **b** $\dfrac{dy}{dt} = -\dfrac{100}{x^2} \dfrac{dx}{dt}$

c **i** 0.25 cm per minute **ii** 0.16 cm per minute

4 **a** 4π m^2 per second **b** 8π m^2 per second

5 **a** $V = \frac{4}{3}\pi r^3$ **b** increasing at 0.375 m per minute

6 **a** decreasing at 0.005 m^3 per minute

b decreasing at 0.16 m^3 per minute

7 $\frac{20}{3}$ cm per minute **8** decreasing at $\frac{250}{13} \approx 19.2$ m s^{-1}

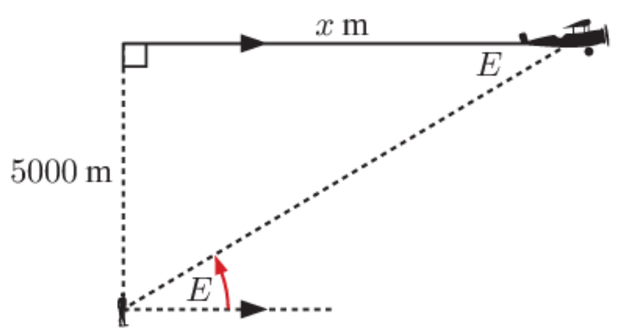
9 **b** ≈ 1.35 cm per minute

10 **a** 0.2 m s^{-1} **b** $\frac{4}{45}$ m s^{-1}

11 **a** increase **b** increasing at $\frac{2\pi}{3\sqrt{3}} \approx 1.21$ cm per minute

12 $\frac{\sqrt{2}}{100}$ radians per second

13 **a**



b **i** decreasing at $\frac{3}{100}$ radians per second

ii decreasing at $\frac{1}{100}$ radians per second

14 **a** $\frac{\sqrt{3}}{2}\pi$ cm s^{-1} **b** 0 cm s^{-1}

REVIEW SET 20A

1 **a** 60 cm **b** **i** ≈ 4.24 years **ii** ≈ 201 years

c **i** 16 cm per year **ii** ≈ 1.95 cm per year

2 **a** £20 000 **b** £146.53 per year

3 **a** $\approx$ \$4930.25

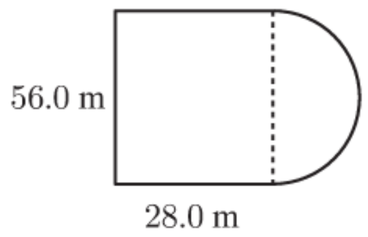
b **i** decreasing at $\approx$ \$1.39 per km h^{-1}

ii increasing at $\approx$ \$2.83 per km h^{-1}

c ≈ 79.4 km h^{-1}

4 **b** C$(\sqrt{3}, 6)$

5 **b** $A = 200x - 2x^2 - \frac{1}{2}\pi x^2$ **c**



6 **a** $y = \dfrac{1}{x^2}, \ x > 0$

c base is $\sqrt[3]{2} \approx 1.26$ m square, height is ≈ 0.630 m

7 **a** $\{t \mid 0 \leqslant t \leqslant 5.9\}$ **b** ≈ 3.33 s **c** ≈ 49.8 m

d ≈ 36.0 m

8 **a** $h(0) = 1.6 \quad \therefore \quad c = 1.6$

b $h'(0) = 16.4 \quad \therefore \quad b = 16.4$ **c** $a = -4.9$

d ≈ 15.3 m at $t \approx 1.67$

9 **a** $A = \frac{\sqrt{3}}{4}x^2$ **b** $\frac{45\sqrt{3}}{2}$ cm^2 per minute

10 **a** $V(r) = \frac{8}{9}\pi r^3$ m^3

b decreasing at $\frac{8}{375\pi} \approx 0.006\,79$ m min^{-1}

REVIEW SET 20B

1 **a** $C'(x) = 2.805x^{-0.15} + 1.4x^{-0.5}$ euros per item

b $C'(1000) \approx$ €1.04, this estimates the cost of making the 1001st item each day.

c $C(1001) - C(1000) \approx$ €1.04, this is the actual cost of making the 1001st item each day. The answer in **b** is a very good estimate.

2 **a** $A = 242\pi(1 - \cos\theta)$ cm^2

b $121\sqrt{2}\pi \approx 538$ cm^2 per radian ≈ 9.38 cm^2 per degree

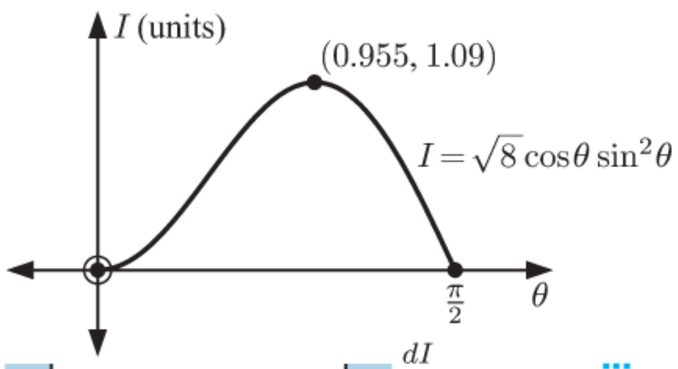
3 6 cm from each end **4** $x \approx 2.11$

5 **a** $\dfrac{dD}{dt} = -3.4476\sin(0.507t)$, this tells us the rate at which the depth of water is increasing or decreasing t hours after midnight.

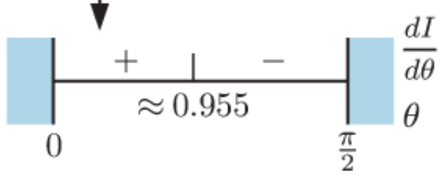
b ≈ 5.15 m **c** rising, when $t = 8$, $\dfrac{dD}{dt} \approx 2.73 > 0$

d midnight and $\approx$ 12:24 pm, maximum depth of 16.1 m

6 **b** **i**

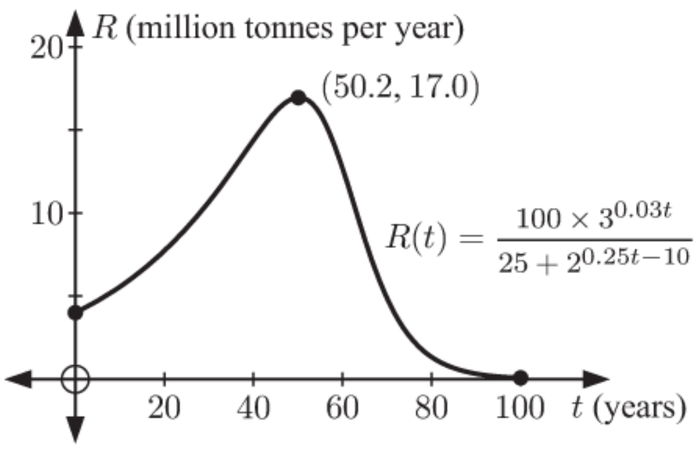


ii

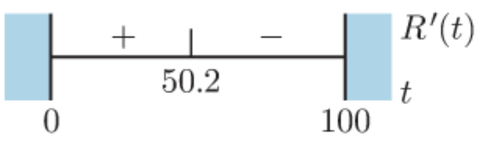


iii ≈ 0.707 m

7 **a**



b



c ≈ 7.72 million tonnes per year

d ≈ 28.0 years and ≈ 63.1 years after mining begins

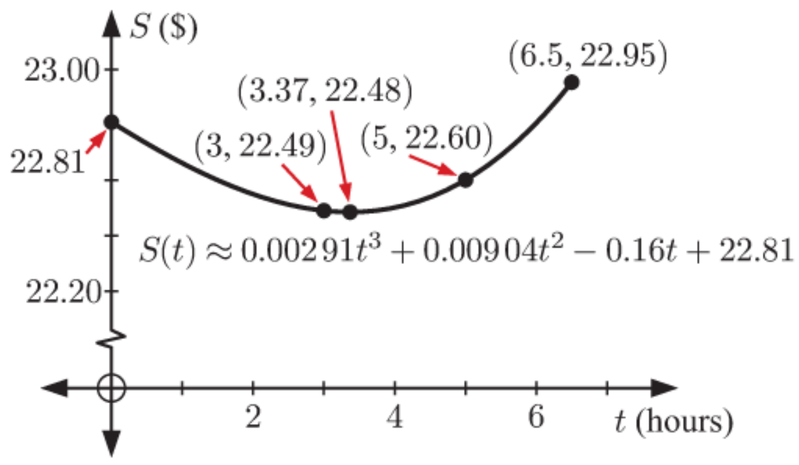
e ≈ 17.0 million tonnes per year, ≈ 50.2 years after mining begins

8 **a** $S(0) = 22.81 \quad \therefore \quad d = 22.81$

b $S'(0) = -0.16 \quad \therefore \quad c = -0.16$

c $a \approx 0.002\,91, \quad b \approx 0.009\,04$

d



e $\approx$ \$22.95 **f** minimum $\approx$ \$22.48 at 12:52 pm, maximum $\approx$ \$22.95 at 4 pm

9 increasing at 0.128 radians per second

10 $\frac{20\sqrt{10}}{3} \approx 21.1$ m per minute

EXERCISE 21A.1

1 **a** **i** 0.4 units^2 **ii** 0.6 units^2 **b** 0.5 units^2

2 **a** ≈ 0.653 units^2 **b** ≈ 0.737 units^2

3

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

A_L and A_U converge to $\frac{7}{3}$.

4 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{2}{3}$ **iv** $\frac{3}{4}$ **c** area $= \frac{1}{a+1}$

5 **a**

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $n = 10\,000$

EXERCISE 21A.2

1 **a** ≈ 2.3479 units^2 **b** ≈ 7.25 units^2

2 **a** $4\frac{1}{2}$ units^2

b $y = 3 - x$ is a straight line.
We are finding the area of a triangle with base and height each 3 units.

3 **a** ≈ 5.2650 units^2 **b** ≈ 0.1346 units^2

c ≈ 0.9920 units^2

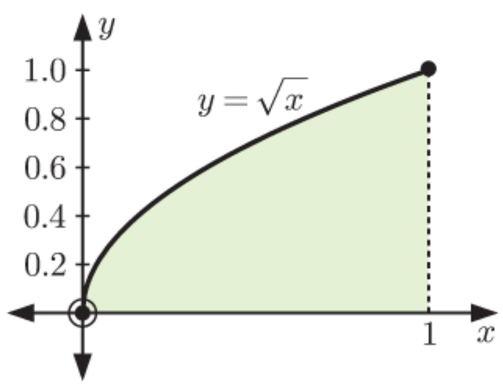
4 **a**

n	*Area estimate*
8	3.0898
40	3.1369
100	3.1404
1000	3.1416

b $n = 1000$

EXERCISE 21B

1 **a**



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\displaystyle\int_0^1 \sqrt{x}\,dx \approx 0.67$

d Using the trapezoidal method with 8 subintervals,

$\displaystyle\int_0^1 \sqrt{x}\,dx \approx 0.6581\,.$

With just 8 subintervals, the trapezoidal method is more accurate than lower and upper rectangles were with $n = 50$.

2 a $A_L = \dfrac{2}{n}\displaystyle\sum_{i=0}^{n-1}\sqrt{1+x_i^{\,3}}$, $A_U = \dfrac{2}{n}\displaystyle\sum_{i=1}^{n}\sqrt{1+x_i^{\,3}}$,

where $x_i = \dfrac{2i}{n}$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

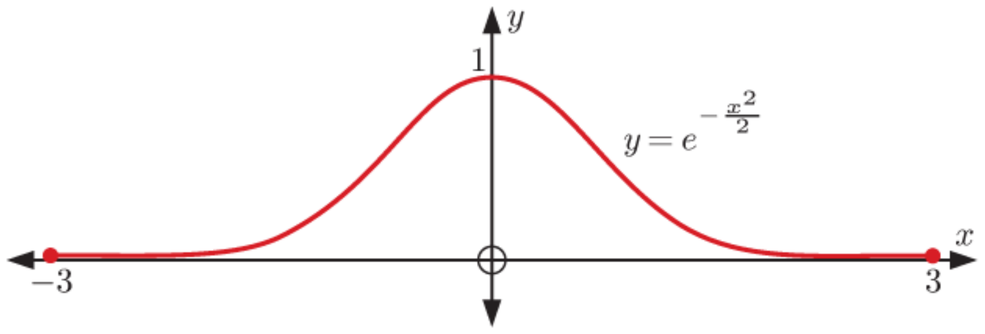
c $\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx 3.24$

d Using the trapezoidal method with 10 subintervals,

$\displaystyle\int_0^2 \sqrt{1+x^3}\,dx \approx 3.2480\,.$

With just 10 subintervals, the trapezoidal method is more accurate than lower and upper rectangles were with $n = 100$.

3 a



b lower ≈ 1.2493, upper ≈ 1.2506

c $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx \approx 2.4999$, $\sqrt{2\pi} \approx 2.5066$

d 7 subintervals

4 a 18 **b** 4.5 **c** 2π **5 b i** 0 **ii** 22

EXERCISE 21C

1 a i $\dfrac{x^2}{2}$ **ii** $\dfrac{x^3}{3}$ **iii** $\dfrac{x^6}{6}$ **iv** $-\dfrac{1}{x}$

v $-\dfrac{1}{3x^3}$ **vi** $\frac{3}{4}x^{\frac{4}{3}}$ **vii** $\frac{3}{2}x^{\frac{2}{3}}$ **viii** $\frac{3}{5}x^{\frac{5}{3}}$

b The antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

2 a i $\frac{1}{3}e^{3x}$ **ii** $\frac{1}{5}e^{5x}$ **iii** $2e^{\frac{1}{2}x}$ **iv** $100e^{0.01x}$

v $\frac{1}{\pi}e^{\pi x}$ **vi** $3e^{\frac{x}{3}}$

b The antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$, where $k \neq 0$ is a constant.

3 a $\dfrac{d}{dx}(x^3+x^2) = 3x^2+2x$

$\therefore$ the antiderivative of $6x^2+4x$ is $2x^3+2x^2$.

b $\dfrac{d}{dx}(x\sqrt{x}) = \frac{3}{2}\sqrt{x}$

$\therefore$ the antiderivative of $\sqrt{x}$ is $\frac{2}{3}x\sqrt{x}$.

c $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x}}\right) = -\frac{1}{2}x^{-\frac{3}{2}} = -\dfrac{1}{2x\sqrt{x}}$

$\therefore$ the antiderivative of $\dfrac{1}{x\sqrt{x}}$ is $-\dfrac{2}{\sqrt{x}}$.

EXERCISE 21D

1 a $\dfrac{d}{dx}(x^2) = 2x$ **b** 8 units^2

$\therefore$ the antiderivative of $2x$ is x^2.

2 a $\frac{2}{3}x^{\frac{3}{2}}$ **b** $\frac{2}{3}$ units^2 **c** $\frac{2}{3} \approx 0.67$

$\therefore$ the answers are the same.

3 a i 4 units^2 **ii** $16\frac{1}{4}$ units^2 **iii** $20\frac{1}{4}$ units^2

b $\displaystyle\int_0^3 x^3\,dx = \int_0^2 x^3\,dx + \int_2^3 x^3\,dx$

4 a $3\frac{3}{4}$ units^2 **b** $8\frac{2}{3}$ units^2 **c** $\dfrac{4\sqrt{2}-2}{3}$ units^2

d 2 units^2

6 $\frac{9\pi}{4}$ units^2

7 c i $\displaystyle\int_0^1 (-x^2)\,dx = -\frac{1}{3}$, the area between $y = -x^2$ and the x-axis from $x = 0$ to $x = 1$ is $\frac{1}{3}$ units^2.

ii $\displaystyle\int_0^1 (x^2-x)\,dx = -\frac{1}{6}$, the area between $y = x^2 - x$ and the x-axis from $x = 0$ to $x = 1$ is $\frac{1}{6}$ units^2.

iii $\displaystyle\int_{-2}^0 3x\,dx = -6$, the area between $y = 3x$ and the x-axis from $x = -2$ to $x = 0$ is 6 units^2.

d $-\pi$

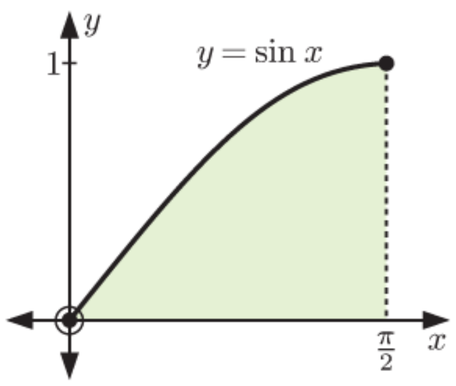
REVIEW SET 21A

1 a $A = 4.25$, $B = 6.25$ **b** ≈ 5.3125

c $\displaystyle\int_0^2 (4-x^2)\,dx = 5\frac{1}{3}$ In **a**, $\dfrac{A+B}{2} = 5.25$

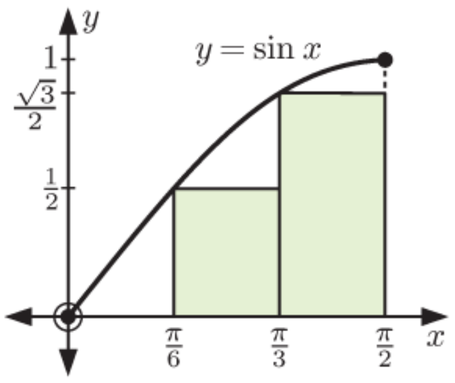
Both answers provide good approximations of the integral.

2 a

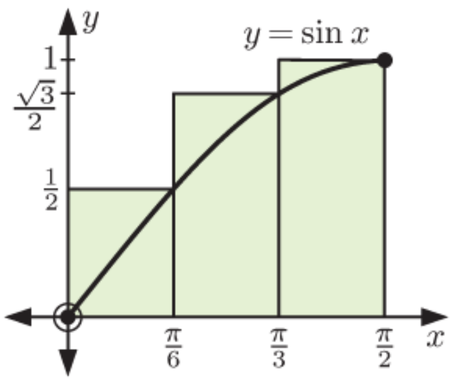


b

lower rectangles



upper rectangles



$\dfrac{\pi(1+\sqrt{3})}{12} < \displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx < \dfrac{\pi(3+\sqrt{3})}{12}$

or $0.715 < \displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx < 1.24$

3 **a** $\frac{x^5}{5}$ **b** $-\frac{1}{2x}$ **c** $-2e^{-\frac{1}{2}x}$ **d** $\sin x$

4 **a** 2π **b** 2

5 **a** **i** $\frac{1}{3}$ units^2 **ii** $2\frac{1}{3}$ units^2 **iii** $2\frac{2}{3}$ units^2

b $\int_0^2 x^2\,dx = \int_0^1 x^2\,dx + \int_1^2 x^2\,dx$

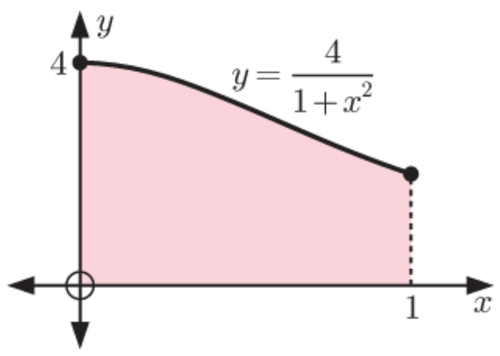
6 **a** ≈ 6.2365

b It is an over estimate. Joining the interval endpoints with straight line segments would give a larger area than the shaded area.

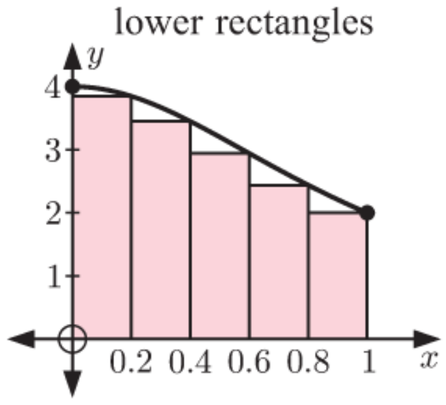
c $\int_{-2}^{-1} x^4\,dx = \frac{31}{5} = 6.2$

REVIEW SET 21B

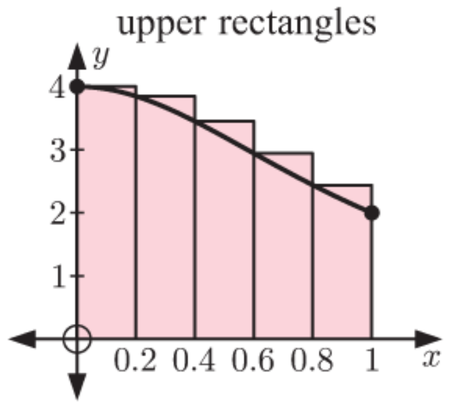
1 **a**



lower rectangles



upper rectangles



b

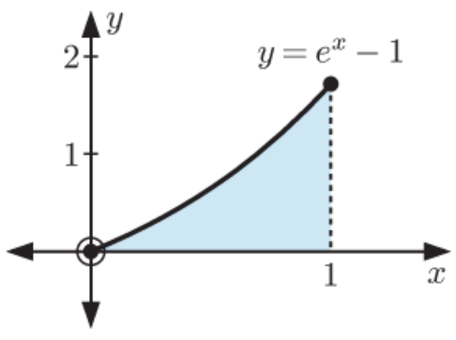
n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\frac{3.1396 + 3.1436}{2} = 3.1416$ **d** ≈ 3.1399

e $\int_0^1 \frac{4}{1+x^2}\,dx = \pi \approx 3.1416$

Both answers in **c** and **d** are good approximations of the integral, but the trapezoidal method required fewer subintervals and is hence more efficient.

2 **a**



b **i** ≈ 0.6338 **ii** ≈ 0.8056 **iii** ≈ 0.7197

c $\int_0^1 (e^x - 1)\,dx = e - 2 \approx 0.7183$

$\frac{0.6338 + 0.8056}{2} = 0.7197$

Our approximations of the integral using the rectangle and trapezoidal methods with 10 subintervals are very accurate.

3 **a** 10 **b** $\frac{\pi}{2}$

4 **a** $\frac{d}{dx}(x^3 - 2x) = 3x^2 - 2$

$\therefore$ the antiderivative of $3x^2 - 2$ is $x^3 - 2x$.

b $\frac{d}{dx}\left(x^{\frac{4}{3}}\right) = \frac{4}{3}x^{\frac{1}{3}} = \frac{4}{3}\sqrt[3]{x}$

$\therefore$ the antiderivative of $\sqrt[3]{x}$ is $\frac{3}{4}x^{\frac{4}{3}}$.

5 **a** 3 **b** $\frac{15}{2}$ **6** **a** 18 units^2 **b** 18 units^2

EXERCISE 22A

1 **a** $\frac{d}{dx}\left(x^7\right) = 7x^6$

$\therefore \int x^6\,dx = \frac{1}{7}x^7 + c$

b $\frac{d}{dx}\left(x^{\frac{3}{2}}\right) = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

$\therefore \int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c$

c $\frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2}x^{-\frac{3}{2}}$

$\therefore \int x^{-\frac{3}{2}}\,dx = -2x^{-\frac{1}{2}} + c$

d $\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n$

$\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$

2 **a** $\frac{d}{dx}\left(e^{4x}\right) = 4e^{4x}$

$\therefore \int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

b $\frac{d}{dx}\left(e^{-\frac{x}{2}}\right) = -\frac{1}{2}e^{-\frac{x}{2}}$

$\therefore \int e^{-\frac{x}{2}}\,dx = -2e^{-\frac{x}{2}} + c$

c $\frac{d}{dx}\left(e^{kx}\right) = ke^{kx}$

$\therefore \int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$

3 **a** $\frac{d}{dx}(\sin x) = \cos x$

$\therefore \int \cos x\,dx = \sin x + c$

b $\frac{d}{dx}(\cos x) = -\sin x$

$\therefore \int \sin x\,dx = -\cos x + c$

c $\frac{d}{dx}(\tan x) = \frac{1}{\cos^2 x}$

$\therefore \int \frac{1}{\cos^2 x}\,dx = \tan x + c$

4 $\dfrac{d}{dx}\left(x^3+x^2\right)=3x^2+2x$

$\therefore \int (3x^2+2x)\,dx = x^3+x^2+c$

5 $\dfrac{d}{dx}\left(3x^4-2x^2\right)=12x^3-4x$

$\therefore \int (3x^3-x)\,dx = \frac{3}{4}x^4-\frac{1}{2}x^2+c$

6 **a** $f(x)+g(x)$

7 **a** $\dfrac{d}{dx}(\sin 3x)=3\cos 3x$

$\therefore \int \cos 3x\,dx = \frac{1}{3}\sin 3x+c$

b $\dfrac{d}{dx}\left(\cos\left(\frac{\pi}{3}-x\right)\right)=\sin\left(\frac{\pi}{3}-x\right)$

$\therefore \int \sin\left(\frac{\pi}{3}-x\right)dx = \cos\left(\frac{\pi}{3}-x\right)+c$

c $\dfrac{d}{dx}\left(e^{3x+1}\right)=3e^{3x+1}$

$\therefore \int e^{3x+1}\,dx = \frac{1}{3}e^{3x+1}+c$

d $\dfrac{d}{dx}\left(\sqrt{5x-1}\right)=\frac{5}{2}(5x-1)^{-\frac{1}{2}}=\dfrac{5}{2\sqrt{5x-1}}$

$\int \dfrac{1}{\sqrt{5x-1}}\,dx = \frac{2}{5}\sqrt{5x-1}+c$

e $\dfrac{d}{dx}\left((2x+1)^4\right)=8(2x+1)^3$

$\therefore \int (2x+1)^3\,dx = \frac{1}{8}(2x+1)^4+c$

f $\dfrac{d}{dx}\left(\left(x^2-x\right)^3\right)=3(2x-1)(x^2-x)^2$

$\therefore \int (2x-1)(x^2-x)^2\,dx = \frac{1}{3}(x^2-x)^3+c$

8 **a** $\dfrac{1}{x}$ **b** $\dfrac{-1}{-x}=\dfrac{1}{x}$

c $\displaystyle\int \frac{1}{x}\,dx = \begin{cases} \ln x+c, & x>0 \\ \ln(-x)+c, & x<0 \end{cases}$

EXERCISE 22B

1 **a** $\frac{1}{3}x^3+\frac{3}{2}x^2-2x+c$ **b** $\frac{2}{3}x^3-\frac{3}{2}x^2+x+c$

c $-\frac{1}{4}x^4+\frac{4}{3}x^3-3x+c$ **d** $\frac{1}{4}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+c$

e $\frac{1}{5}x^5-\frac{1}{3}x^3-\frac{1}{2}x^2+2x+c$ **f** $\frac{4}{3}x^3+\ln|x|+c$

g $\frac{4}{3}x^{\frac{3}{2}}-6x^{\frac{1}{2}}+c$ **h** $\frac{1}{3}\ln|x|+\dfrac{2}{x}+c$

i $\frac{2}{5}x^{\frac{5}{2}}-9x+c$ **j** $-6x^{-\frac{1}{2}}+\frac{4}{5}x^{\frac{5}{4}}+c$

2 **a** $2e^x-\frac{3}{2}x^2+c$ **b** $4\ln|x|+\frac{1}{3}x^3-e^x+c$

c $5e^x+\frac{1}{6}x^3+c$ **d** $-3\cos x-2x+c$

e $2x^2-2\sin x+c$ **f** $-\cos x-2\sin x+e^x+c$

g $\frac{2}{7}x^{\frac{7}{2}}+10\cos x+c$ **h** $\frac{1}{9}x^3-\frac{1}{6}x^2+\tan x+c$

i $\cos x+\frac{4}{3}x^{\frac{3}{2}}+c$

3 **a** $y=6x+c$ **b** $y=\frac{4}{3}x^3+c$ **c** $y=-\dfrac{1}{x}+c$

d $y=3x^{\frac{2}{3}}+c$ **e** $y=\frac{1}{2}x^4-4x+c$

f $y=x^4+x^3+c$ **g** $y=2x-\ln|x|+c$

h $y=-\cos x+2\sin x+c$ **i** $y=2e^x-5x+\frac{1}{2}x^2+c$

4 **a** $2e^t+4\cos t+c$ **b** $3\sin t-\ln|t|+c$

c $-5\cos t-\frac{2}{3}t\sqrt{t}+c$ **d** $\tan x-2\cos x+c$

e $\frac{1}{2}\theta^2+\cos\theta+c$ **f** $2\ln|\theta|-\tan\theta+c$

5 **a** $\frac{4}{3}x^3+2x^2+x+c$ **b** $\frac{1}{3}x^3+2x-\dfrac{1}{x}+c$

c $-\dfrac{2}{\sqrt{x}}-8\sqrt{x}+c$ **d** $\frac{4}{3}x\sqrt{x}-2\sqrt{x}+c$

e $\frac{1}{2}x^2-2x+\ln|x|+c$ **f** $\ln|x|-\frac{1}{2}x^2+c$

g $-\dfrac{4}{x}+4\ln|x|+x+c$ **h** $\dfrac{x^2}{2}-4x+10\ln|x|+c$

i $\frac{3}{2}x^2-2x-\dfrac{5}{x}+c$

6 **a** $f(x)=x-2x^2+\frac{4}{3}x^3+c$

b $f(x)=\frac{2}{3}x\sqrt{x}-4\sqrt{x}+c$ **c** $f(x)=x+\dfrac{5}{x}+c$

7 **a** $\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{2}\sin x+c$ **b** $2e^x+4\cos x+c$

c $3\sin x+\cos x+c$

8 **a** $\frac{1}{5}x^5-x^2-\dfrac{1}{x}+c$ **b** $\frac{2}{5}x^{\frac{5}{2}}-\frac{8}{3}x^{\frac{3}{2}}+4x^{\frac{1}{2}}+c$

c $\frac{18}{7}x^{\frac{7}{2}}-\frac{12}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}+c$

EXERCISE 22C

1 **a** $f(x)=x^2-x+3$ **b** $f(x)=x^3+x^2-7$

c $f(x)=2x+2\sqrt{x}-3$ **d** $f(x)=\frac{1}{2}x^2-4\sqrt{x}+\frac{11}{2}$

e $f(x)=\frac{2}{3}x^{\frac{3}{2}}-2x+\frac{8}{3}$ **f** $f(x)=\ln|x|+1$

2 $y=\dfrac{x^2}{2}-\dfrac{2x^3}{3}+\frac{22}{3}$ **3** $y=x-e^x+2e^3-3$

4 **a** $f(x)=\frac{1}{3}x^3-4\sin x+3$

b $f(x)=2\sin x+3\cos x-2\sqrt{2}$

c $f(x)=\frac{2}{3}x\sqrt{x}-2\tan x-\frac{2}{3}\pi\sqrt{\pi}$

d $f(x)=e^x+3\sin x-e^\pi$

5 $f(x)=-x^2+x+3$ **6** $f(x)=3x^3+1$

7 **a** $f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2+x+\frac{1}{3}$

b $f(x)=4x^{\frac{5}{2}}+4x^{\frac{3}{2}}-4x+5$

c $f(x)=-\cos x-x+4$ **d** $f(x)=\frac{1}{3}x^3-\frac{16}{3}x+5$

8 $f(x)=3e^{-x}-x+1$

EXERCISE 22D

1 **a** $\frac{1}{8}(2x+5)^4+c$ **b** $\frac{1}{32}(4x-3)^8+c$

c $-\frac{3}{5}(1-x)^5+c$ **d** $\dfrac{1}{2(3-2x)}+c$

e $\dfrac{-2}{3(2x-1)^3}+c$ **f** $\frac{2}{9}(3x-4)^{\frac{3}{2}}+c$

g $-4\sqrt{1-5x}+c$ **h** $-2\sqrt{3-4x}+c$

i $-\dfrac{5}{6(3x-2)^2}+c$

2 $y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$ **3** $(-8, -19)$

4 **a** $\frac{1}{2}(2x-1)^3 + c$ **b** $\frac{1}{12}(4x-5)^3 + c$

c $-\frac{1}{12}(1-3x)^4 + c$ **d** $-\frac{1}{15}(2-5x)^3 + c$

e $-\frac{8}{3}(5-x)^{\frac{3}{2}} + c$ **f** $\frac{1}{35}(7x+1)^5 + c$

5 $y = \frac{1}{2}x^2 - \dfrac{5}{1-x} - 7$

6 **a** $-\frac{1}{3}\cos 3x + c$ **b** $-\frac{1}{2}\sin(-4x) + x + c$

c $\frac{1}{2}\tan 2x + c$ **d** $6\sin\frac{x}{2} + c$

e $-\frac{3}{2}\cos 2x + e^{-x} + c$ **f** $-\cos\left(2x + \frac{\pi}{6}\right) + c$

g $3\sin\left(\frac{\pi}{4} - x\right) + c$ **h** $\frac{1}{2}e^{2x} - 4\tan\frac{x}{2} + c$

i $-2\tan\left(\frac{\pi}{3} - 2x\right) + c$ **j** $\frac{1}{2}\sin 2x - \frac{1}{2}\cos 2x + c$

k $-\frac{2}{3}\cos 3x + \frac{5}{4}\sin 4x + c$ **l** $\frac{1}{16}\sin 8x + 3\cos x + c$

7 **a** $2e^x + \frac{5}{2}e^{2x} + c$ **b** $\frac{3}{5}e^{5x-2} + c$

c $-\frac{1}{3}e^{7-3x} + c$ **d** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

e $-\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$

f $\frac{1}{3}e^{3x} - 10e^x - 25e^{-x} + c$

8 $y = x - 2e^x + \frac{1}{2}e^{2x} + \frac{11}{2}$

9 $f(x) = -e^{-2x} + 4$ **10** $p = -\frac{1}{4}$, $f(x) = \frac{1}{2}\cos\frac{x}{2} + \frac{1}{2}$

12 $y = \frac{2}{3}x\sqrt{x} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

13 **a** $6\ln|x+4| + c$ **b** $\frac{1}{2}\ln|2x-1| + c$

c $-\frac{5}{3}\ln|1-3x| + c$ **d** $4x + \frac{1}{5}\ln|5x-2| + c$

e $x - x^2 + 4\ln|x-3| + c$

f $-\frac{1}{2}\cos 2x + \frac{3}{2}\ln|1-2x| + c$

14 Both are correct.

$\dfrac{d}{dx}(\ln(Ax)) = \dfrac{d}{dx}(\ln A + \ln x) = \dfrac{1}{x}, \quad A, x > 0$

15 $\displaystyle\int \frac{3x-1}{x+2}\,dx = 3x - 7\ln|x+2| + c$

16 $f(x) = x^2 + 2\ln|1-x| + 2 - 2\ln 2$

EXERCISE 22E

1 $\dfrac{d}{dx}\left((2x^2-5x)^3\right) = 3(4x-5)(2x^2-5x)^2$

$\therefore \displaystyle\int (4x-5)(2x^2-5x)^2\,dx = \frac{1}{3}(2x^2-5x)^3 + c$

2 $\dfrac{d}{dx}\left(\sin(x^2)\right) = 2x\cos(x^2)$

$\therefore \displaystyle\int x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2) + c$

3 $\dfrac{d}{dx}\left(\ln(5-3x+x^2)\right) = \dfrac{2x-3}{5-3x+x^2}$

$\therefore \displaystyle\int \frac{4x-6}{5-3x+x^2}\,dx = 2\ln\left|5-3x+x^2\right| + c$

4 **a** $\frac{1}{5}(x^3+1)^5 + c$ **b** $\frac{1}{3}e^{x^3+1} + c$ **c** $\frac{1}{5}\sin^5 x + c$

d $\sin(x^2-3) + c$

5 **a** $\frac{1}{4}(2+x^4)^4 + c$ **b** $2\sqrt{x^2+3} + c$

c $-\dfrac{1}{3(2x^3-1)^3} + c$ **d** $\frac{1}{5}(x^3+2x+1)^5 + c$

e $\dfrac{1}{8(1-x^2)^4} + c$ **f** $-\dfrac{1}{2(x^2+4x-3)} + c$

6 **a** $e^{1-2x} + c$ **b** $e^{x^2} + c$ **c** $2e^{\sqrt{x}} + c$

7 **a** $\ln\left|x^2+1\right| + c$ **b** $-\frac{1}{2}\ln\left|2-x^2\right| + c$

c $\ln\left|x^2-3x\right| + c$

8 **a** $-\frac{1}{9}(3-x^3)^3 + c$ **b** $-\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$

c $-\frac{1}{2}e^{1-x^2} + c$ **d** $\frac{1}{4}(\ln x)^4 + c$

e $-e^{x-x^2} + c$ **f** $-\frac{1}{3}\ln\left|x^3-3x\right| + c$

9 **a** $\frac{1}{8}\sin^8 x + c$ **b** $-\frac{1}{6}\cos^6 x + c$

c $-2(\cos x)^{\frac{1}{2}} + c$ **d** $-\ln|\cos x| + c$

e $\frac{2}{3}(\sin x)^{\frac{3}{2}} + c$ **f** $-(2+\sin x)^{-1} + c$

g $\ln|1-\cos x| + c$ **h** $\frac{1}{2}\ln|\sin 2x - 3| + c$

i $-\frac{1}{2}\cos(x^2) + c$

10 **a** $e^{\tan x} + c$ **b** $-\frac{1}{3}\sin^3 x + \sin x + c$ **c** $\frac{1}{8}\sin^4 2x + c$

REVIEW SET 22A

1 $\dfrac{d}{dx}\left(x^4 - x^2\right) = 4x^3 - 2x$

$\therefore \displaystyle\int (2x^3 - x)\,dx = \frac{1}{2}x^4 - \frac{1}{2}x^2 + c$

2 $\dfrac{d}{dx}\left(\sin(\frac{\pi}{3} - 2x)\right) = -2\cos(\frac{\pi}{3} - 2x)$

$\therefore \displaystyle\int \cos(\frac{\pi}{3} - 2x)\,dx = -\frac{1}{2}\sin(\frac{\pi}{3} - 2x) + c$

3 **a** $\frac{2}{3}x\sqrt{x} + \dfrac{2}{x} + c$ **b** $x^2 - \frac{9}{2}x^{\frac{2}{3}} + c$

c $4x\sqrt{x} + 10\sqrt{x} + c$

4 **a** $8\sqrt{x} + c$ **b** $\dfrac{x^4}{12} + x^2 + c$ **c** $-\dfrac{1}{2x^2} + \dfrac{2}{x} + c$

5 **a** $-\frac{3}{5}x^5 + 2x^3 + c$ **b** $\frac{3}{2}x^2 - x + \dfrac{1}{x} + c$

c $\frac{4}{3}x^3 - \frac{8}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$ **d** $4e^x - 3\ln|x| + c$

e $-\frac{1}{4}\cos(4x-5) + c$ **f** $-\frac{1}{3}e^{4-3x} + c$

6 **a** $y = -3e^{-x} - 2\cos(\frac{\pi}{2} - x) + c$

b $y = \frac{1}{4}\sin 4x - \frac{1}{6}x^3 + c$

7 $f(x) = x^3 - 2x^2 + x + 2$ **8** $y = -2x^2 + 3x + 2$

9 $f(x) = \frac{3}{2}e^{2x} + \frac{1}{2}$

10 **a** $\frac{1}{2}x^2 - 7\ln|x| + c$ **b** $\frac{1}{2}e^{2x-3} - \frac{2}{3}\ln|3x-1| + c$

c $-\frac{1}{12}(4-3x)^4 + \frac{1}{2}\cos(-2x) + c$

11 $a = 6\sqrt{2}$, $f(x) = 2\sqrt{2}\sin 3x - 1$

12 $\dfrac{d}{dx}\left(\sqrt{x^2-4}\right) = \dfrac{x}{\sqrt{x^2-4}}$

$\therefore \displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx = \sqrt{x^2-4} + c$

13 $-\frac{1}{2}\cos(x^2 + \frac{\pi}{3}) + c$

14 **a** $\frac{1}{2}\ln\left|x^2+4x\right| + c$ **b** $e^{x^2-1} + c$

c $\frac{1}{10}\sin^{10} x + c$ **d** $-\frac{1}{2}\ln|\cos 2x| + c$

REVIEW SET 22B

1 $\dfrac{d}{dx}\left(6e^{-2x}\right) = -12e^{-2x}$

$\therefore \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + c$

2 $\dfrac{d}{dx}\left(\ln(2x+1)\right) = \dfrac{2}{2x+1}$

$\therefore \int \dfrac{1}{2x+1}\,dx = \frac{1}{2}\ln|\,2x+1\,| + c$

3 **a** $x + \dfrac{2}{x} + c$ **b** $3x^3 - 12x^2 + 16x + c$ **c** $4x - \dfrac{2x^3}{3} + c$

4 **a** $\frac{3}{4}x^{\frac{4}{3}} + 3x + c$ **b** $x^3 - 2x + c$ **c** $9x + 6x^2 + \frac{4}{3}x^3 + c$

5 $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + \frac{13}{6}$

6 **a** $y = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c$ **b** $y = 400x - 40x^{\frac{1}{2}} + c$

7 **a** $\frac{1}{2}x^4 - \frac{5}{2}x^2 + 7x + c$ **b** $\frac{3}{2}x^2 - \ln|\,x\,| + c$

c $-2e^{-x} + 3x + c$ **d** $2\sin 2x + c$

e $9x + 3e^{2x-1} + \frac{1}{4}e^{4x-2} + c$ **f** $-\tan\left(\frac{\pi}{4} - x\right) + c$

8 $f(x) = 2\ln|\,x\,| - x + e + 2 - \ln 4$

9 $y = -\frac{3}{68}x^3 + \frac{21}{68}(x-1)^{\frac{3}{2}} + \frac{275}{68}$

10 $f(x) = -2\sqrt{4-3x} + 8$

11 **a** $-\frac{1}{2}\ln|\,3-2x\,| + c$ **b** $\frac{4}{5}\ln|\,5x+1\,| + c$

12 $\dfrac{d}{dx}\left((3x^2+x)^3\right) = 3(3x^2+x)^2(6x+1)$

$\therefore \int (3x^2+x)^2(6x+1)\,dx = \frac{1}{3}(3x^2+x)^3 + c$

13 **a** $2\sqrt{x^2-5} + c$ **b** $\dfrac{1}{3\cos^3 x} + c$ **c** $-2e^{-x^2} + c$

d $-\cos x + \frac{1}{3}\cos^3 x + c$

14 **a** $\frac{1}{2}\ln\left|\,x^2 - 9\,\right| + c$ **b** $\ln\left|\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)\right| + c$

EXERCISE 23A

1 **a** $\displaystyle\int_1^4 \sqrt{x}\,dx = \frac{14}{3}, \quad \int_1^4 (-\sqrt{x})\,dx = -\frac{14}{3}$

b $\displaystyle\int_0^1 x^7\,dx = \frac{1}{8}, \quad \int_0^1 (-x^7)\,dx = -\frac{1}{8}$

2 **a** $\frac{1}{3}$ **b** $\frac{7}{3}$ **c** $\frac{8}{3}$ **d** 1

3 **a** -4 **b** $\frac{25}{4}$ **c** $\frac{9}{4}$ **4** **a** $\frac{2}{3}$ **b** $\frac{2}{3}$ **c** $\frac{4}{3}$

5 **a** $\frac{1}{4}$ **b** $\frac{2}{3}$ **c** 18 **d** $\frac{3}{2}$ **e** $\frac{101}{6}$ **f** $\frac{20}{3}$

g $\frac{2}{3}$ **h** $\frac{61}{3}$ **i** $21 + \ln 4$

6 $m = -1$ or $\frac{4}{3}$ **7** **a** $\frac{341}{5}$ **b** 2 **c** $\frac{14}{3}$

8 **a** $e - 1$ **b** $2e^3 - 11$ **c** $\frac{1}{3}(e^6 - 1)$

d $e - 1$ **e** $\frac{3}{2}$ **f** $-\dfrac{1}{2e^4} - \dfrac{3}{2e^2} + \dfrac{2}{e} + 1$

9 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** $\sqrt{3} - 1$ **d** $\frac{1}{3}$ **e** 1 **f** $\frac{1}{\sqrt{2}}$

10 $-\frac{1}{6}$ **11** $\ln 4$

12 **a** $-\ln 3$ **b** $\ln 3$ **c** $\frac{4}{3}\ln 2$ **d** $2\ln\left(\frac{13}{5}\right)$

14 **a** ≈ 1.30 **b** ≈ 1.49 **c** ≈ -0.189

d ≈ 0.363 **e** ≈ 0.797 **f** ≈ 6.27

EXERCISE 23B

1 **a** $\frac{81}{4}$ **b** $\frac{1}{12}$ **c** $\frac{1}{3}(e^2 - e)$ **d** $\frac{61}{3}$

e $\frac{1}{4}(e^{-2} - e^{-8})$ **f** $\frac{1}{2}\ln\left(\frac{2}{7}\right)$

2 **a** $2 - \sqrt{2}$ **b** $\frac{1}{24}$ **c** $\frac{1}{2}\ln 3$

EXERCISE 23C

1 **a** 6 units^2 **b** 6 units^2

2 **a** 30 units^2 **b** $4\frac{1}{2}$ units^2 **c** $13\frac{1}{2}$ units^2

3 **a** $\frac{1}{3}$ units^2 **b** $63\frac{3}{4}$ units^2 **c** $2\frac{1}{6}$ units^2

4 **a** A$(-2, 0)$, B$(3, 0)$ **b** $20\frac{5}{6}$ units^2

5 **a** $4\frac{1}{2}$ units^2 **b** 9 units^2 **c** $4\sqrt{3}$ units^2 **7** 1 unit^2

8 $\frac{2}{3}$ units^2 **9** **a** $\frac{\pi}{4}$ **b** $-\ln|\,\cos x\,| + c$ **c** $\ln\sqrt{2}$ units^2

10 **a** $\frac{1}{2}$ units^2 **b** $(e-1)$ units^2 **c** $4\frac{1}{2}$ units^2

d 18 units^2 **e** $\left(2e - \dfrac{2}{e}\right)$ units^2

11 **a** x-coordinate of A ≈ -1.44, x-coordinate of B ≈ 4.69

b ≈ 7.08 units^2

12 **b** $\dfrac{\sin 2}{1 + \cos 2} \approx 1.56$ units^2

13 **a** $b \approx 1.3104$ **b** $a = \sqrt{3}$ **c** $k \approx 1.7377$ **d** $k = 2\sqrt{5}$

EXERCISE 23D

1 **a** $4\frac{1}{2}$ units^2 **b** $(1 + e^{-2})$ units^2 **c** 2 units^2

d $2\frac{1}{4}$ units^2

2 **a** $40\frac{1}{2}$ units^2 **b** 8 units^2

3 **a** $\displaystyle\int_3^5 f(x)\,dx = -\,(\text{area between } x = 3 \text{ and } x = 5)$

b $\displaystyle\int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

4 Region B is larger.

$\displaystyle\int_{-2}^4 f(x)\,dx = \text{area of region } A + (-\text{area of region } B)$

$\{\text{region } B \text{ is below the } x\text{-axis}\}$

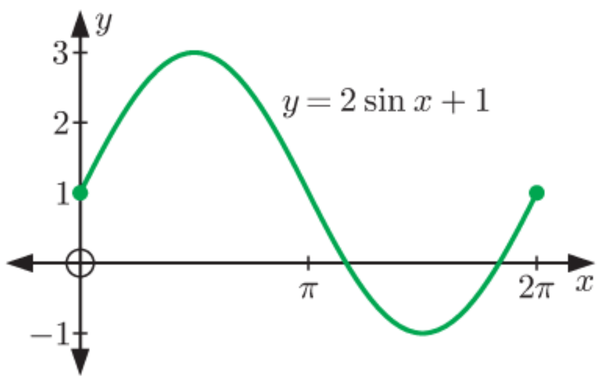
$= \text{area of region } A - \text{area of region } B$

$= -6$

$\therefore$ area of region $B >$ area of region A

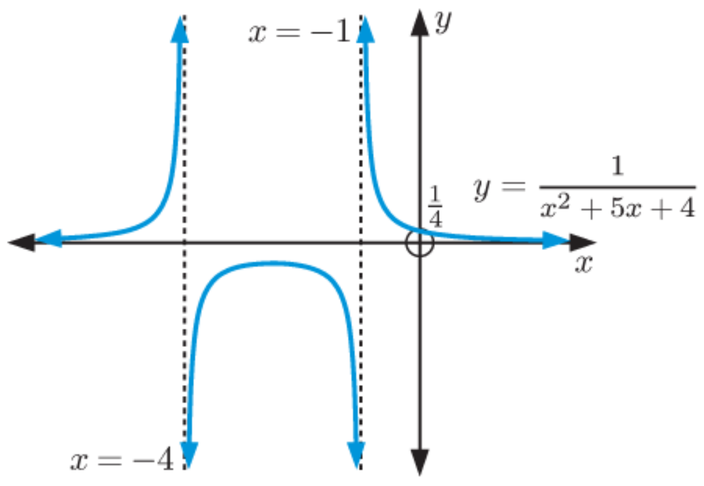
5 $k = \frac{3}{2}$ or 6

6 **a**



b $2\sqrt{3} - \frac{2\pi}{3} \approx 1.37$ units^2

7 **a**



b ≈ 0.462 units^2

8 **a** $33\frac{1}{3}$ cm^2 **b** ≈ 66.7 L

EXERCISE 23E

1 **a** $f^{-1}(y) = \dfrac{1}{y^2}$ **b** $\frac{1}{2}$ units^2

2 **a** $\dfrac{45}{2 \times \sqrt[3]{2}}$ units^2 **b** 12 units^2 **c** $\left(\dfrac{e^3 - 1}{e}\right)$ units^2

3 **a** $3\frac{1}{3}$ units^2 **b** $12\frac{2}{3}$ units^2

4 **a** $6\frac{1}{3}$ units^2 **b** $2\ln 3$ units^2 **c** $(6 - 3\ln\frac{5}{2})$ units^2

EXERCISE 23F.1

1 **a** 36π units^3 **b** 8π units^3 **c** $\frac{127\pi}{7}$ units^3

d $\frac{255\pi}{4}$ units^3 **e** $\frac{992\pi}{5}$ units^3 **f** $\frac{250\pi}{3}$ units^3

g $\frac{\pi}{2}$ units^3 **h** $\frac{40\pi}{3}$ units^3

2 **a** 186π units^3 **b** $\frac{146\pi}{5}$ units^3 **c** $\frac{\pi}{2}(e^8 - 1)$ units^3

d ≈ 17.0 units^3

3 **a** a cone of base radius r and height h

b $y = -\left(\dfrac{r}{h}\right)x + r$ **c** $V = \frac{1}{3}\pi r^2 h$

4 **a** a sphere of radius r

5 **a** 2π units^3 **b** $\pi\sqrt{3}$ units^3

6 **a** ≈ 15.3 units^3 **b** ≈ 19.7 units^3 **c** ≈ 18.6 units^3

d ≈ 30.2 units^3

EXERCISE 23F.2

1 **a** 8π units^3 **b** $\frac{1023\pi}{5}$ units^3 **c** $\frac{\pi}{2}(e^4 - 1)$ units^3

d $\frac{483\pi}{5}$ units^3 **e** $\frac{256\pi}{5}$ units^3

2 **a** $\frac{26\pi}{3}$ units^3 **b** $2\pi\ln 5$ units^3 **c** 52π units^3

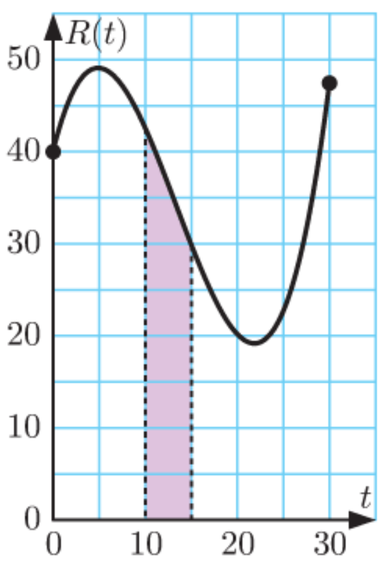
3 48π units^3 **4** 32π units^3

5 **a** 63π units^3 **b** $63\pi \approx 198$ cm^3 **6** ≈ 38.8 cm^3

EXERCISE 23G

1 **a** 20 cars per minute **b** $\approx$ 8:05 am

c



$\int_{10}^{15} R(t)\,dt$ represents the total number of cars going past the pedestrian crossing from 8:10 am to 8:15 am.

d 1031 cars

2 **a** **i** ≈ 1.65 L per minute **ii** ≈ 1.09 L per minute

b The rate of water leaking into the kayak is greater than the rate of water being bailed from the kayak after 2 minutes. So, the amount of water in the kayak is increasing after 2 minutes.

c **i** $\int_0^3 R_1(t)\,dt \approx 3.72$

About 3.72 litres of water have leaked into the kayak in the first 3 minutes.

ii $\int_2^5 R_2(t)\,dt \approx 5.27$

About 5.27 litres of water have been bailed out of the kayak from $t = 2$ minutes to $t = 5$ minutes.

iii $\int_0^8 [R_1(t) - R_2(t)]\,dt \approx 5.09$

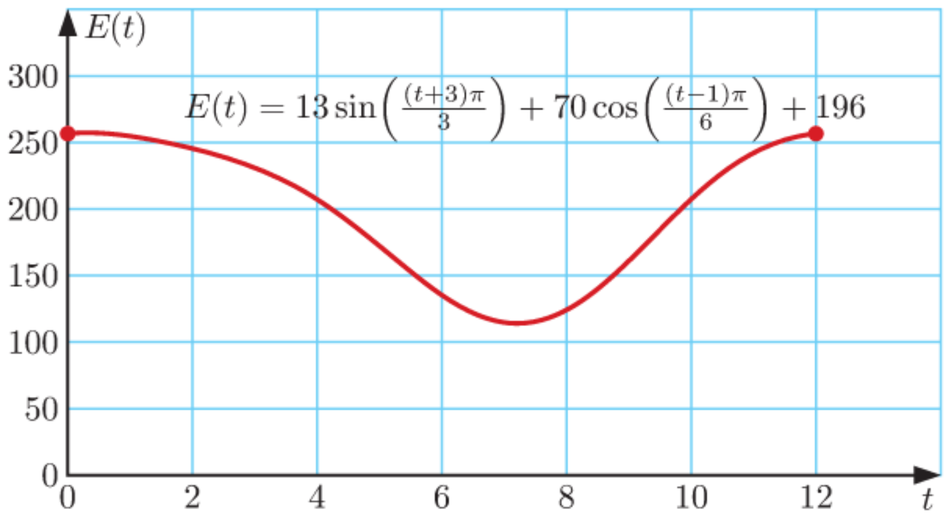
There are about 5.09 litres of water in the kayak 8 minutes after striking the rock.

d ≈ 6.31 litres

3 **a** C_1 is $y = 3\sin\frac{\pi t}{10}$, C_2 is $y = \sin\frac{\pi t}{10}$ **b** $\frac{40}{\pi}$ units

c The area in **b** represents the difference in the total amount of energy that enters the greenhouse in the first 10 hours on a sunny day versus an overcast day.

4 **a**



b **i** $\int_3^4 E(t)\,dt \approx 220.12$

The power consumption of the United Kingdom in April is about 220.12 TWh.

ii $\int_5^8 E(t)\,dt \approx 392.96$

The power consumption of the United Kingdom from June 1st to September 1st is about 392.96 TWh.

iii $\int_0^{12} E(t)\,dt = 2352$

The yearly power consumption of the United Kingdom is 2352 TWh.

REVIEW SET 23A

1 **a** 8 **b** $\frac{1}{8} - \frac{1}{3\sqrt{2}}$ **c** $\frac{178}{15}$

2 **a** $\frac{112}{9}$ **b** $\sqrt{2}$ **c** $2\ln 3$

3 $\dfrac{d}{dx}\left(e^{-2x}\sin x\right) = e^{-2x}(\cos x - 2\sin x)$

$\therefore \int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx = e^{-\pi}$

4 **a** $\approx 1.236\,17$ **b** $\approx 1.952\,49$ **c** $\approx 0.772\,652$

5 **a** $\frac{1}{4}(x^2+1)^4 + c$ **b** **i** $\frac{15}{4}$ **ii** $-\frac{609}{8}$ **6** $\dfrac{e-1}{3}$

7 **a** $(3\pi + 1)$ units^2 **b** 36 units^2 **c** $(3\ln 3 - 2)$ units^2

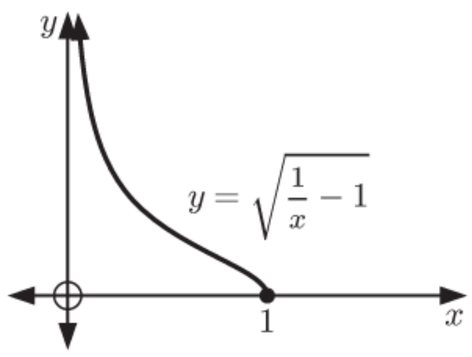
8 **a** 39 units^2 **b** $\left(\frac{16}{3} - 2\sqrt{3}\right)$ units^2 **c** 3 units^2

9 No, total area shaded $= \int_{-1}^{1} f(x)\,dx - \int_1^3 f(x)\,dx$.

10 $k = \sqrt[3]{16}$

11 **a** $2\frac{2}{3}$ units^2 **b** $(15\ln 5 - 30 + 3e)$ units^2

12 **a**



b ≈ 0.262 units^2

13 **a** 312π units^3 **b** 402π units^3 **c** 2π units^3

14 **a** $\pi\left(1 + \frac{1}{\sqrt{3}}\right)$ units^3 **b** ≈ 124 units^3

15 **a** $\frac{31\pi}{5}$ units^3 **b** $\frac{65\pi}{4}$ units^3

16 **a** **i** $\int_0^{\frac{1}{2}} R_2(t)\,dt \approx 0.655$

About 655 millilitres of water leak from the watering can in the first 30 seconds.

ii $\int_0^1 [R_1(t) - R_2(t)]\,dt \approx 5.03$

There are about 5.03 litres of water in the watering can after 1 minute.

b ≈ 199 seconds

REVIEW SET 23B

1 **a** $2 - \frac{2\sqrt{2}}{\sqrt{3}}$ **b** -3 **c** $\frac{8}{15}$

2 **a** $\frac{2}{3}(\sqrt{5} - \sqrt{2})$ **b** $4\ln\left(\frac{4}{3}\right)$ **c** $\frac{\pi}{4} + \sqrt{2}$

3 $b = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

4 **a** ≈ 3.528 **b** ≈ 2.963 **c** ≈ 0.9566 **5** $\frac{5}{18}$

6 **a** 22 units^2 **b** $\ln 3$ units^2 **c** $\frac{1}{2}(e^4 - 1)$ units^2

d $\frac{4\pi}{3}$ units^2

7 **a** x-coordinate of P ≈ -6.41, x-coordinate of Q ≈ 1.41

b ≈ 14.0 units^2

8 $k = \frac{4}{3}$ **9** $m = \frac{\pi}{3}$

10 **a** $\ln 2$ units^2 **b** $\left(3\sqrt[3]{4} - \frac{3}{4}\right)$ units^2

11 **a** A$(0, 2)$, B$(4, 0)$

b **i** $5\frac{1}{3}$ units^2 **ii** $5\frac{1}{3}$ units^2

We get the same result using both methods.

12 **a** 8π units^3 **b** 18π units^3

13 **a** ≈ 4.93 units^3 **b** ≈ 29.6 units^3

14 **a** $-\frac{1}{\sin^2 x}$ **b** 2π units^3

15 **a** 16 cm **b** $\frac{1280\pi}{9}$ cm^3 **c** ≈ 192 mL

16 **a** $\int_5^{12} E(t)\,dt \approx 10.7$

The solar energy transferred into Callum's solar panels from 5 am to 12 pm is about 10.7 kWh.

b $\int_{12}^{20} E(t)\,dt \approx 12.9$

The solar energy transferred into Callum's solar panels from 12 pm to 8 pm is about 12.9 kWh.

c $\int_5^{20} E(t)\,dt \approx 23.5$

The solar energy transferred into Callum's solar panels from 5 am to 8 pm is about 23.5 kWh.

EXERCISE 24A

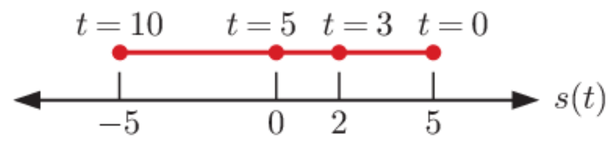
1 **a** 5 cm to the right of the origin

b **i** 2 cm to the right of the origin **ii** 5 cm to the left of the origin

c at $t = 5$ s

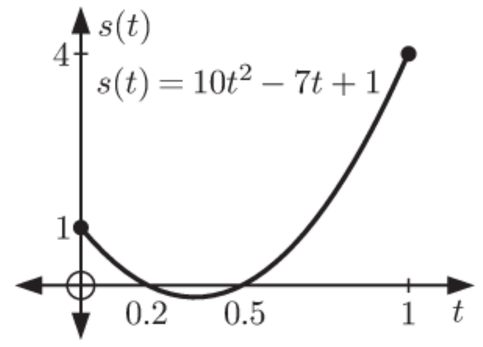
d No, the displacement function $s(t)$ is linear, so it has no turning points.

e



2 **a** 1 m to the right of the origin

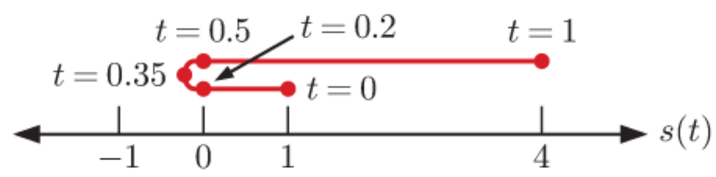
b



c $t = 0.35$ s, 0.225 m to the left of the origin

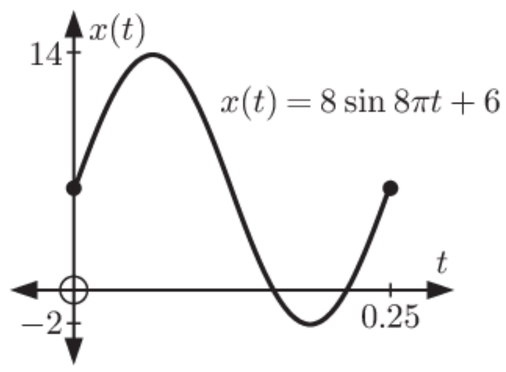
d at $0 \leqslant t < 0.2$ s and $0.5 < t \leqslant 1$ s

e



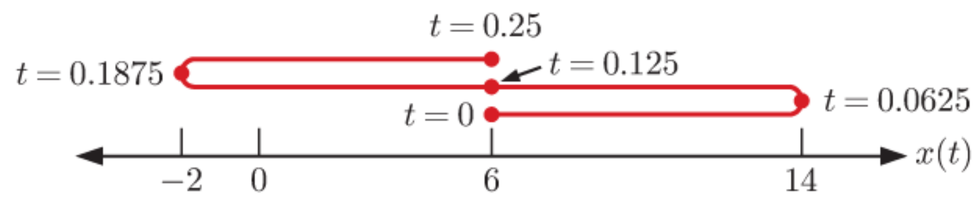
3 **a** at $t = 0$ s, $t = 0.125$ s, and $t = 0.25$ s

b



c 14 cm to the right of the origin, at $t = 0.0625$ s, and 2 cm to the left of the origin, at $t = 0.1875$ s

d



EXERCISE 24B.1

1 **a** **i** 3 m **ii** -3 m **b** -2 m s^{-1} **c** $-\frac{2}{3}$ m s^{-1}

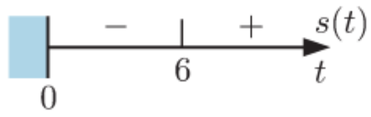
2 **a** -2 m s^{-1} **b** $v(t) = 2t - 6$ m s^{-1}

c **i** -4 m s^{-1} **ii** 4 m s^{-1}

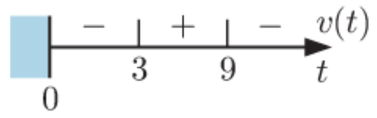
3 **a** 2 cm to the left of the origin **b** $t = 6$ s

c to the right **d** $t = 3$ s and $t = 9$ s

e **i**



ii



f 1 cm s^{-1}

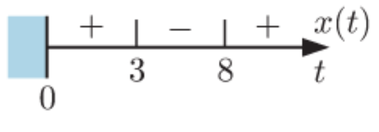
4 **a** $\frac{2}{3}$ cm s^{-1} **b** 3 cm to the right of the origin

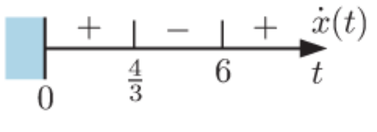
c $v(t) = \frac{1}{\sqrt{t}}$ cm s^{-1} **d** **i** $\frac{1}{2}$ cm s^{-1} **ii** $\frac{1}{4}$ cm s^{-1}

5 **a** $\dot{x}(t) = 3t^2 - 22t + 24$ m s^{-1}

b The object is initially at the origin, moving to the right at 24 m s^{-1}, with decreasing speed.

c



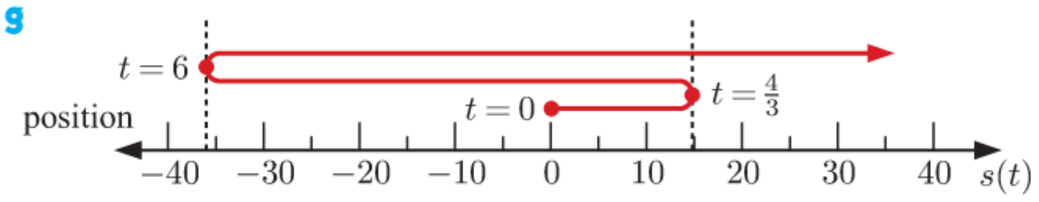


d at 0 s, 3 s, and 8 s

e at $\frac{4}{3}$ s and 6 s; $s(\frac{4}{3}) \approx 14.8$ m, $s(6) = -36$ m

f The object starts at O, and moves towards the right at 24 m s^{-1}. Its velocity is decreasing. After $\frac{4}{3}$ seconds, when it is 14.8 m to the right of O, it changes direction and moves to the left, passing O after 3 seconds. After 6 seconds, when it is 36 m to the left of O, it changes direction again and moves towards the right, passing O once more after 8 seconds.

g



6 **b** **i** 69.58 m s^{-1} **ii** ≈ 247 m

EXERCISE 24B.2

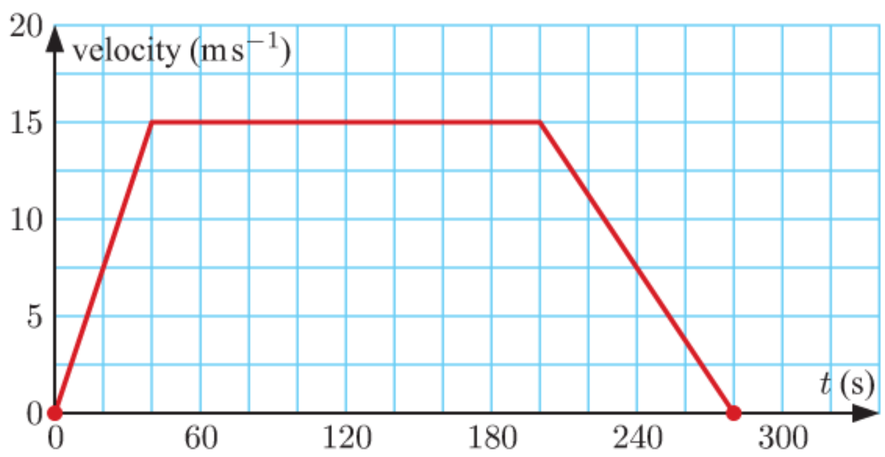
1 110 m

2 **a** **i** travelling forwards

ii travelling backwards (opposite direction)

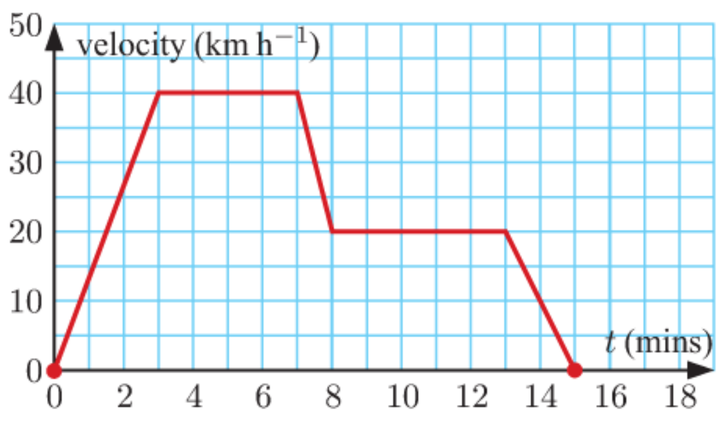
b 16 km **c** 8 km from starting point (on positive side)

3 **a**



b 3.3 km

4 **a**



b $6\frac{1}{6}$ km

5 **a** at $t = \frac{1}{2}$ second **b** $\frac{1}{2}$ cm **c** 0 cm

6 **a** $x(t) = \frac{1}{3}t^3 - \frac{1}{2}t^2 - 2t$ cm **b** $5\frac{1}{6}$ cm

c $1\frac{1}{2}$ cm left of the origin

7 **a** $s(t) = 29.4t - 4.9t^2 + 1$ m **b** 45.1 m

8 **a** $s(t) = 32t + 2t^2 + 16$ m

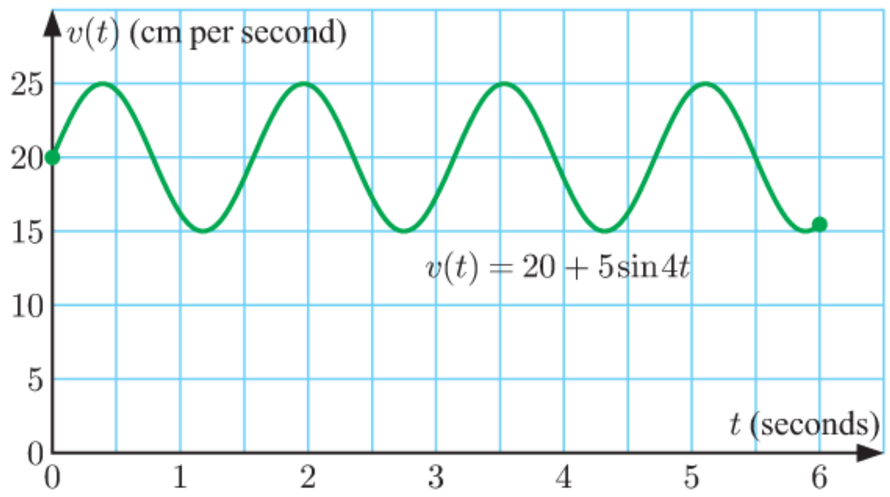
b There is no change in direction,

so displacement $= s(\tau) - s(0) = \int_0^\tau (32 + 4t)\,dt$

c 160 m

9 **a** 1 m **b** $s(\frac{\pi}{3}) = \frac{\sqrt{3}+2}{4}$ m

10 **a**



b ≈ 16.2 cm s^{-1} **c** ≈ 41.4 cm

11 **a** $x(t) = -4t + \frac{2}{3}t^{\frac{3}{2}}$ m **b** $t = 16$ s

c ≈ 10.5 m left of the origin **d** ≈ 32.2 m

12 **a** **i** 10 m s^{-1} **ii** $10\sqrt{2}$ m s^{-1} **b** $s(t) = \frac{20}{3}t^{\frac{3}{2}}$ m

c $\int_0^2 v(t)\,dt \approx 18.9$ The motorcyclist travels about 18.9 m in the first 2 seconds.

d **i** 4 seconds

ii Yes, he only needs $53\frac{1}{3}$ m to reach the required speed.

13 **a** ≈ 513 m **b** ≈ 0.637 cm

EXERCISE 24C

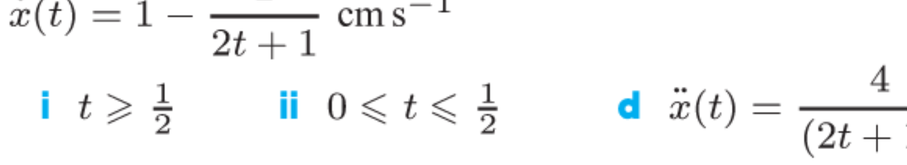
1 **a** 16 cm s^{-1} **b** 6 cm s^{-2}

c $a(t) = 10 - 2t$ cm s^{-2} **d** 4 cm s^{-2}

2 **a** $x(2) = -1$ m, $\dot{x}(2) = 8$ m s^{-1}, $\ddot{x}(2) = 10$ m s^{-2}

b $t = \frac{1}{3}$ s

3 **a** $v(t) = 98 - 9.8t$ m s^{-1} $a(t) = -9.8$ m s^{-2}



b $s(0) = 0$ m above the ground, $v(0) = 98$ m s^{-1} upward

c At $t = 5$ s, the stone is 367.5 m above the ground and moving upward at 49 m s^{-1}. Its speed is decreasing.

At $t = 12$ s, the stone is 470.4 m above the ground and moving downward at 19.6 m s^{-1}. Its speed is increasing.

d 490 m **e** 20 seconds

4 **a** $v(t) = 100 - 40e^{-\frac{t}{5}}$ cm s^{-1}, $a(t) = 8e^{-\frac{t}{5}}$ cm s^{-2}

b $s(0) = 200$ cm to the right of the origin

$v(0) = 60$ cm s^{-1}, $a(0) = 8$ cm s^{-2}

c as $t \to \infty$, $v(t) \to 100$ cm s^{-1} from below

d as $t \to \infty$, $a(t) \to 0$ cm s^{-2} from above

5 **b** $\dot{x}(t) = 1 - \dfrac{2}{2t+1}$ cm s^{-1}

c **i** $t \geqslant \frac{1}{2}$ **ii** $0 \leqslant t \leqslant \frac{1}{2}$ **d** $\ddot{x}(t) = \dfrac{4}{(2t+1)^2}$

f $\frac{4}{25}$ cm s^{-2} **g** $1 + \ln\left(\frac{4}{5}\right) \approx 0.777$ cm

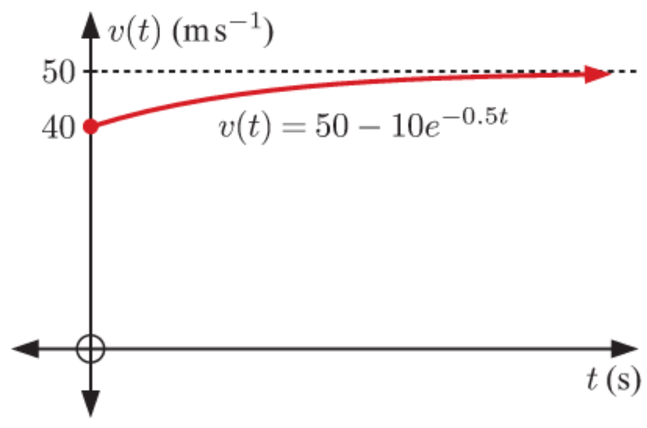
6 **a** 40 m s^{-1} **b** ≈ 47.8 m s^{-1} **c** $2\ln 2 \approx 1.39$ seconds

d as $t \to \infty$, $v(t) \to 50$ from below

e $a(t) = 5e^{-0.5t}$ and as $e^x > 0$ for all x, then $a(t) > 0$ for all t.

f $t = 2\ln\left(\frac{5}{2}\right)$ seconds

g



h ≈ 134 m

7 **a** $v(t) = \dfrac{t^2}{20} - 3t + 45$ m s^{-1}

b $\int_0^{60} v(t)\,dt = 900$ The train travels a total of 900 m in the first 60 seconds.

8 **a** **Hint:** Show that $\dot{x}(t) = 100 - 80e^{-\frac{1}{20}t}$ m s^{-1} and as $t \to \infty$, $\dot{x}(t) \to 100$ m s^{-1}.

b ≈ 370 m

9 **a** $v(t) = -\dfrac{1}{(t+1)^2} + 1$ m s^{-1}

b $s(t) = \dfrac{1}{t+1} + t - 1$ m

c The particle is $\frac{4}{3}$ m to the right of the origin, moving to the right at $\frac{8}{9}$ m s^{-1}, and accelerating at $\frac{2}{27}$ m s^{-2}.

EXERCISE 24D

1 **a** $v(t) = 2t - 6$ m s^{-1} $\quad a(t) = 2$ m s^{-2}

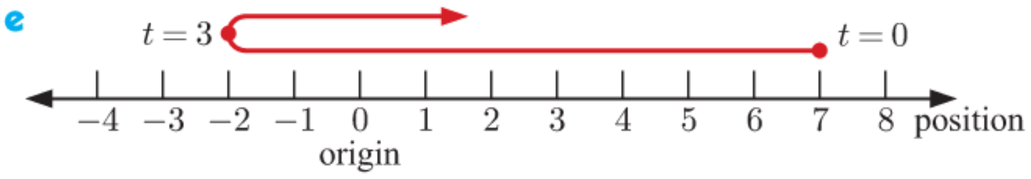


b **i** 4 m s^{-1} **ii** 2 m s^{-1}

c $s(0) = 7$ m, $v(0) = -6$ m s^{-1}, $a(0) = 2$ m s^{-2}
Initially, the object is 7 m to the right of O, moving to the left at 6 m s^{-1}, with acceleration 2 m s^{-2}.

d 2 m to the left of O

e



f $0 \leqslant t \leqslant 3$

2 **a** 1.2 m

b $s'(t) = 28.1 - 9.8t$ represents the instantaneous velocity of the ball.

c ≈ 41.5 m

d **i** 28.1 m s^{-1} **ii** 8.5 m s^{-1} **iii** 20.9 m s^{-1}

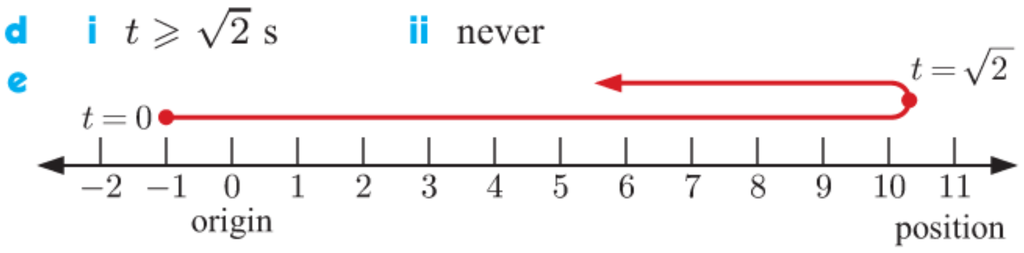
3 **a** $\dot{x}(t) = 12 - 6t^2$ cm s^{-1}, $\ddot{x}(t) = -12t$ cm s^{-2}

b $x(0) = -1$ cm, $\dot{x}(0) = 12$ cm s^{-1}, $\ddot{x}(0) = 0$ cm s^{-2}
The particle started 1 cm to the left of the origin and was travelling to the right at a constant speed of 12 cm s^{-1}.

c $t = \sqrt{2}$ s, $x(\sqrt{2}) = 8\sqrt{2} - 1$ cm

d **i** $t \geqslant \sqrt{2}$ s **ii** never

e



4 **a** $v(t) = -\dfrac{1}{2\sqrt{t+1}}$ m s^{-1} $\quad a(t) = \dfrac{1}{4(t+1)^{\frac{3}{2}}}$ m s^{-2}



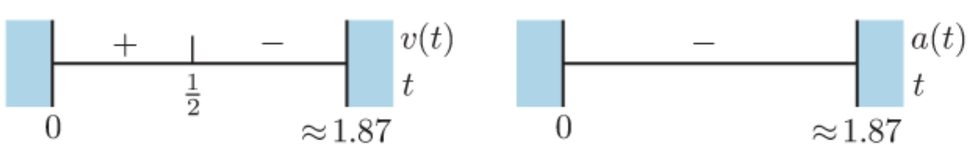
b $s(0) = 3$ m, $v(0) = -\frac{1}{2}$ m s^{-1}, $a(0) = \frac{1}{4}$ m s^{-2}
Initially, the particle is 3 m to the right of O, moving to the left at $\frac{1}{2}$ m s^{-1} with acceleration $\frac{1}{4}$ m s^{-2}.

c After 3 seconds, the particle is 2 m to the right of O, moving to the left at $\frac{1}{4}$ m s^{-1}, with acceleration $\frac{1}{32}$ m s^{-2}.

d The particle's speed is continuously decreasing.

5 **a** $k \approx 1.87$

b $v(t) = -9.8t + 4.9$ m s^{-1} $\quad a(t) = -9.8$ m s^{-2}



c **i** decreasing **ii** increasing

6 **a** $x(0) = -1$ cm, $\dot{x}(0) = 0$ cm s^{-1}, $\ddot{x}(0) = 2$ cm s^{-2}

b At $t = \frac{\pi}{4}$ seconds, the particle is $(\sqrt{2} - 1)$ cm to the left of O, moving to the right at $\sqrt{2}$ cm s^{-1}, with acceleration $\sqrt{2}$ cm s^{-2}.

c changes direction when $t = \pi$ s, $x(\pi) = 3$ cm

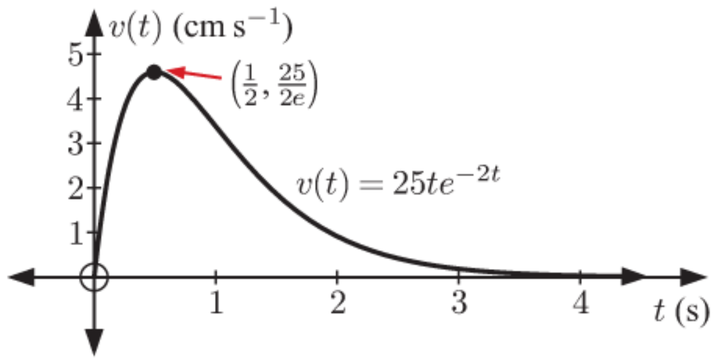
d increasing for $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

7 **a** **i** right **ii** left **b** $\dot{x}(t) = 4\cos\frac{t}{2}$ m s^{-1}

c **i** left **ii** right **d** $\ddot{x}(t) = -2\sin\frac{t}{2}$ m s^{-2}

e decreasing

8 **a**

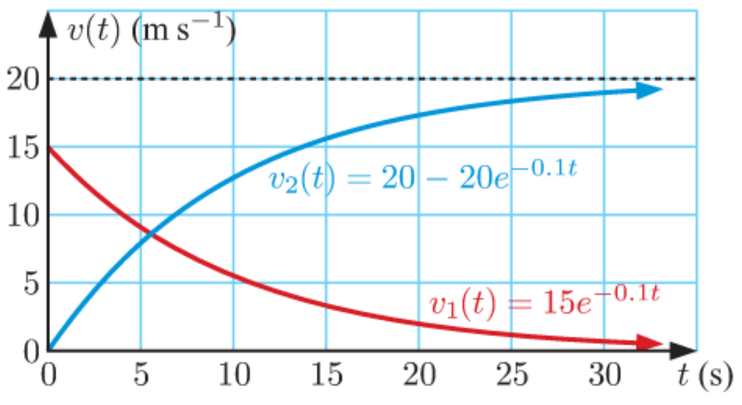


c $0 \leqslant t \leqslant \frac{1}{2}$

d $t \geqslant \frac{1}{2}$

9 **a** lion: ≈ 13.6 m s^{-1}, zebra: ≈ 1.90 m s^{-1}

b



The lion's speed $v_1(t)$ decreases over time whereas the zebra's speed $v_2(t)$ increases over time.

c $\displaystyle\int_0^3 v_1(t)\,dt = 150 - 150e^{-0.3} \approx 38.9$

The lion has travelled about 38.9 m in the first 3 seconds.

d $\displaystyle\int_0^3 [v_1(t) - v_2(t)]\,dt = 290 - 350e^{-0.3} \approx 30.7$

In the first 3 seconds, the lion has gained about 30.7 m on the zebra.

e At the time when $v_1(t) = v_2(t)$, the lion and the zebra will be moving at the same speed. Since the lion's speed decreases over time and the zebra's speed increases over time, the zebra will be faster than the lion after that time. So, they will be closest at the point when their speeds are equal.

f $-10\ln\left(\frac{4}{7}\right) \approx 5.60$ s

g No, the lion was about 1.92 m from the zebra at their closest point.

EXERCISE 24E

1 $a = 48s^5$ m s^{-2}

2 **a** $a = 18s^3$ m s^{-2}

b velocity $= -48$ m s^{-1}, acceleration $= 1152$ m s^{-2}

3 **a** $a = -\dfrac{36}{s^3}$ m s^{-2} **b** $-\frac{32}{3}$ m s^{-2}

4 $a = v\dfrac{dv}{ds} = k\sqrt{s} \times \dfrac{k}{2\sqrt{s}} = \dfrac{k^2}{2}$, where k is a constant

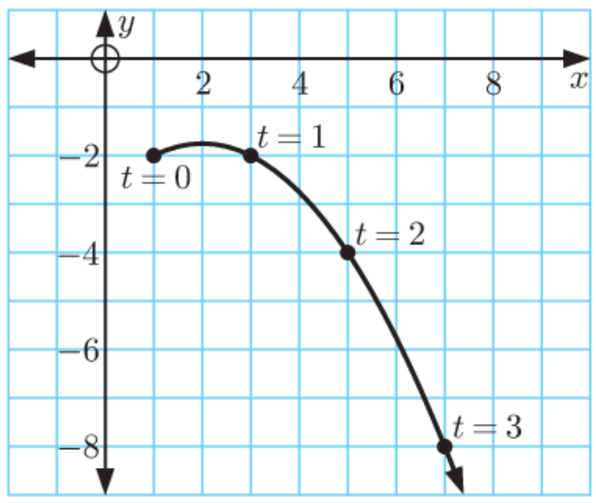
5 **a** $a = s(2s^2 - 9s + 9)$ m s^{-2}

b **i** initial velocity $= -10$ m s^{-1},
initial acceleration $= 70$ m s^{-2}

ii The particle initially moves to the left, and while $3 < s < 5$, $v < 0$ and $a > 0$. Once the particle reaches $s = 3$, $v = a = 0$, so the particle remains at $s = 3$.

EXERCISE 24F

1 **a** $(1, -2)$ **b**



c $\begin{pmatrix} 2 \\ -2t+1 \end{pmatrix}$

d **i** $\sqrt{5}$ m s^{-1} **ii** $\sqrt{13}$ m s^{-1}

e **i** $t = 5.5$ s **ii** $t = 3$ s

2 **a** $(11, -6)$ **b** $\begin{pmatrix} 3 \\ \hline \sqrt{t+1} \\ 4 \end{pmatrix}$ **c** 5 m s^{-1}

d The truck decreases to a speed of 4 m s^{-1}. **e** $t = \frac{19}{17}$ s

3 **a** $(-6, 7)$ **b** $\begin{pmatrix} -4e^{-t} \\ -3e^{-t} \end{pmatrix}$

c **i** $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{3}{4}$, which is constant, so the cat is moving in a straight line.

ii The velocity vector changes over time, so the cat is not moving with constant velocity.

d The cat's speed decreases over time as it moves in a straight line to $(-10, 4)$.

e $\tau \approx 3.91$

4 **a** $(2, -6)$ **b** $\begin{pmatrix} 1 \\ t-2 \end{pmatrix}$ **c** $\sqrt{2}$ m s^{-1}

d **ii** $k = 4$ **iii** $4\sqrt{5}$ m **iv** when $t = 4 + \sqrt{3}$ s

5 **a** $x(t) = t^2 - 2t + 2$, $y(t) = t^3 - 3t + 5$, $t \geqslant 0$

b $(2, 7)$ **c** $4\sqrt{37}$ m s^{-1}

6 **a** 10 m s^{-1}

c highest: ≈ 10.8 m s^{-1}, lowest: ≈ 9.90 m s^{-1}

d $x(t) = 6t - 2\cos t + 2$, $y(t) = 8t + 2\cos t - 2$, $t \geqslant 0$

e ≈ 99.4 m

7 **a** $(2, 0)$ **b** $(0, 2)$

c $(2\cos t)^2 + (2\sin t)^2 = 4(\cos^2 t + \sin^2 t) = 4$ ✓

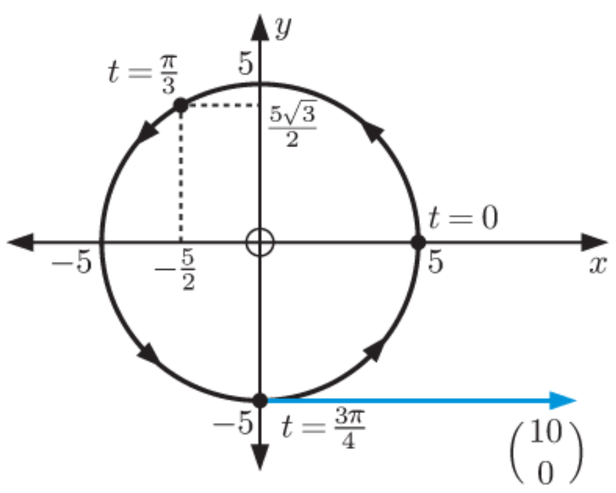
P moves in a circle centred at $(0, 0)$ with radius 2 m.

d $\begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix}$ **e** 2 m s^{-1}

8 **a** **i** $(5, 0)$ **ii** $(-\frac{5}{2}, \frac{5\sqrt{3}}{2})$

b position: $(0, -5)$, velocity: $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

c Starting from $(5, 0)$, P moves anticlockwise in a circle centred at $(0, 0)$ with radius 5 m.



d 10 m s^{-1}

9 **a** 30 cm **b** 12π seconds **c** $\begin{pmatrix} -5\sin\frac{t}{6} \\ 5\cos\frac{t}{6} \end{pmatrix}$

d 5 cm s^{-1}

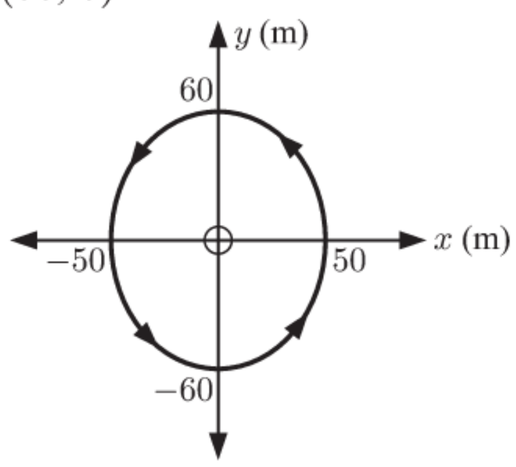
e **i** $x(t) = 30\cos\left(\frac{t-4}{6}\right)$, $y(t) = 30\sin\left(\frac{t-4}{6}\right)$

ii first train: $(30\cos 2, 30\sin 2)$, second train: $(30\cos\frac{4}{3}, 30\sin\frac{4}{3})$

iii $60\sin\frac{1}{3} \approx 19.6$ cm

10 **a** $(50, 0)$

c



The path Kristen runs along is an ellipse with width 100 m and height 120 m.

d 80 seconds **e** $\frac{3\pi}{2} \approx 4.71$ m s^{-1}

EXERCISE 24G

1 **a** $(-0.6, 1.8)$

b initial velocity $= \begin{pmatrix} 20.2 \\ 22.2 \end{pmatrix}$, initial speed ≈ 30.0 m s^{-1}

c 0.6 m **d** ≈ 4.61 seconds **e** ≈ 26.9 m

f ≈ 93.1 m **g** $\approx 47.7°$ **h** $\approx 48.7°$

2 **a** **i** $(3, 1.3)$ **ii** $(28.4, 8.1)$ **b** ≈ 2.79 seconds

c ≈ 35.4 m **d** ≈ 10.2 m **e** ≈ 14.2 m s^{-1}

f $\approx 46.1°$ **g** $\approx 48.1°$

3 **a** initial speed ≈ 62.6 m s^{-1}, initial angle of trajectory $\approx 16.7°$

b $\frac{90}{49} \approx 1.84$ seconds **c** ≈ 16.5 m

d No, the golf ball would be slowed by air resistance and would have a lower maximum height.

4 **a** **i** $5\sqrt{2} \approx 7.07$ m s^{-1} **ii** $\sqrt{25.01} \approx 5.00$ m s^{-1}

b ≈ 3.04 seconds **c** ≈ 15.2 m

5 **a** $x(t) = at$, $y(t) = 2 + bt - 4.9t^2$, $t \geqslant 0$

b $b = a\sqrt{3}$ **c** $a \approx 3.24$ **d** ≈ 6.48 m s^{-1}

e ≈ 3.61 m **f** ≈ 4.74 m s^{-1}

6 **a** **i** $t = \dfrac{b}{g}$ **ii** maximum height $= \dfrac{b^2}{2g}$ **iii** $t = \dfrac{2b}{g}$

iv horizontal distance travelled $= \dfrac{2ab}{g}$

b **i** $D = \sqrt{(at)^2 + (bt - \frac{1}{2}gt^2)^2}$

ii $\dfrac{d}{dt}\left(D^2\right) = 2a^2t + 2b^2t - 3bgt^2 + g^2t^3$

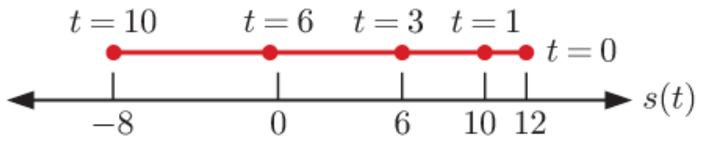
REVIEW SET 24A

1 **a** 12 m to the right of the origin

b **i** 10 m to the right of the origin **ii** 6 m to the right of the origin **c** $t = 6$ s

d No, the displacement function is linear, so it has no turning points.

e



2 **a** $v(t) = 6t^2 - 18t + 12$ cm s^{-1}, $a(t) = 12t - 18$ cm s^{-2}

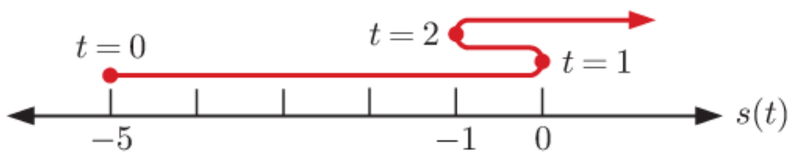


b $s(0) = -5$ cm (5 cm to the left of origin)
$v(0) = 12$ cm s^{-1} towards origin
$a(0) = -18$ cm s^{-2} (decreasing speed)

c At $t = 2$, the particle is 1 cm to the left of the origin, is instantaneously stationary, and is beginning to accelerate.

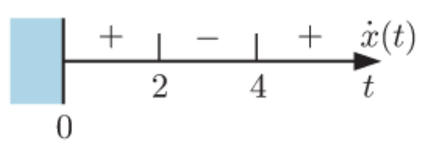
d at $t = 1$ s, $s = 0$ cm, and at $t = 2$ s, $s = -1$ cm

e



f $1 \leqslant t \leqslant \frac{3}{2}$ and $t \geqslant 2$

3 **a**



b The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$, it is $6\frac{2}{3}$ m from its starting point again.

c $6\frac{2}{3}$ m **d** $9\frac{1}{3}$ m

4 **a** **i** 2.75 m s^{-1} **ii** ≈ 1.62 m s^{-1}

b **Hint:** Show that $a(t)$ and $v(t)$ are opposite in sign for all $0 \leqslant t \leqslant 6$.

c $\int_0^2 v(t)\,dt \approx 4.54$ The kayak travels approximately 4.54 m in the first 2 seconds after the kayaker stops paddling.

5 **a** $v(t) = 15 + \dfrac{120}{(t+1)^3}$ cm s^{-1}, $a(t) = -\dfrac{360}{(t+1)^4}$ cm s^{-2}

b At $t = 3$, the particle is 41.25 cm to the right of O, moving to the right at 16.875 cm s^{-1}, with decreasing speed ($a(3) \approx -1.41$ cm s^{-2}).

c The particle's speed is never increasing.

6 **a** $x(0) = 3$ m, $\dot{x}(0) = 2\pi$ m s^{-1}, $\ddot{x}(0) = 0$ m s^{-2}

b $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, 4\frac{1}{2}$ s **c** 20 m

7 **a** 1 m **b** 6 m **c** 4 m

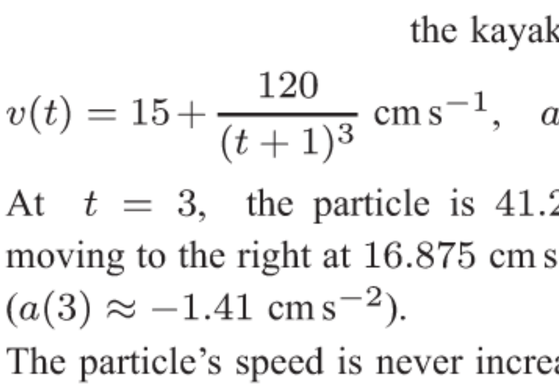
8 **a** $v(t) = -2t + 65$ m s^{-1} **b** $s(t) = -t^2 + 65t$ m

c **i** 31 s **ii** 1054 m

9 **a** $v(0) = 25$ m s^{-1}, $v(3) = 4$ m s^{-1}

b as $t \to \infty$, $v(t) \to 0$ from above

c



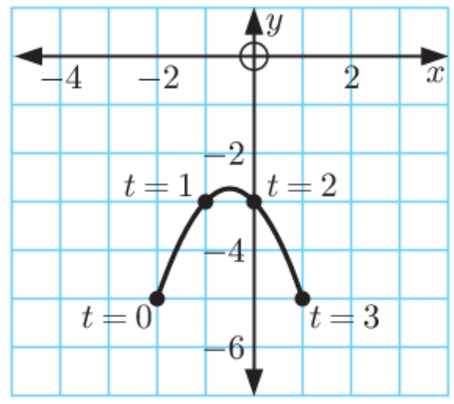
d $\int_0^2 v(t)\,dt = 25$ The boat travels a total distance of 25 m in the first 2 seconds after its engine is turned off.

e 3 seconds

10 **a** $a = 2s^3$ m s^{-2} **b** initial velocity $= 4$ m s^{-1}, initial acceleration $= -16$ m s^{-2}

11 **a** $(-2, -5)$

b



c $\begin{pmatrix} 1 \\ -2t + 3 \end{pmatrix}$

d **i** $\sqrt{10}$ m s^{-1} **ii** $\sqrt{26}$ m s^{-1}

12 **a** **i** (10, 0) **ii** (0, 10) **b** anticlockwise

c 4π seconds **d** $\begin{pmatrix} -5\sin\frac{t}{2} \\ 5\cos\frac{t}{2} \end{pmatrix}$ **e** 5 m s^{-1}

13 **a** $\sqrt{181} \approx 13.5$ m s^{-1} **b** $\approx 42.0°$ **c** ≈ 5.63 m

d ≈ 1.99 seconds **e** ≈ 19.9 m

REVIEW SET 24B

1 **a** 1 m to the right of the origin **b** 8 m s^{-1}

c $v(t) = 2t + 4$ m s^{-1} **d** 6 m s^{-1}

2 200 m

3 **a** $v(t) = 3t^2 - 30t + 27$ cm s^{-1}

b -162 cm (162 cm to the left of the origin)

4 **a** $\dot{x}(t) = 3 - \frac{3}{2}\sqrt{t}$ cm s^{-1} $\ddot{x}(t) = -\dfrac{3}{4\sqrt{t}}$ cm s^{-2}



b $x(0) = 0$, $\dot{x}(0) = 3$
The particle is initially at the origin, moving to the right at 3 cm s^{-1}.

c The particle is ≈ 3.17 cm to the right of the origin, travelling to the right at ≈ 0.879 cm s^{-1}, with decreasing speed ($\ddot{x}(2) \approx -0.530$ cm s^{-2}).

d at $t = 4$ s, 4 cm to the right of O **e** $0 \leqslant t \leqslant 4$

f



g ≈ 4.70 cm

5 **a** **i** 7.2 m s^{-2} **ii** 9.6 m s^{-2} **iii** 0 m s^{-2}
iv -12 m s^{-2}

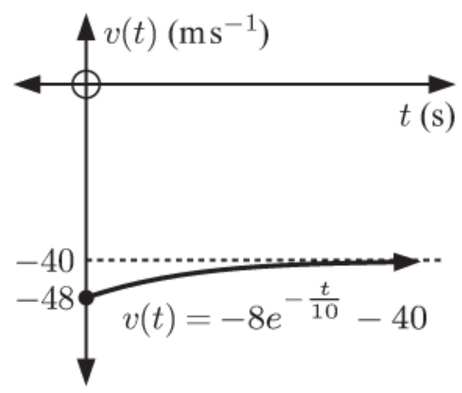
b $\int_0^3 v(t)\,dt = 27$ The human cannonball travels 27 m in the first 3 seconds.

c ≈ 3.14 s

6 **a** $v(t) = -8e^{-\frac{t}{10}} - 40$ m s^{-1}
$a(t) = \frac{4}{5}e^{-\frac{t}{10}}$ m s^{-2} $\{t \geqslant 0\}$

b $s(0) = 80$ m,
$v(0) = -48$ m s^{-1},
$a(0) = 0.8$ m s^{-2}

c



d $t = 10 \ln 2$ seconds

7 **a** $v(0) = 0$ cm s^{-1}, $v(\frac{1}{2}) = -\pi$ cm s^{-1}, $v(1) = 0$ cm s^{-1}, $v(1\frac{1}{2}) = \pi$ cm s^{-1}, $v(2) = 0$ cm s^{-1}

b $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, and so on
So, for $2n \leqslant t \leqslant 2n + 1$, $n \in \{0, 1, 2, 3, ...\}$

8 **a** $\approx 6.76\ \text{m s}^{-1}$

b $a(t) = 0.15(t^{1.1} + 3t)^{0.5}(1.1t^{0.1} + 3)\ \ \text{m s}^{-2}$

c $\approx 1.79\ \text{m s}^{-2}$ **d** $\approx 109\ \text{m}$

9 **a** $x(t) = -\frac{1}{96}t^4 - \frac{1}{24}t^2 + 2\ \text{m}$ **b** $\approx 3.46\ \text{s}$

10 **a** Tyson

b $\int_0^5 v_1(t)\,dt \approx 42.0$ Tyson has travelled about 42.0 m in the first 5 seconds of the race.

c $s_1(t) = 10t + 8e^{-1.25t} - 8\ \text{m}$ **d** Tyson

$s_2(t) = 10.5t + 10.5e^{-t} - 10.5\ \text{m}$

e $s_1(10.8) \approx 100$ **f** Maurice

12 **b** **i** $\approx 5.14\ \text{m s}^{-1}$

ii $x(t) = \dfrac{6\sin\frac{t}{5}}{e^{\frac{t}{5}}}, \quad y(t) = -40e^{-\frac{t}{8}} + 40$

iii $\approx 1.64\ \text{m}$ **iv** $\approx 1.45\ \text{m s}^{-1}$

EXERCISE 25A

1 $\dfrac{dP}{dt} = kP, \quad k > 0$ **2** $\dfrac{dG}{dt} = k\sqrt{G}, \quad k > 0$

3 $\dfrac{dT}{dt} = -k(T - \tau), \quad k > 0$

4 **a** **i** mg **ii** $-kv^2$

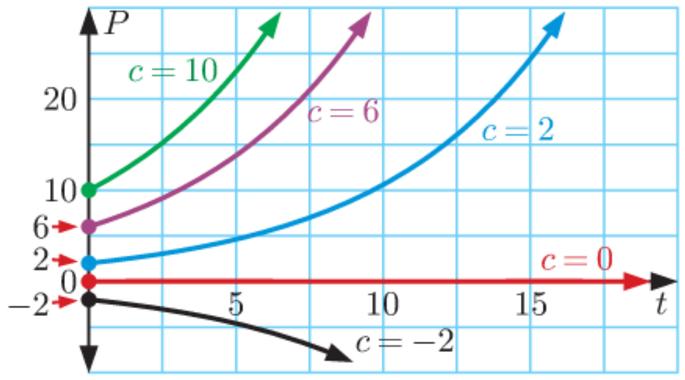
b Air resistance *reduces* the resultant force of the parachutist.

5 **b** $g \approx 9.95\ \text{m s}^{-2}$

EXERCISE 25B

2 **a** **B** **b** **C** **c** **A**

4 **b**

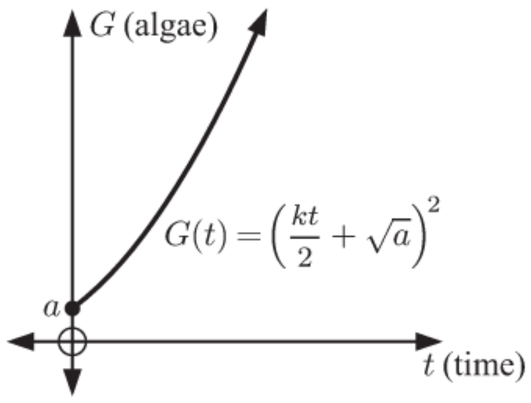


c c must be positive so that the population of guinea pigs is positive.

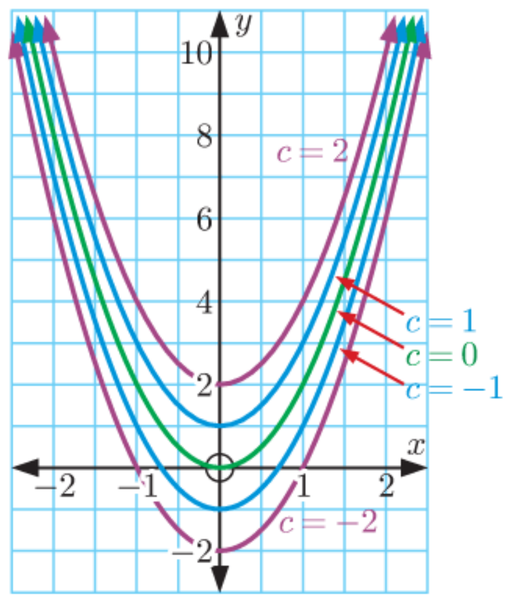
d $P = 30e^{\frac{t}{6}}$

5 **b** $G(0) = a$, a is the initial amount of algae.

d



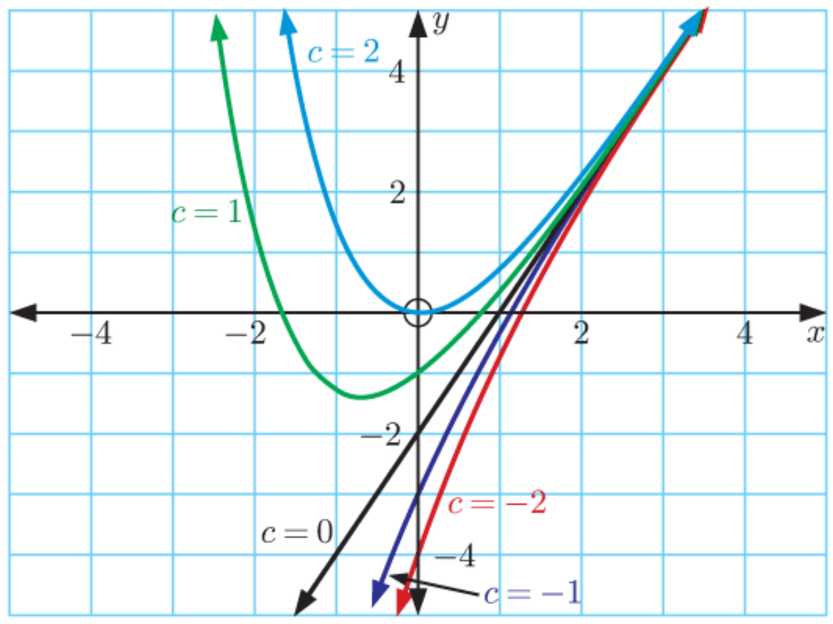
6 **b**



c $y = 2x^2 - \frac{3}{2}$

d $y = 4x - \frac{7}{2}$

7 **b**



c $y = 2x - 2 + 3e^{-x}$ **d** $y = -x + 1$

8 **b** $\lambda = \pm iw$

EXERCISE 25C

1 **a** $y = x^4 + c$ **b** $y = \frac{1}{3}x^3 + 3x^2 + c$

c $y = \frac{1}{3}e^{3x} + 4x + c$ **d** $y = \sin x - \frac{1}{2}\cos 2x + c$

e $y = \ln|x + 4| + c$ **f** $y = \frac{2}{3}x^{\frac{3}{2}} - 2\ln|x| + c$

2 **a** $M = \ln\left|t^3 - 4\right| + c$ **b** $y = -\sqrt{25 - t^2} + c$

c $f(t) = -\frac{1}{2}e^{-t^2+1} + 2t + c$

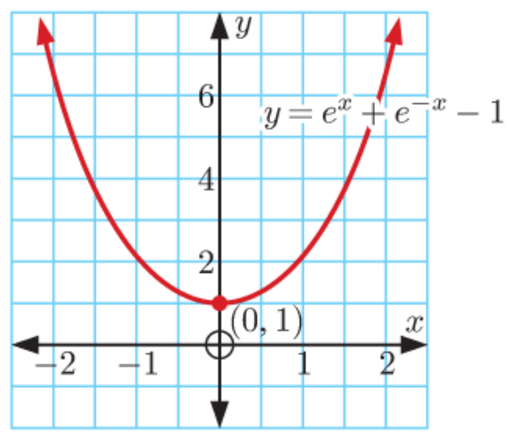
3 **a** $y = \frac{3}{2}x^2 - 2x + 5$ **b** $y = \frac{1}{3}e^{3x} + x - \frac{1}{3}$

c $y = \ln|x| + \ln 6$

4 **a** $y = e^{2t} + e^{-t} - 2$ **b** $M = \frac{1}{2}\sin 2\alpha + 3\cos\alpha + 2$

5 $f(-2) = 2$

6 **a** $y = e^x + e^{-x} - 1$ **b**



c $3x - 2y = 3\ln 2 - 3$

7 $f(x) = \frac{1}{2}x^2 + \dfrac{1}{x} - \frac{3}{2}$

8 **a** $f(x) = x^3 - 2x^2 + 4x - 1$ **b** $y = -\frac{1}{4}\sin 2x + 4x$

9 £4250

10 **a** $P(x) = 15x - 0.015x^2 - 650$ pounds

b maximum profit is £3100, when 500 plates are made

c $46 \leqslant x \leqslant 954$ (you cannot produce part of a plate)

11 $\approx 225°\text{C}$

12 **a** $\approx 400°\text{C}$ **b** $\approx 0.387\ \text{m}$ **c** $\approx 0.187\ \text{m}$

13 **a** $y = \left(\frac{0.01}{3}x^3 - \frac{0.005}{12}x^4 - \frac{0.08}{3}x\right)$ metres

b ≈ 3.33 cm; yes, it seems reasonable that the maximum sag occurs when $x = 2$.

c 2.375 cm **d** $\approx 1.05°$

EXERCISE 25D

1 **a** $y = \sqrt[3]{\frac{3}{2}x^2 + c}$ **b** $y = \ln(x^2 + c)$ **c** $y = Ae^{\frac{3}{2}x^2}$

d $y = \left(\dfrac{x^2}{2} + c\right)^2$ **e** $y = Ae^{-\cos x}$

f $y = \left(-\frac{1}{4}x^2 + c\right)^2 - 1$ **g** $y = Ax$

h $y = -\ln(c - x^3)$ **i** $y = A(x - 1) - 2$

2 **a** $y = Ae^x$ **b** $y = \pm\sqrt{2x + c}$ **c** $y = Ae^t + 4$
d $P = \left(\frac{3}{2}t + c\right)^2$ **e** $Q = Ae^{2t} - \frac{3}{2}$ **f** $t = Q^2 + 3Q + c$

3 $y = \dfrac{1}{c - x}$, defined for $x \neq c$

4 **a** $y = \sqrt[3]{\frac{9}{2}x^2 + 1}$ **b** $y = \frac{1}{36}(x - 26)^2$
c $y = e^{x + \frac{1}{3}x^3}$ **d** $y = \arcsin\left(\frac{3}{2}x^2 - \frac{3}{2}\right)$
e $y = \left(\frac{9}{2}\sin 2x + 3\sqrt{3}\right)^{\frac{2}{3}}$

5 $y = (x + 3)^2$, defined for $x \geqslant -3$

6 **a** $P = 40e^{\frac{t}{2}}$ **b** ≈ 803 rabbits

7 **a** $I = 350e^{-0.4t}$ **b** ≈ 47.4 milliamps
c ≈ 7.16 milliseconds

8 ≈ 5.19 minutes **9** 0.8%

10 $A(t) = \frac{16}{9}\left((\sqrt{2} - 1)t + 3\right)^2$

11 **a** $(V_0 - V)$ is the amount of water remaining

so $\dfrac{dV}{dt} \propto (V_0 - V)$

$\therefore \dfrac{dV}{dt} = k(V_0 - V)$ for some constant k

b $\approx 17.7\%$

12 **a** ≈ 12.5 minutes **b** midnight

13 **a** $I(t) = Ae^{-\frac{100}{3}t} + 2$ **b** $I(t) = 2 - 2e^{-\frac{100}{3}t}$
c 2 amps **d** ≈ 0.138 seconds

14 4 hours

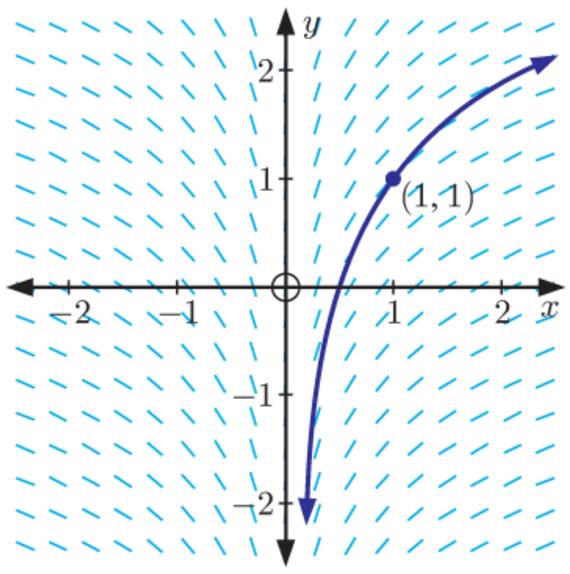
15 **a** $\dfrac{dh}{dt} = \dfrac{r^2}{\pi h^2 - 2\pi r h}$

b **ii** ≈ 14.4 hours **iii** ≈ 20.9 hours

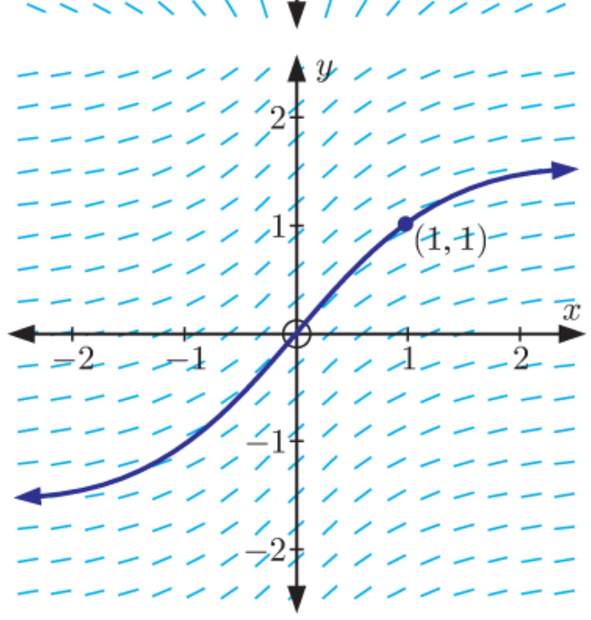
16 **b** ≈ 35.6 minutes

EXERCISE 25E

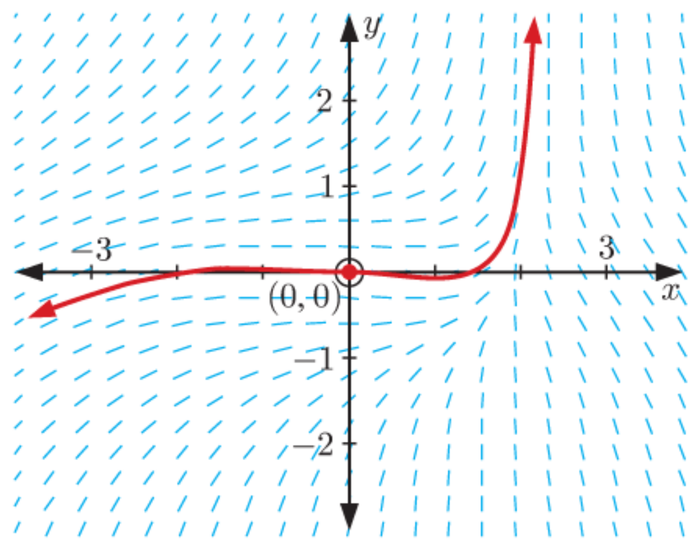
1 **a**



b

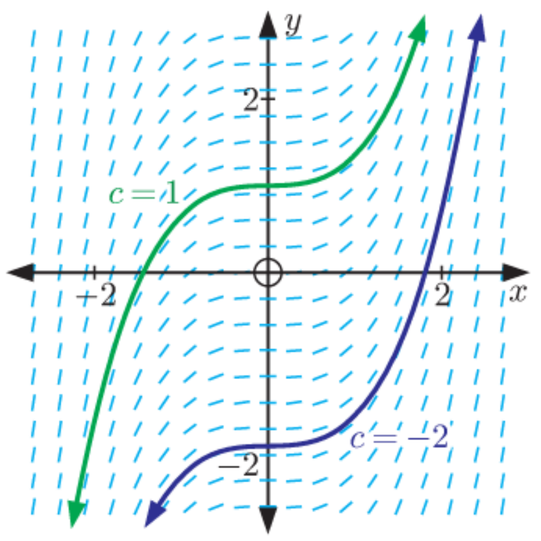


2 **a** $-\frac{1}{10}$ **b**

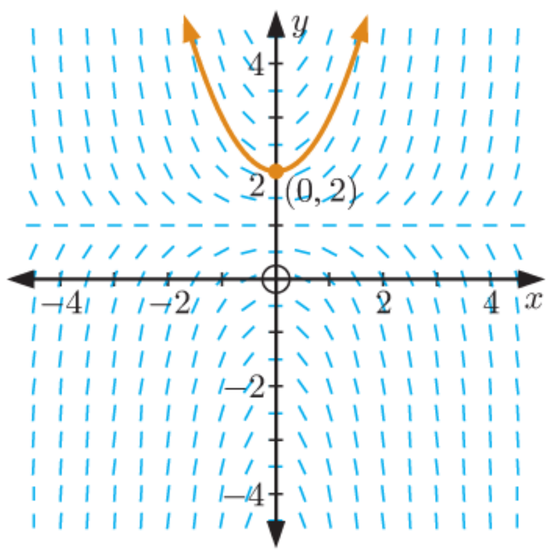


3 **a** $y = \frac{1}{3}x^3 + c$

b **i** $y = \frac{1}{3}x^3 + 1$ **ii** $y = \frac{1}{3}x^3 - 2$

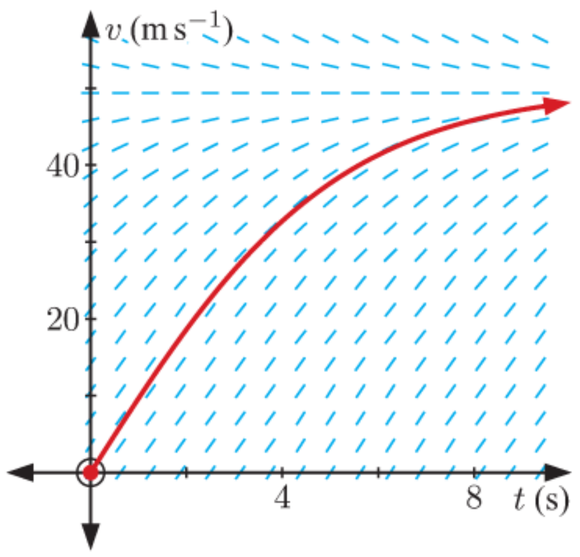


4 **a**



b $y = e^{\frac{1}{2}x^2} + 1$

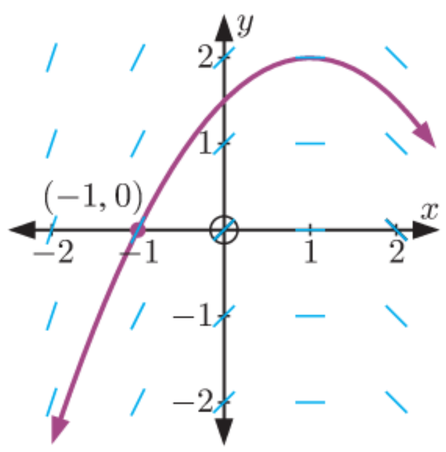
5 **a**



b The skydiver approaches a terminal velocity of 50 m s^{-1}.

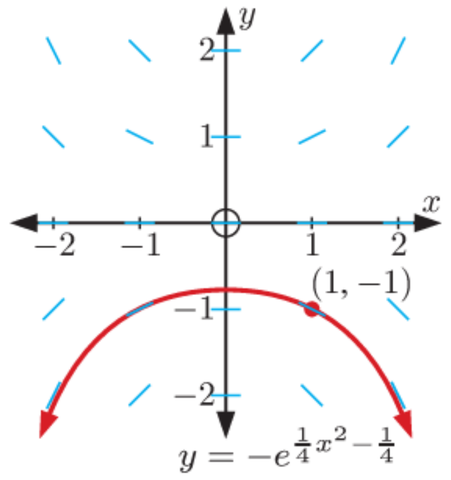
6 **a** $y = x - \frac{1}{2}x^2 + \frac{3}{2}$

b



7 **a** $y = -e^{\frac{1}{4}x^2 - \frac{1}{4}}$

b

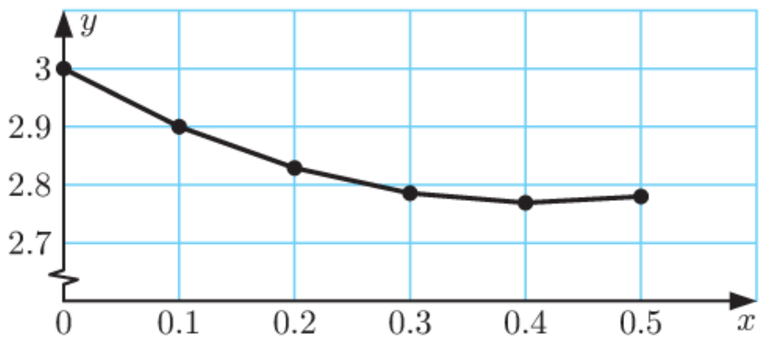


EXERCISE 25F

1 a

Iteration	x_{i-1}	y_{i-1}	$\frac{dy}{dx}$	x_i	y_i
1	0	3	-1	0.1	2.9
2	0.1	2.9	-0.71	0.2	2.829
3	0.2	2.829	-0.4342	0.3	2.7856
4	0.3	2.7856	-0.1643	0.4	2.7691
5	0.4	2.7691	0.1077	0.5	2.7799

b



2 a **i** ≈ 3.4865 **ii** ≈ 3.8283

b $y(0.5) = \frac{3}{2}e \approx 4.0774$

The accuracy of Euler's method was improved by decreasing the step size.

3 $y(1) \approx 0.7869$

4 a $y(0.5) \approx -0.4851$ **b** $y(0.5) \approx -0.4795$

c $y(0.5) = -\sin(0.5) \approx -0.4794$

The accuracy of Euler's method was improved by decreasing the step size.

5 a ≈ 1 **b** ≈ 1.3021 **c** ≈ 1.3018

REVIEW SET 25A

1 $\dfrac{dM}{dt} = -\dfrac{k}{M^2}, \quad k > 0$

3 a $y = \frac{1}{2}\sin 2x + \cos x + c$ **b** $y = 3x + \frac{1}{2}e^{-2x} + c$

c $y = \frac{1}{2}\ln|2x+1| + 2$ **d** $y = \frac{1}{2}(t^2 + e^{t^2} + 3e - 1)$

4 a **i** 160 spectators per minute

ii ≈ 249 spectators per minute

b $\dfrac{d}{dt}\left(\dfrac{1}{1+4e^{-0.05t}}\right) = \dfrac{0.2e^{-0.05t}}{(1+4e^{-0.05t})^2}$

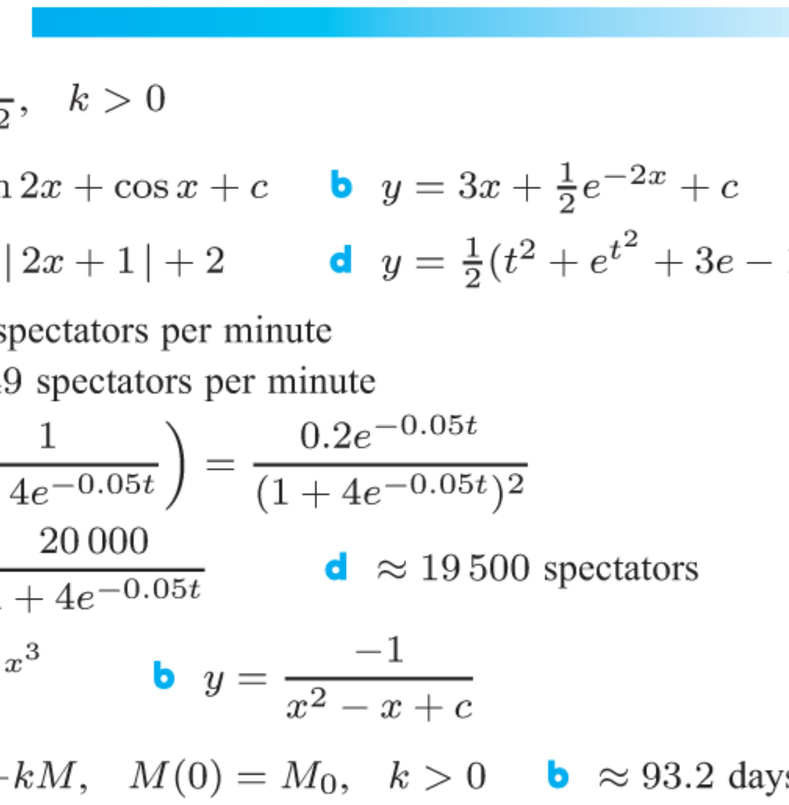
c $S(t) = \dfrac{20\,000}{1+4e^{-0.05t}}$ **d** $\approx 19\,500$ spectators

5 a $y = Ae^{\frac{5}{3}x^3}$ **b** $y = \dfrac{-1}{x^2 - x + c}$

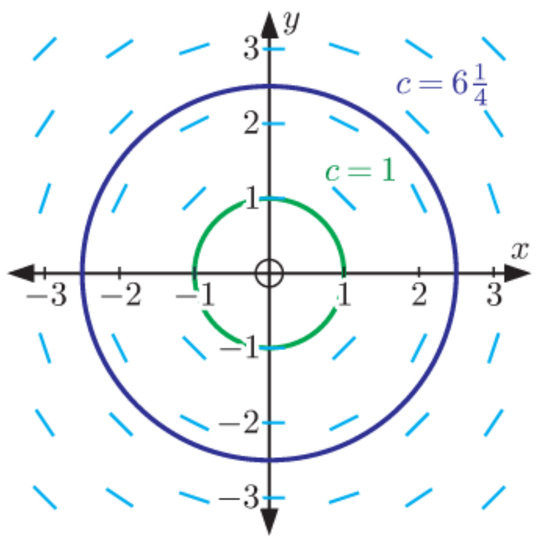
6 a $\dfrac{dM}{dt} = -kM, \quad M(0) = M_0, \quad k > 0$ **b** ≈ 93.2 days

7 $y = \ln(e - 3x)$

8 a 1 **b**



9 b

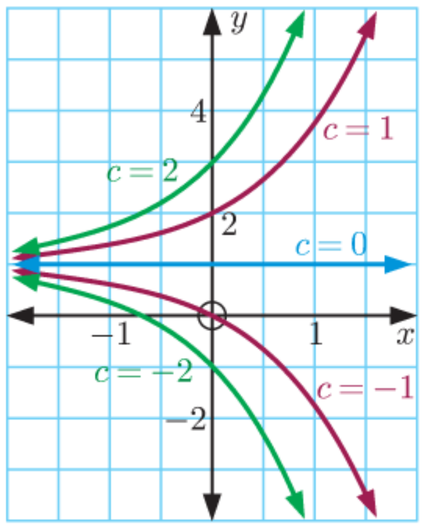


10 $y \approx 1.0088$

REVIEW SET 25B

1 $\dfrac{dD}{dt} = k\sqrt{D}, \quad k > 0$

3 b



c $y = 3e^x + 1$

d $y = 3x + 4$

4 a $y = \ln|e^x - 2| + c$ **b** $y = -\frac{1}{4}\sin\left(\frac{\pi}{3} - 2x\right) - \frac{\sqrt{3}}{8}$

5 a $y = \dfrac{1}{\sqrt[3]{c - 6x}}$ **b** $P = A\sqrt{t^2+1}$

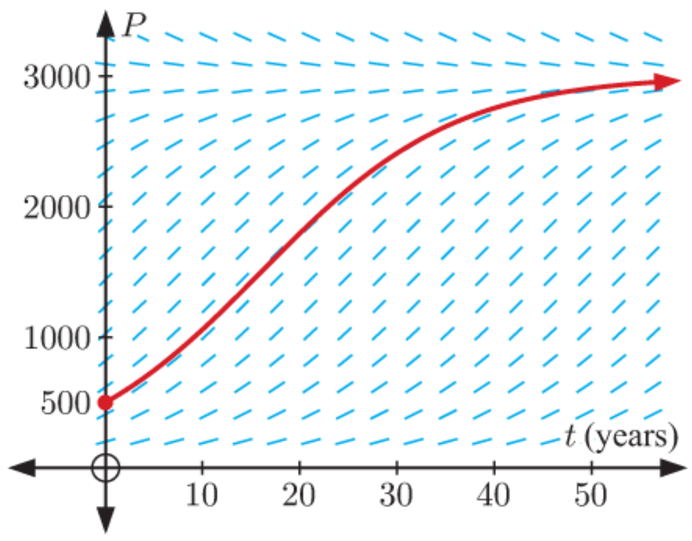
6 a $y = (\frac{1}{2}x \pm 2)^2$ **b** $y = e^{\sin x - 3}$

7 a $\dfrac{dV}{dt} = -k\sqrt{h}, \quad k > 0$

b $V = 2 \times 2 \times h, \quad \dfrac{dh}{dt} = -\dfrac{k}{4}\sqrt{h}, \quad k > 0$ **c** 20 minutes

8 a **B** **b** **C** **c** **A**

9 a



b The population of rodents approaches the limiting population of 3000.

10 $y(0.5) \approx 0.8555$

EXERCISE 26A

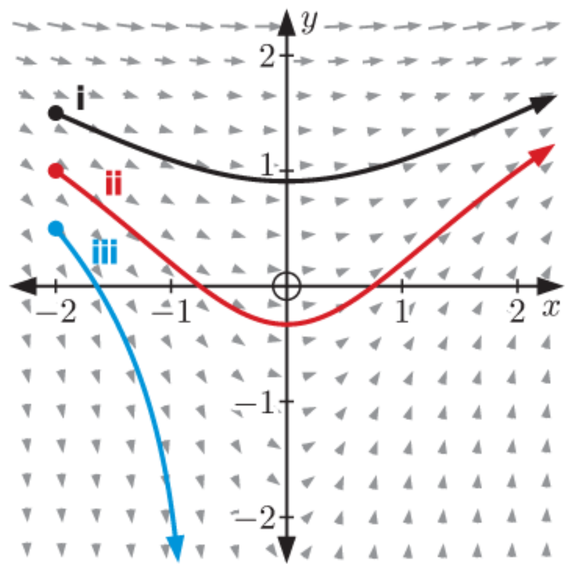
1 a **i** $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

On the phase portrait, the vector at $(1, 0)$ has the direction $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

ii $\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dx} \end{pmatrix} = \begin{pmatrix} e \\ -1 \end{pmatrix}$

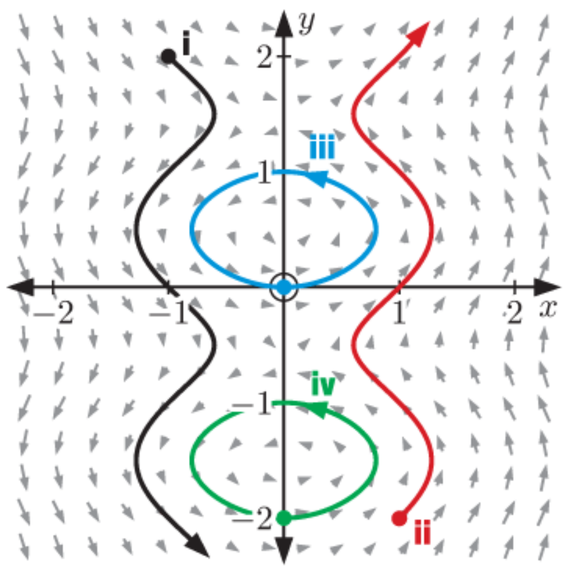
On the phase portrait, the vector at $(-1, 1)$ has direction $\begin{pmatrix} e \\ -1 \end{pmatrix}$.

b



2 **a** **i** centre
ii saddle point

b



3 **a** $(0, 0)$, unstable spiral **b** $(-1, -1)$, saddle point
c $(1, 1)$, unstable fixed **d** $(0, 0)$, centre

4 $(-\frac{1}{2}, -\frac{1}{4})$, stable spiral

5 $(1, 1)$ and $(-1, -1)$ are centres,
$(1, -1)$ and $(-1, 1)$ are saddle points.

6 **a** $(\frac{3}{2}, \frac{1}{2})$, centre

b Anticlockwise, for example at $(0, 0)$ the vector has direction $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

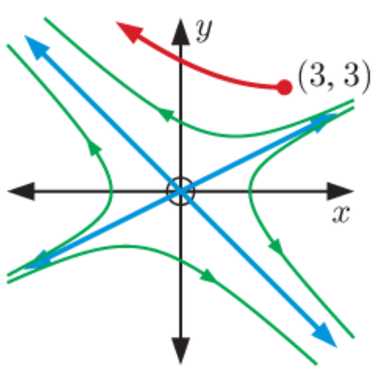
EXERCISE 26B

1 **a** $\mathbf{x} = Ae^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + Be^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b **i** $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ **ii** $\mathbf{x} = 2e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

c saddle point

d

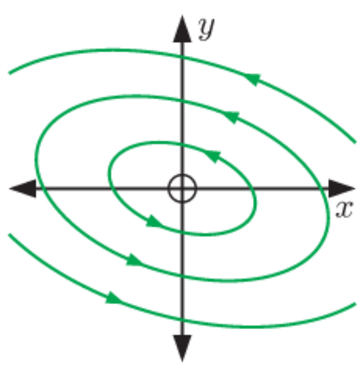


e The line $k\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ will be an asymptote as $t \to \infty$.

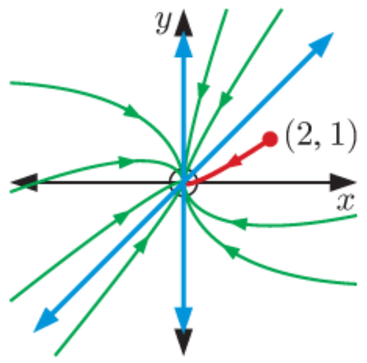
2 **a** The eigenvalues of $\mathbf{A}$ are purely imaginary.

b **i** $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$
ii anticlockwise

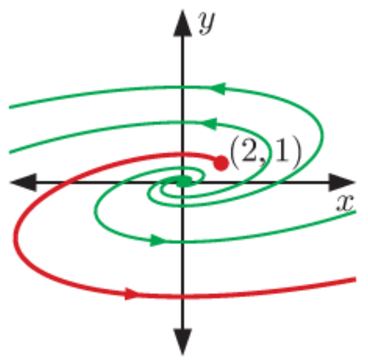
c



3 **a**



b

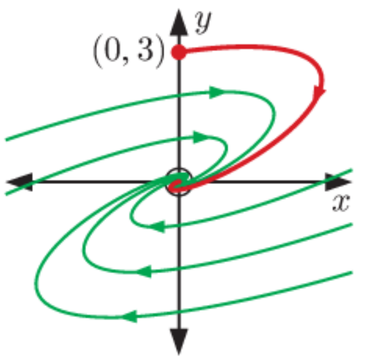


4 **a** $-2 \pm i\sqrt{5}$

b stable spiral

c $\begin{pmatrix} 21 \\ 3 \end{pmatrix}$, the trajectories of the phase portrait rotate clockwise.

d



5 **a** eigenvalues are $\lambda_1 = -1$, $\lambda_2 = -5$

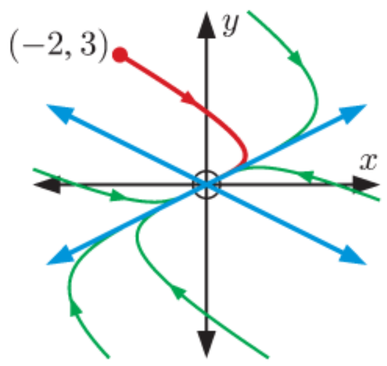
eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

b stable fixed

c **i** $\begin{pmatrix} 18 \\ -11 \end{pmatrix}$ **ii** $\mathbf{x} = e^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2e^{-5t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

d $\approx (0.709, 0.381)$

e



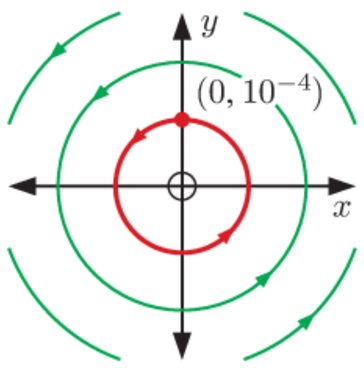
f The trajectory approaches O along $k\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ as $t \to \infty$.

6 **a** **i** V is proportional to $\frac{dI}{dt}$, and acts in the opposite direction.

ii I is proportional to $\frac{dV}{dt}$.

b **i** $\begin{pmatrix} \frac{dI}{dt} \\ \frac{dV}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -25 \\ 10^5 & 0 \end{pmatrix}\begin{pmatrix} I \\ V \end{pmatrix}$ **ii** $\pm 500i\sqrt{10}$

c **i**



ii Over time the voltage and current oscillate, such that when one is zero, the other is at its maximum or minimum value.

EXERCISE 26C

1 **a** $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 x \end{cases}$ **b** $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\mathbf{x}$

2 **a** $\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\omega^2 x - ky \end{cases}$ **b** $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -k \end{pmatrix}\mathbf{x}$

d If $k = 0$, the equilibrium point is a centre, and the spring oscillates indefinitely.

e If $0 < k < 2\omega$, the equilibrium point is a stable spiral, so the spring eventually comes to rest.

3 $\begin{cases} \dot{x} = v \\ \dot{v} = -g - \dfrac{k}{m} v^2 \end{cases}$

4 **a** $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\frac{1}{50} y \end{cases}$ **b** $y(t) = Ae^{-\frac{t}{50}}$ for some constant A.

c $x(t) = -50Ae^{-\frac{t}{50}} + c$ for constants A, c.

d **i** $x(t) \approx 8.59e^{-\frac{t}{50}} + 16.4$

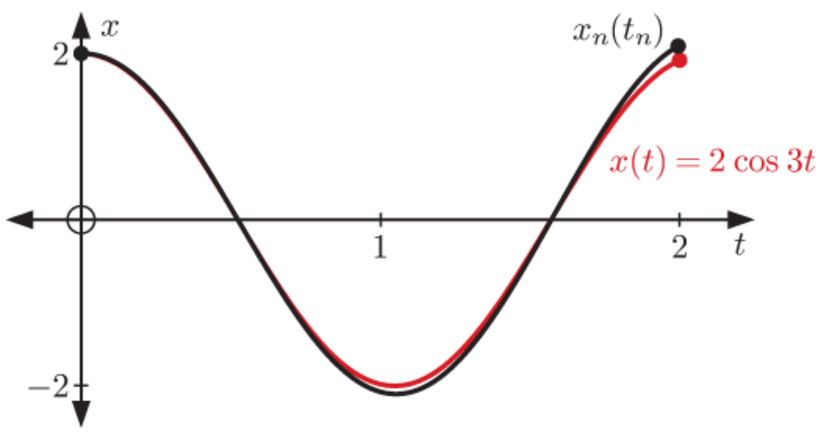
ii The temperature of the drink will approach $\approx 16.4°\text{C}$.

EXERCISE 26D

1 **b** $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -9x \end{cases}$ **c** $\begin{aligned} t_i &= t_{i-1} + h \\ x_i &= x_{i-1} + hy_{i-1} \\ y_i &= y_{i-1} - 9hx_{i-1} \end{aligned}$

d $(2, -0.18)$

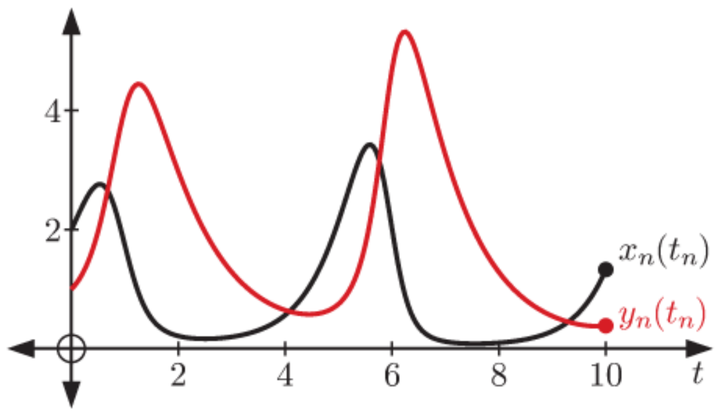
e



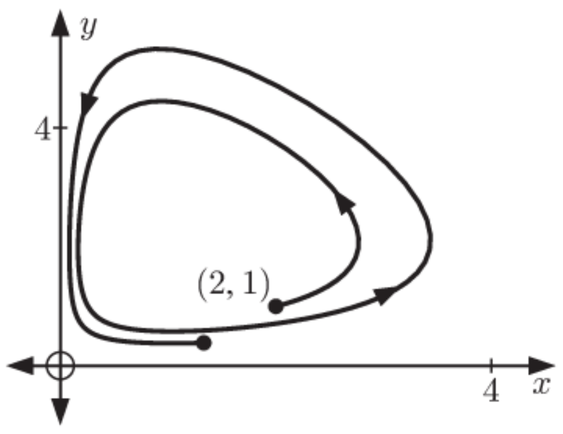
The points (t_i, x_i) approximate a smooth curve similar to the analytic solution $x(t) = 2\cos 3t$. Euler's method is less accurate at times when the curve changes gradient faster.

2 **a** $\begin{aligned} t_i &= t_{i-1} + h \\ x_i &= x_{i-1} + h(2x_{i-1} - x_{i-1}y_{i-1}) \\ y_i &= y_{i-1} + h(x_{i-1}y_{i-1} - y_{i-1}) \end{aligned}$

b **i**

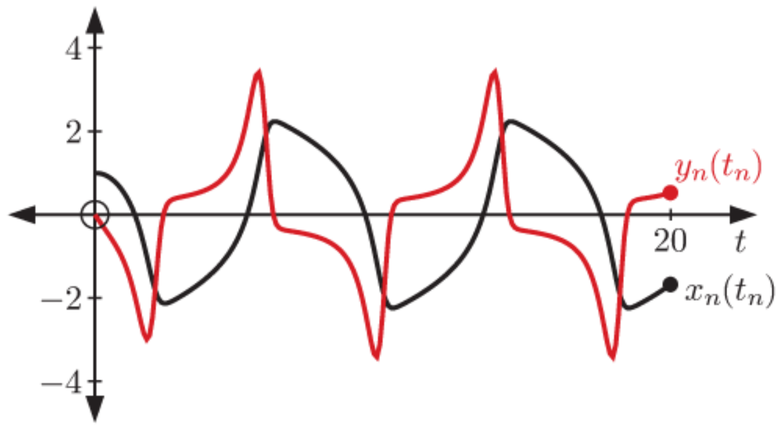


ii

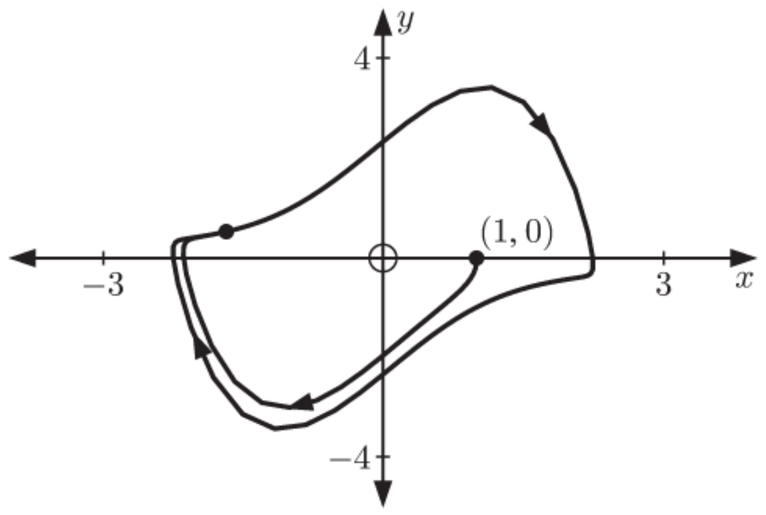


iii The exact solution indicates that the system repeats itself in a closed loop, whereas the numerical solution moves in an unstable spiral. The numerical solution appears to suggest one or both of the populations will become extinct in the long term, which is not the case.

3 **a**



b



4 **a** $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 9.81 - 0.375y^2 \end{cases}$ **b** ≈ 23.7 m

c $\approx 5.1147\ \text{m s}^{-1}$

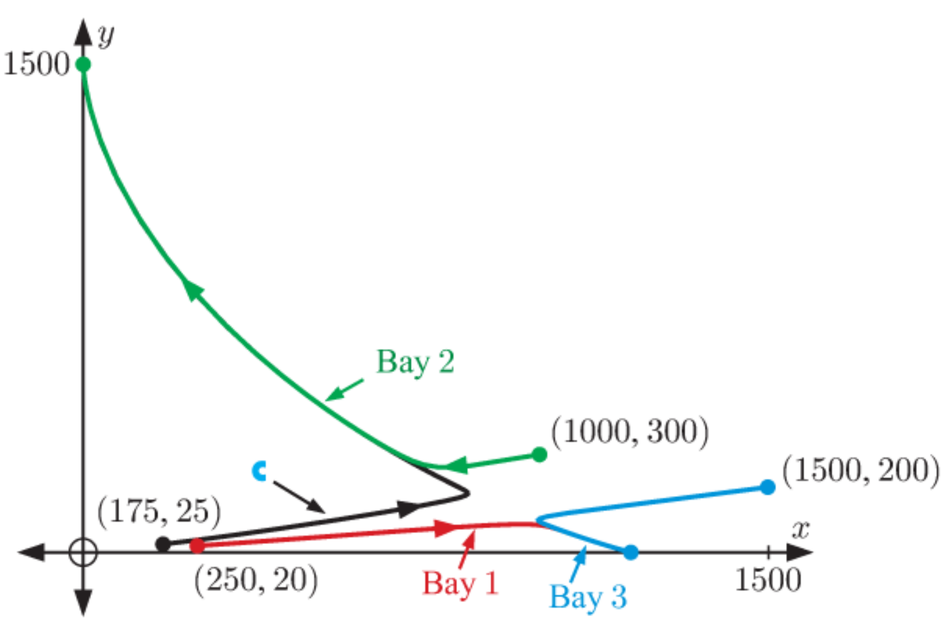
d $\approx 5.1147\ \text{m s}^{-1}$, which is consistent with the answer to **c**.

5 **a** $x, y \geqslant 0$ **b** $(0, 0)$, $(0, 1500)$, $(1200, 0)$, $(900, 150)$

c **i** 820 common hermit crabs, 170 yellow-footed hermit crabs.

ii In the long term, there will be no common hermit crabs and 1500 yellow-footed hermit crabs.

d



e No, the only stable equilibrium points are $(0, 1500)$ and $(1200, 0)$. Unless the populations are initially $x_0 = 900$, $y_0 = 150$, one of the species will become extinct in the long term.

6 **c** **i**

t (seconds)	*Position*	t (seconds)	*Position*
1	(0.998, 0.047)	11	(8.213, 6.135)
2	(1.986, 0.202)	12	(8.531, 7.083)
3	(2.947, 0.474)	13	(8.794, 8.048)
4	(3.867, 0.865)	14	(9.011, 9.024)
5	(4.729, 1.371)	15	(9.190, 10.008)
6	(5.518, 1.984)	16	(9.337, 10.997)
7	(6.224, 2.691)	17	(9.457, 11.990)
8	(6.844, 3.475)	18	(9.556, 12.985)
9	(7.378, 4.320)	19	(9.637, 13.982)
10	(7.832, 5.211)	20	(9.703, 14.979)

ii No, the distance between the fox and the rabbit appears to converge to ≈ 5.03 units.

d **i**

t (seconds)	*Position*
1	(1.996, 0.102)
2	(3.964, 0.451)
3	(5.852, 1.102)
4	(7.571, 2.117)
5	(8.967, 3.538)

t (seconds)	*Position*
6	(9.828, 5.332)
7	(10.000, 6.917)
8	(10.000, 7.917)
9	(10.000, 8.917)
10	(10.000, 9.917)

ii Yes, the fox appears to catch the rabbit after approximately 6.7 seconds.

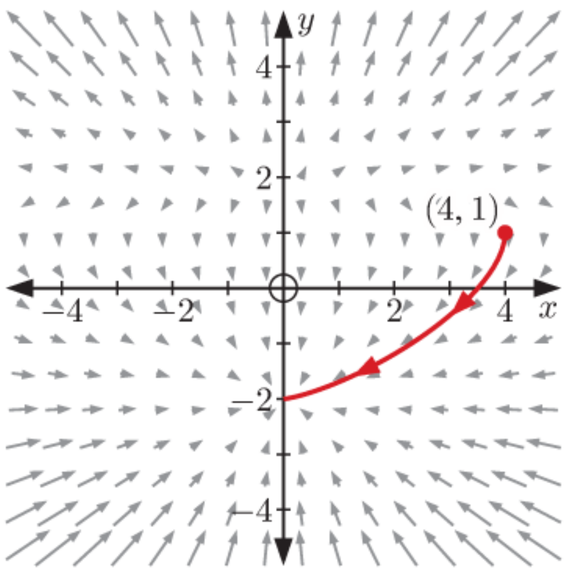
REVIEW SET 26A

1 $(-1, -2)$, unstable spiral

2 **a** $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

b $(0, 2)$ is an unstable fixed point,
$(0, -2)$ is a stable fixed point.

c

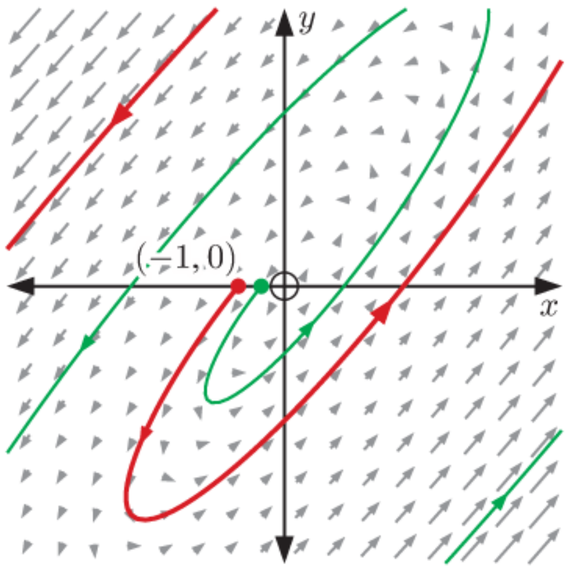


3 **a** $\dfrac{1 \pm i\sqrt{11}}{2}$

b The eigenvalues are complex with positive real part.

c **i** $\begin{pmatrix} -4 \\ -5 \end{pmatrix}$ **ii** anticlockwise

d



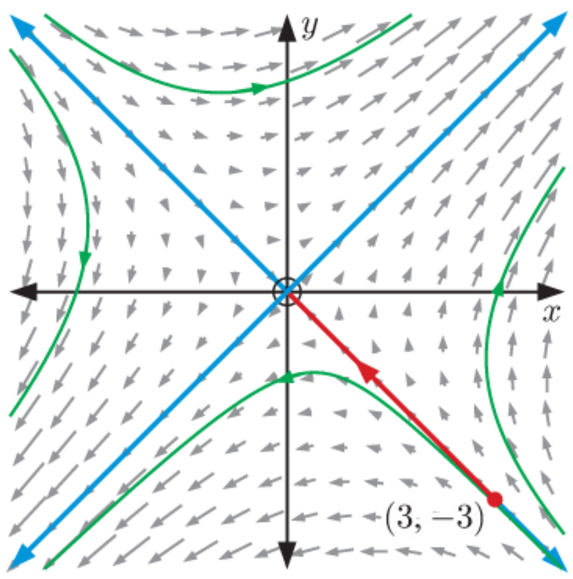
4 $\begin{cases} \dfrac{dP}{dt} = S \\ \dfrac{dS}{dt} = 200 - 30P - 8S \end{cases}$

5 **a** saddle point **b** $\mathbf{x} = Ae^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + Be^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c $\mathbf{x} = -3e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

d The trajectory approaches O along $k\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as $t \to \infty$.

e



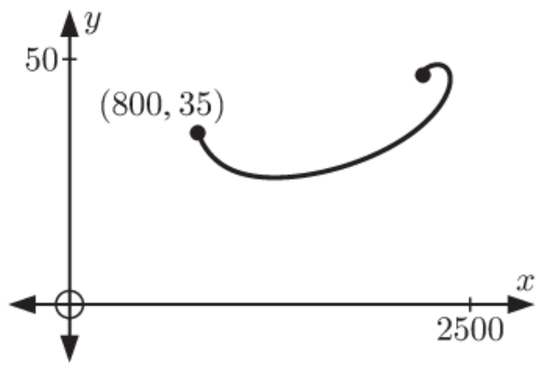
6 **a** (0, 0) - both populations have died out
(10 000, 0) - the lions have died out
$\approx$ (2200, 46.8) - the zebras and lions coexist in equilibrium

b **i** $\approx$ 1010 zebras, 27 lions

ii In the long term, there will be $\approx$ 2200 zebras and 47 lions.

iii $\approx$ 26 lions, after $\approx$ 2 years

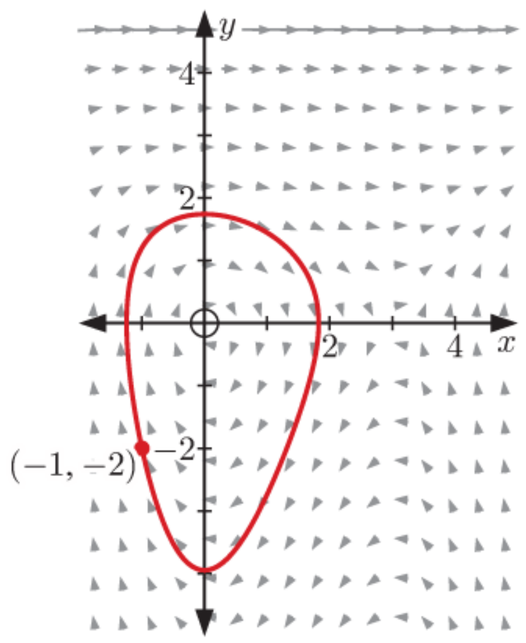
iv



REVIEW SET 26B

1 **a** **i** centre
ii saddle point

b



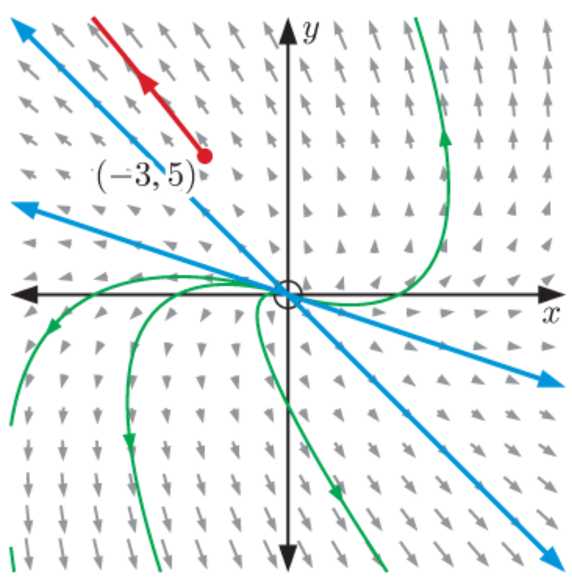
2 **a** A$(\frac{1}{3}, \frac{1}{2})$ **b** Inwards, the trajectory at (0, 0) is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3 **a** $\mathbf{x} = Ae^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + Be^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b **i** $\begin{pmatrix} -21 \\ 27 \end{pmatrix}$ **ii** $\mathbf{x} = -e^{3t}\begin{pmatrix} -3 \\ 1 \end{pmatrix} + 6e^{5t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

c unstable fixed

d



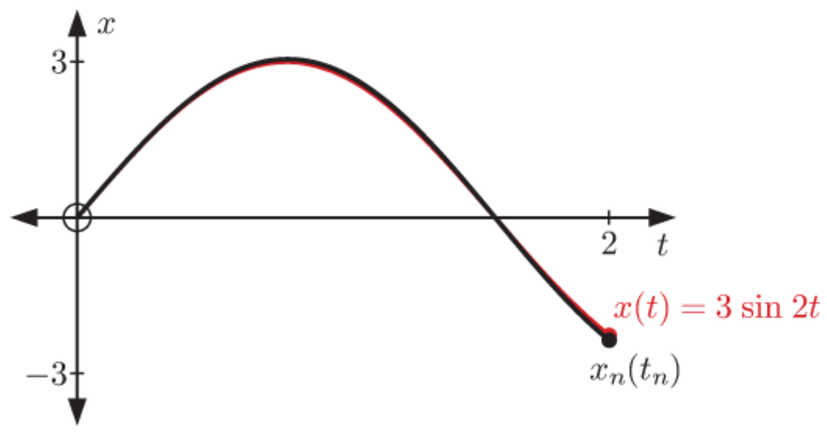
e The trajectory becomes parallel to $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ as $t \to \infty$.

4 **b** $\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -4x \end{cases}$

c $t_i = t_{i-1} + h$
$x_i = x_{i-1} + hy_{i-1}$
$y_i = y_{i-1} - 4hx_{i-1}$

d $(0.03, 3)$

e



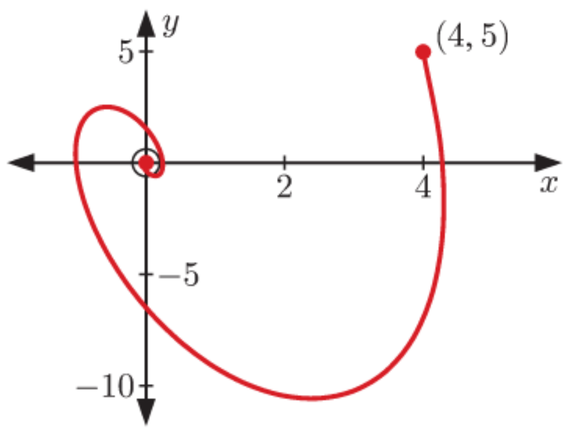
The points (t_i, x_i) approximate a smooth curve similar to the analytic solution $x(t) = 3\sin 2t$. Euler's method is less accurate at times when the curve changes gradient faster.

5 **a** $\begin{cases} \dfrac{dI}{dt} = J \\ \dfrac{dJ}{dt} = -16I - 4J \end{cases}$ **b** $\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ -16 & -4 \end{pmatrix}\mathbf{x}$

c $-2 \pm 2i\sqrt{3}$, stable spiral

d **i** $\begin{pmatrix} 5 \\ -84 \end{pmatrix}$ **ii** when $t \approx 0.6$ ms

iii



iv In the long term, the current converges to zero.

6 **a** $\dot{x} = -R\omega\sin(\omega t + \phi)$, $\dot{y} = R\omega\cos(\omega t + \phi)$

b $\dot{\mathbf{x}} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\mathbf{x}$

c $\pm i\omega$, the equilibrium point of the system is a centre.

d $\begin{pmatrix} -R\omega\sin\phi \\ R\omega\cos\phi \end{pmatrix}$

e **i** If $\omega > 0$, the angle (measured anticlockwise) increases, so the rotation is anticlockwise.
If $\omega < 0$, the rotation is clockwise.

ii If $\mathbf{A}$ has the form $\begin{pmatrix} 0 & - \\ + & 0 \end{pmatrix}$, the rotation is anticlockwise.
If $\mathbf{A}$ has the form $\begin{pmatrix} 0 & + \\ - & 0 \end{pmatrix}$, the rotation is clockwise.

f **ii** bc must be negative.

iii If $b < 0$, $c > 0$, the rotation is anticlockwise.
If $b > 0$, $c < 0$, the rotation is clockwise.

EXERCISE 27A

1 **a** continuous **b** discrete **c** continuous **d** continuous
e discrete **f** discrete **g** continuous **h** continuous

2 **a** **i** X = the height of water in the rain gauge
ii continuous **iii** $0 \leqslant X \leqslant 400$ mm

b **i** X = stopping distance **ii** continuous
iii $0 \leqslant X \leqslant 50$ m

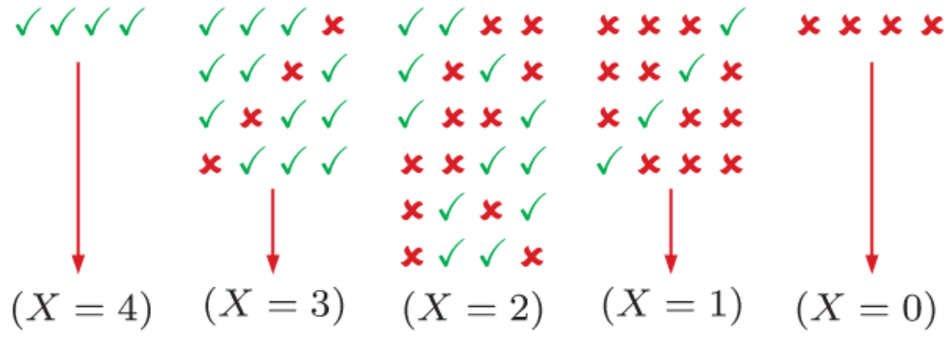
c **i** number of switches until failure
ii discrete **iii** any integer $\geqslant 1$

3 **a** X has a set of distinct possible values.
b $X = 2, 3, 4, 5, 6, 7, 8, 9$, or 10

4 **a** $X = 4, 5, 6$, or 7 **b** **i** $X = 5$ **ii** $X = 6$ or 7

5 **a** $X = 0, 1, 2, 3$, or 4

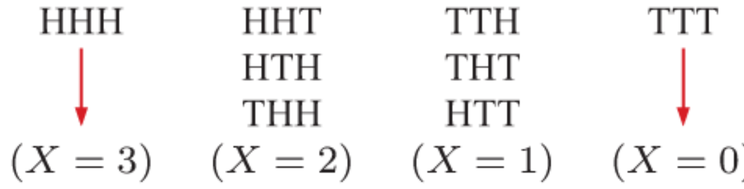
b



c **i** $X = 2$ **ii** $X = 2, 3$, or 4

6 **a** $X = 0, 1, 2$, or 3

b



c No, for example there is probability $\frac{1}{8}$ that $X = 3$, and probability $\frac{3}{8}$ that $X = 2$.

EXERCISE 27B

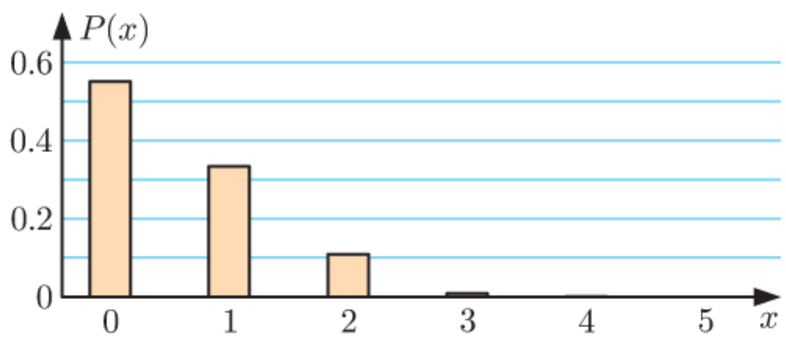
1 **a** **i** yes **ii** no **iii** yes **iv** no
b For **a iii**, X is a uniform random variable.

2 **a** $k = 0.2$ **b** $k = \frac{1}{7}$

3 **a** $a = 0.2$
b No, as the probabilities of each outcome are not all equal.
c 2 **d** $\mathrm{P}(X \geqslant 2) = 0.65$

4 **a** $P(2) = 0.1088$
b $a = 0.5488$ is the probability that Jason does not hit a home run in a game.
c $P(1) + P(2) + P(3) + P(4) + P(5) = 0.4512$ and is the probability that Jason will hit one or more home runs in a game.

d



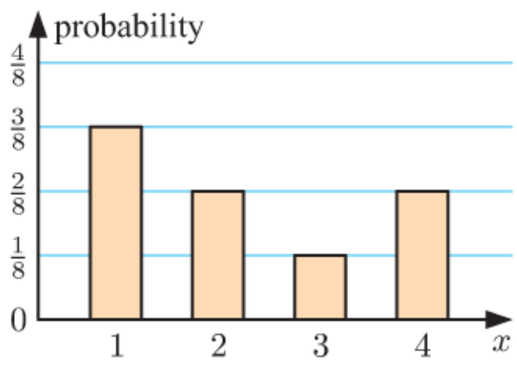
e mode = 0 home runs, median = 0 home runs

5 **a** $k = 0.04$ **b** 0 tyres
c $\mathrm{P}(X > 1) = 0.12$ which is the probability that more than 1 tyre will need replacing on a car being inspected.

6 **a**

x	1	2	3	4
$\mathrm{P}(X = x)$	$\frac{3}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{2}{8}$

b



c mode = 1
median = 2

d $\mathrm{P}(X \leqslant 3) = \frac{3}{4}$

7 **a** $X = 1, 2, 3$, or 4

b

x	1	2	3	4
$P(X = x)$	0.24	0.35	0.27	0.14

c mode $= 2$ bedrooms, median $= 2$ bedrooms

8 **a** $X = 1, 2, 3$, or 4

b

x	1	2	3	4
$P(X = x)$	0.48	0.28	0.08	0.16

c mode $= 1$ shot, median $= 2$ shots

10 **a** $k = \frac{1}{12}$ **b** $k = \frac{12}{25}$

11 **a** $a = 10$ **b** $P(X = 1) = \frac{3}{10}$ **c** 2

EXERCISE 27C.1

1 **a** $E(X) = 1.7$ **b** $E(X) = 2.5$ **c** $E(X) = 3.85$
d $E(X) = 30$

2 **a** $a = \frac{1}{2}$ **b** 3 **c** $\mu = 2\frac{2}{5}$ **3** ≈ 11.7 points

4 1.57 fish **5** **a** $a = 0.25$ **b** 4 books **c** 3.13 books

6 5.25 lollies **7** **a** $\frac{4}{15}$ **b** ≈ 8.93 pins

8 **a** offensive strategy: $P(\text{draw}) = 0.15$
defensive strategy: $P(\text{draw}) = 0.5$
b offensive strategy: 1.05 points per game
defensive strategy: 1.1 points per game
c defensive strategy
d Yes, an offensive strategy would then be better.

9 $a = 0.1$, $b = 0.4$

10 **a** **i** car park B **ii** car park A **iii** car park B
b Zoe should choose car park A as the expected cost for car park A is \$14.80 whereas the expected cost for car park B is \$15.25.

11 \$390

EXERCISE 27C.2

1 fair **2** **a** \$3.50 **b** −\$0.50 **c** no

3 **a** $\approx -\$0.05$ **b** lose $\approx$ \$5.41 **4** −\$0.75

5 **a** Expected gain $\approx -\$0.67 \neq \$0$ **b** \$30

6 **a** **i** 0.3 **ii** 0.1 **b** $E(X) = 2.5$ tokens
c No, as the player can expect to lose half a token on average per game.

7 \$4.75

EXERCISE 27D

1 **a** **i** $\mu = 2$ **ii** $\sigma^2 = 0.6$ **iii** $\sigma \approx 0.775$
b **i** $\mu = 1.5$ **ii** $\sigma^2 = 1.25$ **iii** $\sigma \approx 1.12$

2 **a** $k = 0.15$ **b** 20 **c** $\mu = 13$ **d** $\sigma \approx 6.88$

3 **b** Michelle: $\sigma = 1$ ace, $\sigma^2 = 1$ ace^2 **c** Amanda
Amanda: $\sigma \approx 1.30$ aces, $\sigma^2 = 1.7$ aces^2

4 **a** $k = 0.03$ **b** $\mu = 0.74$ **c** $\sigma \approx 0.996$

5 **a**

x	1	2	3
$P(X = x)$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{3}{6}$

b $\mu \approx 2.17$
c $\sigma \approx 0.898$

6 **a**

x_i	1	2	3	4	5
p_i	0.1	0.2	0.4	0.2	0.1

b $\mu = 3$, $\sigma \approx 1.10$

8 **a** **i** Y would have the greater mean as, on average, we are more likely to obtain higher values from two rolls than one roll.
ii X would have the greater standard deviation as the probabilities of obtaining each outcome are more spread out.

b

x	1	2	3	4
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

y	1	2	3	4
$P(Y = y)$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{5}{16}$	$\frac{7}{16}$

c X: $\mu = 2.5$, $\sigma \approx 1.12$, Y: $\mu = 3.125$, $\sigma \approx 0.927$

EXERCISE 27E

1 $E(aX + b) = E(aX) + E(b) = a\,E(X) + b$

2 **a** 13 **b** −5 **c** $3\frac{1}{3}$

3 mean $= 17$, standard deviation $= 4$

4 **a** $E(Y) = 13$, $\text{Var}(Y) = 16$
b $E(Y) = -22$, $\text{Var}(Y) = 100$
c $E(Y) = 0$, $\text{Var}(Y) = 1$

5 **a** $E(X) = 2$ **b** $\text{Var}(X) = 1$ **c** $\sigma(X) = 1$
d $E(X + 1) = 3$ **e** $\text{Var}(3X + 1) = 9$ **f** $\sigma(5 - X) = 1$
g $E\left(\frac{2X + 5}{3}\right) = 3$ **h** $\text{Var}(20 - 4X) = 16$

6 **a** $a = \frac{1}{4}$
b **i** $E(X) = 2\frac{11}{12}$, $\text{Var}(X) \approx 1.91$, $\sigma(X) \approx 1.38$
ii $E(X + 4) = 6\frac{11}{12}$, $\text{Var}(X + 4) \approx 1.91$, $\sigma(X + 4) \approx 1.38$
iii $E(3X - 1) = 7\frac{3}{4}$, $\text{Var}(3X - 1) \approx 17.2$, $\sigma(3X - 1) \approx 4.15$

7 **a** **i** $E(X) = 1.85$ **ii** $\text{Var}(X) \approx 1.73$ **iii** $\sigma(X) \approx 1.31$
b $Y = 25X + 100$
c **i** $E(Y) = 146.25$ **ii** $\text{Var}(Y) \approx 1080$
iii $\sigma(Y) \approx 32.9$

EXERCISE 27F

1 **a** The binomial distribution applies, as tossing a coin has two possible outcomes (H or T) and each toss is independent of every other toss.
b The binomial distribution applies, as this is equivalent to tossing one coin 100 times.
c The binomial distribution applies as we can draw out a red or a blue marble with the same chances each time.
d The binomial distribution does not apply as the result of each draw is dependent upon the results of previous draws.
e The binomial distribution does not apply, assuming that ten bolts are drawn without replacement. We do not have a repetition of independent trials. However, since there is such a large number of bolts in the bin, the trials are approximately independent, so the distribution is approximately binomial.

2

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

3 **a** $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$
b $4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{4}$
c $6\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$

4 **a** $5\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{5}{32}$ **b** $10\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{5}{16}$
c $\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right) = \frac{1}{32}$

5 **a** $\left(\frac{2}{3}\right)^4 = \frac{16}{81}$ **b** $6\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 = \frac{8}{27}$ **c** $\frac{8}{9}$

6 **a** $15\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2 = \frac{1215}{4096}$ **b** $\frac{347}{2048}$ **c** $\frac{347}{2048}$

EXERCISE 27G

1 **a** ≈ 0.0305 **b** ≈ 0.265

2 **a** ≈ 0.476 **b** ≈ 0.840 **c** ≈ 0.160 **d** ≈ 0.996

3 **a** ≈ 0.0280 **b** $\approx 0.002\,46$ **c** ≈ 0.131 **d** ≈ 0.710

4 $\approx 0.000\,864$ **5** **a** ≈ 0.998 **b** ≈ 0.807

6 **a** ≈ 0.0388 **b** ≈ 0.405 **c** ≈ 0.573 **7** ≈ 0.0341

8 **a** ≈ 0.863 **b** ≈ 0.475 **9** **a** $\frac{1}{36}$ **b** ≈ 0.846

10 **a** ≈ 0.0905 **b** ≈ 0.622

c Yes, the probability that Shelley is on time for work each day of a 5 day week is now $\approx 87.2\%$.

11 **a** ≈ 0.0388 **b** 25 solar components

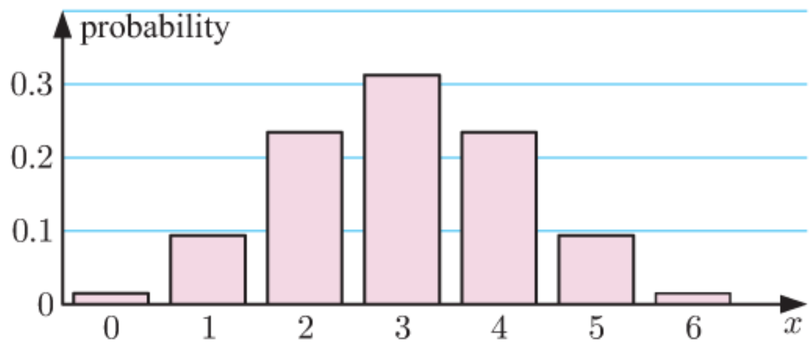
EXERCISE 27H

1 **a** **i** $\mu = 3,\ \sigma \approx 1.22$

ii

x_i	0	1	2	3
$P(x_i)$	0.0156	0.0938	0.2344	0.3125

x_i	4	5	6
$P(x_i)$	0.2344	0.0938	0.0156



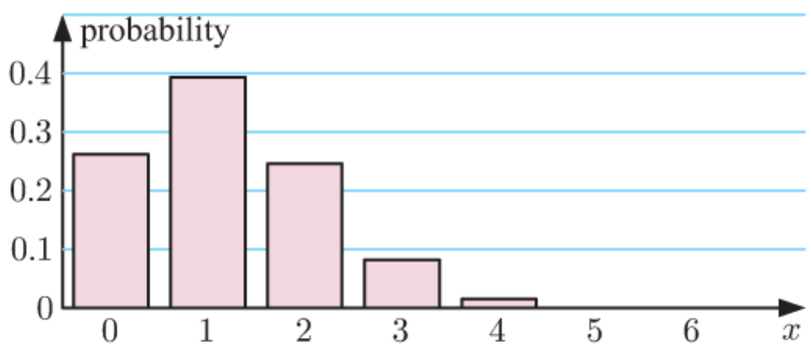
iii The distribution is symmetric.

b **i** $\mu = 1.2,\ \sigma \approx 0.980$

ii

x_i	0	1	2	3
$P(x_i)$	0.2621	0.3932	0.2458	0.0819

x_i	4	5	6
$P(x_i)$	0.0154	0.0015	0.0001



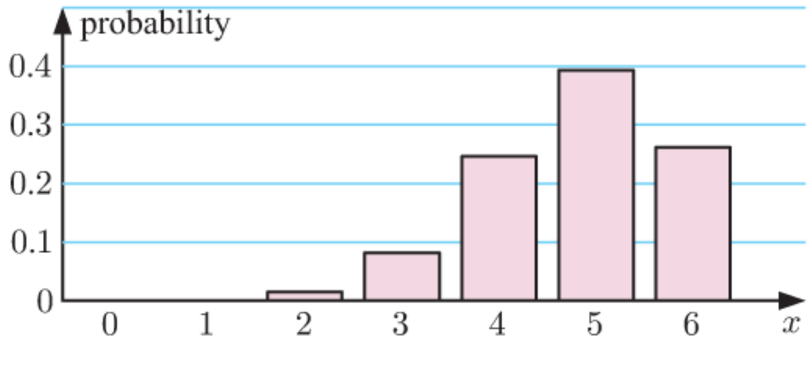
iii The distribution is positively skewed.

c **i** $\mu = 4.8,\ \sigma = 0.980$

ii

x_i	0	1	2	3
$P(x_i)$	0.0001	0.0015	0.0154	0.0819

x_i	4	5	6
$P(x_i)$	0.2458	0.3932	0.2621



iii The distribution is negatively skewed and is the exact reflection of **b**.

2 $\mu = 5,\ \sigma^2 = 2.5$

3 **a** $\mu = 1.2,\ \sigma \approx 1.07$ **b** $\mu = 28.8,\ \sigma \approx 1.07$

4 $\mu = 3.9,\ \sigma \approx 1.84$

5 **a** $\mu = 28.5,\ \sigma \approx 2.67$ **b** ≈ 0.740

6 **a** $\mu_X = np = 100 \times \frac{1}{2} = 50$ $\quad \mu_Y = np = 300 \times \frac{1}{6} = 50$

b $\sigma_X = 5,\ \sigma_Y \approx 6.45$

c X is more likely to lie between 45 and 55 inclusive because the standard deviation of X is lower than that of Y, which means there are more values of X which lie close to the mean.

d **i** ≈ 0.729 **ii** ≈ 0.606

EXERCISE 27I

1 **a** $\mathrm{P}(X = x) = \dfrac{(7.12)^x e^{-7.12}}{x!},\quad x = 0, 1, 2, \ldots.$

b **i** $\mathrm{E}(X) = 7.12$ **ii** $\mathrm{Var}(X) = 7.12$ **iii** $\sigma(X) \approx 2.67$

c **i** ≈ 0.0205 **ii** ≈ 0.0757 **iii** ≈ 0.838 **iv** ≈ 0.974

2 **a** ≈ 0.0498 **b** ≈ 0.577 **c** ≈ 0.185 **d** ≈ 0.440

3 **a** **i** ≈ 0.150 **ii** ≈ 0.967

b $\mu = 1.7,\quad \sigma^2 = 1.7,\quad \sigma \approx 1.30$

4 **a** ≈ 0.271 **b** ≈ 0.271 **c** ≈ 0.677

5 **a** ≈ 0.990

b 4 years, assuming that the probability of a fatal crash remains constant.

6 **a** $\lambda = \dfrac{3 + \sqrt{33}}{2} \approx 4.37$

b **i** $\mathrm{E}(X) \approx 4.37$ **ii** $\sigma(X) \approx 2.09$ **iii** ≈ 0.812 **iv** ≈ 0.524

7 **a** $\lambda = 3 + \sqrt{15} \approx 6.8730$ **b** **i** ≈ 0.177 **ii** ≈ 0.417

8 **a** $\mu = 1.694$ **b** $\lambda = 1.694$

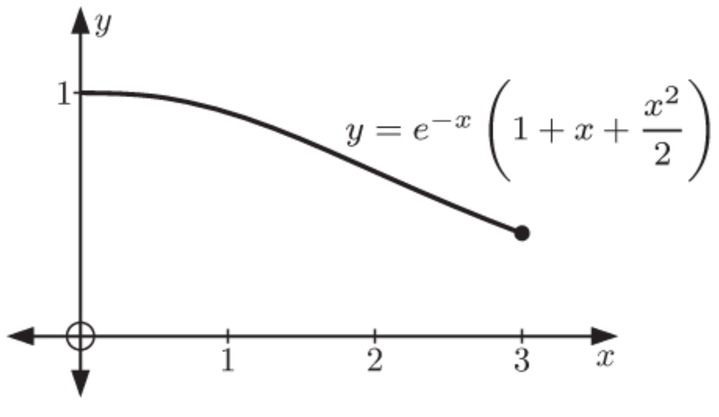
c σ^2 for the data ≈ 1.67

In **b**, $\sigma^2 = \lambda = 1.694$.

The two values are very close.

9 **a** $y = e^{-x}\left(1 + x + \dfrac{x^2}{2}\right)$

b



c $\dfrac{dy}{dx} = -\dfrac{x^2 e^{-x}}{2} < 0$ for all $x > 0$

10 **a** Y eggs are laid, and each egg has probability $\frac{2}{3}$ of hatching independently of one another.

b ≈ 0.0211 **c** $\frac{2}{3}Y$ **d** $\frac{40}{3}$

REVIEW SET 27A

1 **a** discrete **b** continuous **c** discrete

2 **a** **i** yes **ii** no **iii** no **iv** yes **v** yes **vi** yes
b the distribution in **a iv**

3 **a** $a = \frac{5}{9}$ **b** $\frac{4}{9}$

4 **a** $k = 0.05$ **b** 0.15 **c** 2 **d** $\mathrm{E}(X) = 1.7$

5 **a** X has a set of distinct possible values.

b $X = 0$, 1, or 2

c

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{3}{10}$

d 1.2 green balls

6 ≈ 3.83

7 **a** \$7 **b** No, she would lose \$1 per game in the long run.

8 **a** $\mu = 3$ **b** $\sigma^2 = 1.1$ **c** $\sigma \approx 1.05$

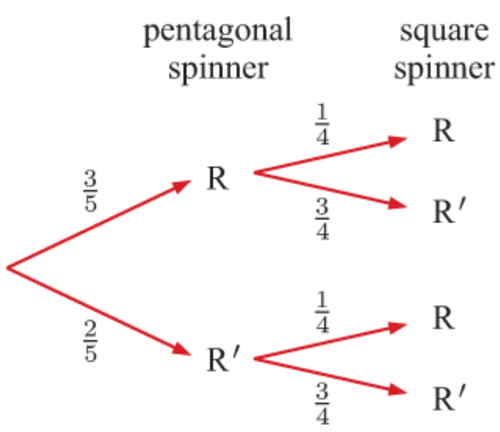
9 **a** $a = -\frac{1}{84}$ **b** 4 marsupials **c** 3

10 mean $= 27$, standard deviation $= 8$

11 **a** $\frac{64}{3125} = 0.020\,48$ **b** $\frac{128}{625} = 0.2048$

12 **a** ≈ 0.544 **b** ≈ 0.456

13 **a**



b $\frac{11}{20}$

c **i** $X \sim \mathrm{B}(10, \frac{11}{20})$

ii $\mathrm{P}(X = 1) = \binom{10}{1}\left(\frac{11}{20}\right)\left(\frac{9}{20}\right)^9 \approx 0.004\,16$

$\mathrm{P}(X = 9) = \binom{10}{9}\left(\frac{11}{20}\right)^9\left(\frac{9}{20}\right) \approx 0.0207$

It is more likely that exactly one red will occur 9 times.

iii $\mu = 5.5, \quad \sigma \approx 1.57$

14 **a** ≈ 0.891 **b** ≈ 0.0293 **c** ≈ 0.450

REVIEW SET 27B

1 **a** X is the number of hits that Sally has in each match.
$X = 0$, 1, 2, 3, 4, or 5

b **i** $k = 0.23$ **ii** $\mathrm{P}(X \geqslant 2) = 0.79$

iii $\mathrm{P}(1 \leqslant X \leqslant 3) = 0.83$

c mode $= 3$ hits, median $= 3$ hits

2 **a** 2 **b** 3 **c** 2.7

4 **a** **i** Naomi **ii** Rosslyn **b** Rosslyn

5 **a** **i** $\frac{2}{5}$ **ii** $\frac{1}{10}$ **iii** $\frac{1}{10}$ **b** \$2.70 per game

6 **a** The probability of rolling a two is not the same for each die.
So X is not a binomial random variable.

b

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{15}{24}$	$\frac{1}{3}$	$\frac{1}{24}$

c $\frac{5}{12}$

7 **a** $a = 0.15, \quad b = 0.35$ **b** $\mathrm{Var}(X) = 1.26$

8 **a** **i** $\mathrm{E}(X) = 2.1$ **ii** $\mathrm{Var}(X) = 1.29$

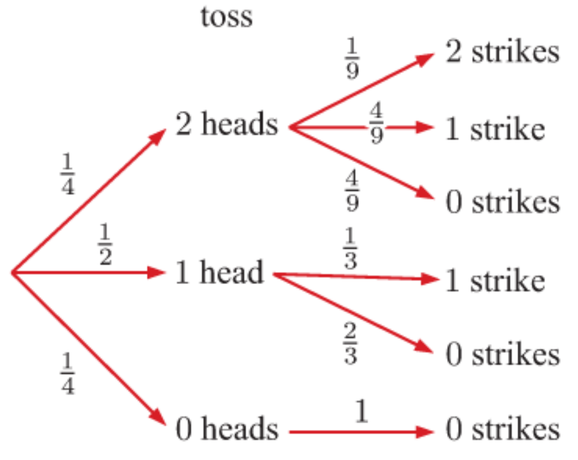
b **i** $Y = 4 - X$ **ii** $\mathrm{E}(Y) = 1.9, \quad \mathrm{Var}(Y) = 1.29$

9 **a** The probability of spinning a 3 is the same for each spin.
b $\mu = 4, \quad \sigma \approx 1.79$

10 **a** 42 donations **b** **i** ≈ 0.334 **ii** ≈ 0.0931

11 **a** **i** ≈ 0.0751 **ii** ≈ 0.166 **b** ≈ 4.97 games

12 **a**



b

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

c $\approx$ \$3.33

d $\approx -$ \$1.67, Suvi should not play the game many times.

13 **a** ≈ 0.0516 **b** no

14 **a** $\mu \approx 1.28, \quad \sigma \approx 1.13$ **b** ≈ 0.366

EXERCISE 28A.1

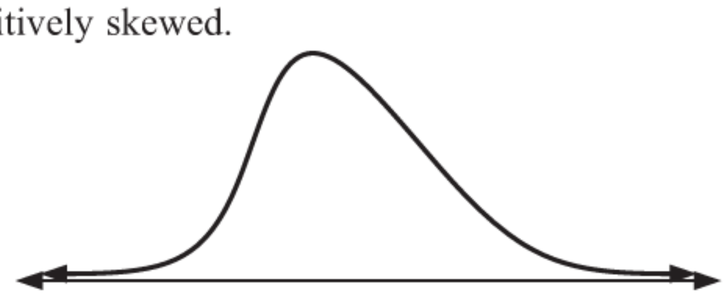
1 **B**, **D**, and **F**

2 **a** The diameters may be affected by:
- the type of lathe used
- the steadiness of the woodworker's hand
- the operating speed of the lathe.

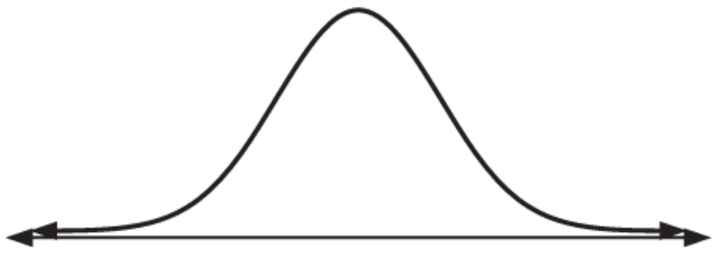
b The scores may be affected by:
- the time spent studying
- natural ability (for example, memory, learning ability)
- general knowledge.

c The times may be affected by:
- weather conditions
- physical fitness
- walking speed
- the terrain.

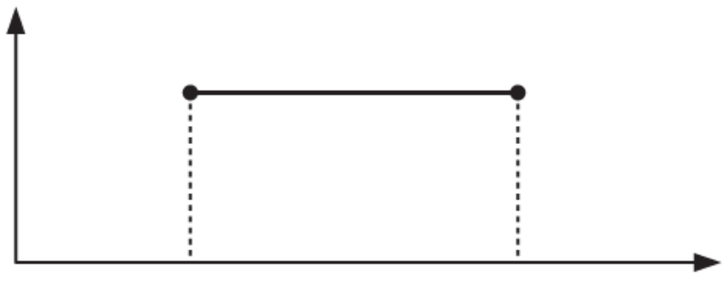
3 **a** The variable is not likely to be normally distributed as it is more likely that there would be more people younger than the mean age than there are older. The distribution may be positively skewed.



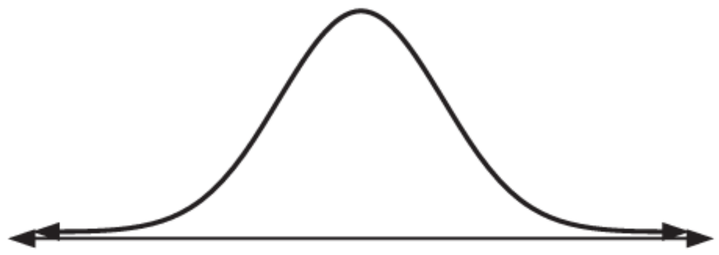
b The variable is likely to be normally distributed as the long jumper is likely to jump the same distance consistently, but it will vary due to factors such as the speed at which the long jumper runs before the jump, and the positioning of their body before hitting the sand.



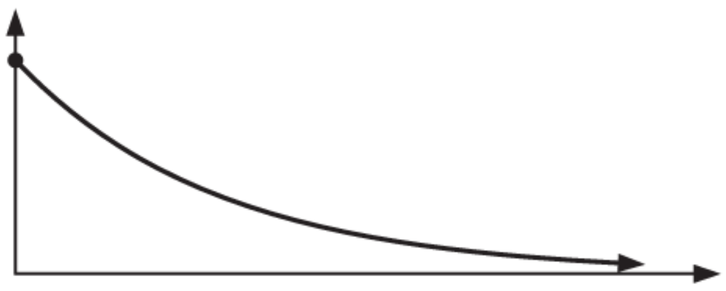
c The variable is not likely to be normally distributed as each number has the same chance of being drawn. The distribution should be uniform.



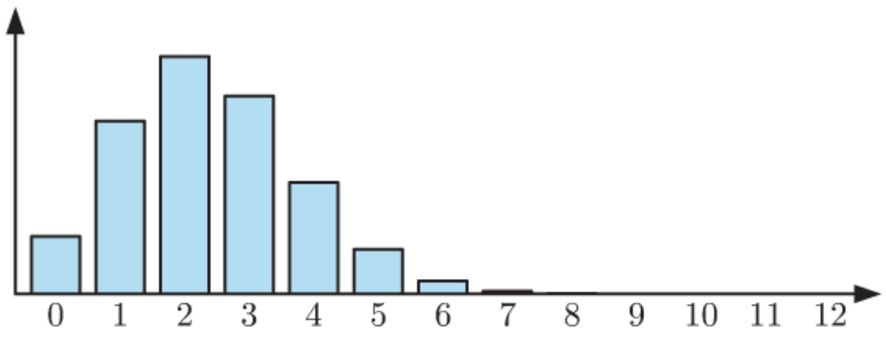
d The variable is likely to be normally distributed as the lengths of the carrots will be generally centred around the mean, but will vary due to factors such as soil quality, different weather conditions, harvest times, and so on.



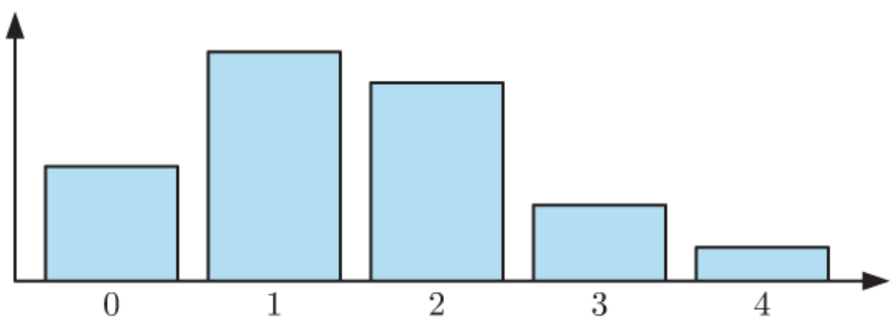
e The variable is not likely to be normally distributed. People are most likely to be served quite quickly. The distribution is likely to be negatively skewed.



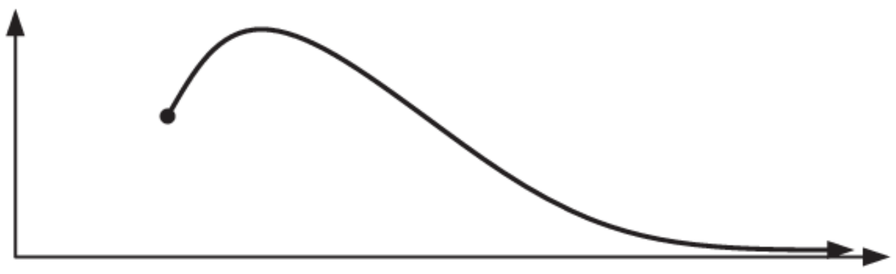
f The variable is not likely to be normal as it is a discrete variable. Each egg has the same probability of being brown, so the distribution is binomial.



g The variable is not likely to be normally distributed as it is a discrete variable. Most families will have 0 - 2 children, and there will be much fewer families with more than 2 children. The distribution will be positively skewed.



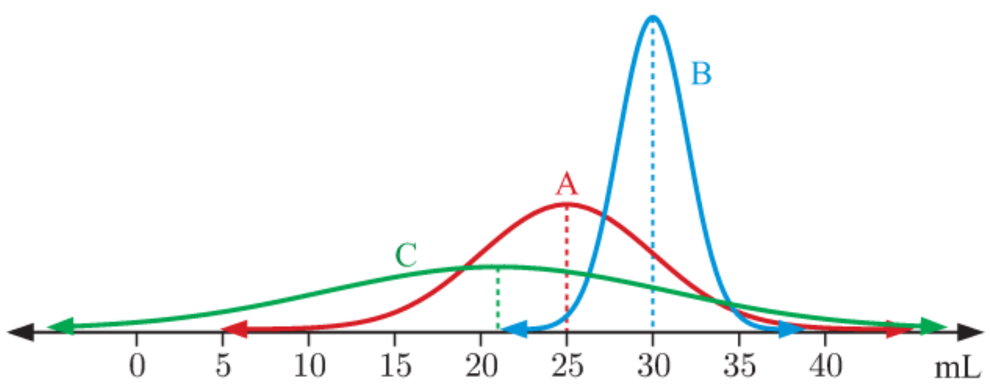
h The variable is not likely to be normally distributed as there will tend to be many more shorter buildings than tall buildings in a city. The distribution will be positively skewed.



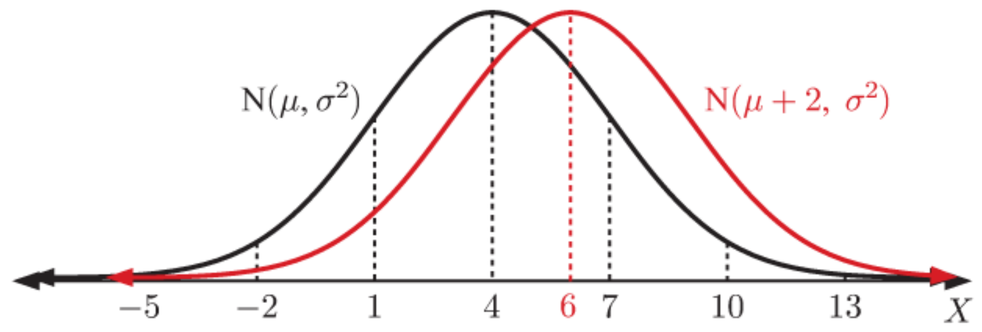
EXERCISE 28A.2

1 **a** B **b** D **c** A **d** C

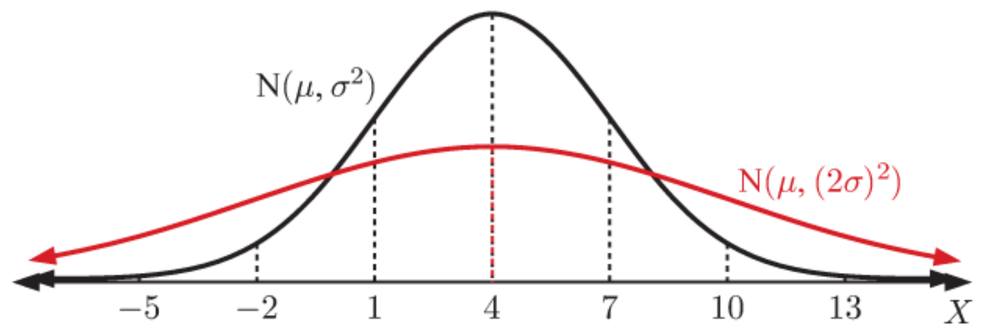
2



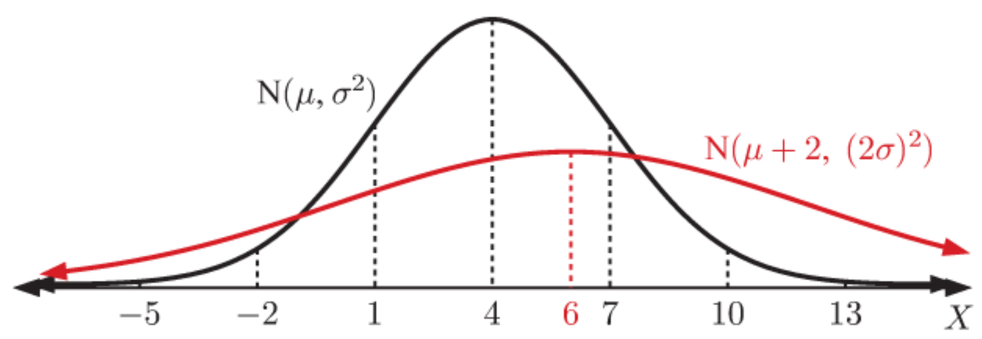
3 **a**



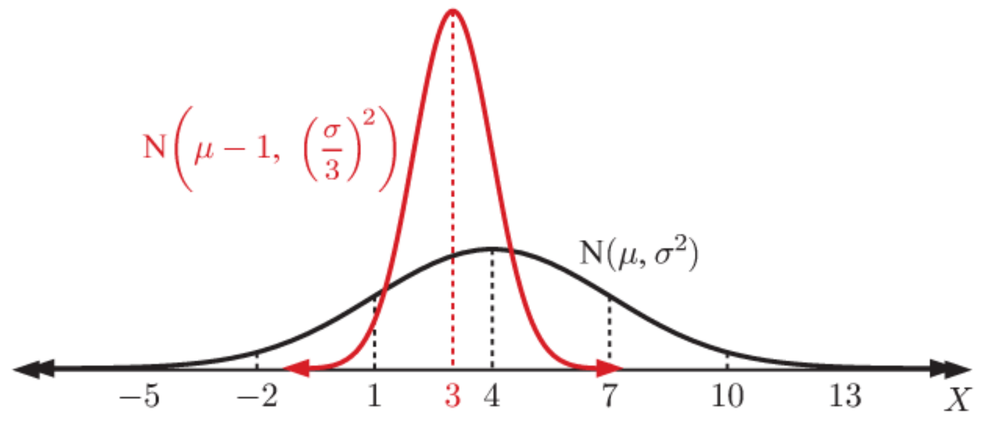
b



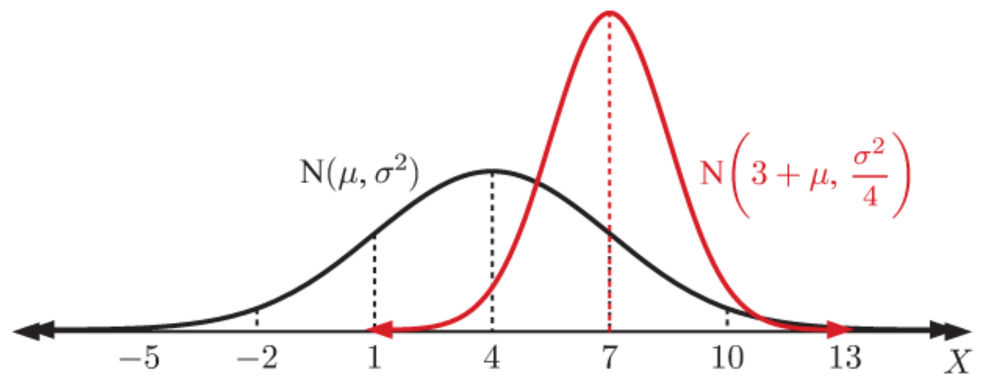
c



d



e



EXERCISE 28B.1

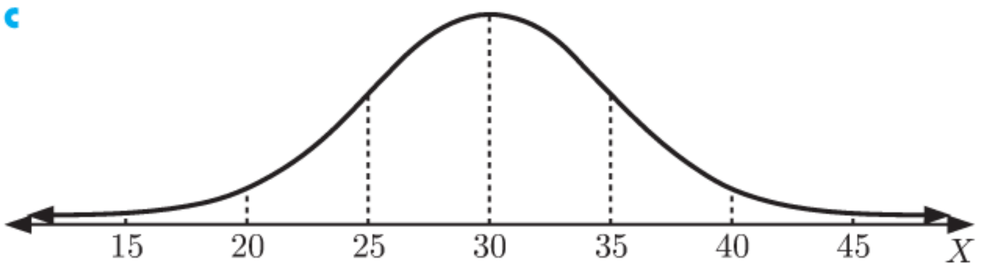
1 **a** **i** 40 **ii** 25

b **i** 1 standard deviation above the mean

ii 2 standard deviations below the mean

iii 3 standard deviations above the mean

c

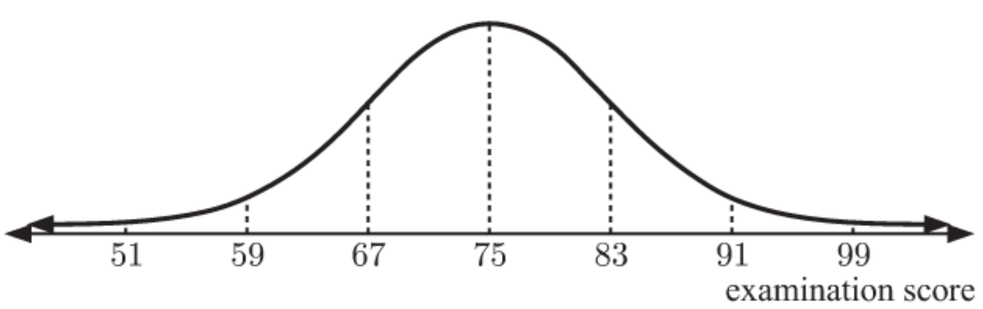


d $\approx 34.13\%$ **e** ≈ 0.1359

2 **a** $\mu = 20, \quad \sigma = 4$

b **i** $\approx 34.13\%$ **ii** $\approx 13.59\%$ **iii** $\approx 2.28\%$

3 **a**



b **i** $\approx 15.87\%$ **ii** $\approx 2.28\%$ **iii** $\approx 81.85\%$

4 **a** ≈ 0.6826 **b** ≈ 0.0228

5 **a** **i** $\approx 34.13\%$ **ii** $\approx 47.72\%$

b **i** ≈ 0.0228 **ii** ≈ 0.8413

c ≈ 68 students **d** $k \approx 178$

6 **a** ≈ 0.8413 **b** ≈ 0.8185

7 **a** ≈ 459 babies **b** ≈ 446 babies

8 **a** ≈ 41 days **b** ≈ 254 days **c** ≈ 213 days

9 **a** ≈ 5 competitors **b** ≈ 32 competitors

c ≈ 137 competitors

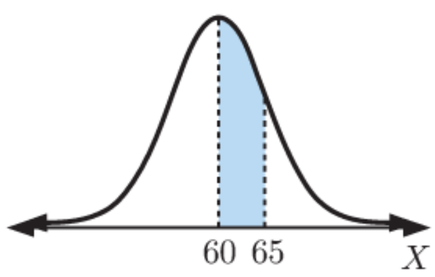
10 **a** $\mu = 176$ g, $\sigma = 24$ g **b** $\approx 81.85\%$

11 **a** **i** $\approx 84.13\%$ **ii** $\approx 2.28\%$

b **i** ≈ 0.0215 **ii** ≈ 0.9544 **c** ≈ 0.0223

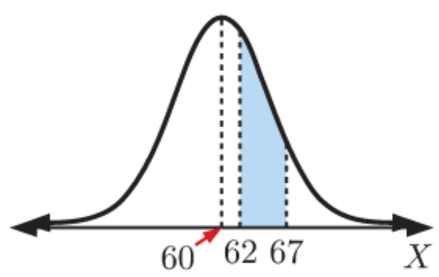
EXERCISE 28B.2

1 **a**



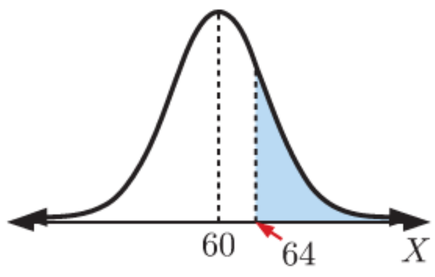
$P(60 \leqslant X \leqslant 65) \approx 0.341$

b



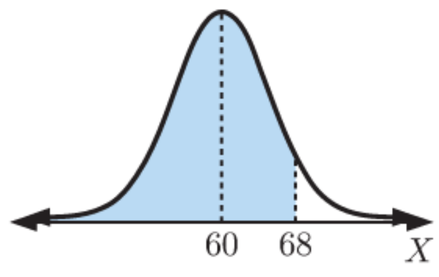
$P(62 \leqslant X \leqslant 67) \approx 0.264$

c



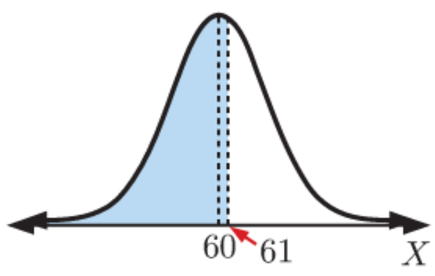
$P(X \geqslant 64) \approx 0.212$

d



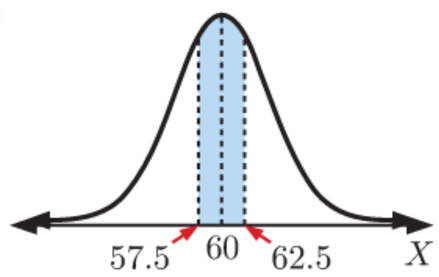
$P(X \leqslant 68) \approx 0.945$

e



$P(X \leqslant 61) \approx 0.579$

f



$P(57.5 \leqslant X \leqslant 62.5) \approx 0.383$

2 **a** ≈ 0.334 **b** ≈ 0.166 **3** ≈ 0.378

4 **a** ≈ 0.303 **b** ≈ 0.968 **c** ≈ 0.309

5 **a** ≈ 0.0509 **b** $\approx 52.1\%$ **c** ≈ 47 eels

6 **a** **i** $\approx 90.4\%$ **ii** $\approx 4.78\%$ **b** \$4160

7 **a** **i** $\approx 12.7\%$ **ii** $\approx 52.0\%$

b **i** 21.6 kL **ii** ≈ 76 customers

8 **a** **i** $\approx 21.5\%$ **ii** $\approx 95.2\%$

b **i** Enrique **ii** Damien

9 **a** $\approx 10.3\%$ **b** ≈ 0.456 **10** **a** $\approx 84.1\%$ **b** ≈ 0.880

11 **a** ≈ 0.133 **b** ≈ 0.971

EXERCISE 28C.1

1 **a** Emma's z-scores:
English ≈ 1.82, Mandarin ≈ 2.33, Geography ≈ 1.61,
Biology $= 0.9$, Mathematics ≈ 2.27

b Mandarin, Mathematics, English, Geography, Biology

c The scores in each of Emma's classes are normally distributed.

2 **a** Sergio's z-scores:
Physics ≈ -0.463, Chemistry ≈ 0.431,
Mathematics ≈ 0.198, German ≈ 0.521,
Biology ≈ -0.769

b German, Chemistry, Mathematics, Physics, Biology

3 **a** Frederick's z-scores:
50 m freestyle ≈ 1.95, 100 m backstroke ≈ -1.07,
200 m breaststroke ≈ -0.578, 100 m butterfly ≈ 0.345

b Lower times are better as they indicate that the person swims faster.

c 100 m backstroke, 200 m breaststroke, 100 m butterfly, 50 m freestyle

EXERCISE 28C.2

1 **a** ≈ 0.683 **b** ≈ 0.840 **c** ≈ 0.341 **d** ≈ 0.977

e ≈ 0.841 **f** ≈ 0.159

2 **a** $a = -1$, $b = 2$ **b** $a = -0.5$, $b = 0$

c $a = 0$, $b = 3$

3 **a** ≈ 0.150 **b** ≈ 0.431 **c** ≈ 0.922

d ≈ 0.885 **e** ≈ 0.298 **f** ≈ 0.0968

g ≈ 0.919 **h** $\approx 3.17 \times 10^{-5}$ **i** ≈ 0.383

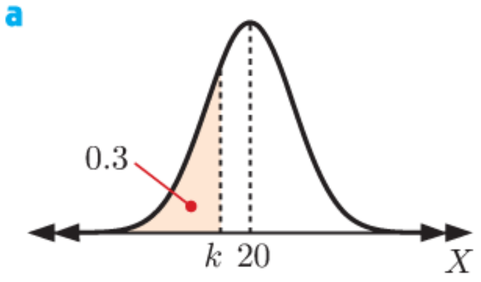
j ≈ 0.950 **k** ≈ 0.900 **l** ≈ 0.100

4 **a** **ii** ≈ 0.976 **b** **i** ≈ 0.910 **ii** ≈ 0.302

5 **a** **i** $z_1 \approx -0.859$, $z_2 \approx 1.18$ **ii** ≈ 0.687

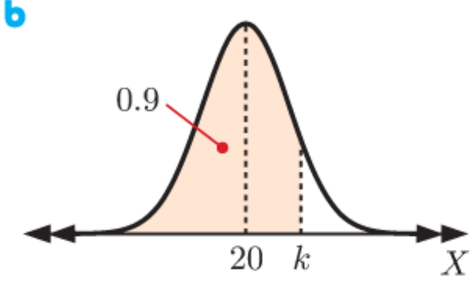
EXERCISE 28D

1 **a**



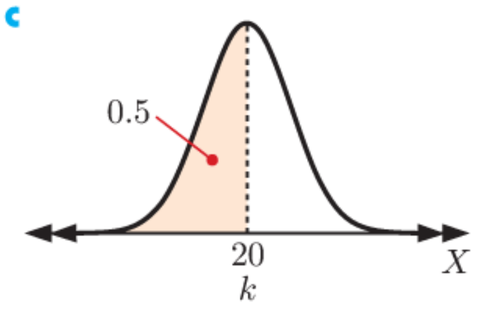
$k \approx 18.4$

b



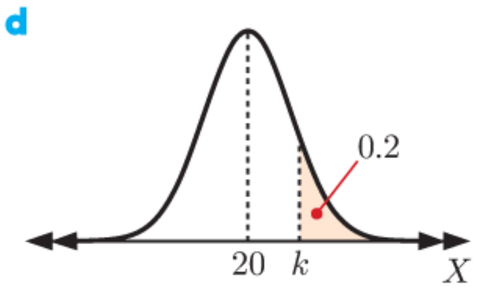
$k \approx 23.8$

c



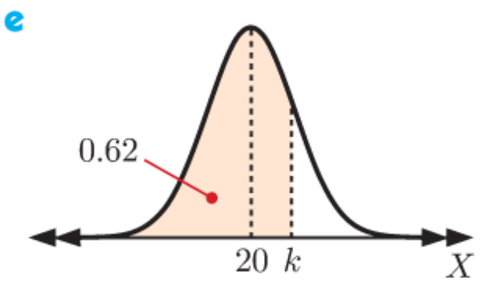
$k = 20$

d



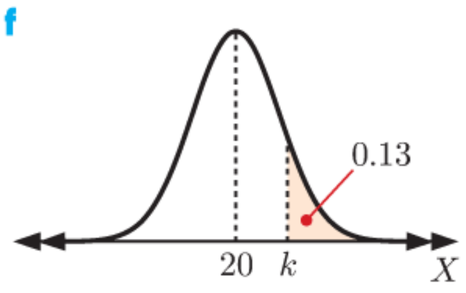
$k \approx 22.5$

e



$k \approx 20.9$

f

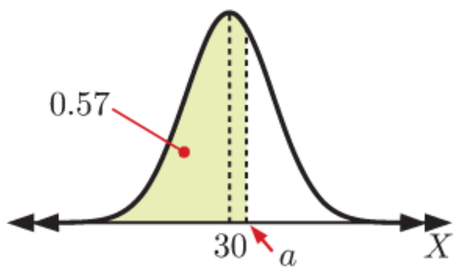


$k \approx 23.4$

2 **a** $k \approx 0.878$ **b** $k \approx 0.202$ **c** $k \approx -0.954$

d $k \approx -1.64$ **e** $k \approx -1.28$ **f** $k \approx 0.228$

3 **a**



$\therefore \quad a > 30$

b $a \approx 30.9$

c **i** 0.43

ii 0.07

4 **a** $k \approx 12.5$ **b** $k \approx 18.8$ **c** $k \approx 4.93$

5 **a** ≈ 0.212 **b** $k \approx 75.1$

6 **a** $a \approx 42.0$ **b** $a \approx 46.7$ **c** $a \approx 40.1$

7 ≈ 24.7 cm **8** ≈ 75.2 mm

9 **a** **i** $k \approx 54.5$ **ii** $l \approx 69.8$

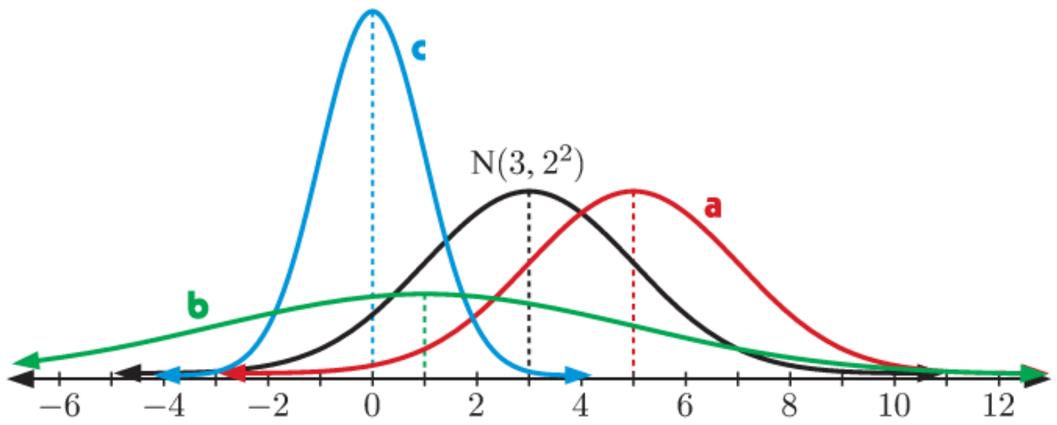
10 ≈ 501.8 mL to 504.0 mL **11** $\approx 31.0°$C

REVIEW SET 28A

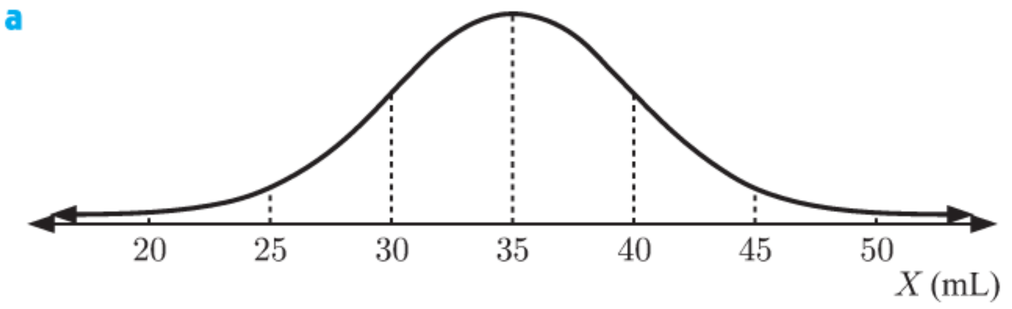
1 **a** The distribution of times taken for students to read a novel is likely to be positively skewed, and hence not normal.

b The mean amount spent on groceries at a supermarket is likely to occur most often, with variations around the mean occurring symmetrically as a result of random variation in the prices of items bought and/or the quantities of items bought (for example weights of fruits and vegetables). So the distribution is likely to be normal.

2



3 **a**



b **i** $\approx 47.7\%$ **ii** $\approx 2.28\%$

4 **a** $\approx 2.28\%$ **b** $\approx 68.26\%$ **c** $\approx 95.44\%$

5 **a** $\approx 50.2\%$ **b** ≈ 7 oysters

6 **a** Harri's test score is 2 standard deviations below the mean.

b $\approx 97.7\%$ **c** 7

7 **a** ≈ 0.364 **b** ≈ 0.356 **c** $k \approx 18.2$

8 **a** $\approx 6.68\%$ **b** ≈ 0.854

9 **a** ≈ 0.758 **b** ≈ 0.115 **c** ≈ 0.285

10 **a** $k \approx 19.7$ **b** $k \approx 28.0$

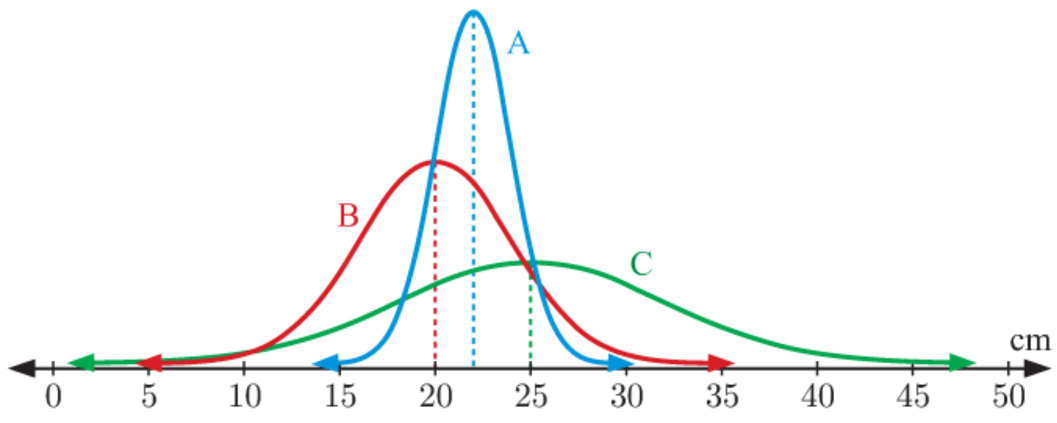
11 **a** ≈ 0.260 **b** ≈ 29.3 weeks

12 **a** **i** ≈ 0.0736 **ii** ≈ 0.0406 **b** ≈ 0.644

13 **a** $k \approx 28.1$ **b** $k \approx 26.5$ **c** $k \approx 25.0$

REVIEW SET 28B

1



2 **a** $\mu = 32,\ \sigma = 5$

b **i** $\approx 34.13\%$ **ii** $\approx 84.13\%$ **iii** $\approx 2.28\%$

3 $k \approx 1.96$

4 **a** **i** $\approx 2.28\%$ **ii** $\approx 84.0\%$ **b** ≈ 0.3413

5 **a** **i** $\approx 76.1\%$ **ii** $\approx 96.0\%$ **b** ≈ 0.598

c $x \approx 61.9$

6 **a** ≈ 0.479 **b** ≈ 0.0766 **c** $k \approx 55.2$

7 ≈ 162 seconds

8 **a** $a \approx 9.05$ **b** $a \approx 13.7$ **c** $a \approx 10.4$

9 **a** ≈ 0.258 **b** ≈ 243 suitcases **c** 23 kg

10 **a** The relative difficulty of each test is not known.

b z-score for English $= 1$, z-score for Chemistry $= 1$

$\therefore$ Kerry's performance relative to the rest of the class is the same in both tests.

11 **a** **i** ≈ 0.0362 **ii** ≈ 0.610 **iii** ≈ 0.566

b $k \approx 74.4$ **c** $a \approx 81.0,\ b \approx 102$ **d** ≈ 0.506

12 **a** $\approx 68.3\%$ **b** ≈ 0.0884

13 **a** **i** ≈ 0.722 **ii** ≈ 0.798 **b** ≈ 0.0563

EXERCISE 29A

1 **a** 5 **b** 4 **c** $\frac{11}{4}$ **d** $\frac{69}{4}$

2 **a** $\mathrm{E}(X + 2Y) = 15.2,\quad \sigma(X + 2Y) = \sqrt{4.403} \approx 2.10$

b $\mathrm{E}(Y - X) = 1.9,\quad \sigma(Y - X) = \sqrt{1.343} \approx 1.16$

c $\mathrm{E}(3X - 2Y) = 0,\quad \sigma(3X - 2Y) = \sqrt{6.987} \approx 2.64$

3 **a** **i**

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

ii

y	1	2	3	4	5	6
$\mathrm{P}(Y = y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

b **i** $\mathrm{E}(X) = 1,\quad \mathrm{Var}(X) = \frac{1}{2}$

ii $\mathrm{E}(Y) = \frac{7}{2},\quad \mathrm{Var}(Y) = \frac{35}{12}$

iii $\mathrm{E}(X + Y) = \frac{9}{2},\quad \mathrm{Var}(X + Y) = \frac{41}{12}$

iv $\mathrm{E}(4X - 2Y) = -3,\quad \mathrm{Var}(4X - 2Y) = \frac{59}{3}$

4 **a** As $\sum P(x) = 1$ in each distribution, each is a well defined probability distribution.

b **i** $\mathrm{E}(X) = \$2.50,\quad \sigma(X) = \sqrt{5.25} \approx \$2.29$

ii $\mathrm{E}(Y) = \$2.20,\quad \sigma(Y) = \sqrt{9.16} \approx \$3.03$

c $-\$0.30$ or -30 cents

d No; the expected gain from playing this game is negative.

5 **a** $\mathrm{E}(X) = p,\quad \mathrm{Var}(X) = p(1 - p)$

b **i** The outcome of each trial of the experiment is a Bernoulli random variable with probability of success p, $X_i \sim \mathrm{B}(1, p)$. So the total number of successes out of n independent trials is $\sum_{i=1}^{n} X_i$. That is $Y = \sum_{i=1}^{n} X_i$, a sum of n independent Bernoulli random variables.

EXERCISE 29B

1 **a** $T \sim \mathrm{Po}(27)$ **b** ≈ 0.307

2 **a** $Y \sim \mathrm{Po}(4)$ **b** ≈ 0.908

3 **a** ≈ 0.175

b $\mathrm{P}(Y \leqslant 1 \cap (X + Y) = 4)$
$= \mathrm{P}(Y = 1)\,\mathrm{P}(X = 3) + \mathrm{P}(Y = 0)\,\mathrm{P}(X = 4)$

c ≈ 0.0314 **d** ≈ 0.179

4 ≈ 0.0251 **5** $\lambda = \ln 4$

6 **a** **i** $(X_1 + X_2) \sim \mathrm{Po}(\lambda_1 + \lambda_2)$

ii $(X_1 + X_2 + X_3) \sim \mathrm{Po}(\lambda_1 + \lambda_2 + \lambda_3)$

b $Y \sim \mathrm{Po}\left(\sum_{k=1}^{n} \lambda_k\right)$; $\sum_{k=1}^{n-1} X_k$ is a Poisson random variable with rate $\sum_{k=1}^{n-1} \lambda_k$ which is independent from $X_n \sim \mathrm{Po}(\lambda_n)$.

So, Y can be written as the sum of 2 independent Poisson random variables.

EXERCISE 29C

1 **a** ≈ 0.0259 **b** ≈ 0.0724 **c** ≈ 0.669

2 **a** ≈ 0.888

b The weights of each adult and each child are independent of one another.

3 The manager needs to adjust the machine.

4 **a** $\mathrm{E}(U) = 20, \quad \sigma(U) = \sqrt{109} \approx 10.4$ **b** ≈ 0.0277

5 **a** $\approx 0.004\,01$ **b** $\approx 0.001\,10$

6 **a** ≈ 0.303 **b** ≈ 0.233

EXERCISE 29D

1 **a** mean $= 70$, standard deviation $= 1.5$

b Yes, 64 should be a sufficiently large sample for the Central Limit Theorem to apply.

2 **a** **i** 50 **ii** $\frac{9}{\sqrt{5}}$

b Yes, as X is normally distributed.

3 **a** mean $= 80$, standard deviation $= 1.5$

b Yes, 36 is a large enough sample.

c **i** ≈ 0.747 **ii** ≈ 0.0912 **iii** ≈ 0.252

d No, as we do not know if X is normally distributed.

4 **a** mean $= 3.6$, standard deviation ≈ 0.124

b Yes, 32 is a large enough sample.

c **i** ≈ 0.581 **ii** ≈ 0.210 **iii** $\approx 0.007\,67$

5 **a** Yes, as X is normally distributed.

b X, as it has a larger standard deviation.

c **i** ≈ 0.309 **ii** ≈ 0.132

6 **a** Histogram B as it is closer to the normal distribution, and the values are spread out over a much smaller range.

b $\frac{65}{300} \approx 0.217$

c $\frac{205}{300} \approx 0.683$

For a normal distribution, the probability that a value is within one standard deviation of the mean is ≈ 0.6826

$\therefore$ our estimate is good.

7 **a** ≈ 0.369 **b** ≈ 0.0680

8 **a** ≈ 0.609 **b** ≈ 0.934 **9** ≈ 0.864

10 ≈ 0.908 **11** **a** ≈ 0.252 **b** ≈ 0.0228

12 **a** $\approx 14.5\%$

b **i** $\overline{X}_{64}$ is normally distributed with mean 267 days and standard deviation 1.875 days.

ii $\approx 0.000\,094\,5$

c The answer to **a** may be affected but the answers to **b** remain the same.

EXERCISE 29E.1

1 $70.8 \leqslant \mu \leqslant 75.2$

2 **a** $z_{\frac{\alpha}{2}} \approx 1.75$ **b** $79.7 \leqslant \mu \leqslant 83.5$

3 **a** **i** $40.7 \leqslant \mu \leqslant 44.3$ **ii** $40.3 \leqslant \mu \leqslant 44.7$

iii $39.6 \leqslant \mu \leqslant 45.4$

b Increasing the confidence level increases the margin of error.

4 **a** $\overline{x} = 29.7$ **b** $26.9 \leqslant \mu \leqslant 32.5$

5 **a** $240.75 \leqslant \mu \leqslant 244.45$ mg **b** width ≈ 3.7 mg

c increase the sample size

EXERCISE 29E.2

1 **a** $n \approx 711$ **b** $n \approx 17\,800$ **c** $n \approx 1\,780\,000$

2 50 crayfish **3** **a** 136 packets **b** 305 packets

4 **a** The width will decrease by a factor of $\sqrt{2}$.

b 4 times larger

5 **a** $1000.2 \leqslant \mu \leqslant 1001.8$ mL **b** width ≈ 1.6 mL

c **i** 326 bottles

ii It would waste a lot of time and money sampling this many bottles.

EXERCISE 29E.3

1 **a** $47.3 \leqslant \mu \leqslant 50.9$

b Since 50 lies within the confidence interval, we do not have enough evidence to reject the claim.

2 **a** $98.9 \leqslant \mu \leqslant 99.9$ grams

b 100 g lies outside the confidence interval so we reject the claim that $\mu = 100$ g.

3 **a** $13.28 \leqslant \mu \leqslant 13.32$ dollars

b Yes, as the entire confidence interval lies below the old mean price of \$13.45 .

4 **a** $12.01 \leqslant \mu \leqslant 12.35$ seconds

b **i** Joan's time to run 100 m has improved as the entire confidence interval lies below the old mean time of 12.46 seconds.

ii 12.26 seconds is within the confidence interval so we do not have enough evidence to accept this claim.

EXERCISE 29F

1 $8.38 \leqslant \mu \leqslant 9.02$ minutes **2** $511 \leqslant \mu \leqslant 517$ grams

3 $37.0 \leqslant \mu \leqslant 39.4$ days

4 **a** $\overline{x} \approx 82.2$ hours, $s \approx 3.71$ hours

b $80.8 \leqslant \mu \leqslant 83.5$ hours

5 **a** $7.04 \leqslant \mu \leqslant 8.96$ minutes

b Yes, as the entire confidence interval lies below the old mean time of 12 minutes.

6 **a** $3.12 \leqslant \mu \leqslant 3.21$ seconds

b **i** Yes, the confidence interval lies entirely below the old mean stopping time.

ii No, 3.1 seconds is below the confidence interval.

REVIEW SET 29A

1 **a** $\mathrm{E}(X_1+2X_2+3X_3) = 6\mu, \quad \sigma(X_1+2X_2+3X_3) = \sigma\sqrt{14}$

b $\mathrm{E}(2X_1-3X_2+X_3) = 0, \quad \sigma(2X_1-3X_2+X_3) = \sigma\sqrt{14}$

2 **a** ≈ 0.380 **b** ≈ 0.395

3 $a = 1, \ b = 1$

4 **a** mean $= 35$, standard deviation $= 4$

b mean $= 35$, standard deviation $= \frac{8}{\sqrt{30}}$

5 ≈ 0.819

6 **a** Histogram B as it is not normally distributed.

b mean $= 11.4$ g kg^{-1}, standard deviation $= 3.07$ g kg^{-1}

7 $594.5 \leqslant \mu \leqslant 598.9$ grams

We are 95% confident that the mean weight of cereal packets is between 594.5 g and 598.9 g.

8 **a** $824.2 \leqslant \mu \leqslant 832.2$ grams **b** 260 cans

9 **a** $s \approx 4.05$ **b** **i** $124.75 \leqslant \mu \leqslant 129.24$ **ii** $k \approx 93.0$

REVIEW SET 29B

1 **a** **i** $2a - 8$ **ii** $4b + \frac{3}{8}$ **b** $a = 4, \ b = \frac{5}{32}$

c Y is a linear combination of normal variables. Consequently Y is normal.

So, $Y \sim \mathrm{N}(0, 1^2)$.

d ≈ 0.106

2 **a** $T \sim \mathrm{Po}(355)$ **b** $\approx 0.008\,79$

3 **a** mean $= 50$, standard deviation $= \frac{6}{\sqrt{60}}$

b Yes, 60 is a large enough sample. **c** ≈ 0.902

4 **a** mean $= 500$ g, standard deviation $= \frac{3.5}{\sqrt{20}}$ g

b Yes, as X is normally distributed.

5 **a** ≈ 0.296

b mean $= 63.4$ g, standard deviation $= \frac{6.33}{\sqrt{10}}$ g

c ≈ 0.0447, this is the probability that the mean weight of 10 sausages will be 60 g or less.

6 **a** $\overline{x} \approx 118$ points, $s \approx 31.0$ points

b $109 \leqslant \mu \leqslant 128$ points **c** width ≈ 19 points

7 **a** Histogram A, as it is more symmetric and has a smaller spread than Histogram B.

b **i** Using the Central Limit Theorem,

$$\overline{T} \sim \text{N}\left(4.3,\ \left(\frac{1.2}{\sqrt{100}}\right)^2\right) \sim \text{N}(4.3,\ (0.12)^2)$$

ii ≈ 0.662 **iii** ≈ 0.662

8 **a** $24.3 \leqslant \mu \leqslant 27.9$ units per kg

b No, the mean protein content of free range chickens, 24.9 units per kg, lies within the confidence interval.

9 **a** ≈ 0.0515 **b** ≈ 0.0299 **c** 601.2 mL

EXERCISE 30A

1 **a** A Type I error involves rejecting a true null hypothesis.

b A Type II error involves accepting a false null hypothesis.

c The null hypothesis is a statement of *no difference*.

d The alternative hypothesis is a statement that there is a difference.

2 **a** a Type I error **b** a Type II error

3 **a** a Type II error **b** a Type I error

4 **a** The alternative hypothesis (H_1) is that the person on trial is guilty.

b a Type I error **c** a Type II error

5 **a** A Type I error would result if X and Y are determined to have different effectiveness, when in fact they have the same.

b A Type II error would result if X and Y are determined to have the same effectiveness, when in fact they have different effectiveness.

6 **a** H_0: $\mu = 80$ {new globe lasts as long as old globe}
H_1: $\mu > 80$ {new globe lasts longer than old globe}

b H_0: $\mu = 80$ {new globe lasts as long as old globe}
H_1: $\mu < 80$ {new globe does not last as long as old globe}

7 H_0: $\mu = 26.3$ {new top speed is the same as old top speed}
H_1: $\mu > 26.3$ {new top speed is greater than old top speed}

8 H_0: $\mu = 80$ {mean weight of paper is 80 g per m^2}
H_1: $\mu \neq 80$ {mean weight of paper is *not* 80 g per m^2}

9 H_0: $\mu = 27$ {mean travel time is the same as before}
H_1: $\mu < 27$ {mean travel time is lower than before}

10 H_0: $\mu = 2.7$ {fat content of Brand B's muesli bars is 2.7 g}
H_1: $\mu > 2.7$ {fat content of Brand B's muesli bars is *greater* than 2.7 g}

11 $p =$ the proportion of smokers in the British population
H_0: $p = 0.21$, H_1: $p < 0.21$

12 $p =$ the proportion of cancer patients who survive more than 5 years after diagnosis on the new treatment
H_0: $p = 0.3$, H_1: $p > 0.3$

13 $p =$ the proportion of Party A supporters

a H_0: $p = 0.42$
H_1: $p > 0.42$

b H_0: $p = 0.42$
H_1: $p \neq 0.42$

EXERCISE 30B

1 **a** **i** $z \approx -1.89$ **ii** p-value ≈ 0.0294

b As the p-value < 0.05, we reject H_0 in favour of H_1.

2 **a** H_0: $\mu = 80$ and H_1: $\mu > 80$

b $z \approx 3.40$, the null distribution is $Z \sim \text{N}(0, 1^2)$

c p-value $\approx 0.000\,339$

d As the p-value < 0.01, we reject H_0.

e We conclude that $\mu > 80$ at the 1% level of significance.

3 $z \approx -2.37$, p-value $\approx 0.008\,85$

Since p-value < 0.05, the customer's claim is valid at the 5% level of significance.

4 $z \approx -3.40$, p-value $\approx 6.63 \times 10^{-4}$

Since p-value < 0.05, we conclude that the herd fineness has changed between 2013 and 2017, at the 5% level of significance.

EXERCISE 30C

1 **a** In a Z-test, the null-distribution is $Z \sim \text{N}(0, 1^2)$, and does not depend on μ_0. For a given significance level α, the critical value, critical region, and acceptance region are the same for each set of hypotheses.

b Yes, the null distribution is still $Z \sim \text{N}(0, 1^2)$. The critical value, critical region, and acceptance region will be the same.

2 **a** H_0: $\mu = 1.6$ and H_1: $\mu < 1.6$

b **i** ≈ -1.96 **ii** $\mathcal{C} = \{z \mid z \leqslant -1.96\}$

iii $\mathcal{A} = \{z \mid z > -1.96\}$

c $z \approx -0.569 \notin \mathcal{C}$, so there is insufficient evidence to claim that the mean tread depth does not meet the legal requirement.

3 **a** **i** $\mathcal{C} = \{z \mid z \leqslant -2.33 \text{ or } z \geqslant 2.33\}$

ii $\mathcal{A} = \{z \mid -2.33 < z < 2.33\}$

b yes; $z \approx -2.59 \in \mathcal{C}$

4 H_0 will be rejected if $\overline{x} \leqslant -23.8$ or $\overline{x} \geqslant -22.2$.

5 The quality controller should not adjust the machine for 502 mL $< \overline{x} <$ 506 mL.

EXERCISE 30D

1 **a** **i** $t \approx -1.53$ **ii** $T \sim t_{23}$ **iii** p-value ≈ 0.141

b **i** Since p-value > 0.05, we do not reject H_0 as there is insufficient evidence to do so. We accept at a 5% level that $\mu = 18.5$.

ii critical region $\mathcal{C} = \{t \mid t \leqslant -2.069 \text{ or } t \geqslant 2.069\}$
Since $t \notin \mathcal{C}$, we do not reject H_0 as there is insufficient evidence to do so. We accept at a 5% level that $\mu = 18.5$.

2 $t \approx -11.8$, p-value ≈ 0

Since p-value < 0.02, we conclude that the mean price has fallen at a 2% level of significance.

3 $t \approx -2.64$, p-value ≈ 0.0136

Since p-value > 0.01, we conclude that the machine is not underfilling the bottles at a 1% level of significance.

4 $t \approx 1.72$, p-value ≈ 0.107

Since p-value > 0.02, it is not justified to adjust the machine at a 2% level of significance.

5 $t \approx 1.67$, p-value ≈ 0.931

Since p-value > 0.01, we conclude that the machine is not underfilling the bags at a 1% level of significance.

6 **a** Since the growth of carrots depends on many factors such as genetic makeup and environment, it is reasonable to assume the weight of carrots is normally distributed.

b $t \approx 2.38$, p-value ≈ 0.0140
Since p-value < 0.05, we conclude that the mean weight is greater than 50 grams at a 5% level of significance. So, the buyer will purchase the crop.

7 **a** $t \approx 0.982$, p-value ≈ 0.164
Since p-value > 0.02, there is insufficient evidence to support the management's claim.

b Type II error

EXERCISE 30E

1 **a**

Athlete	A	B	C	D	E	F	G
d_i	-0.1	-0.2	0.2	-0.3	0	-1.4	0

Athlete	H	I	J	K	L	M
d_i	0	-0.2	-0.2	-0.4	-0.2	0.1

Athlete	N	O	P	Q	R	S	T
d_i	-0.2	0	0.1	0.1	0.2	-0.2	-0.4

b **i** H_0: $\mu_D = 0$, H_1: $\mu_D < 0$
ii H_0: $\mu_D = 0$, H_1: $\mu_D \neq 0$

c $t \approx -2.01$, p-value ≈ 0.0292
Since p-value < 0.05, we conclude that the sprint times of the athletes have improved at a 5% level of significance.

2 $t \approx 1.17$, p-value ≈ 0.134
Since p-value > 0.05, there is insufficient evidence to support that there has been improvement at a 5% level of significance.

3 **a**

Child	A	B	C	D	E	F	G	H	I	J	K
d_i	3	1	7	5	3	2	-1	11	6	5	4

b $t \approx -0.843$, p-value ≈ 0.419
Since p-value > 0.05, there is insufficient evidence to reject the sports commission's claim at a 5% level of significance.

EXERCISE 30F

1 **a** Let μ_1 be the population mean weight of the tomatoes from last year, and
μ_2 be the population mean weight of the tomatoes from this year.
H_0: $\mu_1 = \mu_2$ and H_1: $\mu_1 < \mu_2$

b $t \approx -1.99$, p-value ≈ 0.0244
Since p-value > 0.01, there is insufficient evidence to support the claim that the weight of the crop increased at a 1% level of significance.

2 $t \approx 1.31$, p-value ≈ 0.0967
Since p-value > 0.05, there is insufficient evidence to support the claim that high school students sleep less than middle school students at a 5% significance level.

3 $t \approx -0.198$, p-value ≈ 0.423
Since p-value > 0.05, there is insufficient evidence to support the claim that *Brand B* is better than *Brand A* at a 5% significance level.

4 **a** $t \approx 2.39$, p-value ≈ 0.0251
Since p-value < 0.05, there is sufficient evidence to support the claim that there is a significant difference between the runners' times at a 5% level.

b The mean time for Jesiah ≈ 12.6 s and the mean time for Billy ≈ 11.7 s. So, Billy is faster.

EXERCISE 30G

1 The test statistic $T = \sum_{i=1}^{n} X_i$ is a Poisson random variable which can only take non-negative integer values, so the observed test statistic $t = \sum_{i=1}^{n} x_i$ must also be a non-negative integer.

2 **a** H_0: $\lambda = 2$, H_1: $\lambda < 2$

b Since p-value $\approx 0.458 > 0.05$, there is insufficient evidence to conclude that the specimen is older than 5000 years old at the 5% level of significance.

3 **a** $t = 86$

b p-value $\approx 0.328 > 0.05$, so H_0 should not be rejected.

4 **a** H_0: $\lambda = 30$, H_1: $\lambda > 30$

b Since p-value $\approx 6.38 \times 10^{-11} < 0.1$, there is sufficient evidence to support upgrading the intersection at a 10% level of significance.

c We assume that the number of cars passing through the intersection per minute during peak time has the same distribution every day and does not change during a particular peak time. Thus the 30 minute "sample" may not be entirely representative of the situation. If the data was collected on a different day, or earlier or later in this particular peak time, we may get very different results.

5 **a** Let λ be the mean number of siblings per student.
H_0: $\lambda = 1.5$, H_1: $\lambda > 1.5$

b Since p-value $\approx 0.693 > 0.05$, there is insufficient evidence to support the claim that the mean number of siblings per student is greater than 1.5 on a 5% level of significance.

c $t \approx -0.566$, p-value ≈ 0.712
Since p-value > 0.05, there is insufficient evidence to support the claim that the mean number of siblings per student is greater than 1.5 on a 5% level of significance.

d Both tests give similar results and the same conclusion. However, because the *number of siblings* is a discrete variable and the sample size may not be sufficiently large enough for the Central Limit Theorem to apply, the test in **b** is more appropriate for Emily.

6 **a** **i** $\mathcal{C} = \{0, 1\}$ **ii** $\mathcal{A} = \{t \mid t \geqslant 2\}$ **iii** $c = 1$

b **i** $\mathcal{C} = \{t \mid t \geqslant 16\}$ **ii** $\mathcal{A} = \{0, 1, ..., 15\}$ **iii** $c = 16$

c **i** $\mathcal{C} = \{0, 1, 2, 3, 4, 5\}$ **ii** $\mathcal{A} = \{t \mid t \geqslant 6\}$ **iii** $c = 5$

7 critical region $\mathcal{C} = \{t \mid t \geqslant 26\}$
Since $t = 20 \notin \mathcal{C}$, we accept H_0 at a 5% level of significance.

8 $t = 3$ is the value in $\mathcal{C}$ with the largest p-value ≈ 0.0424. So the significance level $\alpha \geqslant 0.0424$ because all of the values in $\mathcal{C}$ must result in rejecting H_0. Similarly, $t = 4$ is the value in the acceptance region $\mathcal{A}$ with the smallest p-value ≈ 0.0996. So, the significance level $\alpha < 0.0996$ because all of the values in $\mathcal{A}$ must result in retaining H_0. So, $0.0424 \leqslant \alpha < 0.0996$.

EXERCISE 30H

1 **a** p = the probability the magician rolls a six with a fair die
H_0: $p = 0.9$, H_1: $p < 0.9$

b significance level ≈ 0.0159

2 **a** p = the probability of rolling a four with the die
H_0: $p = \frac{1}{6}$, H_1: $p > \frac{1}{6}$

b $\alpha = 0.05$

c Test statistic:
X = the number of fours rolled in 30 rolls of the die
Null distribution: $X \sim \text{B}(30, \frac{1}{6})$

d p-value ≈ 0.114 **e** no

3 **a** p = the probability of a head when the coin is flipped
H_0: $p = 0.5$, H_1: $p < 0.5$

b Test statistic:
X = the number of heads in 80 tosses of the coin
Null distribution: $X \sim \text{B}(80, 0.5)$

c p-value ≈ 0.288 **d** no

4 Since p-value $\approx 0.0259 < 0.05$, there is sufficient evidence to say that the new medication is better than the old on a 5% level of significance.

5 Since p-value $\approx 0.327 > 0.02$, there is insufficient evidence to support the consumer group's suspicions on a 2% significance level.

6 Since p-value $\approx 0.295 > 0.01$, there is insufficient evidence to support the claim that people's feelings about their future financial situation have improved on a 1% level of significance.

7 **a** **i** $\mathcal{C} = \{0\}$ **ii** $\mathcal{A} = \{1, 2, 3, 4, 5\}$ **iii** $c = 0$

b **i** $\mathcal{C} = \{7, 8\}$ **ii** $\mathcal{A} = \{0, 1, 2, 3, 4, 5, 6\}$ **iii** $c = 7$

c **i** $\mathcal{C} = \{0, 1, 2, 3, 4, 5\}$ **ii** $\mathcal{A} = \{6, 7, 8, 9, 10\}$ **iii** $c = 5$

8 **a** **i** critical region $= \varnothing$

ii acceptance region $= \{0, 1, 2, 3, 4\}$

b no **c** Toss the coin more times.

EXERCISE 30I

1 **a** H_0: $\rho = 0$ and H_1: $\rho > 0$

b test statistic ≈ 1.65, p-value ≈ 0.0804

c There is insufficient evidence to reject H_0 in favour of H_1 on a 5% significance level. The variables are not correlated.

2 p-value ≈ 0.0334

a yes; p-value < 0.05 **b** no; p-value > 0.01

3 Since p-value $\approx 1.53 \times 10^{-4} < 0.01$, there is sufficient evidence to support the researcher's claim at the 1% level.

4 Since p-value $\approx 0.0256 > 0.02$, we conclude that the data is not linearly correlated at the 2% level of significance.

5 **a** ≈ 2.29 **b** $\alpha = 0.05$ and $\alpha = 0.1$

c p-value ≈ 0.0477, which is less than 0.05 and 0.1.

EXERCISE 30J.1

1 **a** **i** $\mathcal{C} = \{z \mid z \geqslant 1.64\}$ **ii** $\mathcal{C} = \{z \mid z \geqslant 2.33\}$

b **i** ≈ 0.147 **ii** ≈ 0.356

2 **a** ≈ 0.285 **b** ≈ 0.400 **c** ≈ 0.9999

3 **a** H_0: $\mu = 3.5$, H_1: $\mu \neq 3.5$

b $\{z \mid z \leqslant -2.58$ or $z \geqslant 2.58\}$

c **i** ≈ 0.343 **ii** ≈ 0.657

4 **a** $\alpha \approx 0.006\,21$ **b** **i** ≈ 0.0228 **ii** ≈ 0.977

5 **a** ≈ 0.0455 **b** **i** ≈ 0.0228 **ii** ≈ 0.977

6 **a** **i**

α	0.1	0.05	0.01	0.001
β	$\approx 0.009\,26$	≈ 0.0207	≈ 0.0772	≈ 0.239

ii β increases as α decreases.

b **i**

True value of μ	β
± 1	≈ 0.484
± 2	≈ 0.0207
± 5	≈ 0
± 10	≈ 0

ii As the true value of μ gets further and further away from 0, the value assumed under H_0,
$\beta = \text{P(Type II error)} \to 0$. So the power of the test $1 - \beta = \text{P(correctly rejecting } H_0) \to 1$.
That is, as μ gets further away from the null value, the more likely it is that a genuine difference will be detected.

EXERCISE 30J.2

1 **a** H_0: $p = \frac{1}{4}$, H_1: $p > \frac{1}{4}$

b **i** A Type I error is rejecting H_0 when it is true. This means deciding it is biased when it is in fact fair.

ii P(Type I error) ≈ 0.0496, significance level $\approx 4.96\%$

c ≈ 0.146

2 **a** H_0: $p = 0.5$, H_1: $p > 0.5$

b **i** critical region

ii P(Type I error) ≈ 0.0193, significance level $\approx 1.93\%$

c **i** P(Type II error) ≈ 0.917, power ≈ 0.0834

ii Type II error

3 **a** ≈ 0.0263 or $\approx 2.63\%$ **b** ≈ 0.391

4 **a** $\{t \mid t \geqslant 15\}$ **b** ≈ 0.275

REVIEW SET 30A

1 H_0: $\mu = 0$ {the bus arrives on time}
H_1: $\mu > 0$ {the bus is late}

2 **a** There is 7.94% chance of observing this result if the null hypothesis is true.

b For a 10% significance level, we reject H_0 if there is less than 10% chance of observing this result.

c As the p-value $< 0.1 = \alpha$, we reject H_0 in favour of H_1.

d Type II error

3 $\mathcal{C} = \{z \mid z \leqslant -2.33$ or $z \geqslant 2.33\}$
The test statistic $z \approx -0.685$ does not lie in the critical region. There is insufficient evidence to reject the manufacturer's claim at the 2% level of significance.

4 £428 000 $< \overline{x} <$ £448 000

5 **a** H_0: $\mu = 90$ and H_1: $\mu < 90$ **b** one-sample t-test

c critical region $\mathcal{C} = \{t \mid t \leqslant -2.54\}$
Since $t \approx -3.59 \in \mathcal{C}$, we conclude that Rosario's concerns are justified at a 1% level of significance.

6 **a** Let μ_1 be the population mean number of fish caught by Joe per fishing trip, and
μ_2 be the population mean number of fish caught by Ruben per fishing trip.
H_0: $\mu_1 = \mu_2$ and H_1: $\mu_1 > \mu_2$

b $t \approx 0.644$, p-value ≈ 0.263
Since p-value > 0.05, there is insufficient evidence to support Joe's claim at a 5% level of significance.

7 **a** two sample t-test; the data is not paired *or* there are two different samples of customers.

b $t \approx -1.80$, p-value ≈ 0.0863
Since p-value < 0.1, there is sufficient evidence to support the claim that there is a significant difference between the time spent shopping by customers at the supermarkets at a 10% level.

8 $t \approx -3.06$, p-value $\approx 0.006\,77$
Since p-value < 0.05, we reject the claim that the fuel economy of E10 fuel is just as good as regular unleaded at a 5% level of significance.

9 Since p-value $\approx 0.0302 < 0.05$, there is sufficient evidence to conclude that the new exercises are better at a 5% level of significance.

10 Since p-value $\approx 0.177 > 0.01$, there is insufficient evidence to support hiring a new receptionist at a 1% level of significance.

11 Since p-value $\approx 0.113 > 0.05$, we conclude that the *height* and *weight* are not linearly correlated at a 5% level of significance.

12 **a** $\{z \mid z \leqslant -1.94$ or $z \geqslant 1.94\}$ **c** ≈ 0.651

13 **a** **i** $\{x \mid x \geqslant 13\}$ **ii** $\{x \mid x \leqslant 12\}$

b ≈ 0.0253 or $\approx 2.53\%$ **c** ≈ 0.910

REVIEW SET 30B

1 **a** A Type I error would occur if we concluded that the seafood company is supplying undersize fish when they are in fact not.

b A Type II error would occur if we concluded that the seafood company is not supplying undersize fish when they actually are.

2 $t \approx -11.2$, p-value ≈ 0
Since p-value $\approx 0 < 0.05$, there is sufficient evidence to conclude that Yarni's resting pulse rate has decreased at a 5% level of significance.

3 $z \approx -1.02$, p-value ≈ 0.307
Since p-value > 0.05, there is insufficient evidence to suggest that the mean weight has changed from 2011 to 2012 at a 5% level of significance.

4 $t \approx 1.95$, p-value ≈ 0.0295
Since p-value < 0.05, there is sufficient evidence to justify the company's concerns at a 5% level of significance.

5 $t \approx 1.40$, p-value ≈ 0.0858
Since p-value > 0.05, there is insufficient evidence to support the claim that Arthur has improved at the 5% level of significance.

6 $t \approx 1.18$, p-value ≈ 0.242
Since p-value > 0.1, we conclude that there is not a significant difference between the average points totals of the two suburbs at a 10% level of significance.

7 **a**

	Mean	*Standard deviation*
Revision course	$\overline{x}_1 \approx 34.6$	$s_1 \approx 2.82$
No revision course	$\overline{x}_2 = 32.1$	$s_2 \approx 4.48$

b $t \approx 1.71$, p-value ≈ 0.0508,
Since p-value < 0.1, there is sufficient evidence to support the claim that the revision course improved results at a 10% level of significance.

8 **a** paired t-test; Josie only has 1 sample of individuals, but 2 measurements for each individual, so the data is paired.

b $t \approx -4.89$, p-value $\approx 0.000\,118$
Since p-value < 0.1, there is sufficient evidence to support Josie's claim that yelling helps people to withstand pain at a 10% level of significance.

c
- Each *Silent* measurement is independent of every other *Silent* measurement. Similarly for the *Yelling* measurements.
- The order in which the *Silent* and *Yelling* measurements were taken does not matter.

9 Since p-value $\approx 0.0447 < 0.05$, there is sufficient evidence to conclude that the die is biased towards 6 at a 5% significance level.

10 **a** $p =$ the probability that the train is behind schedule
H_0: $p = 0.3$, H_1: $p < 0.3$

b Test statistic:
$X =$ the number of times the train is late in the 10 trial runs
Null distribution: $X \sim \text{B}(10, 0.3)$

c critical region $= \{0\}$ **d** no

11 Since p-value $\approx 1.22 \times 10^{-4} < 0.01$, the variables are negatively correlated at the 1% level of significance.

12 **a** $\{z \mid z \leqslant -1.64\}$ **b** ≈ 0.639 **c** $\mu \approx 35.5$

13 **a** $\{t \mid t \leqslant 33\}$ **b** ≈ 0.0383 **c** ≈ 0.738

EXERCISE 31A

1 **a** Let p_1 be the population proportion of "heads", and p_2 be the population proportion of "tails".
H_0: $p_1 = 0.5$, $p_2 = 0.5$
H_1: $p_1 \neq 0.5$ and $p_2 \neq 0.5$

b 42 **c** 48 heads, 48 tails **d** $\chi^2_{\text{calc}} = 1.5$

e 1 **f** ≈ 0.221

g Since $p > 0.05 = \alpha$, there is insufficient evidence to suggest that the coin is biased at a 5% significance level.

2 $\chi^2_{\text{calc}} \approx 5.97$, df $= 2$, p-value ≈ 0.0505
Since p-value > 0.01, there is insufficient evidence to suggest that there has been a change in the proportion of voters supporting each party at a 1% significance level.

3 $\chi^2_{\text{calc}} \approx 10.3$, df $= 4$, p-value ≈ 0.0351
Since p-value < 0.1, there is sufficient evidence that the proportions of each ice cream flavour sold are not all the same at a 10% significance level. Brian should change the amounts of each ice cream flavour that he makes.

4 $\chi^2_{\text{calc}} \approx 5.70 \times 10^5$, df $= 4$, p-value ≈ 0
Since p-value < 0.05, there is sufficient evidence to support the claim that there was a significant change in London's demographics between 2001 and 2011 at a 5% significance level.

5 **a**

Band	10	9	8	7	6	5 and below
Expected frequency	11.85	25.05	44.7	44.55	20.25	3.6

b $\chi^2_{\text{calc}} \approx 19.1$, df $= 5$, p-value $\approx 0.001\,81$
Since p-value < 0.01, there is sufficient evidence to support the claim that there is a substantial difference between the school's results and the rest of the nation at a 1% significance level.

c Since the expected frequency for "Band 5 and below" is less than 5, the sample size is not large enough for χ^2 to be distributed appropriately. By combining "Band 6" and "Band 5 and below" we can obtain more reliable results.

d $\chi^2_{\text{calc}} \approx 18.9$, df $= 4$, p-value $\approx 0.000\,807$
Since p-value < 0.01, there is still sufficient evidence to support the claim that there is a substantial difference between the school's results and the rest of the nation at a 1% significance level. Although we have obtained the same result both times, the result is more reliable now that each of the expected frequencies is sufficiently large.

6 **a** $X \sim \text{B}(10, \frac{1}{4})$

b It is unlikely that a person will have to guess the answer to *every* question. Some questions are more likely to be answered correctly than others. So the statistician's model is not completely appropriate.

c Using the p-value of a χ^2 goodness of fit test.
$\chi^2_{\text{calc}} \approx 8.89$, df $= 5$, p-value ≈ 0.114
Since p-value > 0.1, we conclude that the statistician's model is appropriate at a 10% significance level.

7 $\chi^2_{\text{calc}} \approx 7.60$, df $= 4$, p-value ≈ 0.107
Since p-value > 0.01, there is insufficient evidence to reject H_0 at a 1% significance level. We conclude that the data is from a Poisson distribution with mean 5.

EXERCISE 31B

1 $\chi^2_{\text{calc}} \approx 135$, df $= 5$, p-value $\approx 2.43 \times 10^{-27}$
Since p-value < 0.05, we conclude that the data is not binomially distributed at the 5% level of significance.

2 **a** ≈ 0.885 accidents per week

b $\chi^2_{\text{calc}} \approx 11.5$, df $= 2$, p-value ≈ 0.0325
Since p-value < 0.05, we conclude that the data does not come from a Poisson distribution at a 5% significance level.

3 **a** $\sigma \approx 3.35$

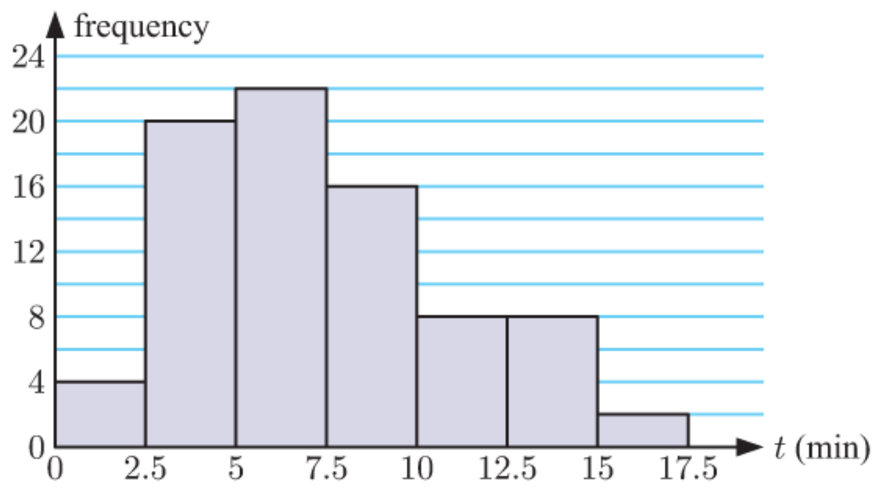
b $\chi^2_{\text{calc}} \approx 61.7, \quad \text{df} = 4, \quad p\text{-value} \approx 1.26 \times 10^{-12}$

Since p-value < 0.05, we conclude that the data is not from a normal distribution with mean 5.

4 **a**

Time (t min)	*Frequency*
$0 \leqslant t < 2.5$	4
$2.5 \leqslant t < 5$	20
$5 \leqslant t < 7.5$	22
$7.5 \leqslant t < 10$	16
$10 \leqslant t < 12.5$	8
$12.5 \leqslant t < 15$	8
$15 \leqslant t < 17.5$	2
Total	80

b



c slightly positively skewed

d $\mu \approx 7.325$ minutes, $\sigma \approx 3.64$ minutes

e $\chi^2_{\text{calc}} \approx 9.43, \quad \text{df} = 4, \quad p\text{-value} \approx 0.0512$

Since p-value > 0.01, there is insufficient evidence to reject H_0 at a 1% level of significance. We conclude that the data is normally distributed.

EXERCISE 31C

1 **a** $\chi^2_{\text{crit}} = 9.49$

b As $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, we reject H_0 in favour of H_1.

2 **a** Let p_1, p_2, p_3, and p_4 be the population proportions of red, yellow, green, and blue lollies respectively.

H_0: $p_1 = \frac{1}{4}, \ p_2 = \frac{1}{4}, \ \ldots, \ p_4 = \frac{1}{4}$

H_1: at least one of $p_1, p_2, p_3, p_4 \neq \frac{1}{4}$

b $\chi^2_{\text{calc}} \approx 2.02$ **c** $\chi^2_{\text{crit}} = 6.25$

d As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, there is insufficient evidence to reject H_0.

e As p-value $\approx 0.569 > 0.1 = \alpha$, there is insufficient evidence to reject H_0 at a 10% significance level.

3 As $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$ ($\chi^2_{\text{calc}} \approx 28.8, \ \chi^2_{\text{crit}} = 13.28$), there is sufficient evidence to support the claim that the ISP's changes were effective.

4 As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$ ($\chi^2_{\text{calc}} \approx 3.54, \ \chi^2_{\text{crit}} = 9.24$), there is insufficient evidence to reject H_0. The data is Poisson with rate $\lambda = 2.2$ at a 10% significance level.

5 As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$ ($\chi^2_{\text{calc}} \approx 4.23, \ \chi^2_{\text{crit}} = 5.99$), there is insufficient evidence to reject H_0. The data is binomially distributed at a 5% significance level.

EXERCISE 31D.1

1 **a**

	Junior school	*Middle school*	*High school*
Plays sport	38.28	56.76	69.96
Does not play sport	19.72	29.24	36.04

b

	Drove to work	*Cycled to work*	*Public transport*
Male	25.3	7.7	11
Female	20.7	6.3	9

c

	Wore hat and sunscreen	*Wore hat or sunscreen*	*Wore neither*
Sunburnt	10.92	6.16	3.92
Not sunburnt	28.08	15.84	10.08

2 **a**

	Pass Maths test	*Fail Maths test*
Male	30	20
Female	30	20

b In a sample of 100 students, we would expect 30 to be male and pass the Maths test.

c

f_{obs}	f_{exp}	$f_{\text{obs}} - f_{\text{exp}}$	$(f_{\text{obs}} - f_{\text{exp}})^2$	$\frac{(f_{\text{obs}} - f_{\text{exp}})^2}{f_{\text{exp}}}$
24	30	-6	36	1.2
26	20	6	36	1.8
36	30	6	36	1.2
14	20	-6	36	1.8
			Total	6

$\chi^2_{\text{calc}} = 6$

3 $\chi^2_{\text{calc}} \approx 6.61, \quad \text{df} = 2, \quad p\text{-value} \approx 0.0368$

Since p-value < 0.05, we reject H_0, and conclude that the variables *weight* and *suffering diabetes* are dependent.

4 **a** $\chi^2_{\text{crit}} = 4.61$

b $\chi^2_{\text{calc}} \approx 8.58$

Since $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, we reject H_0. So at a 10% level, we conclude that *age* and the *party they wish to vote for* are dependent.

5 **a** $\chi^2_{\text{calc}} \approx 23.6, \quad \text{df} = 3, \quad \chi^2_{\text{crit}} = 7.81$

Since $\chi^2_{\text{calc}} > \chi^2_{\text{crit}}$, we reject H_0. So at a 5% level, *reason for travelling* and *rating* are dependent.

b Guests travelling for a holiday are more likely to give a higher rating.

6 $\chi^2_{\text{calc}} \approx 7.94, \quad \text{df} = 6, \quad p\text{-value} \approx 0.242$

Since p-value > 0.1, there is insufficient evidence to reject H_0. So at a 10% level, *position* and *injury type* are independent.

7 **a**

Age	*Owns a pet?* Yes	No
0 - 19	≈ 4.02	≈ 3.98
20 - 29	≈ 27.1	≈ 26.9
30 - 49	≈ 50.2	≈ 49.8
50+	≈ 36.7	≈ 36.3

b Yes, 4.02 and 3.98.

c

Age	*Owns a pet?* Yes	No
0 - 29	37	25
30 - 49	42	58
50+	39	34

d $\chi^2_{\text{calc}} \approx 5.22, \quad \text{df} = 2, \quad p\text{-value} \approx 0.0735$

Since p-value > 0.05, we do not reject H_0. So at a 5% level, *age* and *owning a pet* are independent.

8 **a**

		Intelligence level			
		Low	*Average*	*High*	*Very high*
Smoking habits	*Non smoker*	≈ 262	≈ 383	≈ 114	≈ 4.69
	Medium level smoker	≈ 133	≈ 194	≈ 57.7	≈ 2.38
	Heavy smoker	≈ 107	≈ 157	≈ 46.6	≈ 1.93

b $\chi^2_{\text{calc}} \approx 16.9$, $\text{df} = 6$, $p\text{-value} \approx 0.009\,59$

Since $p\text{-value} < 0.01$, we reject H_0. So at a 1% level, we conclude that *intelligence level* and *smoking habits* are not independent.

c

		Intelligence level			
		Low	*Average*	*High/Very high*	*Sum*
Smoking habits	*Non smoker*	279	386	98	763
	Medium level smoker	123	201	63	387
	Heavy smoker	100	147	66	313
	Sum	502	734	227	1463

d $\chi^2_{\text{calc}} \approx 13.2$, $\text{df} = 4$, $p\text{-value} = 0.0104$

Since $p\text{-value} > 0.01$, we do not reject H_0. So at a 1% level, *intelligence level* and *smoking habits* are independent. This is a different conclusion from the one in **b**.

EXERCISE 31D.2

1 **a**

		Result	
		Heads	*Tails*
Guess	*Heads*	49.4	54.6
	Tails	45.6	50.4

b $\chi^2_{\text{calc}} \approx 1.35$

c As $\chi^2_{\text{calc}} < 3.84$, there is insufficient evidence to reject H_0. So at a 5% level, Horace's *guess* and *result* are independent.

d According to this test, Horace's claim is not valid.

2 **a**

		Result	
		Pass	*Fail*
Country	*France*	63.8	21.2
	Germany	168.2	55.8

b $\chi^2_{\text{crit}} = 2.71$

c $\chi^2_{\text{calc}} \approx 4.62$

d As $\chi^2_{\text{calc}} > 2.71$, we reject H_0. So at a 10% level, *motorbike test result* and *country* are dependent.

REVIEW SET 31A

1 **a** 12.59

b As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, there is insufficient evidence to reject H_0.

2 **a** Let p_1, p_2, p_3, p_4, and p_5 be the population proportions of shirts which are small, medium, large, X-large, and XX-large respectively.

i H_0: $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.35$, $p_4 = 0.25$, $p_5 = 0.1$

ii $\text{df} = 4$

b ≈ 0.0134

c Yes, the p-value < 0.05, so we reject H_0 in favour of H_1 on a 5% level of significance. We conclude that the store should change its distribution.

3 $\chi^2_{\text{calc}} \approx 42.1$, $\text{df} = 2$, $p\text{-value} = 7.37 \times 10^{-10}$

As $p\text{-value} < 0.1$, we reject H_0 at a 10% significance level. The variables *age of a driver* and their *opinion* are dependent.

4 **a**

Item rarity	super rare	rare	uncommon	common
Expected frequency	12.5	25	62.5	150

b $\chi^2_{\text{crit}} = 11.34$, $\text{df} = 3$, $\chi^2_{\text{calc}} \approx 10.0$

As $\chi^2_{\text{calc}} < \chi^2_{\text{crit}}$, there is insufficient evidence to reject H_0 at a 1% significance level. Emmanuel's suspicions are not justified.

5 **a** 4.02 biros per packet **b** ≈ 0.67

c $\chi^2_{\text{calc}} \approx 2.71$, $\text{df} = 3$, $p\text{-value} \approx 0.438$

Since $p\text{-value} > 0.1$, there is insufficient evidence to reject H_0 at a 10% significance level. We conclude that the number of red biros is binomially distributed.

6 $\chi^2_{\text{calc}} \approx 27.6$, $\text{df} = 9$, $p\text{-value} \approx 0.001\,12$

Since $p\text{-value} < 0.05$, we reject H_0. So at a 1% level, *social media* and *exercise* are related.

REVIEW SET 31B

1 $\chi^2_{\text{calc}} = 5.215$, $\text{df} = 3$, $p\text{-value} \approx 0.157$

Since $p\text{-value} > 0.05$, there is insufficient evidence to reject H_0. We accept the manufacturer's claim.

2 **a** **i** $\text{P}(X = x) = \underbrace{0.5 \times 0.5 \times \ldots \times 0.5}_{x-1 \text{ heads}} \times \underbrace{0.5}_{1 \text{ tail}}$

$= (0.5)^x$

ii

Number of tosses required	*Frequency*
1	50
2	25
3	12.5
4	6.25
5	3.125
6	1.5625
7	0.781 25
$\geqslant 8$	0.781 25

b $\chi^2_{\text{calc}} \approx 11.4$, $\text{df} = 4$, $p\text{-value} \approx 0.0220$

Since $p\text{-value} < 0.05$, we conclude that the distribution is not a suitable model for the data.

3 **a**

Score (x)	*Observed frequencies*	*Expected frequencies*
$x < 70.5$	10	3
$70.5 \leqslant x < 80.5$	45	48
$80.5 \leqslant x < 90.5$	287	291
$90.5 \leqslant x < 100.5$	641	698
$100.5 \leqslant x < 110.5$	725	667
$110.5 \leqslant x < 120.5$	250	253
$120.5 \leqslant x < 130.5$	40	38
$x \geqslant 130.5$	2	2

b $\chi^2_{\text{calc}} \approx 10.2$, $\text{df} = 5$, $p\text{-value} \approx 0.0697$

Since $p\text{-value} > 0.05$, there is insufficient evidence to reject H_0. We conclude that the data is from a normal distribution with mean 100 and variance 100.

4 $\chi^2_{\text{calc}} \approx 13.0$, $\text{df} = 6$, $p\text{-value} \approx 0.0433$

a Since $p\text{-value} < 0.05$, we reject H_0. So, at a 5% level, P and Q are dependent.

b Since $p\text{-value} > 0.01$, we do not reject H_0. So, at a 1% level, P and Q are independent.

5 $\chi^2_{\text{calc}} \approx 6.06$, $\text{df} = 2$, $p\text{-value} \approx 0.0485$

Since $p\text{-value} < 0.05$, we conclude that the data is not normally distributed at a 5% level of significance.

6 $\chi^2_{\text{calc}} \approx 25.6$

Since $\chi^2_{\text{calc}} > 21.67$, we reject H_0. So at a 1% level, *education level* and *business success* are dependent.

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